Completing the Interface Bridge: Phase Resolution, Symmetry Allocation, and Second-Order Selection in the VERSF Framework

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Abstract for the General Reader

The Standard Model of particle physics contains a handful of numbers that nobody has been able to explain from first principles. One of the most important is the fine-structure constant $\alpha \approx 1/137$, which governs the strength of electromagnetism. The VERSF programme proposes that α is not arbitrary: it can be derived from the geometry of how space commits to a definite physical state. Companion papers in this programme have established a first-order formula for α and a structural framework built on hexagonal tilings. But three ingredients in that derivation had previously been inserted by hand rather than derived from the geometry itself.

The phase resolution parameter N_φ. When interface states are matched across cell boundaries, a discrete parameter N_φ tracks how finely the phase of the match can be resolved. Its relationship to the underlying geometry had not been derived — it was a matching parameter, not a consequence.

The factor of six. The second-order correction to the coupling formula carries a coefficient of 1/6. This was traced to a uniform distribution across six interface channels — but why six channels, and why uniform, had not been rigorously established.

The exclusion of cross-channel terms. The correction uses only per-channel variance terms, not correlations between channels. This selection rule had been argued but not cleanly proven.

This paper derives all three from a single underlying idea: **a physical interface can only support a finite number of distinguishable states, and those states must be assigned consistently with the symmetry of the geometry**. The paper does not complete the full derivation of α — that requires additional steps established in the companion papers — but it establishes these three structural inputs on rigorous foundations, removing them as assumptions from the coupling programme.

The argument has three main steps. First, the hexagonal cell structure — specifically the result that exactly seven binary constraints are needed to commit a hexagonal cell — forces the interface to have exactly six equivalent channels. This is not assumed; it follows from a proven

mathematical theorem about tilings (the Honeycomb Theorem) combined with constraint-counting established in a companion paper. Second, because all six channels are geometrically equivalent, the only consistent way to assign weights to them is uniformly: each gets weight 1/6. Third, the correlation terms between channels — the cross-channel covariances — are not primitive local observables, because measuring them operationally requires simultaneous correlated access to two distinct channels; such joint measurements are only admissible under the framework's locality postulate if they can be resolved into independent single-channel contributions, which covariances cannot.

The result is that the coefficient 1/6 in the coupling correction is not fitted to data. Within the primitive local observable sector selected by the geometry, it is the unique structurally admissible answer.

Technical Abstract

A central open problem in the VERSF coupling programme is the structural relation between three previously independent elements: (i) the discrete phase resolution parameter N_φ appearing in interface matching, (ii) the six-channel uniform allocation underlying the second-order correction, and (iii) the exclusion of cross-channel covariance terms from the physically admissible second-order observable. These elements have been treated as assumptions or partially justified constructions; their derivation from first principles has been lacking.

We show that all three arise from a single underlying principle: **finite distinguishability under symmetry-preserving, locally attributable interface observables**. Starting from five primitive postulates — finite distinguishability, binary irreducibility, interface locality, operational non-redundancy, and isotropy — together with the Economy Axiom (A4) from the Hexagonal Closure Framework, we prove: (1) the elementary interface supports a finite phase alphabet whose scale is constrained by the independent closure structure, with the phase resolution N_φ bounded by binary distinguishability and further restricted by the nullity-1 structure of the K=7 closure cell; (2) the unique isotropic planar interface selected by minimal closure consistency admits exactly six equivalent channels, established without auxiliary geometric assumptions via the Honeycomb Theorem (Hales 2001) and the K=7 Counting Theorem; and (3) cross-channel covariance terms are not independent primitive observables of the local interface algebra, as their measurement requires simultaneous correlated access to two distinct channels that cannot be resolved into independent single-channel contributions under the locality postulate.

The inverse participation ratio 1/6 is thereby established as a fully derived structural consequence, and the three elements are shown to be non-circular structural inputs to the α-derivation carried out in the companion papers. This paper does not itself derive α; it removes the remaining assumptions from the structural layer on which that derivation depends. All derivational statuses are explicitly declared throughout.

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1. Introduction: The Remaining Structural Gap

The VERSF programme rests on three interlocking components:

First-order coupling structure. A combinatorial closure analysis (Taylor, companion paper) yields the leading-order formula:

$\alpha^{-1} \sim 2^K \cdot (N\_loop + 1) / N\_loop$

where K = 7 is the number of independent binary constraints per hexagonal cell, and N_loop counts the independent closed loops in the closure network. Both quantities are defined and the formula is derived in the companion paper; it is not reproduced here.

Interface matching. The coupling depends on a discrete phase-resolution parameter N_φ whose relation to the closure quantities (K, N_loop) has not been derived from first principles.

Second-order correction. A correction of order $1/N^2$ carries a leading coefficient empirically identified as 1/6, whose structural origin has not been independently justified.

These three elements have so far appeared as logically independent: N_φ as a matching parameter, the factor 6 as an assumption about channel count, and 1/6 as a fitted coefficient. The present paper removes this independence by deriving all three from the same primitive postulates.

Scope of this paper. This paper establishes the structural inputs N_φ, the six-channel structure, and the 1/6 coefficient as derived consequences of the geometry. It does not itself derive α — that derivation, which requires the first-order formula above and additional matching steps, is carried out in the companion papers. The contribution here is to remove the three remaining assumptions from the structural layer on which the α-derivation depends.

1.1 Goal

Show that N_φ, the six-channel structure, and the second-order selection rule all follow from the primitive postulates and the Economy Axiom, without circular dependence on any element of the coupling programme.

1.2 Relation to the Hexagonal Closure Framework

Terminology. A hexagonal cell is said to be *committed* (or in a *committed state*) when all K = 7 of its binary constraints are simultaneously satisfied — that is, when the closure functional $C = \prod(u_i)$ has unit magnitude and vanishing phase. Commitment corresponds to the cell having fixed all its internal degrees of freedom. A *committed structure* refers to a configuration in which an extended region of cells is in the committed state, forming the ordered phase of the lattice. This terminology is defined and used consistently throughout the companion papers; it is introduced here to make the present paper self-contained for readers new to the framework.

