Born Rule as Entropic Unfolding — A First-Principles Derivation

Relationship to the Broader VERSF-RAL Framework

Context: This paper is part of the Void Energy-Regulated Space Framework (VERSF) and Resonant Assembly Language (RAL) program. It sits alongside a companion document ("The Pre-Entropic and Entropic Domains") that develops the full framework encompassing measurement, time emergence, gravity, and cosmology.

Scope Comparison:

This Paper (Born Rule as Entropic Unfolding):

- **Narrow focus**: Rigorous derivation of Born rule probabilities
- **Two complementary approaches**: (1) MaxCal with entropic costs, (2) Pure symmetry (Gleason)
- **Key result**: $P_i \propto |c_i|^2 e^{(-\lambda\Delta S_i)}$, reducing to Born rule when ΔS_i = const
- **Emphasis**: Mathematical rigor, uniqueness proofs, testable deviations
- **Status**: Theorem-level results with clear experimental predictions

Companion Paper (Pre-Entropic and Entropic Domains):

- **Broad scope**: Unified framework for quantum mechanics, measurement, time, gravity
- **Core mechanism**: Alignment functional $\mathcal{A}$ controlling boundary coupling $\Gamma(\mathcal{A})$
- **Key results**: Born rule from flux conservation, measurement as phase transition at $\mathcal{A} = \mathcal{A}_c$, gravity from entropy gradients
- **Emphasis**: Conceptual unification, explanatory power, paradigm shift
- **Status**: Framework-level with multiple research frontiers

Complementary Roles:

The companion paper establishes the *foundational picture*: reality has pre-entropic (timeless, reversible) and entropic (temporal, irreversible) domains, with measurement occurring when alignment $\mathcal{A}$ crosses a critical threshold $\mathcal{A}_c$, triggering entropy flow and time emergence. This provides the physical motivation for why measurement involves entropy costs.

This paper provides *mathematical precision* for one specific claim: that Born rule probabilities emerge from energy conservation at the measurement boundary, with entropy costs ΔS_i producing calculable deviations. The MaxCal derivation (Sections 1-

13) shows how to compute these deviations, while the symmetry derivation (Section 14) anchors the $|c_i|^2$ form on purely kinematic grounds.

Unified Picture:

1. **Pre-measurement** ($\mathcal{A} < \mathcal{A}_c$): System exists in pre-entropic domain with amplitudes $c_i = \sqrt{a_i}\, e^{i\varphi_i}$
2. **Alignment buildup**: Phase relationships strengthen, $\mathcal{A}$ increases toward $\mathcal{A}_c$
3. **Critical threshold**: $\mathcal{A} \to \mathcal{A}_c$ triggers boundary coupling $\Gamma(\mathcal{A}) > 0$
4. **Entropic unfolding**: Each outcome i requires entropy export ΔS_i to stabilize
5. **Probability assignment**: $P_i = (a_i e^{-\lambda\Delta S_i})/(\Sigma_j a_j e^{-\lambda\Delta S_j})$ from MaxCal
6. **Born rule recovery**: When apparatus achieves iso-entropic design (ΔS_i = const), $P_i = a_i = |c_i|^2$

Key Distinction:

- The companion paper asks: *What is measurement?* Answer: A phase transition from pre-entropic to entropic domains
- This paper asks: *What are measurement probabilities?* Answer: Gibbs-weighted flux conservation, reducing to $|c_i|^2$ in the iso-entropic limit

For Readers:

- *Start with companion paper* for conceptual framework, physical intuition, and broad scope
- *Read this paper* for rigorous probability derivation, uniqueness theorems, and experimental protocols
- Together they demonstrate how quantum mechanics, thermodynamics, and probability theory unite at the measurement boundary

Critical Difference in Approach:

The companion paper treats alignment $\mathcal{A}$ as the *primary* variable controlling measurement via $\Gamma(\mathcal{A})$. This paper treats alignment readiness $a_i = |c_i|^2$ as a *geometric* quantity (control-theoretic reachability) and derives how entropy costs ΔS_i modulate the resulting probabilities. Both are consistent: $\mathcal{A}$ controls *when* measurement happens (threshold crossing), while a_i and ΔS_i control *which* outcome emerges and with what probability.

Abstract

We derive a thermodynamically-generalized probability law for quantum measurement that reduces to the Born rule in the iso-entropic limit. Beginning from minimal assumptions—Hilbert structure for reversible evolution, conservation of global entropy balance, and maximum caliber for outcome paths under an unfolding action—we show that the probability of outcome *i* takes the Gibbs-biased form $P_i \propto |c_i|^2 e^{(-\lambda\Delta S_i)}$. When measurement processes export equal entropy across all outcomes (ΔS_i = const), the standard Born rule $P_i = |c_i|^2$ is recovered.

Intuitively: Amplitudes ($|c_i|^2$) tell you how ready each outcome is geometrically. Real detectors must also dump heat to make a result stick in time. When the heat cost is the same for every outcome, you get the usual Born rule. If one outcome is cheaper to make real—because the detector for it wastes less energy—it wins slightly more often. That tiny, testable tilt is what we model.

We prove that the action functional $M_i = -\ln a_i + \lambda\Delta S_i$ is uniquely determined by four operational requirements: independent composition, operational stability, gauge neutrality, and thermodynamic extensivity.

How to Read This Paper:

This paper is written with multiple audiences in mind:

- **Expert physicists/mathematicians:** Read the main technical content. Skip the "Plain Language" sections—they're for others.
- **Physics graduate students:** Read everything. The technical sections give you the rigor; the plain language sections give you the intuition.
- **Interested non-specialists:** Focus on the Abstract, "Plain Language" sections (marked with horizontal rules), and the final Plain Language Summary. You can skim or skip the heavy mathematics.
- **Skeptical reviewers:** We've made every assumption explicit, every proof rigorous, and every claim falsifiable. The plain language sections don't water down the math—they explain *why* the math has the form it does.

The paper proceeds in two arcs: Sections 1-13 develop the entropic-unfolding derivation with testable predictions, while Section 14 returns to pure symmetry principles to show Born rule emerges from geometry alone.

Plain Language Sections appear after technical content, marked by horizontal rules (---). Key locations:

- After Section 3 intro: What are geometric vs thermodynamic costs?
- After Section 3.1: Why these four physical requirements?
- After Theorem 3.0: What did we just prove?
- After Section 3.3: Understanding the core probability formula
- After Section 4: What the small-bias expansion means
- After Section 11: The formal results translated
- Section 14 intro: The symmetry route explained
- After Section 14.2: Why probabilities must be quadratic
- After Section 14.7: How the two derivations connect
- After Section 13: Plain Language Summary of the entire paper We also derive a control-theoretic ceiling on alignment readiness that naturally explains practical limitations in finite-bandwidth control, and outline experimental protocols to test thermodynamic deviations from Born statistics.

The paper proceeds in two complementary arcs: **Section 14** establishes that the Born rule follows from pure symmetry and conservation principles (Gleason's theorem), providing the foundational quadratic core. **Sections 1–13** then develop the entropic-unfolding framework as a thermodynamic refinement of this core, modeling how real measurements with non-ideal entropy costs can produce testable deviations. Together, these demonstrate that the Born rule is both fundamentally grounded in symmetry and practically subject to thermodynamic corrections in realistic apparatus.

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1. Minimal Axioms and Physical Commitments

We adopt four minimal axioms consistent with quantum theory and the VERSF framework:

A1 (Reversible Kinematics): Between observations, evolution is reversible and represented by a unitary group U(t) on a complex Hilbert space $\mathscr{H}$.

A2 (Outcome Basis): A measurement context defines an orthonormal basis $\{|i\rangle\}$ in which macroscopic records are stable.

A3 (Unfolding Cost): Each candidate outcome *i* requires exporting entropy $\Delta S_i \geq 0$ to stabilize as a temporal record (void-symmetric → time-embedded).

A4 (Maximum Caliber — Inference Principle): Realized outcome frequencies $\{P_i\}$ maximize path entropy subject to normalization and a fixed expected unfolding action M_i.

Axioms A1–A2 encode standard quantum kinematics. A3 expresses the VERSF claim that time-embedding demands entropy export. A4 selects outcome statistics via rational inference when microscopic measurement dynamics are not fully specified—this is an **inference principle** (the dynamical analogue of Jaynes' maximum-entropy reasoning), not a teleological physical law.

VERSF-RAL Primer. In the VERSF view, configurations reside in a void-symmetric sector (fully reversible, zero net entropy production). A measurement is the unfolding of one branch into time-embedded dynamics, which requires entropy export to stabilize macroscopic records. RAL (Resonant Assembly Language) is a bookkeeping layer treating amplitudes and detector couplings as alignment resources under control constraints. The entropic framework developed in §1–13 refines (not replaces) the symmetry-only Born core formalized in §14.

Note on Maximum Caliber (A4). The use of maximum-caliber in A4 should not be read as a claim that nature literally maximizes path entropy. It is an inference principle used by observers to assign probabilities when only partial constraints (normalization and expected action) are known. If the microscopic dynamics were known in full, no maximization would be required. Thus A4 is a rational inference rule for assigning probabilities given incomplete information, not a fundamental dynamical law.

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2. Kinematic Setup and Alignment Readiness

Let the pre-measurement state be $|\psi\rangle = \sum_i c_i |i\rangle$ with $a_i := |c_i|^2$ and $\sum_i a_i = 1$. We interpret a_i as **alignment readiness**: a geometric measure of how prepared branch *i* is to unfold coherently in the given context.

Critical Note on $a_i = |c_i|^2$. We treat $a_i = |c_i|^2$ as a geometric overlap—not a probability—available from Hilbert kinematics (state tomography), independent of the Born rule. This is the transition amplitude squared, measurable via repeated state preparation and basis projections, without invoking any probability interpretation. The derivation below shows that *probabilities equal these geometric overlaps* under specific physical conditions.

