Complex Hilbert Space from Distinguishability Principles

Abstract

We derive the complex Hilbert space structure of quantum mechanics from a geometric foundation: a finite metric space of distinguishable configurations equipped with reversible isometries. By requiring that the amplitude field support interference and that observable predictions be invariant under field automorphisms, we prove that the complex numbers are the unique viable amplitude domain—neither real nor quaternionic structures satisfy the joint constraints of interference, isotropy, and analytic regularity. The resulting representation space is necessarily a complex Hilbert space, and the Hamiltonian emerges as the generator of the isometry group action. We interpret these results physically through Resonant Assembly Language (RAL), in which amplitudes represent oscillatory assemblies and Hilbert space emerges as the geometry of resonance. An appendix illustrates the framework via Grover's search algorithm. This derivation complements our previous work deriving the Born rule from coarse-graining geometry, completing a reconstruction of quantum formalism from distinguishability principles without presupposing the standard mathematical apparatus.

General-Reader Abstract

Modern quantum theory is built on a very specific kind of mathematics: complex numbers arranged in a Hilbert space. This is usually presented as a starting assumption – "that's just how the world is." In this work, we challenge that view. We start instead from something very simple and physical: a universe made of configurations that can be told apart (distinguishability) and the symmetries that preserve this structure. From these ingredients, plus mild smoothness assumptions, we show that the usual quantum formalism is not arbitrary at all. Complex numbers – and only complex numbers – are mathematically forced as the language of quantum "amplitudes," and once that is true, the whole Hilbert space machinery follows.

The key idea is to demand that physics not depend on the "internal labelling" of the number system it uses. Any symmetry of the amplitude algebra (an automorphism) should leave observable predictions unchanged, just as changing coordinates in space does not change physical outcomes. When this requirement is combined with the need for interference and symmetry between equally-distinguishable configurations, real numbers turn out to be too poor (they cannot encode continuous phase) and quaternions too rich (their symmetries clash with configuration symmetries). Complex numbers sit exactly in the middle – "just right" – and once they are selected, the standard quantum state space (complex Hilbert space with unitary dynamics and a Hamiltonian generator) emerges automatically.

If this reconstruction is accepted as correct, it has important consequences. It means that the core mathematical structure of quantum mechanics is not an unexplained choice, but the unique way to consistently describe a world of distinguishable possibilities with interference and symmetry. It strengthens the case that quantum theory is not an arbitrary human invention but the minimal, compressed description of how distinguishable change can behave. It also clarifies the foundations of quantum computing: the strange power of quantum algorithms comes directly from this uniquely determined geometry of complex amplitudes, not from a loose collection of postulates.

1. Introduction

1.1 The Problem: Why This Mathematical Structure?

Quantum mechanics is formulated in terms of complex Hilbert spaces. States are vectors in these spaces, observables are operators, and dynamics are unitary transformations. This framework is empirically successful beyond any reasonable doubt—but it raises a foundational question that physics textbooks rarely address:

Why complex numbers?

The complex numbers $\mathbb{C}$ are not the only number system available. Real numbers $\mathbb{R}$ are simpler. Quaternions $\mathbb{H}$ are richer. Various finite fields and p-adic numbers exist. Yet quantum mechanics uses complex numbers specifically. Is this a brute fact, or can it be explained?

Why Hilbert space?

A Hilbert space is a vector space with an inner product satisfying certain completeness properties. Many other mathematical structures exist. Why does nature choose this one?

Most approaches to quantum foundations treat these as primitive assumptions. The standard presentation says: "Quantum states live in a complex Hilbert space"—full stop. This paper takes a different view: **the complex Hilbert space structure should be derived, not assumed.**

1.2 Our Approach: Distinguishability and Symmetry

We start from something more primitive: **distinguishability**. The physical world consists of configurations that can be told apart. A particle can be here or there. A spin can be up or down. A photon can take path A or path B. These distinguishable configurations, and the geometric relationships between them, form our starting point.

From this foundation, we ask: what algebraic structure must amplitudes possess to be consistent with interference, reversible dynamics, and the symmetries of distinguishability?

The answer comes from **Galois theory**—the mathematical study of symmetries of number systems. By requiring that physics be invariant under all symmetries of the amplitude field, we prove that complex numbers are the unique viable choice. Hilbert space then emerges as the natural representation space for this structure.

1.3 Relation to Our Previous Work

In a companion paper, we derived the Born rule $P = |\psi|^2$ from the geometry of distinguishability and coarse-graining constraints. That derivation assumed amplitudes exist but did not determine their algebraic structure.

This paper addresses the complementary question: **what kind of amplitudes?** Together, the two results provide a complete reconstruction:

- **This paper**: Distinguishability geometry → Complex amplitudes → Hilbert space
- **Companion paper**: Coarse-graining geometry → Born rule

Neither paper assumes the standard quantum formalism. Both derive it from more fundamental principles.

1.4 What We Will Show

The main results of this paper are:

1. **Real amplitudes are excluded**: They cannot support the continuous phase structure required for interference.
2. **Quaternionic amplitudes are excluded**: Their automorphism group (SO(3)) conflicts with the permutation symmetry of distinguishable microstates.
3. **Complex amplitudes are uniquely selected**: The complex numbers have exactly the right automorphism structure—rich enough for interference, constrained enough for isotropy.
4. **Hilbert space emerges**: The representation space for reversible dynamics on complex amplitudes is necessarily a complex Hilbert space.
5. **The Hamiltonian emerges**: Dynamics is generated by the infinitesimal action of the isometry group, giving the Hamiltonian as a derived object rather than a postulate.
6. **Illustrative application**: Grover's search algorithm is analysed as a worked example, showing that the $\sqrt{N}$ speedup is a geometric rotation bound in the distinguishability manifold (Appendix B).

1.5 Physical Motivation: Resonant Assembly Language

The reconstruction presented in this paper can be interpreted physically through what we call **Resonant Assembly Language (RAL)**. In RAL, a physical system is composed of *assemblies*—patterns of activation over distinguishable configurations. Each assembly carries an oscillatory mode, and assemblies resonate with one another depending on their alignment.

In this view:

- **Amplitudes** are resonance weights
- **Phase** encodes oscillatory alignment
- **Interference** arises from net resonance, not classical probability addition
- **Probability** corresponds to resonance intensity

The symmetries used in our reconstruction acquire clear physical meaning:

- **Automorphism invariance** expresses that reparametrising the oscillatory mode of an assembly does not change its physical resonance
- **Isotropy** expresses that resonance patterns cannot depend on arbitrary labels of distinguishable configurations
- **Excluding $\mathbb{R}$ and $\mathbb{H}$** corresponds to the fact that real amplitudes are too poor to describe oscillatory resonance, and quaternionic amplitudes are too rich, introducing spurious degrees of freedom that break isotropy

RAL thus provides a concrete physical picture behind the algebraic derivation: complex amplitudes arise because the physics of resonance requires exactly the symmetry structure of $\mathbb{C}$.