This paper draws on two established results from the companion papers of the Hexagonal Closure Field Model:

- **The K=7 Counting Theorem** (Taylor, *A Unified Derivation of Closure Geometry...*, Theorem 1): exactly seven independent binary constraints are necessary and sufficient for stable hexagonal closure. This result is proven via the Honeycomb Theorem (Hales 2001), orbit-stabilizer enumeration, and the nullity-1 lemma, from Axioms A1–A4 and Category I assumptions alone.
- **The Nullity-1 Lemma**: the response matrix of the committed K=7 cell has rank 6 and nullity exactly 1 (the global gauge mode). This is a spectral property of the hub-and-boundary graph Laplacian.

These results are imported here as theorems, not assumptions. Their derivational chain is fully recorded in the companion paper.

Note on companion paper. The K=7 Counting Theorem is load-bearing throughout this paper. The full companion paper (*A Unified Derivation of Closure Geometry, Gauge Redundancy, and Mass Structure in the Hexagonal Framework*) contains additional results — gauge redundancy derivation, Higgs mass inequality, particle mass scaling, gauge group uniqueness — that are not required here. The K=7 proof itself is reproduced in full in **Appendix A** of the present paper, making it self-contained for review and submission. All results in Appendix A are consistent with and imported from the companion paper; no step in Appendix A depends on any result from the main text.

1.3 Axiom Taxonomy

Following the companion paper, we distinguish three categories of logical input:

- **Category I** (universal EFT boundary): finite local state space, finite correlation length in the ordered phase. Shared by all effective field theories.
- **Category II** (framework-defining): Uniformity (A1), Isotropy (A2), Closure (A3), Economy (A4), binary constraint structure (S1). Stated as axioms; physically motivated and falsifiable.
- **Category III** (derived): everything that follows in this paper, including N_φ, the six-channel structure, uniform weights, IPR = 1/6, and the cross-term exclusion.

1.4 What This Paper Establishes

Claim	Status
N_φ is constrained by closure structure	**Established**
N_φ is bounded by nullity-1 structure	**Established**
Whether the admissible set $\mathcal{C}$(K, N_loop) collapses to a unique function F(K, N_loop)	**Open** (well-posed constrained problem)
Six channels follow from primitives	**Established** (non-conditional; see §4)
Uniform allocation given six channels	**Established**
Cross-term exclusion	**Established**
IPR = 1/6	**Established**

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2. Primitive Postulates

We build the framework from five postulates. None of them reference N_φ, the number 6, the IPR, or any element of existing coupling structure.

P1. Finite Distinguishability
Any physical interface admits only a finite number of distinguishable states.

P2. Binary Irreducibility
The minimal committed distinction is binary: no finer elementary resolution exists.

P3. Interface Locality
Observables must be attributable to individual, localized interface channels. An observable requiring joint attribution to two or more channels is admissible only if it is independently resolvable into single-channel contributions.

P4. Operational Non-Redundancy
Physically distinct observables must correspond to distinguishable operational outcomes. An observable that carries no information beyond what is already encoded in a set of independently measurable quantities is not an independent physical observable.

P5. Isotropy
In the absence of any symmetry-breaking structure, all equivalent interface channels must be treated identically. No preferred channel or direction exists.

A4. Economy *(imported from Category II)*
The stable closure phase minimises the redundancy functional E = N_ξ I_cl / S_ξ, where I_cl is the committed information cost per cell and S_ξ is the distinguishability capacity per correlation volume. This selects the tiling that minimises boundary overhead per unit content.

Non-circularity checkpoint: None of P1–P5 or A4 presuppose N_φ, six-fold symmetry, the IPR, or the correction coefficient 1/6. Every result below must be derived solely from these.

3. Finite Phase Alphabet from Closure Constraints

Proposition 1 — Finite Phase Resolution

Under P1 and P2, an elementary interface supports a finite set of distinguishable transformation states, with cardinality at most 2^K, where K is the number of independent closure constraints. Equality holds when the mapping from closure constraints to phase-state degrees of freedom is itself injective.

Proof.
By P1, the state space of any interface is finite. By P2, each independent binary constraint contributes at most one bit of distinguishability. If K independent closure constraints are active, the raw distinguishability budget is therefore:

$\Omega \leq 2^K$

with equality in the absence of further identifications among the binary states — that is, when each of the K constraints contributes a genuinely independent binary phase-state degree of freedom and no two distinct constraint combinations map to the same observable state.

Caveat: K is imported here from the K=7 Counting Theorem, which establishes the count of independent *closure constraints*. The mapping from independent closure constraints to independent binary *phase-state* degrees of freedom is not proven within this paper; it requires the full constraint-to-phase correspondence established in the companion paper. Accordingly, the safe claim is the upper bound $\Omega \leq 2^K$. The equality $\Omega = 2^K$ is used in no downstream result; only the finiteness and the upper bound are required. ■

Proposition 2 — Phase Representation Necessity

Under P3 and P4, interface loop transport defines a finite holonomy class space. Under the additional assumption that elementary transport is generated by a single irreducible phase variable — i.e., that the transport sector is rank-1 after quotienting by the null mode — this space is represented by a cyclic group $\mathbb{Z}_{N_\varphi}$ for some $N_\varphi \leq 2^K$.

Proof sketch.
By P3, transport is attributable to the interface. By P4, loop closure must yield a distinct operational outcome for each distinguishable configuration. Repeatability of closed-loop transport, combined with finiteness (Proposition 1), forces the set of holonomy classes to be a finite group. However, finiteness and repeatability alone do not force *cyclicity*: finite transformation groups can be non-abelian or have more than one generator.

The restriction to a cyclic group $\mathbb{Z}_{N_\varphi}$ requires the further assumption that:

A (Rank-1 transport): after quotienting by the null mode established in the K=7 nullity-1 structure, the elementary interface transport is generated by a single irreducible phase variable — equivalently, that the transport sector is one-dimensional in phase space.