Physical Motivation for a_i. Let H(t) be the bounded control Hamiltonian implementing the pre-measurement rotation toward the record basis $\{|i\rangle\}$. For a two-level slice, the reachable overlap after time τ with $\|H\| \leq \Omega_{max}$ obeys the quantum speed-limit bound $\theta \leq 2\Omega_{max} \tau$, giving $a_r = \cos^2(\theta/2)$. Thus a_i quantifies geometric reachability of outcome *i* under finite control—equivalently, the Fisher-geometry overlap the controller can establish before readout. The "64% cap" discussed in §6 arises when θ saturates the available bandwidth-time product. Thermodynamics then adds a separate bias via ΔS_i.

Notation:

$$|\psi\rangle = \sum_i c_i |i\rangle,\ a_i := |c_i|^2,\ \sum_i a_i = 1 \quad (2.1)$$

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3. Maximum-Caliber Refinement on a Quadratic Core

Plain Language - What Are We Doing Here?

Imagine you're designing a measurement device. Each possible measurement outcome *i* has two "costs":

1. **Geometric cost** ($-\ln a_i$): How hard is it to prepare the quantum state so outcome *i* is "ready" to happen? This is like aiming—some targets are easier to line up than others.
2. **Thermodynamic cost** (ΔS_i): How much heat must the detector dump to "lock in" outcome *i* as a permanent record? This is like developing a photograph—some images require more chemical reactions than others.

The total "effort" or "action" M_i combines both costs. We're about to prove this is the *only* way to combine them that makes physical sense.

Define the unfolding action for outcome *i* as the sum of a geometric alignment penalty and a thermodynamic cost. We choose the unique additive, convex form

$M_i := -\ln a_i + \lambda\Delta S_i, \lambda > 0$ (3.1)

Domain and Regularity. We restrict to $a_i \in (0,1)$ with $\sum_i a_i = 1$. Boundary cases $a_i \to 0,1$ are treated by continuity; convexity of $-\ln a_i$ ensures regular behavior. The functional M_i is well-defined on the interior of the probability simplex and extends continuously to the closure.

where the first term encodes control-theoretic preparation cost and the second encodes entropy export required for record stabilization.

Units and Physical Interpretation. We work with physical entropy ΔS_i in J/K. To keep the Gibbs exponent dimensionless, write the bias as $e^{(-\beta W_i)}$ with W_i the outcome-conditioned dissipated work and $\beta = 1/(k_B T_{eff})$. Using Landauer's bound $W_i \geq T_{eff} \Delta S_i$, the minimal model takes $W_i = T_{eff} \Delta S_i$, giving $e^{(-\beta W_i)} = e^{(-\Delta S_i/k_B)}$. Thus the coupling may be written either as $\lambda = 1/k_B$ or equivalently $\lambda = \beta = (k_B T_{eff})^{(-1)}$, depending on whether we emphasize the entropy form or work form. Eq. (3.2) can be written either as

$P_i \propto a_i e^{(-\Delta S_i/k_B)}$ (entropy form)

or equivalently

$P_i \propto a_i e^{(-\beta W_i)}$ (work form).

Convention: We set $k_B = 1$ except where dimensions are shown explicitly. This allows us to write $e^{(-\Delta S_i)}$ rather than $e^{(-\Delta S_i/k_B)}$ throughout, with the understanding that ΔS_i is measured in units of k_B (nat or bit units for information-theoretic contexts).

Plain Language - The Two Forms:

The "entropy form" $e^{(-\Delta S_i/k_B)}$ emphasizes heat flow: how many joules of heat per kelvin must leave the system. The "work form" $e^{(-\beta W_i)}$ emphasizes energy dissipation: how many joules of irreversible work must be performed. Landauer's principle says these are the same (at minimum): you can't create a permanent information record without dumping heat. The parameter $\beta = 1/(k_B T_{eff})$ is just inverse temperature—it tells you how "expensive" heat dumping is. At low temperatures (high β), even small entropy costs matter a lot. At high temperatures (low β), you can dump heat cheaply.

3.1 Physical Postulates Underlying the Action Form

We motivate the functional form (3.1) with four operational requirements:

P1 (Independent composition ⇒ multiplicative readiness). If two subsystems are independently prepared for the same measurement context, the joint alignment readiness factors: $a_{ij} = a_i a_j$. This is the standard tensor-product rule for overlaps $|\langle\psi\otimes\varphi|i\otimes j\rangle|^2 = |\langle\psi|i\rangle|^2|\langle\varphi|j\rangle|^2$ under independence, and it is the same multiplicativity used in log-likelihood and information addition laws. To keep "effort" extensive under product composition, the geometric part of the action must be additive on products, hence a Cauchy-type equation for $F(a)$. **In brief:** Independent prep ⇒ multiplicative overlaps; lab reproducibility ⇒ continuous, order-preserving effort ⇒ $F(ab) = F(a) + F(b)$ with regularity ⇒ $-\ln a$.

P2 (Operational stability ⇒ regularity). Small changes in state preparation should not cause discontinuous jumps in cost or preference. This is the lab-level requirement that M be continuous, monotone, and bounded on compact readiness intervals; otherwise arbitrarily tiny control noise could flip outcome rankings and violate reproducibility. Pathological (non-measurable) solutions of the Cauchy equation are therefore excluded on physical—not merely mathematical—grounds.

P3 (Gauge neutrality in the iso-entropic limit). If all branches export the same entropy (ΔS_i = const), the thermodynamic part is an additive gauge that cancels in the Gibbs normalizer. **Operationally:** if we engineer measurement apparatus A and B such that $\Delta S_i^{(A)} = \Delta S_i^{(B)}$ = const for all outcomes, Born frequencies must be identical for A and B. Any residual bias would be empirically detectable as a relabeling asymmetry under equalized erasure work. Thus neutrality is not an extra assumption—it is the statement that adding the same constant heat-to-temperature to every branch cannot change relative frequencies.

P4 (Thermodynamic extensivity). When outcome channels are aggregated (coarse-grained), entropy costs add: $\Delta S_{I\cup J} = \Delta S_I + \Delta S_J$ (assuming *I* and *J* are macroscopically distinguishable channels; quantum interference between branches is suppressed by decoherence). Linearity in ΔS then follows from the same extensivity and bounded-regularity logic (Cauchy on $\mathbb{R}_+$ under mild regularity) we use for $F(a)$. **Interference caveat:** If channels are not macroscopically distinct, interference corrections may render ΔS non-additive; our coarse-grained, decohered regime excludes that case.

Theorem 3.0 (Uniqueness of the Unfolding Action). Suppose an outcome "effort" functional M satisfies: (i) Extensivity on products: $M(ab) = M(a) + M(b)$ for independent readiness $a, b \in (0,1]$; (ii) Operational regularity: continuity at $a = 1$, monotonicity in a, measurability on compacts; (iii) Iso-entropic neutrality: adding a constant to all outcome costs cannot change relative probabilities; (iv) Thermodynamic additivity: $B(\Delta S_{I \cup J}) = B(\Delta S_I) + B(\Delta S_J)$ for macroscopically distinct channels. Then

$$M_i = -\kappa \ln a_i + \lambda \Delta S_i,$$

with $\kappa, \lambda > 0$. Up to overall scale, this is unique.

Proof. **(Forward direction)** Suppose $M_i = -\kappa \ln a_i + \lambda \Delta S_i$. Then:

- (i) holds: $M(ab) = -\kappa \ln(ab) + \lambda(\Delta S_a + \Delta S_b) = -\kappa \ln a - \kappa \ln b + \lambda \Delta S_a + \lambda \Delta S_b = M(a) + M(b)$
- (ii) holds: $-\kappa \ln a$ is continuous at $a = 1$, monotone decreasing for $\kappa > 0$, and measurable
- (iii) holds: Adding constant β to all ΔS_i shifts $M_i \to M_i + \lambda\beta$, which cancels in exponential normalization
- (iv) holds: $\lambda(\Delta S_I + \Delta S_J) = \lambda \Delta S_I + \lambda \Delta S_J$ by linearity

(Reverse direction) Suppose M satisfies (i)-(iv). Decompose $M_i = F(a_i) + B(\Delta S_i)$.

- From (i), F must satisfy $F(ab) = F(a) + F(b)$. Combined with (ii), Lemma 3.1 gives $F(a) = -\kappa \ln a$ for $\kappa > 0$.
- From (iv), B must satisfy $B(\Delta S_I + \Delta S_J) = B(\Delta S_I) + B(\Delta S_J)$. This is Cauchy's equation on $\mathbb{R}_+$. With regularity from (ii), this forces $B(\Delta S) = \lambda \Delta S + \beta$ for constants λ, β.
- From (iii), the constant β must cancel in relative probabilities, so we may set $\beta = 0$ without loss of generality.
- Thus $M_i = -\kappa \ln a_i + \lambda \Delta S_i$. Rescaling sets $\kappa = 1$. ■

Convexity and Stability. The action M_i is convex in a_i:

$$\partial^2 M_i/\partial a_i^2 = \kappa/a_i^2 > 0$$

ensuring a unique maximum of the caliber functional and stability of the variational problem.

Plain Language - What We Just Proved:

We asked: "If measurement has both geometric preparation costs and thermodynamic recording costs, how should they combine?"

We showed there's exactly ONE answer: $M_i = -\ln a_i + \lambda \Delta S_i$.

The proof had two parts:

1. **Forward:** "If we use this formula, all four physical requirements (P1-P4) are satisfied." We checked each one explicitly.

2. **Reverse:** "If all four requirements must be satisfied, this is the ONLY formula that works." We used Cauchy's functional equation (a classical result from 1821) to show logarithms are forced, then showed additivity forces linearity in ΔS.

The convexity result says this formula has a nice "bowl shape"—there's one clear minimum effort, not multiple competing solutions. This guarantees measurement outcomes are deterministic given the state and apparatus (no weird ambiguities).

What this means: The form of quantum probabilities isn't arbitrary. It's the unique mathematical structure consistent with (1) independent systems multiplying, (2) lab equipment being stable, (3) only differences in cost mattering, and (4) energy conservation.

Connection to Control Theory (§2). The factorization $a_i = |c_i|^2$ used in P1 was independently motivated in Section 2 as the quantum speed-limit reachability under bounded control. Thus the geometric term $-\ln a_i$ encodes control-theoretic "distance" from the target outcome, while the thermodynamic term $\lambda \Delta S_i$ encodes the entropy cost of stabilizing it. The action M_i unifies preparation geometry and measurement thermodynamics.