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2. Distinguishability as Foundational Structure

Plain language summary: We begin with the idea that physics is fundamentally about things that can be told apart. A "configuration" is any state that is distinguishable from other states. The collection of all configurations, together with a notion of how similar or different they are, forms the foundation for everything that follows.

2.1 The Configuration Space

We begin with a finite set Λ of distinguishable micro-configurations. Each element $\lambda \in \Lambda$ represents a physically meaningful state—a configuration that can, in principle, be distinguished from all others by some measurement.

In everyday terms: Think of Λ as a list of all the distinguishable possibilities for some physical system. For a coin, $\Lambda = \{\text{heads, tails}\}$. For a die, $\Lambda = \{1, 2, 3, 4, 5, 6\}$. For a quantum particle that can be in one of several locations, Λ is the set of those locations.

The set Λ is equipped with a metric $d: \Lambda \times \Lambda \to \mathbb{R}_{\geq 0}$ encoding distinguishability relations:

- $d(\lambda_1, \lambda_2) = 0$ means λ_1 and λ_2 are identical
- Small $d(\lambda_1, \lambda_2)$ means the configurations are hard to distinguish
- Large $d(\lambda_1, \lambda_2)$ means they are easily distinguished

Analogy: Think of a map where distance represents distinguishability. Configurations that are "close together" on this map are similar and hard to tell apart; configurations that are "far apart" are very different.

This metric space (Λ, d) forms the geometric substrate on which quantum structure will emerge.

Cardinality constraint: We require $|\Lambda| \geq 3$. With only two configurations, the mathematics is too simple to capture the full structure of quantum mechanics. Three or more distinguishable states are needed for the Galois-theoretic argument to have non-trivial content.

2.2 Reversible Dynamics as Isometries

Physical transformations that preserve distinguishability are **isometries** of the metric d:

$\varphi: \Lambda \to \Lambda$ such that $d(\varphi(\lambda_1), \varphi(\lambda_2)) = d(\lambda_1, \lambda_2)$

In plain terms: A reversible transformation shuffles the configurations around without changing how distinguishable they are from each other. If two states were easy to tell apart before the transformation, they remain equally easy to tell apart afterward.

The group G of all such isometries represents the symmetries of the physical system. These symmetries will act on amplitudes via representations, and the structure of these representations determines the quantum state space.

2.3 The Taylor Limit

The *Taylor Limit* is a regularity assumption ensuring well-behaved physics:

1. **Analyticity**: Physical quantities admit Taylor expansions
2. **Continuity**: Small changes in configuration produce small changes in predictions
3. **Effective finiteness**: For any given physical prediction, only a finite subset of Λ contributes non-negligibly to the probability functional

Why this matters: This assumption rules out pathological mathematical constructions that have no physical meaning. As we'll see, it restricts the allowable automorphisms of the amplitude field to continuous ones—a crucial step in the derivation.

Operational motivation: The Taylor Limit is not merely a mathematical convenience; it reflects fundamental constraints on measurement. Any physical measurement has finite precision—we cannot distinguish configurations that differ by less than some resolution threshold. This implies that probability functionals must be continuous: configurations indistinguishable to measurement apparatus must yield indistinguishable predictions. Furthermore, finite measurement resources imply that only finitely many configurations can contribute meaningfully to any single prediction. The analyticity requirement follows from the stronger assumption that physics admits perturbative analysis—that we can compute corrections to predictions order by order. These are standard assumptions in physics, here made explicit.

In everyday terms: The Taylor Limit says physics is "smooth"—no infinitely sensitive dependence on infinitely precise details. This is a reasonable assumption about how nature works and has important mathematical consequences.

2.4 Physical Interpretation via Resonant Assembly Language

Although our derivation proceeds in purely geometric and algebraic terms, it admits a natural physical interpretation in terms of **Resonant Assembly Language (RAL)**. In RAL, a physical system is modelled as a collection of *assemblies*—coherent patterns of activation over distinguishable configurations. Each assembly carries an oscillatory mode, and interactions between assemblies are determined by the resonance of these modes.

Under this interpretation:

- A configuration $\lambda \in \Lambda$ corresponds to a **primitive assembly**
- An amplitude $\psi(\lambda)$ encodes the **resonant response strength** of the assembly λ
- The magnitude $|\psi(\lambda)|$ represents its **participation strength**
- The phase $\arg \psi(\lambda)$ corresponds to the **oscillatory alignment** of λ within a coherent superposition
- Interference phenomena arise from the **net resonance** of assemblies whose oscillatory modes align or misalign

This provides a physical meaning to the algebraic constraints used in the paper. In particular:

Interference requires that oscillatory alignment be parameterised continuously, forcing amplitudes to come from a field supporting continuous phase—already pointing toward $\mathbb{C}$.

Galois invariance expresses the physical requirement that rephasing or reparametrising the internal oscillatory mode of an assembly does not change observable resonance intensities. Thus, automorphisms of the amplitude domain correspond to redundancies in resonance description.

Isotropy expresses the requirement that resonance behaviour cannot introduce preferred directions tied to arbitrary labelling of configurations; the SO(3) automorphisms of $\mathbb{H}$ fail this condition.

The **Hilbert space inner product** emerges naturally as the quantity that measures net resonance:

$$\langle\psi|\varphi\rangle = \Sigma\_\{\lambda\in\Lambda\}\ \psi^*(\lambda)\ \varphi(\lambda)$$

representing the total resonance overlap of assemblies ψ and φ.

RAL therefore provides a physical substrate interpretation of the mathematical structures derived in this paper: complex amplitudes represent oscillatory assemblies, and the Hilbert-space inner product represents their resonance structure. This perspective reinforces the uniqueness of the complex field as the only amplitude domain compatible with resonant assembly behaviour, Galois invariance, and distinguishability geometry.

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3. Amplitudes and the Requirement of Interference

Plain language summary: To describe quantum interference—where possibilities can reinforce or cancel each other—we need "amplitudes" that carry phase information. The question is: what number system should these amplitudes use? We don't assume the answer; we'll derive it.

3.1 Why Amplitudes?

Classical probability is additive: if something can happen two ways, you add the probabilities. But quantum mechanics exhibits **interference**: amplitudes from different paths combine, and the combination can be larger (constructive) or smaller (destructive) than either alone.

The double-slit experiment illustrates this dramatically. An electron can reach a point on the screen via two paths. Classically, more paths means more electrons. Quantum mechanically, the amplitudes can cancel, creating dark bands where *fewer* electrons arrive than if one slit were blocked.

To describe interference mathematically, we need amplitudes with:

- **Magnitude**: How "big" the amplitude is
- **Phase**: A directional property that allows cancellation

The question is: what algebraic structure F provides these features?

3.2 Candidate Amplitude Fields

We consider three natural candidates:

Real numbers ($\mathbb{R}$): The simplest choice. Real numbers have magnitude (absolute value) but only discrete phase (positive or negative).

Complex numbers ($\mathbb{C}$): Numbers of the form a + bi where $i^2 = -1$. They have continuous phase—any angle from 0° to 360°.

Quaternions ($\mathbb{H}$): A four-dimensional system with three imaginary units (i, j, k). Even richer structure than complex numbers.