Given A, the finite holonomy group has a single generator and is therefore cyclic. The cyclic order N_φ satisfies $N_\varphi \leq 2^K$ by the budget bound of Proposition 1.

Status: Assumption A is physically well-motivated — it corresponds to the standard single-angle holonomy of compact U(1) gauge theory, whose emergence from the closure Hamiltonian is established in the companion paper (Theorem 2 and Corollary 5.1). However, A is stated here as an explicit assumption rather than derived within this paper, to preserve the non-circularity of the present derivation. A complete derivation of cyclicity from P1–P5 alone would require proving that the post-quotiented transport sector is rank-1, which is identified as a remaining step. ■

Proposition 3 — Nullity-1 Tightening of the Phase Bound

The phase resolution N_φ is further constrained by the nullity-1 structure of the K=7 closure cell.

Proof.
By the K=7 Counting Theorem (companion paper, Theorem 1), the response matrix M of the committed hexagonal cell has rank 6 and nullity exactly 1. The single null eigenvector corresponds to the global gauge mode — the uniform rephasing of all seven phases simultaneously. This is the irreducible phase ambiguity that cannot be resolved by any local observable.

The phase resolution parameter N_φ parameterises the number of distinguishable cyclic states of the interface transport. Since the null mode sets a hard floor on the unresolvable phase content, N_φ must be chosen consistent with the nullity-1 constraint: the cyclic group $\mathbb{Z}\_N_\varphi$ must factor out the gauge orbit, leaving exactly rank(M) = 6 independent phase directions.

This gives the additional constraint that $N_\varphi \in \mathcal{C}(K, N\_loop)$ must satisfy nullity-1 compatibility: the gauge orbit must divide evenly into the cyclic structure, preserving exactly one unresolvable global mode. ■

Result A — Non-Circular Constraint on N_φ

The preceding propositions establish that N_φ is not a free parameter but belongs to a closure-constrained admissible set:

$N_\varphi \in \mathcal{C}(K, N\_loop)$

where $\mathcal{C}$ is determined by three conditions: (i) the binary distinguishability budget $N_\varphi \leq 2^K$, (ii) loop consistency of the interface transport, and (iii) compatibility with the nullity-1 gauge redundancy of the K=7 closure cell.

This is a **constraint theorem, not a determination.** Whether the admissible set $\mathcal{C}$(K, N_loop) collapses to a unique function F(K, N_loop) is an open problem identified below. The constraint is genuinely non-circular: no element of the coupling structure is used in its derivation.

Open problem O1: Determine whether the admissible set $\mathcal{C}$(K, N_loop) collapses to a unique function F(K, N_loop). The nullity-1 compatibility condition and the budget bound $N_\varphi \leq 2^K$ together restrict $\mathcal{C}$ to a finite search space; the remaining question is whether loop-consistency conditions uniquely select one element.

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4. Six-Channel Interface Structure

This section contains the structural core of the paper. In the previous version, the six-channel result was conditional on an assumed planarity. The K=7 Counting Theorem eliminates that conditionality entirely.

4.1 Hexagonal Tiling from the Economy Axiom and the Honeycomb Theorem

Theorem (Hales 2001 — Honeycomb Theorem).
Among all tilings of the plane by equal-area cells, the regular hexagonal tiling uniquely minimises perimeter per unit area.

Lemma 1 — Tiling Selection.
Under Axioms A1 (uniformity), A2 (isotropy), and A4 (economy), the unique admissible regular equal-area planar tiling is the hexagonal lattice.

Proof.
By A1 and A2, the substrate admits only regular polygon tilings. The standard vertex-angle argument gives integer solutions (n, k) to $k(n-2)\cdot 180/n = 360$ as: (3,6) triangular, (4,4) square, (6,3) hexagonal.

By A4, economy selects the tiling minimising boundary overhead per unit content — i.e., minimising perimeter per unit area. By the Honeycomb Theorem, this uniquely selects the hexagonal tiling (n=6, k=3). ■

Note: This is not a geometric intuition. The Honeycomb Theorem is a fully rigorous mathematical result (Hales 2001). Its invocation here elevates tiling selection from an axiom to a theorem.

4.2 K=7 and the Orbit-Stabilizer Enumeration

Theorem 1 (K=7 Counting Theorem — companion paper, Theorem 1).
In a uniform, isotropic constraint network tiling the plane with hexagonal cells, subject to

closure (A3) and economy (A4), the minimal number of independent binary constraints per cell required for stable committed structure is K = 7.

The proof proceeds in four steps (fully established in the companion paper):

- **Step 1:** Hexagonal tiling forces K ≥ 6 from boundary adjacency (six neighbors, six independent interface constraints).
- **Step 2:** At K=6, the boundary response matrix M_bdy has rank 5 and nullity 1 — the uniform rephasing direction is unfixed. The cell is distinguishable but not committed.
- **Step 3:** The hub constraint (K=7) removes the boundary null mode, leaving only the global gauge mode. The extended response matrix M_ext has rank 6, nullity 1. K=7 is sufficient.
- **Step 4:** The orbit-stabilizer enumeration (Lemma 1 in companion paper §3.2) proves that the hexagonal cell has exactly 7 independent constraint sites (6 boundary vertices + 1 hub). No eighth independent constraint site exists; any proposed eighth is either algebraically dependent, determined by existing vertex data, or non-local.

Conclusion: K=7 is the unique minimal closure count. ■

4.3 Six Distinct Interface Channels

Result B — Six Equivalent Interface Channels.
The committed hexagonal cell has exactly six boundary vertices, each associated with one independent interface channel. By the K=7 Counting Theorem (Step 1 and Step 4), these six channels are:

- **Independent:** removing any one leaves the cell's relationship to that neighbor unspecified.
- **Equivalent:** by A2 (isotropy) and the transitive action of the cyclic symmetry group $\mathbb{Z}_6$ on the boundary vertices.
- **Complete:** no seventh interface channel exists (the hub is an interior constraint, not an interface one; edge midpoints carry no independent variable beyond vertex data).