Physical Regularity and Laboratory Stability. The appeal to Cauchy additivity is physically motivated: independent preparations multiply overlaps, so any extensive "effort" functional must add under products. We impose operational regularity—continuity at $a = 1$, monotonicity, and bounded response to bounded input—because otherwise arbitrarily small preparation noise could flip outcome rankings, contradicting laboratory reproducibility. Under these mild, testable laboratory conditions, the unique solution is $F(a) = -\kappa \ln a$ (Lem. 3.1). Pathological solutions are excluded not by mathematical taste but by empirical stability requirements.

Plain Language - Why These Four Requirements Make Sense:

P1 (Composition): If you prepare two independent quantum systems, their "readinesses" multiply. This is just like probabilities: if coin A has 50% heads and coin B has 50% heads, together they have 25% both-heads. Since we want *effort* (not probability) to add up, we need logarithms: $\ln(a \times b) = \ln(a) + \ln(b)$. This forces the $-\ln a$ form.

P2 (Stability): Imagine a super-sensitive scale that gives wildly different readings when you breathe near it. Useless! Same here: if tiny noise in preparing your quantum state completely changes which outcome wins, you can't do reproducible science. Requiring continuity and monotonicity ensures stable, predictable behavior.

P3 (Neutrality): Suppose every outcome costs exactly 10 joules more than you thought. Does that change which outcome is most likely? No—it's the *differences* that matter, not the absolute values. This symmetry removes arbitrary constants from the physics.

P4 (Additivity): When you measure two systems together, the total heat dumped is the sum of individual heats (assuming they're independent). This is just conservation of energy applied to entropy. If channels don't interact, their costs add.

The Punchline: Only $M_i = -\ln a_i + \lambda\Delta S_i$ satisfies all four physical requirements. Any other formula either violates composition, fails stability, introduces spurious gauge freedom, or breaks thermodynamic accounting. The math isn't arbitrary—it's the unique form consistent with laboratory physics.

3.2 Mathematical Proofs

Lemma 3.1 (Cauchy on $\mathbb{R}_+$ with regularity). Let $F: (0,1] \to \mathbb{R}$ be a functional satisfying: (i) $F(ab) = F(a) + F(b)$ for all $a,b \in (0,1]$ (ii) F continuous at $a = 1$ (iii) F is order-preserving: $a < b \Rightarrow F(a) > F(b)$ (iv) $F(1) = 0$ (v) F is measurable on compact subsets of $(0,1]$

Then $F(a) = -k \ln a$ for some $k > 0$.

Domain note: We restrict to $a_i \in (0,1)$. Limits $a_i \to 0,1$ are taken after optimization; $-\ln a_i$ ensures convexity and operational stability on the interior of the probability simplex.

Proof. Let $a = e^x$, define $G(x) = F(e^x)$ for $x \in (-\infty, 0]$. Then $G(x+y) = G(x) + G(y)$ for all $x,y \in (-\infty, 0]$. Continuity at $x = 0$ (corresponding to $a = 1$) and measurability imply G is a continuous additive functional on $\mathbb{R}$. By the fundamental theorem for Cauchy's equation with regularity, $G(x) = kx$ for some constant $k \in \mathbb{R}$. Hence $F(a) = k \ln a$. Monotonicity (iii) requires $k > 0$. Writing $-F(a)$ yields $-k \ln a$. ■

Lemma 3.2 (Thermodynamic part affine in ΔS). If (T1) ΔS is additive on products, (T2) basis-label invariance holds, and (T3) iso-entropic neutrality applies, then any bias functional $B(\Delta S_i)$ must be affine: $B(\Delta S_i) = \alpha\Delta S_i + \beta$. By (T3) β is irrelevant; set $\alpha = \lambda > 0$. Additivity induces Cauchy additivity; boundedness implies linearity. Neutrality removes the constant. ■

Together, Lemmas 3.1–3.2 establish Proposition 3.0.

3.3 The Gibbs-Biased Unfolding Law

The realized distribution P_i maximizes the caliber $C[P] = -\sum_i P_i \ln P_i$ subject to $\sum_i P_i = 1$ and $\sum_i P_i M_i = \text{const}$.

Explicit Derivation via Lagrange Multipliers. Form the Lagrangian:

$$\mathscr{L}[P] = -\sum_i P_i \ln P_i - \alpha(\sum_i P_i - 1) - \beta(\sum_i P_i M_i - \langle M\rangle)$$

Setting $\delta\mathscr{L}/\delta P_i = 0$ yields:

$$-\ln P_i - 1 - \alpha - \beta M_i = 0 \Longrightarrow P_i = e^{(-1-\alpha)} e^{(-\beta M_i)}$$

Normalization $\sum_i P_i = 1$ determines the constant $e^{(-1-\alpha)}$, giving:

$$P_i = e^{(-\beta M_i)} / \sum_j e^{(-\beta M_j)} \quad (3.2a)$$

Partition Function Formulation. Define the partition function:

$$Z(\lambda) := \sum_j a_j e^{(-\lambda\Delta S_j)} \quad (3.2b)$$

where we identify β with λ and substitute $M_i = -\ln a_i + \lambda\Delta S_i$ to obtain:

$$P_i = (a_i e^{(-\lambda\Delta S_i)}) / Z(\lambda) \quad (3.2)$$

Equation (3.2) is the Gibbs-biased unfolding law. It reduces to the Born rule whenever the unfolding costs are equal across outcomes:

$$\Delta S_i = \text{const} \Rightarrow P_i = a_i = |c_i|^2 \quad (3.3)$$

Plain Language - The Core Result:

We just derived the probability formula for quantum measurement outcomes. Here's what it says:

$P_i = (a_i \times e^{(-\lambda\Delta S_i)}) / (\text{sum over all outcomes})$

In words: "The probability of outcome *i* equals its geometric readiness ($a_i = |c_i|^2$) times a thermodynamic penalty factor ($e^{(-\lambda\Delta S_i)}$), normalized so all probabilities add to 1."

Three scenarios:

1. **Perfect detector** (ΔS_i all equal): The exponential factors cancel, leaving $P_i = |c_i|^2$. This is the famous Born rule! It emerges automatically when your detector treats all outcomes equally from a thermodynamic standpoint.
2. **Biased detector** (one ΔS_i smaller): That outcome's probability goes UP (smaller negative exponent = bigger $e^{(-\lambda\Delta S_i)}$). It's "cheaper" to make real, so nature favors it slightly.
3. **Asymmetric detector** (varying ΔS_i): Probabilities tilt toward thermodynamically cheaper outcomes. The tilt is measurable and testable—that's the experiment we're proposing!

The Z(λ) denominator (called the "partition function" in statistical mechanics) is just a normalization constant ensuring probabilities sum to 1. It doesn't change the physics, just ensures proper accounting.

Key insight: The Born rule isn't fundamental—it's a *special case* of a deeper thermodynamic law. It holds exactly when measurement is "iso-entropic" (equal entropy costs). Real detectors with imperfect symmetry should show tiny, calculable deviations.

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4. Small-Bias Expansion and Deviation from Born

Definition of Expectations. We distinguish two types of entropy averages:

$\langle \Delta S \rangle_a := \sum_i a_i \Delta S_i$ (average over geometric weights) $\langle \Delta S \rangle_P := \sum_i P_i \Delta S_i$ (average over realized probabilities) (4.0)

Formal Derivation of First-Order Expansion. From the partition function formulation (3.2b), we have:

$\ln P_i = -\ln a_i - \lambda \Delta S_i - \ln Z(\lambda)$

Taking the derivative with respect to λ:

$\partial_\lambda \ln P_i = -\Delta S_i + (\partial_\lambda Z)/Z = -\Delta S_i + \langle \Delta S \rangle_P$

Integrating from $\lambda = 0$ (where $P_i = a_i$) to small λ:

$\ln P_i - \ln a_i = -\lambda(\Delta S_i - \langle \Delta S \rangle_a) + O(\lambda^2)$

where we use $\langle \Delta S \rangle_P = \langle \Delta S \rangle_a + O(\lambda)$. Exponentiating and expanding:

$P_i(\lambda) = (a_i e^{(-\lambda\Delta S_i)}) / Z(\lambda) = a_i [1 - \lambda(\Delta S_i - \langle \Delta S \rangle_a)] + O(\lambda^2)$ (4.1)

Result: Thus deviations from Born rule are controlled by the relative cost $\Delta S_i - \langle \Delta S \rangle_a$; equal costs recover $P_i = a_i$ exactly.

Variance-Based Deviation Bound. From (4.1), we obtain:

$$|P_i - a_i| \leq \lambda a_i |\Delta S_i - \langle \Delta S \rangle_a| + O(\lambda^2)$$

Summing over all outcomes:

$$\|P - a\|_1 \leq \lambda \sqrt{\mathrm{Var}_a(\Delta S)} + O(\lambda^2) \quad (4.2)$$

where $\mathrm{Var}_a(\Delta S) = \langle (\Delta S - \langle \Delta S \rangle_a)^2 \rangle_a$ is the variance of entropy costs weighted by geometric overlaps. This provides a measurable upper bound on total deviation from Born statistics.

Analytic Properties of the Partition Function. The partition function $Z(\lambda) = \sum_j a_j e^{-\lambda \Delta S_j}$ has several important properties:

1. **Analyticity:** $Z(\lambda)$ is analytic in λ for all real λ (since it is a finite sum of exponentials)
2. **Convexity:** The log-partition function is convex:

   $$(d^2/d\lambda^2) \ln Z(\lambda) = \mathrm{Var}_P(\Delta S) \geq 0 \quad (4.3)$$

   where $\mathrm{Var}_P(\Delta S) = \langle \Delta S^2 \rangle_P - \langle \Delta S \rangle_P^2$ is the variance under the realized distribution.

3. **Monotonicity:** Since $\mathrm{Var}_P(\Delta S) \geq 0$, the bias increases monotonically with λ, and the functional dependence $P_i(\lambda)$ is smooth and well-behaved.

These properties ensure the thermodynamic refinement is analytically controlled and connects directly to measurable statistical quantities (variance of entropy costs).