Why these three? Other algebraic structures exist—finite fields, p-adic numbers, algebraic extensions of $\mathbb{Q}$—but they fail our requirements earlier in the analysis:

- **Finite fields** (e.g., $\mathbb{F}_p$) lack a notion of magnitude compatible with probability. There is no natural ordering or absolute value that could ground the Born rule.

- **p-adic numbers** ($\mathbb{Q}_p$) have ultrametric structure incompatible with the Archimedean property required for probabilities to sum meaningfully. The Taylor Limit (analyticity over $\mathbb{R}$) also excludes them directly.
- **Algebraic extensions of** $\mathbb{Q}$ other than $\mathbb{C}$ (e.g., $\mathbb{Q}(\sqrt{2})$) either lack closure under the operations needed for interference or have Galois groups incompatible with our constraints.
- **Larger division algebras** (octonions, sedenions) are non-associative, breaking the composition requirement for reversible dynamics.

Frobenius's theorem (1878) establishes that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the *only* finite-dimensional associative division algebras over $\mathbb{R}$. Thus our argument is automatically complete with respect to all possible amplitude domains satisfying algebraic closure and analytic structure—there are no other candidates to consider. Our Galois-invariance criterion then uniquely selects $\mathbb{C}$ from this exhaustive list.

We will show that **only $\mathbb{C}$ is viable**. Real numbers are too simple; quaternions are too complicated. Complex numbers are uniquely selected by consistency requirements.

3.3 Requirements on the Amplitude Field

The amplitude field F must support:

1. **Interference**: Amplitudes combine in ways that allow constructive and destructive interference with continuous phase relationships
2. **Reversible composition**: The isometry group G acts on amplitudes coherently
3. **Isotropy**: Relabelling configurations (permuting Λ) shouldn't change the physics
4. **Probability invariance**: Observable predictions must be invariant under internal symmetries of F

Clarification on isotropy: Isotropy applies when the distinguishability metric treats configurations symmetrically—i.e., $d(\lambda_i, \lambda_j)$ depends only on $i \neq j$, not on which specific configurations are involved. This is the natural setting for systems like spin states, energy levels, or computational basis states where no configuration is intrinsically preferred. For systems with asymmetric distinguishability (e.g., position states where $x = 0$ and $x = 1$ are physically different), the isometry group G is correspondingly smaller, and isotropy applies only to those permutations that are genuine isometries. Our argument requires only that *when* configurations are symmetrically distinguishable, *then* permutation invariance holds. The quaternion exclusion uses this: for three symmetrically-placed configurations, isotropy under S_3 conflicts with SO(3) Galois invariance.

The last requirement—**Galois invariance**—is the key to selecting $\mathbb{C}$.

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4. Galois-Theoretic Selection of the Amplitude Field

Plain language summary: This is the heart of the paper. We use a powerful principle: physics shouldn't depend on arbitrary mathematical conventions. Specifically, if the number system has internal symmetries (automorphisms), physical predictions must be invariant under them. This principle eliminates all possibilities except complex numbers.

4.1 What is an Automorphism?

An **automorphism** of a number system is a symmetry—a way of rearranging the system that preserves all its algebraic structure (addition, multiplication, etc.).

Example: For complex numbers, consider complex conjugation:

$\sigma(a + bi) = a - bi$

This swaps i with -i throughout. Remarkably, it preserves arithmetic:

- $\sigma(z_1 + z_2) = \sigma(z_1) + \sigma(z_2)$
- $\sigma(z_1 \cdot z_2) = \sigma(z_1) \cdot \sigma(z_2)$

Complex conjugation is an automorphism. In fact, for continuous maps, it's the **only** non-trivial automorphism of $\mathbb{C}$ (besides the identity).

The automorphism group $\mathrm{Aut}(F/\mathbb{R})$ consists of all automorphisms of F that fix the real numbers pointwise. We fix $\mathbb{R}$ because probabilities are real-valued; any physically meaningful symmetry must leave probability values unchanged.

4.2 The Galois Invariance Principle

We impose a fundamental requirement:

Galois Invariance: Observable predictions must be invariant under all automorphisms of the amplitude field.

Formally: If $\sigma \in \mathrm{Aut}(F/\mathbb{R})$, then all physical predictions for amplitude α must equal those for $\sigma(\alpha)$.

Why should we believe this?

This is a natural extension of a basic principle in physics: **predictions can't depend on arbitrary conventions**.

Consider coordinates: physics doesn't depend on whether you use Cartesian or polar coordinates. The mathematics looks different, but predictions are the same.

Similarly, if our number system has internal symmetries, physics shouldn't depend on which "version" we're using. Two amplitude assignments related by an automorphism represent the same physical situation.

Analogy: Imagine measuring temperature. You could use Celsius or Fahrenheit—different conventions, same physics. Now imagine a number system with an internal symmetry like conjugation. Using z or $\bar{z}$ is analogous to using Celsius or Fahrenheit—if they're related by a symmetry, they must give the same predictions.

4.3 Regularity Constraint: Continuous Automorphisms Only

Technical point: The complex numbers technically admit "wild" discontinuous automorphisms, constructed using the axiom of choice. These are mathematical curiosities with no physical relevance—they're incompatible with any reasonable notion of continuity.

The Taylor Limit rules them out: any automorphism compatible with analytic probability functionals must be continuous.

Upshot: For $\mathbb{C}$, the only physically relevant automorphisms are identity and conjugation.

Automorphism Regularity: We restrict to automorphisms compatible with the analytic structure required by the Taylor Limit. Under this constraint, $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{\text{identity, conjugation}\}$.

4.4 Case 1: Real Numbers Are Excluded

For $F = \mathbb{R}$, the automorphism group $\mathrm{Aut}(\mathbb{R}/\mathbb{R})$ is trivial—it contains only the identity.

Consequence: No non-trivial phase transformations exist.

Why this is fatal: Interference requires continuous phase. In the double-slit experiment, the interference pattern varies smoothly as you move across the screen. This requires amplitudes whose phase can take any value, not just "positive" or "negative."

Real numbers have only discrete phase: +1 or -1. You can get complete cancellation (destructive interference) when one amplitude is +1 and another is -1. But you cannot get the continuous gradation of interference—the smooth transition from bright to dark—that experiments reveal.

In everyday terms: Real numbers are like having only two volume settings: on or off. Complex numbers are like having a continuous dial. Quantum interference requires the continuous dial.

Conclusion: Under our axioms, $F = \mathbb{R}$ cannot support the required interference structure. Real amplitudes are excluded.

Note on real quantum mechanics: Stueckelberg and others have developed "real Hilbert space" formulations of quantum mechanics. These can reproduce interference phenomena, but only through dimension doubling: a complex n-dimensional Hilbert space is re-encoded as a real 2n-dimensional space, with the complex structure implicit in a privileged antisymmetric operator J satisfying $J^2 = -I$. This J effectively reintroduces $\mathbb{C}$ in disguise. Under our Galois-invariance requirement, such formulations are equivalent to complex quantum mechanics—the distinction is representational, not physical. Our exclusion of $\mathbb{R}$ as the *fundamental* amplitude field is thus compatible with this literature: real QM works precisely because it secretly embeds $\mathbb{C}$.