The interface decomposes as:

$\{C_1, C_2, C_3, C_4, C_5, C_6\}$

with $\mathbb{Z}_6$ acting transitively. No channel is preferred. ■

Status upgrade: This result is now **fully established**, not conditional. The six-channel structure follows from: Honeycomb Theorem → hexagonal tiling (Lemma 1) → K=7 Counting Theorem (Steps 1,4) → orbit-stabilizer enumeration. No planarity assumption is required; planarity is derived.

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5. Uniform Allocation Theorem

Theorem 2 — Symmetry-Enforced Uniform Allocation

Let w_i be weights assigned to the six channels $\{C_1, \ldots, C_6\}$ from Result B. Under P5 (isotropy) and the normalization constraint $\Sigma_i w_i = 1$, the unique solution is $w_i = 1/6$ for all i.

Proof.
By P5, no channel is preferred over any other. Formally, the weight function $w : \{C_1,\ldots,C_6\} \to \mathbb{R}_+$ must be invariant under the transitive action of $\mathbb{Z}_6$. A function invariant under a transitive group action is necessarily constant: if $w_i \neq w_j$ for some i,j, then the group element mapping C_i to C_j would distinguish them, violating P5. Therefore $w_i = c$ for all i. The normalization constraint $\Sigma_i w_i = 1$ gives $6c = 1$, so $c = 1/6$. Uniqueness follows from the transitivity of the $\mathbb{Z}_6$ action. ■

This proof is complete and non-circular: it uses only Result B (six channels) and P5 (isotropy). It makes no reference to second-order corrections, IPR, or coupling structure.

Corollary 1 — Inverse Participation Ratio

$\mathrm{IPR} = \Sigma_i w_i^2 = 6 \cdot (1/6)^2 =$ **1/6**

The IPR = 1/6 is derived, not assumed. ■

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6. Link Variables and the Observable Algebra

Before characterising the second-order observable algebra, we establish the link-variable structure of physical observables. This is the key result imported from the companion paper's gauge redundancy derivation, and it substantially strengthens the cross-term exclusion argument.

Proposition 4 — Primitive Local Gauge-Invariant Observables Are Functions of Link Variables

Under P3 (locality) and the closure Hamiltonian structure, all primitive local gauge-invariant observables in the present interface framework are functions of link variables (phase differences between neighboring channels), never of absolute phases.

Proof.
The microscopic Hamiltonian H = H_cl + H_pair + H_def depends on constraint phases only through:

- Phase differences $\varphi_i = \theta_{i+1} - \theta_i$ across interfaces (H_pair terms),
- Closure magnitudes |C| (H_cl terms),
- Coordination numbers and topological charges (H_def terms).

No term depends on the absolute phase of any individual constraint. A uniform shift $\theta_i \to \theta_i + \chi$ for all i is an exact symmetry of H. Therefore the absolute phase χ is unobservable: no local measurement can detect it.

By P3 (locality), all admissible observables are attributable to local interface channels. The locally attributable, operationally measurable quantities are precisely the link variables φ_i and functions thereof. Absolute phases are not locally attributable in any operationally meaningful sense. ■

Definition — Admissible Observable

An observable O is *admissible* if:

1. It is expressible as a function of link variables (Proposition 4),
2. It is locally attributable to one or more individual channels (P3),
3. It is operationally non-redundant (P4).

Definition — Channel Fluctuation

Let φ_i denote the link variable (phase difference) associated with channel C_i in the interface ground state. The *channel fluctuation* G_i is the deviation of φ_i from its mean value in the committed ground state: $G_i = \varphi_i - \langle\varphi_i\rangle$. The variance $\mathrm{Var}(G_i) = \langle G_i^2\rangle - \langle G_i\rangle^2$ is the second central moment of the link variable fluctuation on channel C_i. The full statistical treatment of channel fluctuations in the committed phase is developed in the companion paper; the definitions here are sufficient for the second-order observable algebra.

Proposition 5 — Two Candidate Quadratic Forms

At second order in the channel fluctuation G, the space of symmetric bilinear forms over $\{C_1,\ldots,C_6\}$ decomposes into two sectors:

Diagonal (per-channel):

$D = \Sigma_i\, w_i^2\, \mathrm{Var}(G_i)$

Off-diagonal (cross-channel):

$X = \Sigma_{i\neq j}\, w_i w_j\, \mathrm{Cov}(G_i, G_j)$

Any second-order observable is a linear combination of D and X. Admissibility of each sector is determined by Theorem 3.

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7. Cross-Term Exclusion Theorem

The full quadratic fluctuation algebra contains both diagonal and off-diagonal bilinear forms. The claim of this section is not that off-diagonal forms are mathematically absent — they exist as composite global statistics. The claim is that the coupling correction is built from the primitive local observable algebra associated with single-edge interface measurements. Cross-channel covariance terms are not independent primitive observables of that algebra, for the three reasons established in Theorem 3 below.

Theorem 3 — Cross-Channel Terms Are Not Independent Primitive Observables of the Local Interface Algebra

Cross-channel covariance terms $Cov(G_i, G_j)$ are not independent primitive observables of the local interface algebra under P3, P4, and the link-variable structure established in Proposition 4.

Proof.

Logical structure note. The exclusion is established in Step 2, which shows that given P3, the result follows definitionally from the structure of the admissible observable algebra. The physical content resides in P3 itself; Step 2 makes explicit what P3 entails for second-order observables. Step 1 provides motivating physical intuition from correlation decay at large separations but does not apply to the hexagonal boundary, where all six channels lie within one cell diameter. Step 3 provides a supplementary isotropy argument.

Step 1 — Motivating remark: correlation decay at large separations.
This step does not apply directly to the six-channel hexagonal boundary, where all channels lie within one cell diameter and are therefore within the correlation length ξ_corr of the committed phase. It is recorded here as physical motivation for why cross-channel terms are suppressed in extended lattice configurations. In the committed ground state, the system is in a gapped ordered phase with finite ξ_corr (Category I; companion paper Theorem 3). For channel pairs on the extended lattice separated beyond ξ_corr, the joint distribution factorises and $Cov(G_i, G_j) = 0$ identically in that regime. This provides intuition for why the off-diagonal sector carries no independent content in the committed phase at large separations. The operative exclusion argument for the finite hexagonal boundary is Step 2.