Plain Language - What Sections 4 Just Told Us:

The small-bias expansion (Eq. 4.1) is our prediction for real experiments:

$$P_i \approx |c_i|^2 \times [1 - \lambda(\Delta S_i - \text{average } \Delta S)]$$

This says: "Start with Born rule ($|c_i|^2$). Then apply a correction proportional to how much outcome *i*'s entropy cost differs from the average."

Example: Suppose measuring outcome "up" requires dumping 1.1×10^{-20} J/K of heat, while "down" requires only 0.9×10^{-20} J/K. The average is 1.0×10^{-20} J/K. Then:

- "Up" probability gets *reduced* slightly (its ΔS is above average)
- "Down" probability gets *increased* slightly (its ΔS is below average)
- The shift is proportional to λ and the difference (0.2×10^{-20} J/K)

The variance bound (Eq. 4.2) says: "Total deviation from Born rule is bounded by the spread (variance) of entropy costs."

If all your outcomes have nearly identical entropy costs (small variance), deviations are tiny. If costs vary wildly, deviations can be large. This is testable: measure the variance of ΔS_i, predict the deviation bound, then check if actual deviations match.

The partition function properties prove our formula is mathematically well-behaved:

- **Analytic:** No weird singularities or discontinuities
- **Convex:** The function curves smoothly (like a bowl), ensuring one clear answer
- **Monotonic:** As you increase λ (make thermodynamics more important), the bias increases smoothly

Bottom line: We've given you a formula that (1) reduces to Born rule in the limit, (2) predicts specific deviations when entropy costs vary, and (3) is mathematically rigorous and stable. The deviations are tiny but measurable with modern nano-calorimeters.

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5. Residual Probability as Entropy Flow ('36% Rule' — Phenomenological Example)

Let *r* denote the targeted (aligned) outcome. Define residual probability $R := 1 - P_r$. Identify the experimentally auditable entropy export as $\Phi_E := \sum_i P_i \Delta S_i$. A natural phenomenological calibration in fixed-context experiments is:

$$R \equiv 1 - P_r = \Phi_E / \Phi_{total} \quad (5.1)$$

where Φ_{total} is the entropy budget required to complete unfolding in that context.

Status: Equation (5.1) is a phenomenological calibration relating residual probability to measured entropy flow. It does not enter any formal derivation or proof in this paper; rather, it provides an experimentally convenient parametrization of the relationship between alignment ceiling and entropy export. The '36% rule' label refers to scenarios where $a_r^{(max)} \approx 0.64$, giving $R \approx 0.36$ as a typical example, not a universal constant.

6. Control-Theory Ceiling on Alignment Readiness

Consider a qubit with bounded control Hamiltonian $\|H\| \leq \Omega_{max}$ over duration τ. The maximal Bloch rotation angle is $\theta_{max} = 2\Omega_{max} \tau$. Preparing the measurement eigenstate requires a rotation producing success probability

$$a_r^{(max)} = \cos^2(\Omega_{max} \tau) \quad (6.1)$$

Example: The 64% Limit. Setting $a_r^{(max)} = 0.64$ yields $\cos(\Omega_{max} \tau) = 0.8 \rightarrow \Omega_{max} \tau \approx 36.87° \approx 0.6435$ rad. Thus finite control bandwidth-time budgets can cap alignment readiness near 64% in scenarios where $\Omega_{max} \tau$ saturates available resources. This is one among many possible operating points; different experimental configurations yield different ceilings. Thermodynamic bias then nudges realized P_r via (3.2).

Important Note: The 64% numerical example illustrates a control-limited ceiling $a_r^{(max)} = \cos^2(\Omega_{max} \tau)$; it is not universal and shifts with $\Omega_{max} \tau$. Thermodynamic bias via $e^{(-\Delta S_i/k_B)}$ acts after this geometric ceiling. This is not a universal prediction of "64% always," but rather an illustration that quantum speed limits naturally impose ceilings on a_r that depend on control parameters Ω_{max} and τ. Different apparatus yield different caps.

7. Worked Two-Outcome Example

Plain Language - Setting Up the Example:

Imagine the simplest possible quantum measurement: a qubit with two outcomes, like measuring whether an electron's spin points "up" or "down."

Suppose your quantum state preparation gives you:

- 64% geometric readiness for outcome *r* (the "right" or targeted outcome)
- 36% geometric readiness for the other outcome

Under perfect Born rule, you'd measure *r* exactly 64% of the time and "not-r" exactly 36% of the time.

But now add thermodynamics: suppose your detector is *slightly asymmetric*. Outcome *r* requires dumping slightly *less* heat (easier to lock in), while "not-r" requires slightly *more* heat (harder to lock in). How does this tilt the probabilities?

Let's work it out exactly.

Setup: Consider a two-outcome measurement with:

- Geometric overlaps: $a_r = 0.64$, $a_{\neg r} = 0.36$ (where ¬r denotes "not r")
- Normalization check: $0.64 + 0.36 = 1.00$ ✓
- Entropy costs centered around some average value ΔS, with asymmetry ±δ:
    - $\Delta S_r = \Delta S - \delta$ (outcome *r* costs *less* entropy)
    - $\Delta S_{\neg r} = \Delta S + \delta$ (outcome ¬r costs *more* entropy)

Step 1: Apply the Gibbs-Biased Formula (3.2)

The partition function is:

$$Z(\lambda) = a_r e^{-\lambda\Delta S_r} + a_{\neg r} e^{-\lambda\Delta S_{\neg r}} = 0.64\, e^{-\lambda(\Delta S-\delta)} + 0.36\, e^{-\lambda(\Delta S+\delta)} = e^{-\lambda\Delta S} [0.64\, e^{+\lambda\delta} + 0.36\, e^{-\lambda\delta}]$$

The probability of outcome *r* is:

$$P_r = (a_r e^{-\lambda\Delta S_r}) / Z(\lambda) = (0.64\, e^{-\lambda(\Delta S-\delta)}) / (e^{-\lambda\Delta S} [0.64\, e^{+\lambda\delta} + 0.36\, e^{-\lambda\delta}])$$

Step 2: Simplify

The $e^{-\lambda\Delta S}$ factors cancel (this is why only *differences* in ΔS matter):

$$P_r = (0.64\, e^{+\lambda\delta}) / [0.64\, e^{+\lambda\delta} + 0.36\, e^{-\lambda\delta}]$$

Dividing numerator and denominator by $0.64\, e^{+\lambda\delta}$:

$$P_r = 1 / [1 + (0.36/0.64)\, e^{-2\lambda\delta}] \quad (7.1)$$

Step 3: Analysis of Different Cases

Case 1: Iso-entropic (δ = 0) If entropy costs are equal ($\delta = 0$), then $e^{-2\lambda\delta} = e^0 = 1$:

$$P_r = 1 / [1 + 0.36/0.64] = 1 / [1 + 0.5625] = 1/1.5625 = 0.64$$

This recovers the Born rule exactly: $P_r = a_r = 0.64$. ✓

Case 2: Asymmetric (δ > 0) If outcome *r* is thermodynamically cheaper ($\delta > 0$), then $e^{-2\lambda\delta} < 1$, so:

$P_r = 1 / [1 + (0.36/0.64) \times (\text{something} < 1)] = 1 / [1 + (\text{something} < 0.5625)] > 1/1.5625 = 0.64$

The probability *increases* above the Born value.

Case 3: Asymmetric ($\delta < 0$) If outcome *r* is thermodynamically more expensive ($\delta < 0$), then $e^{-2\lambda\delta} > 1$, so:

$P_r = 1 / [1 + (0.36/0.64) \times (\text{something} > 1)] < 0.64$

The probability *decreases* below the Born value.

Plain Language - What This Example Shows:

The setup: You've prepared a quantum state that's "64% ready" for outcome *r* according to geometry. Under perfect Born rule, you'd measure *r* 64% of the time.

The thermodynamic tilt: Your detector has a slight asymmetry—one outcome is "cheaper" (requires less heat dumping) than the other.

Three scenarios:

1. **Balanced detector ($\delta = 0$):** Both outcomes cost the same entropy. Result: Born rule holds exactly (64%).
2. **Favors r ($\delta > 0$):** Outcome *r* is cheaper (requires less heat). Result: You measure *r more* than 64% of the time. The easier-to-record outcome wins more often!
3. **Disfavors r ($\delta < 0$):** Outcome *r* is more expensive (requires more heat). Result: You measure *r less* than 64% of the time. The harder-to-record outcome loses frequency.

The key formula: $P_r = 1 / [1 + (a_{\neg r}/a_r) e^{-2\lambda\delta}]$

- The ratio $a_{\neg r}/a_r = 0.36/0.64 \approx 0.56$ is the Born-rule baseline
- The exponential factor $e^{-2\lambda\delta}$ modulates this based on entropy asymmetry
- When δ is large and positive, $e^{-2\lambda\delta} \to 0$, and $P_r \to 1$ (always get *r*)
- When δ is large and negative, $e^{-2\lambda\delta} \to \infty$, and $P_r \to 0$ (never get *r*)

Physical interpretation: The detector's thermodynamic asymmetry acts like a "bias" or "weight" on top of the geometric Born probabilities. Small asymmetries produce small tilts; large asymmetries can completely dominate.

Numerical Examples

Let's plug in some realistic values. Suppose:

- $\lambda = 1/k_B \approx 7.24 \times 10^{22}$ J^{-1} K (at room temperature with k_B in SI units, but we use $k_B = 1$)
- $\delta = 10^{-21}$ J/K (a tiny entropy difference, barely measurable)

Then $\lambda\delta \approx 0.007$ (small parameter), giving:

$e^{-2\lambda\delta} \approx e^{-0.014} \approx 0.986$

$P_r = 1 / [1 + 0.5625 \times 0.986] = 1 / 1.555 \approx 0.643$

Result: Instead of 64.0%, you measure outcome *r* about 64.3% of the time—a 0.3 percentage point shift. Tiny but measurable!

For a larger asymmetry, $\delta = 10^{-20}$ J/K:

$\lambda\delta \approx 0.07$, $e^{-2\lambda\delta} \approx 0.87$

$P_r \approx 1 / [1 + 0.5625 \times 0.87] \approx 1/1.49 \approx 0.67$

Result: Now you're at 67% instead of 64%—a 3 percentage point shift, easily measurable.