4.5 Case 2: Quaternions Are Excluded

Theorem (Quaternionic Symmetry Clash). *Let P: $\mathbb{H}^n \to \mathbb{R}$ be analytic, invariant under (i) permutations S_n acting on indices, and (ii) algebra automorphisms Aut($\mathbb{H}/\mathbb{R}$) $\cong$ SO(3) acting diagonally on components. Then P depends only on $\Sigma_i\, |\psi_i|^2$. Hence P cannot encode phase-sensitive interference.*

The remainder of this section proves this theorem constructively.

Quaternions $\mathbb{H}$ are a four-dimensional number system: q = a + bi + cj + dk, where $i^2 = j^2 = k^2 = ijk = -1$.

Note: Quaternions are a non-commutative division algebra, not a field. We consider algebra automorphisms Aut($\mathbb{H}/\mathbb{R}$) rather than Galois groups in the strict sense.

The automorphism group: Aut($\mathbb{H}/\mathbb{R}$) $\cong$ SO(3)

This is the group of three-dimensional rotations, acting on the imaginary subspace {bi + cj + dk}. Every 3D rotation corresponds to a quaternion automorphism.

Why this is fatal: The automorphism group is too large, creating a conflict with isotropy.

Explicit construction of the isotropy violation:

Consider $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ with the maximally symmetric metric $d(\lambda_a, \lambda_\beta) = 1$ for $a \neq b$. Isotropy requires that all permutations of Λ be physical symmetries—relabelling configurations should not change predictions.

Assign quaternionic amplitudes: $\psi(\lambda_1) = i$, $\psi(\lambda_2) = j$, $\psi(\lambda_3) = k$.

Under the permutation $\sigma = (12)$ swapping $\lambda_1 \leftrightarrow \lambda_2$, the amplitudes become (j, i, k).

Under Galois invariance, predictions must also be invariant under Aut($\mathbb{H}/\mathbb{R}$). Consider the automorphism R corresponding to 90° rotation about the k-axis, which maps $i \to j$, $j \to -i$, $k \to k$.

Applying R to the original (i, j, k) gives (j, -i, k).

For the probability functional P to be invariant under both σ and R independently:

- P(i, j, k) = P(j, i, k) [isotropy under σ]
- P(i, j, k) = P(j, -i, k) [Galois invariance under R]

Combined: P(j, i, k) = P(j, -i, k), so P is insensitive to sign changes in the second component.

Applying the full constraint set from $S_3 \times SO(3)$ forces P to depend only on $|\psi(\lambda_1)|^2 + |\psi(\lambda_2)|^2 + |\psi(\lambda_3)|^2$—the total norm. This destroys interference: P cannot distinguish states differing only in relative phase. Any attempt to restore phase-sensitivity violates either isotropy or Galois invariance.

For complex numbers: The automorphism group {id, conjugation} is abelian and acts uniformly on all components. Conjugation commutes with all permutations, so no such conflict arises.

Formal statement:

Exclusion of ℍ: Let $|\Lambda| \geq 3$ with symmetric distinguishability metric. Quaternionic amplitudes cannot simultaneously satisfy Galois invariance under Aut(ℍ/ℝ) ≅ SO(3) and isotropy under $S_{|\Lambda|}$ while preserving phase-sensitive interference. Hence quaternionic amplitudes are excluded.

4.6 Case 3: Complex Numbers Are Uniquely Selected

For F = ℂ, under the regularity constraint:

Aut(ℂ/ℝ) = {identity, conjugation}

This minimal automorphism group:

- **Supports interference**: ℂ has continuous phase
- **Preserves isotropy**: Conjugation commutes with all permutations
- **Is compatible with regularity**: Both automorphisms are continuous

The Goldilocks principle: Complex numbers are "just right."

- Real numbers are **too simple**: trivial automorphism group, no continuous phase
- Quaternions are **too complicated**: SO(3) automorphism group conflicts with permutation symmetry
- Complex numbers are **just right**: two-element automorphism group, compatible with all requirements

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Summary of Assumptions. We now collect the assumptions used in the derivation:

1. **(Λ, d)** is a finite metric space with $|\Lambda| \geq 3$ (the configuration space of distinguishable microstates)
2. **Reversible dynamics** are isometries of (Λ, d), forming a group G
3. **Taylor Limit**: the probability functional is analytic, continuous, and effectively finite
4. **Interference**: the amplitude field F supports continuous phase structure
5. **Galois invariance**: observable predictions are invariant under $\mathrm{Aut}(F/\mathbb{R})$
6. **Isotropy**: physics is invariant under permutations of configuration labels ($S|\Lambda|$)

Under these assumptions, we have:

Theorem 1 (Selection of $\mathbb{C}$). *The complex numbers $\mathbb{C}$ are the unique amplitude field compatible with Assumptions 1–6.*

Proof: The argument is distributed through §§4.4–4.6. Real numbers are excluded (§4.4) because $\mathrm{Aut}(\mathbb{R}/\mathbb{R})$ is trivial, precluding continuous phase. Quaternions are excluded (§4.5) because $\mathrm{Aut}(\mathbb{H}/\mathbb{R}) \cong SO(3)$ conflicts with isotropy under $S|\Lambda|$—the explicit construction shows that joint invariance destroys phase-sensitivity. Complex numbers satisfy all requirements: $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \mathrm{conjugation}\}$ supports interference while commuting with permutations. Other fields are excluded by Frobenius's theorem and the requirements of §3.2. ■

What this theorem says in plain English: If you want amplitudes that allow interference, respect the symmetries of both the number system and the configuration space, and behave smoothly—then complex numbers are your only option. Not by assumption, but by mathematical necessity.

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5. Emergence of Hilbert Space

Plain language summary: Having established that amplitudes must be complex numbers, we now show that the natural mathematical home for quantum states is a complex Hilbert space. This structure wasn't assumed—it emerges from the requirements we've imposed.

5.1 From Amplitudes to State Space

An amplitude assignment is a function $\psi: \Lambda \to \mathbb{C}$, assigning a complex amplitude to each configuration in Λ.

The space of all such assignments is $\mathbb{C}|\Lambda|$—the set of $|\Lambda|$-tuples of complex numbers. This is naturally a **complex vector space**:

- You can add amplitude assignments: $(\psi + \varphi)(\lambda) = \psi(\lambda) + \varphi(\lambda)$

- You can multiply by complex scalars: $(c\cdot\psi)(\lambda) = c\cdot\psi(\lambda)$

This is already half of Hilbert space: a complex vector space.

5.2 The Inner Product from Distinguishability

To complete the Hilbert space structure, we need an **inner product**—a way to measure overlap between states.

The natural choice, arising from the distinguishability geometry, is:

$\langle\psi|\varphi\rangle = \Sigma_{\{\lambda\in\Lambda\}}\ \psi^*(\lambda)\ \varphi(\lambda)$

What does this mean?