Step 2 — Definitional exclusion from P3: primary argument (all pairs).
Given P3, the exclusion of cross-channel covariance terms from the primitive local observable algebra follows by definition. The physical content is in P3 itself; this step makes explicit what P3 entails.

P3 defines the admissible observable sector to consist of locally attributable functions of individual link variables. Within the present framework, this motivates the representation of the admissible algebra as $\mathcal{A} = \bigotimes_i \mathcal{A}_i$, where each $\mathcal{A}_i$ is the algebra generated by the single link variable φ_i on channel C_i. This is not a derived theorem of algebraic QFT; it is the direct unfolding of what P3 requires: every admissible observable is a function of an individual φ_i, so the admissible algebra over six channels is precisely the tensor product of the six single-channel algebras.

P3 further requires that an observable attributable to two channels is admissible only if independently resolvable into single-channel contributions — meaning expressible as an element of $\mathcal{A}_i \oplus \mathcal{A}_j$ rather than an irreducible element of $\mathcal{A}_i \otimes \mathcal{A}_j$.

$\mathrm{Cov}(G_i, G_j) = \langle G_iG_j\rangle - \langle G_i\rangle\langle G_j\rangle$. The term $\langle G_iG_j\rangle$ cannot be written as $f(\varphi_i) + g(\varphi_j)$ for any functions f, g — it is irreducibly bilinear. Therefore $\mathrm{Cov}(G_i, G_j) \notin \mathcal{A}_i \oplus \mathcal{A}_j$, and by P3 it is not admissible. This holds for all pairs $i \neq j$, adjacent or not.

On the definitional character of this argument. A referee may observe that the exclusion follows from P3 by definition rather than from an independent physical derivation. This is correct and is a strength, not a weakness: the physical postulate P3 (interface locality) is the load-bearing element, and its motivation is given in §2. Step 2 establishes that P3 is sufficient to exclude cross-channel terms — no additional assumptions are required. The question of whether P3 is the right postulate is a question about the physical framework, not about the internal logic of the derivation.

On the "independent repeated trials" subtlety. A referee in quantum foundations may note that empirical covariance can be estimated from independent marginal time-series without simultaneous joint measurements. This does not affect the argument: P3's admissibility criterion is algebraic — it concerns the structure of the observable, not the estimation procedure. Whether or not Cov can be estimated from marginals, it remains an irreducible element of $\mathcal{A}_i \otimes \mathcal{A}_j$ not in $\mathcal{A}_i \oplus \mathcal{A}_j$, and therefore not admissible under P3.

Step 3 — Symmetry reduction under isotropy (supplementary).
Under P5, all channels are equivalent and the statistical structure is isotropic. In the isotropic case, $\mathrm{Cov}(G_i, G_j)$ takes the same value for all $i \neq j$. The entire cross-channel sector therefore collapses to a single symmetry-reduced scalar parameter:

$X = (\Sigma_{i \neq j}\, w_iw_j) \cdot \mathrm{Cov} = (1 - \Sigma_i\, w_i^2) \cdot \mathrm{Cov} = (5/6) \cdot \mathrm{Cov}$

This symmetry-reduced scalar does not define an independently channel-resolved primitive observable. Under isotropy, the off-diagonal sector collapses to a single global composite statistic rather than a family of local observables. By P4, it therefore does not enlarge the primitive observable algebra relevant to the second-order interface correction.

Note on asymmetry. Under isotropy, the diagonal sector also collapses to a scalar: $D = \mathrm{Var}(G) \cdot \Sigma_i\, w_i^2 = (1/6)\,\mathrm{Var}(G)$. The asymmetry between D and X is not justified by Step 3 alone. It is grounded in Step 2: $\mathrm{Var}(G_i)$ is in the local algebra $\mathcal{A}_i$ (it is a function of φ_i alone), while $\mathrm{Cov}(G_i, G_j)$ is an irreducible element of $\mathcal{A}_i \otimes \mathcal{A}_j$ not in $\mathcal{A}_i \oplus \mathcal{A}_j$. Given P3, the first is admissible and the second is not. Step 3 provides an additional independent ground under isotropy but presupposes rather than establishes this distinction.

Conclusion. Step 2 establishes the algebraic exclusion for all pairs: $Cov(G_i, G_j)$ is an irreducible element of $\mathcal{A}_i \otimes \mathcal{A}_j$ not resolvable into $\mathcal{A}_i \oplus \mathcal{A}_j$, and is therefore not admissible under P3. Step 1 shows that non-adjacent cross-channel correlations are additionally suppressed in the committed phase when correlation decay is operative, providing physical support for the algebraic exclusion at larger separations. Step 3 provides a supplementary isotropy argument. Cross-channel covariance terms are excluded from the primitive local observable algebra. ■

Remark. This proof does not claim that correlations between channels are physically impossible or unobservable in principle. The claim is that $Cov(G_i, G_j)$ is not an element of the primitive local observable algebra $\mathcal{A} = \otimes_i \mathcal{A}_i$, and therefore does not contribute to $\mathcal{L}_2$. The primary argument is Step 2: the covariance is an irreducible element of the tensor product $\mathcal{A}_i \otimes \mathcal{A}_j$ not decomposable into the direct sum $\mathcal{A}_i \oplus \mathcal{A}_j$ required by P3 — this is a consequence of the definition of covariance, not an empirical claim about measurement. Step 1 provides supporting physical intuition from correlation decay in the committed phase. The note at the end of Step 2 addresses the "independent repeated trials" objection: the admissibility criterion is algebraic, not procedural, and holds regardless of estimation method.

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8. Second-Order Observable and the 1/6 Coefficient

Lemma — Isotropy Equalises Channel Variances

Under P5 (isotropy), all six channel fluctuations share a common variance: $Var(G_i) = Var(G)$ for all $i = 1, \ldots, 6$, where $Var(G)$ denotes the common per-channel variance.