Residual Probability

The residual $R = 1 - P_r$ quantifies how much probability "leaked" to other outcomes due to thermodynamic costs:

- $\delta = 0$: $R = 0.36$ (Born rule residual)
- $\delta = 10^{-21}$ J/K: $R \approx 0.357$ (slightly more residual)
- $\delta = 10^{-20}$ J/K: $R \approx 0.33$ (less residual—more concentrated on *r*)

Via equation (5.1), this residual is proportional to the total entropy exported: $R \approx \Phi_E / \Phi_{total}$.

Experimental Realization

How would you create this asymmetry in the lab?

Superconducting qubit example:

- Measure qubit state ($|0\rangle$ vs $|1\rangle$) via dispersive readout

- Engineer detector asymmetry by using different resistive loads for the two readout branches
- Branch for |0⟩: low resistance → less Joule heating → smaller ΔS_0
- Branch for |1⟩: high resistance → more Joule heating → larger ΔS_1
- Measure the probability ratio P_0/P_1 vs the resistance ratio
- Fit to equation (7.1) to extract λ

Predicted signature: A logarithmic plot of $\ln(P_r / P_{\neg r})$ vs δ should be linear with slope -2λ:

$$\ln(P_r / P_{\neg r}) \approx \ln(a_r / a_{\neg r}) - 2\lambda\delta$$

This is your experimental "smoking gun"—direct evidence that thermodynamic costs modulate quantum probabilities.

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8. Why Not Other Functional Forms?

Why not alternatives? One might consider $M_i = \sqrt{a_i} + \lambda\Delta S_i$ or $M_i = -a_i \ln a_i + \lambda\Delta S_i$. The first violates additivity under products (P1); the second, while entropy-like in form, fails the regularity requirement (P2) at the boundaries $a_i \to 0$ or $a_i \to 1$ where operational stability demands smooth behavior. Only the logarithmic-linear form survives all four constraints P1–P4.

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9. Operational Definitions and Measurement Protocols

Parameter Identification. The thermodynamic coupling λ may be identified via differential fits of $\log(P_i/a_i)$ versus ΔS_i differences. In weak-coupling readout chains described by GKSL equations, $\lambda \equiv (1/(k_B T_{eff})) \cdot (\partial W_{diss}/\partial \Delta S)|_{context}$. Experimentally, vary calibrated detector asymmetry (δR or δt) and fit slopes. Outcome-conditioned entropy ΔS_i is measured through analog ($\int Q_i/T\, dt$) and digital ($W_{erase,i}/T_0$) components, with lock-in modulation to suppress noise.

Methods: Measuring Outcome-Conditioned Entropy (ΔS_i)

$$\Delta S_i = \Delta S_i^{(analog)} + \Delta S_i^{(digital)}$$

Analog heat (calorimetry): $\Delta S_i^{(analog)} = \int Q_i(t)/T(t)\, dt$ using cryo-nanocalorimeters on each branch; lock-in modulation toggles the branch-asymmetry (gain/resistor) at 10–100 Hz to extract $\partial \ln P_i/\partial \Delta S_i$.

Digital erasure (Landauer): $\Delta S_i^{(digital)} \approx W_{erase,i} / T_0$ by counting irreversible bit resets in the readout FPGA/ASIC conditioned on outcome *i*.

Work-form cross-check: Independently estimate W_i (quench work + controller dissipation) and verify $P_i/a_i \propto e^{-\beta W_i}$.

Here Q_i is the conditional heat flux and $W_{erase,i}$ the erasure work in the logic pipeline. Outcome-specific calorimeters isolate each branch.

Experimental Protocols:

A. Superconducting qubit (dispersive readout): Prepare $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, readout via matched JPA/JPC branches with calibrated resistive loads; measure heat Q_0, Q_1 and erasure work per branch.

B. Trapped-ion fluorescence: Measure bright/dark outcomes with differing integration windows, compute ΔS from photon-detector heat and digital erasure.

C. NV-center optical readout: Vary pump power/duration and record calorimetric and erasure contributions per branch.

- **Alignment readiness a_i:** Estimated via state tomography or calibrated control pulses mapping amplitude to population.
- **Unfolding cost ΔS_i:** Inferred from irreversible heat/erasure work in the measurement chain; Landauer erasure and calorimetry provide lower bounds.
- **Entropy budget Φ_{total}:** Fixed by apparatus geometry, quench protocol, and readout pipeline; determined by baseline runs.

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10. Empirical Program and Falsifiability

10.1 Experimental Tests

- Vary control bandwidth Ω_{max} and duration τ to modulate the ceiling $a_r^{(max)}$ (Eq. 6.1) and compare P_r trajectories to (3.2).
- Engineer asymmetric ΔS via detector gains or resistive loads; test the predicted skew $P_i \propto a_i\, e^{-\lambda \Delta S_i}$.
- Audit Φ_E and check proportionality $R \approx \Phi_E/\Phi_{total}$ at fixed context (Eq. 5.1).
- Search for regime where $\lambda \to 0$ (iso-entropic readout) to verify restoration of $P_i = |c_i|^2$.

Null Test (Falsifiability). Engineer ΔS_i asymmetry δ and fit

$\ln[(P_i/a_i)/(P_j/a_j)] = -(\Delta S_i - \Delta S_j)/k_B + O(\delta^2)$

Failure to detect the predicted slope within noise bounds places a quantitative limit on the thermodynamic refinement, effectively restoring the Born limit.

Feasibility and Expected Scales. Modern cryo-nanocalorimeters resolve $\delta Q \approx 10^{-19}$–$10^{-18}$ J on sub-ms windows; at $T \approx 50$–100 mK this gives $\delta S \approx 10^{-20}$–$10^{-19}$ J/K. The fractional skew $(P_i/a_i)/(P_j/a_j) = \exp[-\lambda(\Delta S_i-\Delta S_j)]$ scales linearly with ΔS. Differential protocols toggling δR or δt at 10–100 Hz and lock-in detection can bound $|\lambda| \lesssim \varepsilon/|\Delta S_i-\Delta S_j|$. If no skew is detected above noise ε, that null directly constrains λ, confirming Born recovery in the iso-entropic regime.

10.2 Why Haven't Deviations Been Observed?

The Gibbs-biased law (3.2) predicts $P_i \neq |c_i|^2$ whenever $\lambda(\Delta S_i - \Delta S_j)$ is non-negligible. Why do standard quantum experiments agree with Born statistics to high precision?

Answer: Most precision quantum measurements are effectively iso-entropic:

1. **Detector symmetry:** Well-engineered detectors treat all outcomes equally, so $\Delta S_i \approx$ const.
2. **Small λ:** At typical operating temperatures and short measurement times, $\lambda(\Delta S_i - \Delta S_j) \ll 1$ even when ΔS_i varies.
3. **Averaging:** Standard experiments average over many apparatus configurations, washing out small asymmetries.

The framework predicts Born violations only when:

- ΔS_i deliberately made asymmetric (via engineering detector loads)
- Long integration times allow accumulation of thermodynamic bias
- Single-shot measurements on carefully controlled apparatus

This is precisely the regime where the protocols in §9 are designed to operate.

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11. Formal Results

Theorem 1 (Gibbs-Born Unfolding Law). Under axioms A1–A4 with unfolding action $M_i = -\ln a_i + \lambda\Delta S_i$ (Theorem 3.0), the realized outcome distribution is

$P_i = (a_i e^{-\lambda\Delta S_i}) / Z(\lambda)$, where $Z(\lambda) = \sum_j a_j e^{-\lambda\Delta S_j}$

Proof. Maximize caliber $C[P] = -\sum_i P_i \ln P_i$ subject to $\sum_i P_i = 1$ and $\sum_i P_i M_i = \langle M \rangle$. The Lagrange multiplier method (§3.3) yields the exponential family with sufficient statistics $\{-\ln a_i, \Delta S_i\}$. Normalization determines the partition function $Z(\lambda)$. ■

Corollary 1 (Born Limit). If ΔS_i = const for all i, then $P_i = a_i = |c_i|^2$.

Proof. When $\Delta S_i = \Delta S_0$ (constant), we have $Z(\lambda) = e^{-\lambda \Delta S_0} \sum_j a_j = e^{-\lambda \Delta S_0}$, so $P_i = a_i e^{-\lambda \Delta S_0}/e^{-\lambda \Delta S_0} = a_i$. ■

Corollary 2 (Product Systems). For independent subsystems with factorized amplitudes $a_{ij} = a_i a_j$ and additive costs $\Delta S_{ij} = \Delta S_i + \Delta S_j$, the joint probability factorizes:

$$P_{ij} = (a_i a_j e^{-\lambda(\Delta S_i + \Delta S_j)}) / (Z_i(\lambda) Z_j(\lambda))$$

preserving tensor-product structure.

Proof. Direct substitution using extensivity $M_{ij} = M_i + M_j$ from Theorem 3.0. ■

Proposition 1 (First-Order Deviation). For small λ, the deviation from Born rule satisfies

$$\ln(P_i/a_i) = -\lambda(\Delta S_i - \langle \Delta S \rangle_a) + O(\lambda^2)$$

with total variation bound $\|P - a\|_1 \leq \lambda\sqrt{\mathrm{Var}_a(\Delta S)} + O(\lambda^2)$.

Proof. See derivation in §4 via $\partial_\lambda \ln P_i$. ■

Proposition 2 (Convexity of Log-Partition Function). The function $\ln Z(\lambda)$ is convex in λ with

$$(d^2/d\lambda^2) \ln Z(\lambda) = \mathrm{Var}_P(\Delta S) \geq 0$$

Proof. Standard result from statistical mechanics; the second derivative equals the variance of the observable ΔS under distribution P. ■

Plain Language - The Formal Results Translated:

Theorem 1 says: "Given our four physical requirements (axioms A1-A4), the probability formula MUST be the Gibbs-biased form." This isn't a guess or approximation—it's mathematically forced.

Corollary 1 says: "When entropy costs are equal, Born rule is exact." This explains why standard quantum mechanics works so well: most carefully engineered detectors are approximately iso-entropic (treat all outcomes symmetrically).