The inner product sums over all configurations, multiplying the complex conjugate of one amplitude by the other. This measures "how much" the two amplitude assignments overlap.

Key properties:

- **$\langle\psi|\psi\rangle \geq 0$**: The inner product of a state with itself is non-negative
- **$\langle\psi|\psi\rangle = 0$ implies $\psi = 0$**: Only the zero state has zero norm
- **$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$**: Swapping order conjugates the result
- **Sesquilinearity**: Linear in the second argument, antilinear in the first

Why this inner product?

The inner product is constrained by the requirement that it be **invariant under the isometry group G**. Reversible physical transformations (isometries of the distinguishability metric) must preserve inner products. The standard form above is the unique (up to scaling) inner product with this property.

5.3 Hilbert Space as the Unique Structure

The resulting space $(\mathbb{C}^{|\Lambda|}, \langle\cdot|\cdot\rangle)$ is a **finite-dimensional complex Hilbert space**.

Lemma (Uniqueness of G-Invariant Inner Product). *Let G act transitively on Λ and let V = $\mathbb{C}^{|\Lambda|}$ with basis $\{|\lambda\rangle\}$. Any G-invariant sesquilinear form on V is proportional to the standard inner product $\langle\psi|\varphi\rangle = \Sigma_\lambda\ \psi(\lambda)\varphi(\lambda)$.**

Proof sketch: Let $B(\cdot,\cdot)$ be any G-invariant sesquilinear form. Define $\bar{B}$ by averaging over G: $\bar{B}(\psi,\varphi) = (1/|G|)\Sigma_g B(U_g\ \psi, U_g\ \varphi)$. This $\bar{B}$ is G-invariant by construction. Transitivity of G on Λ forces $\bar{B}(|\lambda\rangle,|\lambda\rangle)$ to be independent of λ. Schur's lemma then implies uniqueness on each irreducible component of the G-representation. For transitive actions, V decomposes with a trivial representation appearing exactly once, fixing $\bar{B}$ up to scaling. ■

Theorem 2 (Emergence of Hilbert Space). *Under Assumptions 1–6, with G acting transitively on Λ:*

1. *The unique amplitude field is ℂ (Theorem 1)*
2. *The natural state space is the complex vector space* $\mathbb{C}^{|\Lambda|}$
3. *The inner product* $\langle\psi|\varphi\rangle = \Sigma\_\lambda\ \psi(\lambda)\varphi(\lambda)$ is the unique G-invariant inner product (up to scaling)*
4. *The resulting structure is (up to isomorphism) a finite-dimensional complex Hilbert space*

Proof: (1) follows from Theorem 1. (2) is immediate: amplitude assignments $\psi: \Lambda \to \mathbb{C}$ form a vector space under pointwise operations. For (3), G-invariance requires $\langle U\_g\ \psi|U\_g\ \varphi\rangle = \langle\psi|\varphi\rangle$ for all $g \in G$. With G transitive on Λ, Schur's lemma implies the invariant sesquilinear form is unique up to scaling; the standard form satisfies this. (4) follows: $\mathbb{C}^{|\Lambda|}$ with this inner product is a finite-dimensional complex inner product space, hence a Hilbert space. ■

What this theorem says in plain English: Start with distinguishable configurations. Require interference, symmetry under field automorphisms, and consistency with the physical symmetries of the system. The mathematical structure that emerges is exactly Hilbert space—complex vector space with inner product. This is the arena where quantum mechanics lives, and we've derived it rather than assuming it.

5.4 Representations of the Symmetry Group

The isometry group G acts on Λ, permuting configurations while preserving distinguishability. This induces an action on $\mathbb{C}^{|\Lambda|}$:

$(U\_g\ \psi)(\lambda) = \psi(g^{-1}\lambda)$

This action is **unitary**: it preserves the inner product.

$\langle U\_g\ \psi \mid U\_g\ \varphi\rangle = \langle\psi|\varphi\rangle$

The upshot: Reversible dynamics are represented by unitary operators on Hilbert space—exactly as in standard quantum mechanics.

5.5 Dynamics: The Hamiltonian as Isometry Generator

Having derived the Hilbert space structure, we now show how dynamics emerges naturally from the isometry group.

The standard picture. In conventional quantum mechanics, the Hamiltonian H appears in the Schrödinger equation:

$i\hbar\ \partial|\psi(t)\rangle/\partial t = H|\psi(t)\rangle$

This formulation assumes continuous time, a fundamental Hilbert space, and complex amplitudes as primitives—all of which we have now derived rather than assumed.

The geometric picture. In our framework, the system occupies a point in the distinguishability manifold Λ. As the system evolves, this point moves along trajectories generated by the isometry group G. The Hamiltonian is simply the **infinitesimal generator** of this geometric flow.

Formally, consider a one-parameter subgroup $\{\varphi_t\} \subset G$ of isometries (a Lie subgroup). Its unitary representation $U(\varphi_t)$ on $\mathbb{C}^{|\Lambda|}$ satisfies $U(\varphi_t)U(\varphi_s) = U(\varphi_{t+s})$, and the generator is:

$$H = i\hbar \, (d/dt) \, U(\varphi_t)|_{t=0}$$

By **Stone's theorem**, any strongly continuous one-parameter unitary group $U(t)$ on a Hilbert space has a unique self-adjoint generator H such that $U(t) = \exp(-iHt/\hbar)$. Thus the Hamiltonian is not postulated but emerges as the Lie-algebra generator of the isometry group's unitary representation.

Interpretation:

- H corresponds to a vector field on the distinguishability manifold
- $e^{-iHt/\hbar}$ corresponds to flow along that vector field
- Energy eigenvalues correspond to stable oscillatory patterns in the flow

Connection to RAL. In a RAL implementation, diagonal entries of H correspond to mode frequencies and off-diagonal entries to couplings between assemblies; standard control primitives (WAIT, DETUNE, COUPLE) implement the corresponding one-parameter unitary families.

Summary. The Hamiltonian is not a fundamental postulate but emerges as the generator of tick-driven geometric isometries on the distinguishability manifold. It is linearised into matrix form for representational convenience and maps directly into resonant hardware via RAL. This completes the dynamical picture: Hilbert space provides the arena, unitary operators provide the allowed moves, and the Hamiltonian generates continuous families of such moves.

Note on novelty: The Hamiltonian's emergence as a Lie algebra generator is standard—this is textbook material once one has a Lie group acting unitarily on Hilbert space. Our contribution is not the dynamics but the *arena*: we derived the complex Hilbert space on which this standard Lie-theoretic machinery operates. The novelty is in proving that this arena is uniquely forced by distinguishability geometry and automorphism invariance.

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6. Relation to Previous Reconstruction Programmes

Plain language summary: We're not the first to ask where Hilbert space comes from. Here we compare our approach to other major programmes, highlighting what's distinctive about our contribution.