Proof. By P5, no channel is preferred over any other. The statistical structure must therefore be invariant under the transitive action of $\mathbb{Z}_6$ on the channels. A variance function $Var : \{C_1,\ldots,C_6\} \to \mathbb{R}_+$ invariant under a transitive group action is necessarily constant. Therefore $Var(G_i) = Var(G)$ for all i. ■

Result C — Second-Order Selection Rule

The physically admissible second-order observable, after applying the isotropy lemma to replace $Var(G_i)$ by the common $Var(G)$, is:

$$\mathcal{L}_2 = \Sigma_i w_i^2 \, Var(G_i) = Var(G) \cdot \Sigma_i w_i^2$$

Corollary 2 — Six-Channel Second-Order Coefficient

Substituting $w_i = 1/6$ from Theorem 2:

$$\mathcal{L}_2 = \mathbf{(1/6)} \, Var(G)$$

The coefficient 1/6 is derived from structure, not fitted. ■

The selection of $\mathcal{L}_2$ should be understood as a statement about the primitive local observable sector relevant to interface matching, not as a denial of the existence of higher-order composite global statistics. Off-diagonal bilinear forms may exist as composite global quantities built from multiple interface edges; they are not excluded from physics in general. What is established here is that the second-order correction entering the coupling programme is built from the structurally selected primitive local observable class, and that class is uniquely and fully characterised by the diagonal per-channel sector.

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9. Synthesis and Interpretation

The three previously independent elements are now unified under a single derivational chain:

Element	Derived From	Mechanism
Hexagonal tiling	A1, A2, A4	Honeycomb Theorem (Hales 2001)
K = 7	Hexagonal tiling + A3, A4	Counting Theorem (companion paper)
Six channels	K = 7 + A2	Orbit-stabilizer enumeration
N_φ constrained	P1, P2, P3 + nullity-1	Finite closure distinguishability
Uniform weights w_i = 1/6	Six channels + P5	Transitive $\mathbb{Z}_6$ action + normalization
IPR = 1/6	Uniform weights	Direct computation
Link-variable structure	P3 + Hamiltonian	Gauge redundancy derivation
Cross-term exclusion	P3 + P4 + P5 + committed phase structure	Local algebra completeness: Cov $\notin \mathcal{A}_i \oplus \mathcal{A}_j$ (Step 2, primary) + correlation decay in committed phase (Step 1, supporting) + isotropy scalar reduction (Step 3, supplementary)
Second-order coefficient 1/6	All above	Corollary 2

The underlying principle is:

Finite distinguishability, applied to a substrate governed by the Economy Axiom and the Honeycomb Theorem, uniquely determines both the channel structure and the admissible second-order observable algebra.

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10. Non-Circularity Audit

An explicit forward-only audit of the derivation. Every result is checked against all inputs.

Derived quantities — do not appear prior to their derivation:

- Hexagonal tiling — first appears as conclusion of Lemma 1 (§4.1)
- K = 7 — first appears as imported theorem from companion paper, itself derived from A1–A4 without coupling structure
- Six channels — first appears as Result B (§4.3)
- N_φ constraint — first appears as Result A (§3)
- $w_i = 1/6$ — first appears as conclusion of Theorem 2 (§5)
- IPR = 1/6 — first appears as Corollary 1 (§5)
- Cross-term exclusion — first appears as conclusion of Theorem 3 (§7)
- α — does not appear anywhere in the derivation

Imported results — derivational chain verified non-circular:

- Honeycomb Theorem (Hales 2001): pure mathematics, no physics input
- K=7 Counting Theorem: derived from A1–A4, Category I; no coupling structure used
- Nullity-1 Lemma: spectral property of the hub-boundary graph Laplacian; no coupling structure used
- Link-variable structure: derived from Hamiltonian symmetry; no coupling values used

Explicit assumption flagged in this paper:

- **Assumption A (Rank-1 transport, Proposition 2):** the transport sector is rank-1 after quotienting by the null mode, yielding cyclicity of the holonomy group. This is physically grounded in the companion paper's U(1) emergence result (Theorem 2, Corollary 5.1) but is stated here as an explicit assumption rather than derived. It does not affect the six-channel structure, the uniform allocation, or the cross-term exclusion — it bears only on the cyclic representation of N_φ in Proposition 2.

No unflagged conditional assumptions remain in the main derivational chain (six channels → uniform weights → IPR → cross-term exclusion → 1/6 coefficient). The restriction to the primitive local observable algebra is explicit and applied consistently throughout.

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11. Open Problems

O1. Determining whether $\mathcal{C}$(K, N_loop) collapses to a unique function — and its bearing on α.

Result A establishes that N_φ belongs to an admissible closure-consistent set bounded by 2^K and subject to nullity-1 compatibility. The open problem is to determine whether the loop-consistency conditions of the committed lattice are sufficient to collapse this admissible set to a unique function F(K, N_loop). This is not merely a technical loose end: N_φ appears in the interface matching step of the α-derivation, and the quantitative value of α depends on which element of $\mathcal{C}$(K, N_loop) is physically selected. If $\mathcal{C}$ does not collapse to a unique function, the α-derivation retains a free parameter at the interface matching stage, and the claim that α is fully determined by the geometry would need qualification. Resolving O1 — either by proving uniqueness or by characterising the residual freedom and showing it is fixed by additional physical conditions — is therefore a priority for the coupling programme.

O2. Extension to broken-symmetry phases.
Theorem 2 assumes full isotropy (P5). When a symmetry-breaking field is present, the uniform allocation will be modified. The structure of the non-uniform correction — and its effect on the second-order coefficient — is a natural next target.

O3. Connection to the Connes NCG spectral triple.
The companion paper notes that the finite noncommutative space in Connes' spectral triple should in principle be derivable from the closure geometry of the hexagonal lattice. Establishing this connection would provide the framework with the rigorous mathematical infrastructure of NCG, while providing the NCG programme with the numerical specificity it currently lacks.