Corollary 2 says: "For multiple independent quantum systems, probabilities multiply properly." If you measure two qubits separately, the joint probability is just the product of individual probabilities (as it should be). Our thermodynamic extension doesn't break this fundamental composition rule.

Proposition 1 gives the **prediction formula**:

- First-order deviation: $\ln(P_i / |c_i|^2) = -\lambda(\Delta S_i - \text{average } \Delta S)$
- Total deviation bound: $\|P - \text{Born}\| \leq \lambda \times \sqrt{(\text{variance of } \Delta S)}$

These are plot-ready, experiment-ready formulas. Measure ΔS_i, compute the variance, predict the deviation, then test whether reality matches.

Proposition 2 is a "sanity check": the math has all the right properties (convexity, smoothness) that well-behaved physics should have. No pathologies or weird edge cases.

The takeaway: We've built a complete mathematical theory with theorems, proofs, and testable predictions. It's not just "here's a formula"—it's "here's why this formula is inevitable, what it predicts, and how to test it."

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12. Discussion and Relation to Standard Quantum Theory

Formal Statement: Our result shows the Born assignment is the unique frame function compatible with unitary covariance and composition (Gleason/Busch). The entropic factor $e^{(-\Delta S_i/k_B)}$ is not a new kinematics; it is a contextual thermodynamic weight on the instrument, vanishing in the iso-entropic limit. **In operational terms:** a_i is fixed by controlled unitary reachability; ΔS_i is fixed by the instrument's nonequilibrium bookkeeping. The first is geometric (determined by state preparation), the second is thermodynamic (determined by measurement apparatus design).

The derivation presupposes only unitary kinematics and a thermodynamic unfolding principle; it does not assume the Born rule. The Born rule emerges as the iso-entropic limit, whereas non-ideal readout introduces controlled, testable biases through ΔS. This reframes measurement as entropic unfolding rather than abrupt collapse, aligning with VERSF's void-to-time transition.

Relationship to Established Results: The symmetry derivation (§14) recovers Gleason's theorem, establishing the quadratic core $P_i = |c_i|^2$ from pure kinematics. The entropic framework (§1–13) then models thermodynamic refinements to this ideal core, showing how real measurements can deviate from Born statistics in a controlled, calculable way.

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13. Outlook

Future work will:

1. Quantify ΔS_i from microscopic detector models (open-system QFT)
2. Integrate Schwinger–Keldysh influence functionals to compute λ from bath couplings
3. Extend to continuous spectra and POVMs with unfolding costs assigned to effect operators
4. Investigate experimental signatures in trapped-ion, superconducting-qubit, and NV-center platforms

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14. Derivation from Symmetry and Conservation Principles (True First Principles)

Plain Language - A Second Route to the Same Destination:

Sections 1-13 derived probabilities from *thermodynamics* (entropy costs). Now we'll derive the same result from pure *symmetry* (geometric structure). This is like reaching a mountain peak by two different trails—if both routes lead to the same summit, you know it's the right peak!

The goal: prove that probabilities MUST be $P_i = |c_i|^2$ without mentioning entropy, heat, or thermodynamics at all. Just using:

- The geometry of quantum states (Hilbert space)
- Symmetry (rotating your measurement apparatus shouldn't change physics)
- Composition (independent systems behave independently)

This was first done by Andrew Gleason in 1957 for dimensions ≥ 3, and extended by Paul Busch in 2003 to include dimension 2 (qubits). We're presenting their argument in VERSF language to show the Born rule has a purely kinematic foundation—the thermodynamic stuff from §1-13 is a *refinement*, not a replacement.

We now derive the quadratic probability law directly from symmetry and conservation, without invoking MaxCal or any thermodynamic postulate. The goal is to show that the only probability assignment compatible with (i) reversible dynamics, (ii) composition of systems, (iii) noncontextuality with respect to orthonormal decompositions, and (iv) invariance under unitary rotations on complex projective space, is the Born rule $P_i = |\langle i|\psi\rangle|^2$. This establishes a first-principles core on which the entropic-unfolding extension in §3 rides as a thermodynamic refinement.

14.1 Assumptions (No Born Rule)

- **H1 (Projective Kinematics):** Physical states are rays in a complex Hilbert space $\mathscr{H}$; reversible transformations act by unitaries or antiunitaries (Wigner).
- **H2 (Outcome Frames):** A measurement context is represented by an orthonormal basis $\mathfrak{B} = \{|i\rangle\}$ with mutually exclusive outcomes.
- **H3 (Frame Function):** A probability assignment is a map $p_\psi(\cdot)$ that to each projector $\Pi_i = |i\rangle\langle i|$ assigns $p_\psi(\Pi_i) \in [0,1]$, with $\sum_{i\in\mathfrak{B}} p_\psi(\Pi_i) = 1$ for every basis $\mathfrak{B}$ containing Π_i.
- **H4 (Noncontextuality for Projectors):** $p_\psi(\Pi)$ depends only on Π and ψ, not on which larger basis $\mathfrak{B}$ contains Π.
- **H5 (Product Composition):** For independent systems, probabilities factor on product projectors $\Pi\otimes\Sigma$.
- **H6 (Unitary Covariance):** $p_{U\psi}(U\Pi U^\dagger) = p_\psi(\Pi)$ for all unitaries U.

These assumptions encode only kinematics, exclusivity, normalization, composition, and symmetry—no dynamics or entropy.

14.2 Quadratic Form from Unitary Invariance

Fix ψ with $\|\psi\| = 1$ and define $f_\psi(\Pi) := p_\psi(\Pi)$. Consider rank-1 projectors $\Pi = |\varphi\rangle\langle\varphi|$. Unitary covariance (H6) implies f_ψ depends only on the invariant angle between rays, i.e., on $|\langle\varphi|\psi\rangle|$. Continuity and normalization over any orthonormal basis $\{|e_k\rangle\}$ require $\sum_k f_\psi(|e_k\rangle\langle e_k|) = 1$ and invariance under rotations of $\{|e_k\rangle\}$. The unique frame function on complex projective space $CP^{(d-1)}$ satisfying these constraints is quadratic in the overlap:

$$f_\psi(|\varphi\rangle\langle\varphi|) = g(|\langle\varphi|\psi\rangle|^2) \quad (14.1)$$

For any orthonormal basis $\{|e_k\rangle\}$, completeness demands $\sum_k g(|\langle e_k|\psi\rangle|^2) = 1$ for all ψ and all bases. By permutation symmetry of coefficients in any basis and continuity, the only solution is $g(x) = x$ (up to a constant fixed by normalization). Hence:

$$p_\psi(|\varphi\rangle\langle\varphi|) = |\langle\varphi|\psi\rangle|^2 \quad (14.2)$$

Sketch of uniqueness: Let $x_k := |\langle e_k|\psi\rangle|^2$ with $\sum_k x_k = 1$. We require $\sum_k g(x_k) = 1$ for all $\{x_k\}$ on the probability simplex and all dimensions $d \geq 2$. Symmetry forces g to be affine-linear on partitions; normalization at vertices ($x_j = 1$) implies $g(1) = 1$ and $g(0) = 0$; Jensen-convexity from basis refinements then collapses g to the identity function $g(x) = x$.

Plain Language - Why Probabilities Must Be Quadratic:

Here's the amazing result: **symmetry alone forces the $|\psi|^2$ formula.**

The argument:

1. **Unitary covariance (H6):** If you rotate your quantum state and rotate your measurement basis by the same amount, probabilities shouldn't change. This means probabilities depend only on the *angle* between the state and the measurement direction—not on any particular coordinate system.
2. **That angle is measured by the overlap:** In quantum mechanics, the "angle" between state $|\psi\rangle$ and measurement direction $|\varphi\rangle$ is captured by $|\langle\varphi|\psi\rangle|$. This is a number between 0 (perpendicular) and 1 (parallel).
3. **Normalization:** Probabilities must sum to 1 over any complete set of outcomes.
4. **Uniqueness:** There's only ONE function $g(|\langle\varphi|\psi\rangle|^2)$ that satisfies: (a) symmetry under rotations, (b) normalization $\sum g = 1$, (c) consistency across all bases, and (d) smoothness.

   That function is: $g(x) = x$.

   Therefore: probability = $|\langle\varphi|\psi\rangle|^2 = |c_i|^2$.

Why is this profound? You didn't need to *assume* the Born rule or invoke measurement collapse or entropy. It falls out of pure geometry: the structure of Hilbert space plus the requirement that probabilities be consistent and basis-independent.

The catch: This works perfectly for dimensions 3 and higher. For qubits (dimension 2), you need one extra assumption—either extend to generalized measurements (POVMs) or embed into a larger space. But the conclusion is the same: $|c_i|^2$ is forced by symmetry.

14.3 Relation to Gleason-Type Results

Relation to Gleason and Busch. For $\dim(\mathscr{H}) \geq 3$, assumptions (H3–H6) coincide with those of Gleason's theorem: every frame function on projectors is quadratic, $p_\psi(\Pi) = \mathrm{Tr}(\rho_\psi \Pi)$.

For $\dim(\mathscr{H}) = 2$ (qubits): Uniqueness is restored by either (i) extending to POVMs (Busch theorem—requires that the probability assignment be noncontextual for *all* effect operators, not just rank-1 projectors), or (ii) embedding two qubits in $d = 4$, applying Gleason there, and restricting back by partial trace (requires accepting the tensor-product extension and assuming consistency under reduction). Both routes force $p_\psi(\Pi) = |\langle\varphi|\psi\rangle|^2$ on rank-1 projectors.

Our symmetry language is therefore a restatement of Gleason/Busch, while the entropic-unfolding law (§3) adds a thermodynamic refinement beyond the quadratic core.

For $\dim(\mathscr{H}) \geq 3$, Gleason's theorem guarantees that any frame function (H3–H4) is of the quadratic form $p_\psi(\Pi) = \mathrm{Tr}(\rho_\psi \Pi)$. Specializing to pure states $\rho_\psi = |\psi\rangle\langle\psi|$ recovers (14.2). For qubits (dim = 2), the conclusion follows by extending to POVMs or by continuity and product-composition (H5) on pairs of qubits (dim = 4), then restricting back.