6.1 The Reconstruction Landscape

Several research programmes have sought to derive quantum structure from more primitive principles:

Programme	Starting Point	Derives ℂ?
Hardy (2001)	Operational axioms	Selects via simplicity
Chiribella–D'Ariano–Perinotti	Purification axiom	Constrains, doesn't derive
Dakić–Brukner (2011)	Information axioms	Constrains, doesn't derive
Masanes–Müller (2011)	Physical principles	Assumes ℂ
This work	Distinguishability geometry	**Derives from Galois invariance**

Solèr's theorem (1995) establishes that under natural orthomodular lattice assumptions, Hilbert spaces must be defined over ℝ, ℂ, or ℍ. Our automorphism-invariance criterion then uniquely selects ℂ within this class. Thus our work complements lattice-theoretic approaches: Solèr restricts the menu to three items; our Galois argument selects the unique viable entry.

6.2 What's New Here

Previous approaches either assume complex amplitudes or select them via simplicity/elegance criteria. Hardy's "simplicity" axiom, for instance, says: among theories that work, pick the simplest. This *selects* ℂ but doesn't *explain* it.

Our approach derives ℂ from a symmetry principle: Galois invariance. Complex numbers aren't the simplest choice (that would be ℝ) or the most general (that might be ℍ). They're the **unique** choice compatible with interference, isotropy, and invariance under field automorphisms.

This is a stronger result. We don't just select ℂ from a menu—we prove the menu has only one item.

6.3 Comparison with Hardy (2001)

Hardy derives quantum theory from five operational axioms, including "simplicity" which effectively picks ℂ over ℝ.

Key differences:

Aspect	Hardy	This work
Starting point	Abstract state spaces	Distinguishability geometry
Selection of $\mathbb{C}$	Simplicity axiom	Galois invariance (derived)
Mathematical character	Operational/axiomatic	Geometric/symmetry-based

Our work can be seen as providing a **geometric underpinning** for Hardy's operational axioms. We explain *why* complex numbers satisfy his axioms better than alternatives.

6.4 Comparison with Chiribella–D'Ariano–Perinotti

The CDP framework uses the "purification axiom"—every mixed state can be seen as part of a pure state on a larger system. This constrains the state space strongly, but doesn't single out $\mathbb{C}$ uniquely.

Our contribution: We provide a Galois-theoretic argument that $\mathbb{C}$ is forced, not merely consistent.

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7. Discussion

7.1 What We've Achieved

We have derived the complex Hilbert space structure from:

1. **Distinguishability geometry**: A metric space of configurations
2. **Interference requirement**: Amplitudes need continuous phase
3. **Galois invariance**: Predictions invariant under field automorphisms
4. **Isotropy**: Compatibility with configuration permutations
5. **Regularity**: Analytic, continuous probability functionals

From these principles, we proved:

- Complex numbers are the unique viable amplitude field
- The state space is necessarily $\mathbb{C}|\Lambda|$
- The natural inner product makes this a Hilbert space
- Reversible dynamics act as unitary operators

None of this was assumed. The standard mathematical apparatus of quantum mechanics—complex Hilbert space with unitary dynamics—**emerged** from more primitive requirements.

7.2 The Role of Symmetry

Our derivation parallels how symmetry principles constrain structure elsewhere in physics:

Theory	Symmetry	Emergent Structure
General Relativity	Local Lorentz invariance	Spacetime curvature
Standard Model	Gauge symmetry	Force fields
This work	Galois invariance + isotropy	Complex Hilbert space

Einstein didn't assume curved spacetime—he derived it from the requirement that physics look the same in all reference frames. Similarly, we don't assume Hilbert space—we derive it from the requirement that physics be invariant under field automorphisms and configuration permutations.

7.3 Relation to the Born Rule

In our companion paper, we derive the Born rule $P = |\psi|^2$ from coarse-graining constraints on the distinguishability geometry. That derivation assumes amplitudes exist but doesn't determine their algebraic structure.

This paper fills the gap: it determines that amplitudes must be complex. Together, the two papers provide:

- **This paper**: Why $\mathbb{C}$ (Galois invariance)
- **Companion paper**: Why $|\psi|^2$ (coarse-graining geometry)

The full reconstruction derives both the algebraic structure and the probability rule from distinguishability principles.

7.4 Open Questions

Several questions remain for future work:

1. **Infinite dimensions**: Our derivation uses finite Λ. How does it extend to infinite-dimensional separable Hilbert spaces?

Proposition (Finite-to-Infinite Extension). *If every finite-dimensional subspace of a separable Hilbert space $\mathscr{H}$ obeys the Galois-invariance and isotropy constraints, then $\mathscr{H}$ is a complex Hilbert space.*

Sketch: The Galois argument (Theorem 1) applies to any finite-dimensional subspace $V \subset \mathscr{H}$, establishing that V must be complex. A separable Hilbert space is the completion of an increasing union of finite-dimensional subspaces $V_1 \subset V_2 \subset \cdots$, each complex. The complex structure is preserved under limits (the field operations are continuous), so the completion inherits it. The inner product extends by continuity. ■

A fully rigorous treatment would invoke the spectral theorem to decompose infinite-dimensional operators into finite-dimensional invariant subspaces where our argument applies directly.

2. **Tensor products**: Can the tensor product structure for composite systems be derived from distinguishability geometry on product configuration spaces?
3. **Relativistic extension**: How does distinguishability geometry connect to spacetime structure, and can relativistic quantum field theory be reconstructed?
4. **Measurement dynamics**: Can the projection postulate be derived from the interaction between system and apparatus within this framework?
5. **Quantum algorithms**: Can the full range of quantum algorithms (QPE, Shor, quantum walks, HHL) be systematically recast in distinguishability geometry? Preliminary analysis suggests yes—these algorithms become procedures for extracting holonomy frequencies, harmonic decomposition, and geometric diffusion over graph-structured manifolds.

Note that the question of dynamics (Schrödinger equation, Hamiltonian) is addressed in §5.5: the Hamiltonian emerges as the generator of the isometry group action on the Hilbert space.

7.5 Assumption Minimality

A natural question is which assumptions are essential and which are conveniences. We distinguish:

Essential assumptions (removing any one breaks the derivation):

- Automorphism invariance: predictions unchanged under field automorphisms
- Interference requirement: amplitudes must support continuous phase
- Isotropy for symmetric metrics: permutation invariance when configurations are equally distinguishable
- Taylor Limit: continuity and analyticity of probability functionals

Convenient but potentially relaxable:

- Finite Λ: extendable to separable Hilbert spaces via limits (see §7.4.1)
- Strong isotropy: full S_n invariance; weaker versions tied to specific isometry groups may suffice
- Associativity of amplitude algebra: assumed via Frobenius, but octonion-based alternatives might be explored (at the cost of non-associative dynamics)

This classification shows that our assumptions are not arbitrary packaging—each does specific work in the derivation. The essential assumptions express physical requirements (symmetry, interference, regularity); the convenient ones are technical choices that simplify the presentation.

7.6 Interpretation via Resonant Assembly Language

An appealing interpretation of our results is provided by **Resonant Assembly Language (RAL)**, a conceptual framework (developed in companion work) in which physical configurations form assemblies with oscillatory modes. Amplitudes represent the strength and phase of these oscillations. Interference corresponds to the net resonance of assemblies.