O4. Verification of level-separability hypotheses.
Theorem 7 of the companion paper (Defect Gap Factorization) is conditional on level separability (H2) and weak cross-coupling (H3). Verifying these on finite clusters would promote the α^{-4} mass scaling from a conditional theorem to a fully derived result, closing the last remaining open problem in the gauge-Higgs-confinement core.

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12. Conclusion

We have shown that three previously independent elements of the VERSF coupling programme — phase resolution, channel allocation, and second-order observable selection — are unified under a single derivational chain from five primitive postulates and the Economy Axiom. This paper establishes these elements as structural inputs to the α-derivation carried out in the companion papers; it does not itself derive α.

The key advances are:

- **Six channels are now fully derived**, not conditional. The Honeycomb Theorem (Hales 2001) establishes hexagonal tiling from the Economy Axiom, and the K=7 Counting Theorem (companion paper) establishes exactly six independent interface channels from the orbit-stabilizer structure.
- **Cross-term exclusion is operationally grounded.** Steps 1 and 2 of Theorem 3 rest on the distinction between single-channel marginal second moments (estimable from one edge, primitive-local) and irreducible joint second moments (requiring correlated multi-channel access, not primitive-local). Step 3 provides a supplementary isotropy argument.

The asymmetry between the admissible diagonal sector and the excluded off-diagonal sector is grounded in this operational distinction, not in isotropy alone.
- **N_φ is constrained, not free.** It belongs to a closure-consistent admissible set $\mathcal{C}$(K, N_loop). Whether this set collapses to a unique function F(K, N_loop) is identified as the primary open problem for the coupling programme, with direct bearing on the quantitative α result.

Specifically:

- **$N_\varphi \in \mathcal{C}$(K, N_loop):** bounded above by 2^K and subject to nullity-1 compatibility. The cyclic representation $\mathbb{Z}$_N_φ follows given the rank-1 transport assumption (Proposition 2, Assumption A), which is grounded in the companion paper's U(1) emergence result but stated explicitly here.
- **The six-channel structure** follows from the Honeycomb Theorem and the K=7 Counting Theorem.
- **The uniform weight w_i = 1/6** follows uniquely from the transitive $\mathbb{Z}_6$ action and normalization.
- **IPR = 1/6** and the second-order coefficient follow as direct corollaries.
- **Cross-channel covariance terms** are excluded primarily by local algebra completeness (Step 2): Cov(G_i, G_j) is an irreducible element of $\mathcal{A}_i \otimes \mathcal{A}_j$ not decomposable into the direct sum $\mathcal{A}_i \oplus \mathcal{A}_j$ required by P3 — a consequence of the definition of covariance. Step 1 provides supporting physical intuition from correlation decay in the committed phase. The "independent repeated trials" objection is addressed in the proof: the admissibility criterion is algebraic, not procedural.

The remaining open problem — whether $\mathcal{C}$(K, N_loop) collapses to a unique function F(K, N_loop) — has direct bearing on the completeness of the α-derivation and is now precisely bounded and computationally tractable.

References

1. Hales, T. C. "The Honeycomb Conjecture." *Discrete and Computational Geometry* 25 (2001): 1–22.
2. Taylor, K. "A Unified Derivation of Closure Geometry, Gauge Redundancy, and Mass Structure in the Hexagonal Framework." VERSF Theoretical Physics Program, AIDA Institute.
3. Taylor, K. "The Standard Model from Hexagonal Geometry." VERSF Theoretical Physics Program, AIDA Institute.
4. Kogut, J. B. "An introduction to lattice gauge theory and spin systems." *Reviews of Modern Physics* 51 (1979): 659.
5. Wilson, K. G. "Renormalization group and critical phenomena." *Physical Review B* 4 (1971): 3174.
6. Connes, A. *Noncommutative Geometry*. Academic Press, 1994.

7. Chamseddine, A. H. and Connes, A. "The spectral action principle." *Communications in Mathematical Physics* 186 (1997): 731–750.
8. Wen, X.-G. "Quantum order from string-net condensation and the origin of light and massless fermions." *Physical Review D* 68 (2003): 065003.
9. Balaban, T. "Propagators and renormalization transformations for lattice gauge theories." *Communications in Mathematical Physics* 95 (1985): 17–40.
10. Particle Data Group. "Review of Particle Physics." *Prog. Theor. Exp. Phys.* 2022, 083C01 (2022).

Appendix A: The K=7 Counting Theorem

This appendix reproduces the proof of Theorem 1 from the companion paper (Taylor, "A Unified Derivation of Closure Geometry, Gauge Redundancy, and Mass Structure in the Hexagonal Framework") in self-contained form, so that the present paper can be reviewed and submitted independently. All notation is consistent with the main text.

A.1 Setup

Cells, constraints, and closure. Consider a regular tiling of the plane by identical cells. Each cell carries K binary constraint variables $s_i \in \{0,1\}$, $i = 1,\ldots,K$, each with an associated compact phase $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$. The complex constraint field is $u_i = s_i \exp(i\theta_i)$. A cell is *committed* if and only if all constraints are simultaneously satisfied: the closure functional $C = \prod(u_i)$ satisfies $|C| = 1$ and $\arg(C) = 0 \mod 2\pi$.

Microscopic Hamiltonian. $H = H\_cl + H\_pair + H\_def$, where $H\_cl = \lambda\, \Sigma\_cells\ (1 - |C|)^2$ enforces closure, $H\_pair = \kappa\, \Sigma\_neighbors\ (1 - \cos(\theta_a - \theta\_b))$ penalises phase mismatch across interfaces, and H_def assigns energy to coordination defects. All three terms depend only on phase differences and magnitudes, never on absolute phases.

Axioms in force. A1 (uniformity), A2 (isotropy), A3 (closure), A4 (economy), binary constraint structure S1. Category I: finite local state space, finite correlation length in the ordered phase.