14.4 Composition and Product Projectors

Assumption (H5) requires $p_{\psi\otimes\chi}(\Pi\otimes\Sigma) = p_\psi(\Pi)p_\chi(\Sigma)$. The quadratic law $p_\psi(\Pi) = \mathrm{Tr}(|\psi\rangle\langle\psi|\ \Pi)$ is uniquely compatible with this factorization because $\mathrm{Tr}(|\psi\rangle\langle\psi|\ \Pi)\mathrm{Tr}(|\chi\rangle\langle\chi|\ \Sigma) = \mathrm{Tr}(|\psi\rangle\langle\psi|\otimes|\chi\rangle\langle\chi| \cdot \Pi\otimes\Sigma)$, i.e., probabilities multiply under tensor products if and only if they arise from a quadratic (Born) form.

14.5 Conservation and the Iso-Entropic Limit

The symmetry derivation yields $P_i = |c_i|^2$ in the absence of extrinsic thermodynamic bias. In the VERSF picture, this corresponds to an iso-entropic readout where the void-to-time transition exports equal entropy across outcomes: $\Delta S_i = \text{const} \Rightarrow \lambda = 0$ in §3, hence (3.3) matches (14.2).

14.6 Uniqueness of the Unfolding Action (Link to §3)

Given the quadratic core, any thermodynamic refinement must preserve (H1–H6) while introducing a scalar bias that (i) adds under composition, (ii) is basis-independent for fixed Π, and (iii) reduces to zero in the iso-entropic limit. Cauchy-additivity on products fixes the functional to be affine in ΔS_i, while information-geometric consistency with the quadratic measure singles out $-\ln a_i$ as the unique geometric contribution. Thus $M_i = -\ln a_i + \lambda\Delta S_i$ in §3 is the unique separable refinement consistent with the symmetry-derived quadratic law.

14.7 Summary of the First-Principles Chain

- Wigner symmetry ⇒ states are rays; transformations are unitary/antiunitary.
- Frame/noncontextual probability on projectors + unitary invariance ⇒ quadratic dependence on overlaps.
- Normalization on every orthonormal basis ⇒ $g(x) = x$ ⇒ Born rule.

- Product composition ⇒ uniqueness and tensor-factorization of Born probabilities.
- VERSF refinement: add entropic unfolding cost to model non-ideal, biased readout; iso-entropic limit recovers pure Born law.

Plain Language - How the Two Derivations Connect:

We now have TWO independent derivations of quantum probabilities:

Route 1 (Sections 1-13): Thermodynamics

- Start with: measurement requires entropy export
- Add: MaxCal inference (maximize uncertainty given constraints)
- Get: $P_i \propto |c_i|^2 e^{(-\lambda\Delta S_i)}$
- Special case: When ΔS_i equal → $P_i = |c_i|^2$

Route 2 (Section 14): Pure Symmetry

- Start with: Hilbert space geometry
- Add: symmetry + consistency requirements
- Get: $P_i = |c_i|^2$ (Gleason/Busch theorem)
- No thermodynamics needed!

How they fit together:

Route 2 tells us the *ideal* form—what probabilities must be in a perfectly symmetric universe with no thermodynamic imperfections. This is the "quadratic core" that's absolutely fundamental.

Route 1 tells us how *real* measurements deviate from the ideal when entropy costs aren't perfectly balanced. The $e^{(-\lambda\Delta S_i)}$ factor is a thermodynamic correction to the geometric $|c_i|^2$ core.

Analogy:

- Gleason's theorem (Route 2) is like Newton's first law: "Objects in motion stay in motion." That's the ideal, frictionless case.
- Our thermodynamic extension (Route 1) adds friction: "Objects in motion slow down proportional to resistance." The ideal case is recovered when friction vanishes.

The upshot: The Born rule is rock-solid, grounded in pure geometry. But real measurements in our messy, thermodynamic universe might show tiny deviations—and those deviations are calculable, not mysterious.

This is the best of both worlds: mathematical inevitability from symmetry, plus practical predictions for real experiments.

Appendix A — The Qubit Case and Restoration of Uniqueness

The two-dimensional Hilbert space (a single qubit) is a well-known loophole in Gleason's theorem:
for dim ($\mathscr{H}$) = 2, there exist frame functions that assign probabilities to projectors in a way that satisfies basis additivity yet *do not* take the Born-quadratic form.
This occurs because, in 2 D, the set of orthonormal bases (great circles on the Bloch sphere) is too small to constrain all possible functions on the sphere consistently—non-quadratic, direction-dependent assignments can exist.

To recover uniqueness, one must **extend** the domain of admissible measurements or embed the qubit in a higher-dimensional context. Two equivalent routes are standard:

A.1 Extension to POVMs (Busch Theorem)

Busch (1999) showed that if we extend the probability assignment

$$p_\psi(E)$$

from rank-1 projectors to *all positive-operator valued measures* (POVMs)—operators $E_i$satisfying $0 \leq E_i \leq I, \sum_i \quad E_i = I$—then any normalized, noncontextual, and additive frame function on this enlarged domain must take the Born form:

$$p_\psi(E) = \mathrm{Tr}(\rho_\psi E), \rho_\psi = | \psi\rangle\langle\psi |.$$

Intuition.
POVMs include not only orthogonal projectors but also unsharp or over-complete measurements (e.g., tetrahedral SIC-POVMs).
Requiring consistency across such overlapping measurement sets introduces enough functional constraints that only the quadratic rule survives.
Thus, extending H3–H4 to POVMs restores the Gleason result even for d = 2.

A.2 Lifting to Higher Dimension (Tensor-Product Embedding)

Alternatively, one can embed two qubits into a composite space $\mathscr{H}\otimes\mathscr{H}$ of dimension 4.
Gleason's theorem *does* apply there, guaranteeing quadratic probabilities for all projectors $\Pi\otimes\Sigma$.
For a product state $| \psi\rangle \otimes| \chi\rangle$,

$$p_{\psi\otimes\chi}(\Pi\otimes\Sigma) = |\langle\varphi\mid\psi\rangle|^2 |\langle\sigma\mid\chi\rangle|^2.$$

Now restrict back to one subsystem by tracing out the second:
$\mathrm{Tr}_2(\,|\psi\rangle\langle\psi|\otimes|\chi\rangle\langle\chi|\,) = |\psi\rangle\langle\psi|.$
The marginal probability on the first qubit then satisfies

$$p_\psi(\Pi) = \mathrm{Tr}(|\psi\rangle\langle\psi|\Pi) = |\langle\varphi\mid\psi\rangle|^2,$$

recovering the Born rule.
Physical meaning: the qubit, though simple, can always be viewed as part of a larger system where standard Gleason constraints hold; projecting back onto the subsystem preserves the quadratic form.

A.3 Connection to the Entropic-Unfolding Framework

In the entropic-unfolding picture, the qubit case corresponds to a two-channel measurement where the alignment readinesses $a_i = |c_i|^2$satisfy $a_0 + a_1 = 1$.
The Gibbs-biased refinement

$$P_i = \frac{a_i e^{-\lambda\Delta S_i}}{a_0 e^{-\lambda\Delta S_0} + a_1 e^{-\lambda\Delta S_1}}$$

remains valid regardless of dimension.
When ΔS_i = const the qubit obeys P_i = a_i, the Born rule; small entropy asymmetries introduce controlled, measurable deviations.
Thus, even though Gleason's theorem alone cannot enforce the quadratic law in 2 D, the **thermodynamic structure** of the VERSF-RAL framework naturally reinstates it in the iso-entropic limit and predicts specific corrections when that symmetry is broken.

A.4 Summary

Route	Key Assumption	Outcome
(i) POVM Extension (Busch)	Additivity for all positive effects E_i	Restores Born rule for d = 2
(ii) Tensor-Product Lifting	Apply Gleason in d ≥ 4, then trace back	Restores Born rule on qubit subsystem
(iii) Entropic-Unfolding	Entropy-balanced measurement	Reduces to Born rule; predicts measurable bias when ΔS varies

Appendix B — Mathematical Refinements for "Born Rule as Entropic Unfolding" (VERSF-RAL)

Prepared: October 14, 2025

B.0 Scope and Notation

This appendix provides expanded mathematical details supporting Sections 3–4 of the main paper, with all symbols and variables explicitly defined. The aim is to make each equation self-contained and physically interpretable. Throughout, $a_i := |c_i|^2$ with $a_i \in (0,1)$ and $\sum_i a_i = 1$. We write ΔS_i for outcome-conditioned entropy export (J/K). Unless stated otherwise, $k_B = 1$ (entropy in nats).

B.1 Domains, Regularity, and Variables

The unfolding action is $M_i = F(a_i) + B(\Delta S_i)$, combining geometric readiness a_i and entropy cost ΔS_i. To ensure reproducibility and exclude pathological functions, both F and B are required to be measurable, continuous, and additive under independent composition.

Variable Summary:

Symbol	Meaning	Units / Domain
a_i	Alignment readiness = $\lvert c_i\rvert^2$	dimensionless, $0 < a_i \leq 1$
ΔS_i	Entropy exported by outcome i	$J\,K^{-1}$ or dimensionless (if $k_B = 1$)
M_i	Unfolding action	dimensionless (scaled by k_B)
λ	Thermodynamic coupling constant	$1/k_B$ or $\beta = 1/(k_B T_{eff})$
$F(a)$	Geometric contribution	$-\ln a$ form (dimensionless)
$B(\Delta S)$	Thermodynamic bias term	linear in ΔS

B.2 Uniqueness of the Unfolding Action (Full Proof of Theorem 3.0)

Theorem. Suppose M satisfies: (i) $M(ab) = M(a) + M(b)$ for independent $a,b \in (0,1]$; (ii) F and B are measurable and continuous; (iii) adding a constant to all ΔS_i does not change relative probabilities; (iv) entropy costs add for distinct channels. Then $M_i = -\kappa \ln a_i + \lambda\Delta S_i$ with $\kappa, \lambda > 0$. Unique up to scale.

Equation: $M_i = -\kappa \ln a_i + \lambda\Delta S_i$

Definitions: M_i — unfolding cost; a_i — geometric readiness; ΔS_i — entropy exported; κ — scaling factor for geometric term; λ — thermodynamic coupling constant.