Note: RAL is presented here as an interpretive framework that gives physical intuition for the mathematical structures, not as established physics. Its value lies in connecting the abstract Galois derivation to concrete physical pictures (oscillators, resonance, mode coupling) that may guide experimental implementation.

Under this picture:

- **Complex numbers** encode oscillatory modes of assemblies
- **Conjugation** corresponds to reversing oscillatory orientation
- **Galois invariance** corresponds to invariance under reparametrisations of assembly oscillations
- **Hilbert space** emerges as the natural geometry of resonance overlap

This provides a physical substrate for the mathematical derivation: quantum mechanics is the unique resonance calculus compatible with the geometry of distinguishability and the symmetry constraints we impose.

7.7 Information-Theoretic Interpretation

The Galois invariance requirement admits an information-theoretic interpretation. Field automorphisms identify amplitude assignments that differ only in physically irrelevant "gauge" structure—details that do not affect observable predictions. Requiring invariance under these automorphisms is equivalent to stripping away redundant representational degrees of freedom.

From this perspective, complex Hilbert space emerges as the *minimal faithful encoding* of distinguishability geometry: it contains exactly enough structure to reproduce interference and measurement statistics, with no superfluous parameters. Real amplitudes are too impoverished (they cannot encode continuous phase); quaternionic amplitudes carry excess structure (the SO(3) worth of imaginary directions) that must be artificially constrained to recover consistent predictions.

This suggests a deeper principle: the mathematical formalism of quantum mechanics may be understood as the Kolmogorov-optimal compression of the pattern of distinguishable change. Galois invariance is not merely an aesthetic constraint but a compression constraint—it demands that the formalism carry no representational baggage beyond what is needed to faithfully encode the physics. The complex Hilbert space, selected uniquely by our axioms, is the shortest code for the distinguishability geometry.

8. Conclusion

We have shown that complex Hilbert space—the mathematical arena of quantum mechanics—emerges from distinguishability geometry and symmetry principles.

The key insight is Galois invariance: physical predictions must be invariant under automorphisms of the amplitude field. This requirement, combined with the need for interference and isotropy under configuration permutations, **uniquely selects complex numbers** as the amplitude field.

Real numbers are excluded because they lack continuous phase—their trivial automorphism group cannot support interference patterns.

Quaternions are excluded because their large automorphism group (SO(3)) conflicts with the permutation symmetry of distinguishable configurations.

Complex numbers are uniquely selected: their two-element automorphism group {identity, conjugation} is rich enough for interference yet constrained enough for isotropy.

With complex amplitudes established, **Hilbert space emerges** as the natural representation space. The inner product is fixed by G-invariance; unitary dynamics follow from the isometry group structure; and the **Hamiltonian emerges** as the infinitesimal generator of this dynamics.

As an illustration, we applied the framework to **Grover's search algorithm** (Appendix B), showing that the $\sqrt{N}$ speedup is a geometric rotation bound arising naturally from the structure of the distinguishability manifold.

This derivation demotes Hilbert space from a primitive postulate to a derived consequence. The complex numbers are not an arbitrary choice but a mathematical necessity. The structure of quantum mechanics is not mysterious—it is forced by the geometry of distinguishability and the requirement of consistent symmetry.

Appendix A: Technical Details

A.1 Continuous Automorphisms of $\mathbb{C}$

The field $\mathbb{C}$ admits discontinuous automorphisms constructed via Zorn's lemma (permuting a transcendence basis of $\mathbb{C}$ over Q). These are nowhere measurable and incompatible with any topology.

Under the Taylor Limit—requiring analytic probability functionals—only continuous automorphisms are admissible. The continuous automorphism group is:

$\mathrm{Aut}(\mathbb{C}/\mathbb{R})$_continuous = {id, conjugation}

Proof sketch: A continuous automorphism σ must fix $\mathbb{R}$ pointwise (by assumption) and preserve the metric topology. Since $\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$, we have $\sigma(i) = \pm i$. If $\sigma(i) = i$, then σ = id. If $\sigma(i) = -i$, then σ = conjugation.

A.2 The Quaternion Automorphism Group

For $\mathbb{H} = \{a + bi + cj + dk\}$, the algebra automorphism group $\mathrm{Aut}(\mathbb{H}/\mathbb{R})$ consists of maps fixing $\mathbb{R}$ and permuting $\{i, j, k\}$ in a way compatible with multiplication.

The constraints $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, $ki = j$ are preserved exactly by orthogonal transformations of the imaginary subspace that preserve orientation. This gives:

$\mathrm{Aut}(\mathbb{H}/\mathbb{R}) \cong SO(3)$

Each rotation $R \in SO(3)$ acts by: $\sigma\_R(a + bi + cj + dk) = a + (R\cdot v)\cdot(i,j,k)$ where $v = (b, c, d)$.

A.3 The Symmetry Clash (Detailed Proof)

Let $|\Lambda| = 3$ with configurations $\{\lambda_1, \lambda_2, \lambda_3\}$ and symmetric metric $d(\lambda_a, \lambda_\beta) = 1$ for $a \neq b$. Isotropy requires invariance under S_3; Galois invariance requires invariance under $\mathrm{Aut}(\mathbb{H}/\mathbb{R}) \cong SO(3)$.

Explicit construction: Consider $\psi = (i, j, k)$.

Let $\sigma = (12) \in S_3$ (swapping $\lambda_1 \leftrightarrow \lambda_2$), giving $\sigma\psi = (j, i, k)$.

Let $R \in SO(3)$ be 90° rotation about the k-axis, so R: $i \to j$, $j \to -i$, $k \to k$. Then $R\psi = (j, -i, k)$.

For probability functional P to satisfy both invariances:

- Isotropy: $P(i, j, k) = P(j, i, k)$
- Galois: $P(i, j, k) = P(j, -i, k)$

Combining: $P(j, i, k) = P(j, -i, k)$.

Repeating with other permutations and rotations, P must be invariant under arbitrary sign changes in any component. Further, rotations mixing components force P to depend only on total magnitude. The only functional satisfying all constraints is $P(\psi) = f(|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2)$—invariant under the full O(3) acting diagonally on magnitudes. (Pathological non-measurable functionals might formally evade this, but such constructions are excluded by the Taylor Limit's analyticity requirement.)

The problem: This P cannot distinguish states with different relative phases. For interference, we need P sensitive to phase relationships (as in $|\psi_1 + \psi_2|^2 \neq |\psi_1|^2 + |\psi_2|^2$). But any phase-sensitive P violates one of the symmetry requirements.

Conclusion: Quaternionic amplitudes cannot simultaneously satisfy Galois invariance, isotropy, and phase-sensitivity. ■

For complex amplitudes, $\mathrm{Aut}(\mathbb{C}/\mathbb{R})$ = {id, conjugation} is abelian and acts identically on all components. Conjugation commutes with all permutations, so no clash occurs. Phase-sensitive functionals like $P = |\psi_1 + \psi_2|^2$ satisfy both invariances.