B.3 Maximum-Caliber Extremization

The realized distribution maximizes the path entropy subject to normalization and fixed mean unfolding cost:

$C[P] = -\sum_i P_i \ln P_i$, with constraints $\sum_i P_i = 1$, $\sum_i P_i M_i = \langle M \rangle$.

Introducing multipliers α, β and setting $\delta L/\delta P_i = 0$ gives:

$P_i = e^{(-1-\alpha)} e^{(-\beta M_i)}$. Normalization yields: $P_i = e^{(-\beta M_i)}/\sum_j e^{(-\beta M_j)}$.

Substituting $M_i = -\ln a_i + \lambda \Delta S_i \rightarrow P_i = (a_i e^{(-\lambda \Delta S_i)}) / Z(\lambda)$, $Z(\lambda) = \sum_j a_j e^{(-\lambda \Delta S_j)}$.

Variable meanings: P_i — probability of outcome i; $Z(\lambda)$ — partition function ensuring normalization; λ — bias strength; ΔS_i — entropy cost; a_i — readiness amplitude.

B.4 Small-Bias Expansion and Expectation Conventions

For small λ, expand P_i around $\lambda = 0$:

$\ln P_i = -\ln a_i - \lambda \Delta S_i - \ln Z(\lambda)$.

Differentiate: $\partial_\lambda \ln P_i = -\Delta S_i + \langle \Delta S \rangle_P$, where $\langle \Delta S \rangle_P = \sum_i P_i \Delta S_i$.

At $\lambda = 0$, $P_i = a_i \Rightarrow \langle \Delta S \rangle_P = \langle \Delta S \rangle_a + O(\lambda)$. Integrate to first order:

$\ln(P_i/a_i) = -\lambda(\Delta S_i - \langle \Delta S \rangle_a) + O(\lambda^2)$. Expanding gives: $P_i = a_i[1 - \lambda(\Delta S_i - \langle \Delta S \rangle_a)] + O(\lambda^2)$.

Definitions: $\langle \Delta S \rangle_a = \sum_i a_i \Delta S_i$ (geometric mean entropy), $\langle \Delta S \rangle_P = \sum_i P_i \Delta S_i$ (realized mean).

B.5 Deviation Bounds and Convexity

$|P_i - a_i| \le \lambda a_i |\Delta S_i - \langle \Delta S \rangle_a| + O(\lambda^2)$. Summing yields: $\|P - a\|_1 \le \lambda \sqrt{(\mathrm{Var}_a(\Delta S))} + O(\lambda^2)$, where $\mathrm{Var}_a(\Delta S) = \langle (\Delta S - \langle \Delta S \rangle_a)^2 \rangle_a$.

Convexity of M_i: $\partial^2 M_i/\partial a_i^2 = \kappa/a_i^2 > 0$ ensures unique maximum of the caliber functional.

Convexity of $\ln Z(\lambda)$: $d^2/d\lambda^2 \ln Z(\lambda) = \mathrm{Var}_P(\Delta S) \ge 0$ ensures monotonic bias behavior.

B.6 Analyticity and Smooth Dependence on λ

$Z(\lambda) = \sum_j a_j e^{(-\lambda \Delta S_j)}$ is analytic for real λ. Therefore, $P_i(\lambda)$ is analytic in λ and admits a power-series expansion near $\lambda = 0$ with convergence radius determined by the maximum $|\Delta S_i - \Delta S_j|$.

B.7 Operator Representation

Let $\hat{E}$ be a diagonal operator with eigenvalues ΔS_i in the measurement basis. Then the thermodynamic refinement can be written:

$\rho' = e^{(-(\lambda/2)\hat{E})} \rho e^{(-(\lambda/2)\hat{E})} / \mathrm{Tr}[e^{(-(\lambda/2)\hat{E})} \rho e^{(-(\lambda/2)\hat{E})}]$.

Outcome probabilities: $P_i = \mathrm{Tr}(\rho' |i\rangle\langle i|)$.

Variable meanings: ρ — pre-measurement density matrix; $\hat{E}$ — entropy-bias operator; λ — bias strength; ρ' — biased, normalized post-map state.

B.8 Units and Calibration

Entropy form: $P_i \propto a_i e^{(-\Delta S_i/k_B)}$ with $\lambda = 1/k_B$. Work form:
$P_i \propto a_i e^{(-\beta W_i)}$ where $\beta = 1/(k_B T_{eff})$ and $W_i \geq T_{eff} \Delta S_i$ (Landauer bound). Experimental calibration plots $\ln[(P_i/a_i)/(P_j/a_j)]$ vs. $(\Delta S_i - \Delta S_j)$; the slope gives $-1/k_B$ (entropy form) or $-\beta$ (work form).

B.9 Edge Cases and Interference Caveat

Thermodynamic additivity $B(\Delta S_{I \cup J})=B(\Delta S_I)+B(\Delta S_J)$ holds only for decohered, distinguishable channels. If channels interfere, non-additive corrections may appear. The current formulation applies after decoherence has suppressed off-diagonal coherences in the record basis.

Appendix C — Conceptual and Empirical Clarifications

C.1 The Inference–Reality Gap

Problem: MaxCal is a Jaynes-style inference principle, not a physical dynamics—so why does it correctly predict empirical frequencies?

Resolution: MaxCal works because reproducible universes must admit an additive, convex information measure that stabilizes long-run frequencies. Inference and dynamics coincide when microscopic evolution is ergodic and information-complete. Three interpretations are consistent:

1. Frequentist: MaxCal probabilities match time-averaged frequencies under ergodic dynamics.
2. Dynamical: A coarse-grained Liouville or path-integral dynamics realizes the MaxCal weighting.
3. Anthropic/Pragmatic:Only universes where inference and dynamics agree permit predictive science.

The VERSF companion paper elaborates option (2), showing that MaxCal weights arise from entropy-regulated void dynamics.

C.2 Operational Meaning of ΔS_i

Problem.Entropy export depends on where one draws the system–apparatus boundary.

Resolution.ΔS_i is defined as the entropy exported across a specified boundary separating the quantum system from the irreversible record channel. Predictions are therefore conditional on that boundary—analogous to ensemble choice in thermodynamics. The natural boundary is the first stage where outcome information becomes thermodynamically irreversible (e.g., amplifier or logic bit whose reversal requires external work). Shifting the boundary redefines ΔS_i and λ together but leaves observable ratios (P_i/P_j) invariant.

C.3 Why Deviations Have Not Been Observed

Problem. If deviations exist, why haven't experiments seen them?

Quantitative Estimate. The predicted fractional deviation scales as $|\Delta P|/P \approx \lambda\Delta S \approx \Delta S/(k_B T_{eff})$. For cryogenic readouts ($T \approx 50$ mK, $\Delta S \approx 10^{-20}$ J K^{-1}), $\lambda\Delta S \approx 10^{-8} - 10^{-7}$. Present Born-rule precision tests:

• Trapped-ion interferometry $\approx 10^{-4}$ (Mazurek 2019)
• Optical polarization $\approx 10^{-6}$ (Neves 2020)

Hence current limits are 2–4 orders of magnitude above predicted effects; null results are expected. Detecting or ruling out $\lambda\Delta S \approx 10^{-8}$ would require single-shot calorimetry below 10^{-20} J resolution or ensemble averaging over 10^8 runs. A null result at 10^{-10} would falsify the model for any realistic ΔS.

C.4 Relationship to Decoherence Theory

Problem. How does the entropy-export picture relate to environment-induced decoherence (Zurek)?

Resolution. Decoherence theory describes how off-diagonal density-matrix elements vanish with rate Γ_i; the present framework quantifies how much entropy is exported in doing so. In weak coupling,

$\Delta S_i \approx \int_0^{t_m} (\Gamma_i(t)/\Gamma_{tot})\, k_B \ln[\rho_{ii}(t)/\rho_{ii}(0)]\, dt$

linking entropy export to cumulative decoherence. The iso-entropic limit corresponds to equal integrated decoherence rates ($\Gamma_i = \Gamma_j$), recovering Born symmetry. Thus decoherence supplies the mechanism; VERSF-RAL supplies the thermodynamic bookkeeping. The two are complementary: einselection identifies stable pointer states, while entropic unfolding quantifies the irreversibility cost of recording them.

C.5 Summary

Issue	Resolution
Inference ↔ Reality	Emergent correspondence under ergodic, reproducible dynamics.
ΔS measurability	Defined relative to chosen system–apparatus boundary.
Lack of observed deviations	Present experiments below 10^{-8} sensitivity.
Relation to decoherence	ΔS tracks integrated decoherence entropy; iso-entropic ≈ equal Γ_i.

C.6 Alternative Interpretations

Problem. Could the predicted deviations from the Born rule simply reflect one of several existing interpretations rather than a distinct thermodynamic refinement?

Clarification. The VERSF-RAL framework interprets the Gibbs-biased term

$$P_i \propto a_i e^{-\lambda \Delta S_i}$$

as an *apparatus-dependent thermodynamic correction*, not as a modification of quantum dynamics itself. Nevertheless, the framework can be contrasted with three major interpretive classes:

Interpretation	Core Mechanism	Relation to VERSF-RAL	Why Distinct
(1) Collapse / Spontaneous Localization (GRW, CSL)	Adds stochastic nonlinear terms to Schrödinger equation, producing objective collapse noise.	VERSF-RAL keeps unitary dynamics intact; entropy export is external, arising at the system–apparatus boundary.	No extra stochastic term; deviations vanish in iso-entropic limit.
(2) Environment-Dependent Dynamics / Consistent Histories	Probabilities arise from decoherence and history-selection within an environment-dependent branching structure.	VERSF-RAL quantifies the *thermodynamic cost* of that decoherence; ΔS_i measures exported entropy, not path-weighting.	Provides measurable prediction ($\lambda \Delta S$ bias) absent from histories formalism.
(3) Observational Selection / Many-Worlds Counting	Born weights reflect branch measure or self-location probabilities.	The model remains single-world at the thermodynamic boundary; branching is not ontological but informational.	Entropy cost is empirically measurable, unlike counting measures in Everettian theory.