Appendix B: Worked Example — Grover's Algorithm

This appendix is not part of the core derivation; it illustrates how the framework applies to a standard quantum algorithm and demonstrates that the reconstruction has computational content.

To illustrate how the framework applies to quantum computation, we analyse Grover's search algorithm through the lens of distinguishability geometry.

B.1 Setup

Let $\Lambda = \{0, 1, ..., N-1\}$ be a database of N distinguishable items, one of which (call it ω) is "marked" as the search target. The distinguishability metric treats all items symmetrically—they are equally distinguishable from each other.

The initial state is the symmetric superposition:

$$|s\rangle = (1/\sqrt{N})\ \Sigma_i\ |i\rangle$$

In geometric terms, this represents the **centre of the distinguishability manifold**—maximal symmetry, no item preferred.

B.2 The Oracle as Local Holonomy

The oracle O identifies the marked item by applying a phase flip:

$$O|\omega\rangle = -|\omega\rangle \quad O|i\rangle = |i\rangle \text{ for } i \neq \omega$$

In matrix form: $O = I - 2|\omega\rangle\langle\omega|$

Geometric interpretation: The oracle introduces a **local holonomy**—a phase distortion in the region corresponding to ω. Distinguishability magnitudes are unchanged; only the phase structure is modified. The system becomes slightly asymmetric, preparing it for amplification.

In RAL terms: This is a PHASE_FLIP instruction applied to mode ω with angle π.

B.3 Diffusion as Reflection About the Mean

The diffusion operator performs "inversion about the mean":

$D = 2|s\rangle\langle s| - I$

Geometric interpretation: This is a **reflection across the symmetric centre** |s⟩. When the oracle has broken symmetry, D reflects the configuration across |s⟩, pulling the marked region outward and increasing its amplitude magnitude.

In RAL terms: This is a REFLECT_ABOUT_MEAN macro, implemented by coupling all modes to a shared bus representing |s⟩, evolving for a calibrated duration, then decoupling.

B.4 Grover Iteration as Geometric Rotation

Although Λ contains N items, the algorithm evolves entirely within the 2D subspace spanned by |ω⟩ and |s⟩.

Let θ satisfy $\sin(\theta) = 1/\sqrt{N}$. Each Grover iteration $G = D\cdot O$ rotates the state toward |ω⟩ by angle 2θ. After k iterations:

$P(\text{success}) = \sin^2((2k+1)\theta)$

Optimal $k \approx (\pi/4)\sqrt{N}$ yields near-unit success probability.

Why √N? The initial overlap with |ω⟩ is $\langle\omega|s\rangle = 1/\sqrt{N}$. Each iteration rotates by $O(1/\sqrt{N})$, so $O(\sqrt{N})$ iterations are needed to reach |ω⟩. This is a **geometric bound**, not a mysterious quantum speedup—it follows directly from the structure of rotations in the distinguishability geometry.

B.5 Matrix Mechanics as Linear Representation

The matrix formalism is a **linear representation** of geometric operations:

Geometric Operation	**Matrix Form**
Local holonomy at ω	$O = I - 2|\omega\rangle\langle\omega|$
Reflection about centre	$D = 2|s\rangle\langle s| - I$
Combined rotation	$G = D\cdot O$

Both O and D are unitary because they preserve the distinguishability geometry. The Hilbert space inner product ensures that geometric magnitudes (which determine probabilities via the Born rule) are preserved under these transformations.

B.6 RAL Implementation

The complete Grover algorithm in Resonant Assembly Language:

```
INIT_UNIFORM Λ                // Prepare symmetric state |s⟩
REPEAT k TIMES:               // k ≈ (π/4)√N
   PHASE_FLIP mode=ω, angle=π  // Oracle: mark target
   REFLECT_ABOUT_MEAN Λ       // Diffusion: amplify
MEASURE register=Λ            // Born projection → classical outcome
```

The underlying resonant hardware performs geometric transformations through wave dynamics. RAL exposes these operations as programmable instructions, bridging the abstract derivation to physical implementation.

B.7 Summary

Grover's algorithm illustrates the framework in action:

- **Λ** = database items (distinguishable configurations)
- **Amplitudes** = complex numbers (derived from Galois invariance)
- **Oracle** = local phase modification (holonomy in the geometry)
- **Diffusion** = reflection about symmetric centre
- **Speedup** = geometric rotation bound, not quantum magic
- **Measurement** = Born-rule projection from geometry to probability

The algorithm works because complex Hilbert space is the unique structure supporting interference, symmetry, and consistent probability—exactly what we derived in the main text.

B.8 Extensions to Other Algorithms

The geometric interpretation extends naturally to other quantum algorithms. Quantum Phase Estimation extracts holonomy frequencies from reversible flow; the Quantum Fourier Transform provides harmonic decomposition of distinguishability geometry; Shor's algorithm detects the resonance period of modular arithmetic loops; quantum walks implement geometric diffusion over graph-structured manifolds; and the HHL algorithm performs geometric inversion via eigenphase extraction. These applications, to be developed in a companion work, demonstrate that the framework provides a unified geometric foundation for quantum computation, not merely an alternative interpretation of individual algorithms.

Glossary

Amplitude: A value assigned to a configuration, used to calculate probabilities. We prove amplitudes must be complex numbers.

Automorphism: A symmetry of an algebraic structure—a map preserving all operations. For $\mathbb{C}$, the continuous automorphisms are identity and conjugation.

Configuration space (Λ): The finite set of distinguishable states of a physical system.

Distinguishability metric (d): A distance function on Λ encoding how distinguishable configurations are.

Galois invariance: The requirement that physical predictions be unchanged under automorphisms of the amplitude field.

Frobenius's theorem: The classification result (1878) showing $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only finite-dimensional associative division algebras over $\mathbb{R}$.

Hamiltonian: The infinitesimal generator of a one-parameter subgroup of isometries acting on Hilbert space. Emerges from the geometric structure rather than being postulated.

Hilbert space: A complex vector space with an inner product. We derive this structure rather than assuming it.

Inner product: A sesquilinear form $\langle\cdot|\cdot\rangle$ measuring overlap between states.

Isometry: A transformation preserving the distinguishability metric. Reversible dynamics are isometries.

Isotropy: The property that physics doesn't depend on how configurations are labelled.

Kolmogorov complexity: The length of the shortest program that produces a given output. Used here informally to suggest that complex Hilbert space is the minimal encoding of distinguishability geometry.

Quaternions ($\mathbb{H}$): A four-dimensional non-commutative division algebra. Excluded by the symmetry clash argument.

Resonant Assembly Language (RAL): A physical interpretation framework in which configurations form assemblies with oscillatory modes, amplitudes encode resonance strengths and phases, and interference arises from net resonance.

Solèr's theorem: The result (1995) showing that infinite-dimensional orthomodular spaces satisfying certain axioms must be Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Stone's theorem: The result (1932) that every strongly continuous one-parameter unitary group has a unique self-adjoint generator (the Hamiltonian).

Taylor Limit: A regularity condition ensuring smooth, well-behaved physics.

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