A.2 Theorem Statement

Theorem A.1 (K=7 Counting Theorem). *In a uniform, isotropic constraint network tiling the plane with identical cells, subject to closure (A3) and economy (A4), the minimal number of independent binary constraints per cell required for stable committed structure is K = 7.*

A.3 Step 1 — Tiling Selection and K ≥ 6

Tiling selection. By A1 and A2, the substrate admits only regular polygon tilings. The standard vertex-angle argument gives integer solutions (n, k) to $k(n-2)\cdot 180/n = 360$ as: (3,6) triangular, (4,4) square, (6,3) hexagonal. By A4, economy selects the tiling minimising boundary overhead per unit content. By the Honeycomb Theorem (Hales 2001), the regular hexagonal tiling uniquely minimises perimeter per unit area among all equal-area tilings. Therefore A4 uniquely selects the hexagonal tiling.

K ≥ 6. A hexagonal cell shares one edge with each of its six neighbours. Each shared edge defines a boundary adjacency relation that must be independently specified for the cell's state to be fully determined relative to its environment. Removing any one of these six relations leaves the cell's relationship to that neighbour unspecified, breaking uniform connectivity. Therefore $K \geq 6$.

A.4 Step 2 — Residual Null Mode at K = 6

Link variables. Let $\theta_1,\ldots,\theta_6 \in \mathbb{R}/2\pi\mathbb{Z}$ be the constraint phases at the six boundary vertices, ordered cyclically. Define link variables $\varphi_i = \theta_{i+1} - \theta_i$ (indices mod 6). All boundary adjacency observables are functions of the φ_i, since H_pair depends only on phase differences.

Boundary closure condition. The net holonomy must vanish: $\Sigma_{i=1}^{6} \varphi_i = 0 \pmod{2\pi}$. This is one constraint on six link variables, giving a five-dimensional solution space.

The null mode. The uniform shift $\theta_i \to \theta_i + \chi$ for all i leaves every link variable $\varphi_i = \theta_{i+1} - \theta_i$ exactly invariant. Since all boundary observables depend only on the φ_i, no boundary measurement can detect the absolute phase χ. This is an exact symmetry of H, not an approximation.

Rank argument. The boundary response matrix M_bdy (Hessian of the boundary energy with respect to vertex phases) has the structure of a graph Laplacian on the hexagonal boundary cycle. The uniform vector $\mathbf{1} = (1,1,\ldots,1)^T$ satisfies M_bdy $\mathbf{1} = 0$, so:

rank(M_bdy) = 5, nullity(M_bdy) = 1.

Physical consequence. A cell with only six boundary constraints is distinguishable but not committed: the absolute phase can be continuously deformed along the null direction without energetic cost. The cell occupies the distinguishable-but-uncommitted level of the hierarchy. K = 6 is therefore insufficient for stable committed structure.

A.5 Step 3 — Sufficiency of K = 7

The hub constraint. Introduce a single additional binary constraint s_7 associated with the interior hub of the cell, carrying phase θ_7. This constraint is independent of the six boundary constraints: it is localised at the cell's centre, not at any boundary vertex, and its satisfaction is not determined by boundary data alone.

Hub-boundary link variables. Define $\psi_i = \theta_i - \theta_7$ for i = 1,…,6. These are gauge-invariant quantities (phase differences). Under the uniform shift $\theta_j \to \theta_j + \chi$ for all j = 1,…,7, every ψ_i is invariant — both θ_i and θ_7 shift by χ. The global gauge mode is preserved.

Effect on the null mode. At K = 6, the boundary null mode was the uniform shift of all six boundary phases — invisible to all boundary observables. The hub-boundary links $\psi_i = \theta_i - \theta_7$ provide new observables: they fix each boundary phase *relative to the hub*, removing the boundary-only underdetermination. The only remaining null mode is the uniform shift of all seven phases simultaneously, which cancels in every link variable and is the irreducible global gauge mode.

Rank argument. The extended response matrix M_ext on the 7-dimensional phase space $(\theta_1,\ldots,\theta_7)$ has the structure of a graph Laplacian on the hub-and-boundary graph G_7: six boundary vertices connected cyclically, each also connected to the central hub. G_7 is connected and has 7 vertices. By standard spectral theory of graph Laplacians:

rank(M_ext) = |V| − (number of connected components) = 7 − 1 = 6, nullity(M_ext) = 1.

The single null eigenvector is the uniform mode $(1,1,\ldots,1)^T$ — the global gauge mode. This is the physically required irreducible redundancy: the overall phase of the cell as a unit, which cancels in every link variable and is therefore unobservable.

Nullity-1 Lemma. Nullity 1 is the unique minimal kernel consistent with gauge redundancy and closure: it corresponds to exactly one unobservable degree of freedom (the global phase), no more and no less. This is used throughout the main paper as the Nullity-1 Lemma.

K = 7 is therefore sufficient for stable committed structure: the cell is fully committed, with all internal degrees of freedom fixed up to the irreducible global gauge mode.

A.6 Step 4 — Exclusion of K > 7

Lemma A.1 (Independent Constraint Support Sites). *The hexagonal cell has exactly 7 independent constraint support sites: 6 boundary vertices and 1 interior hub. No eighth independent constraint site exists.*

Proof. We enumerate all geometrically distinct sites on the hexagonal cell using the orbit-stabilizer theorem. The symmetry group of the hexagonal cell is C_6 (order 6). If a point p has stabilizer subgroup Stab(p), its orbit under C_6 has size $|C_6| / |\mathrm{Stab}(p)| = 6 / |\mathrm{Stab}(p)|$.

The geometrically distinct site types and their independent constraint variables are:

Site type	Stabilizer	Orbit size	Independent variables
Centre (hub)	C_6	1	1 (the hub constraint s_7)
Boundary vertex	{e}	6	6 (one per vertex: $s_1,\ldots,s_6$)
Boundary edge midpoint	{e}	6	0 (determined by adjacent vertices under interface pairing)
Generic interior point	{e}	6 (continuous family)	0 (no independent variable at cell resolution)

Boundary edge midpoints carry no independent variable: each edge midpoint lies on the shared interface between two cells, and under interface pairing (S3) the interface state is fully determined by the link variable $\varphi_i = \theta_{i+1} - \theta_i$, which is a function of the adjacent vertex data.

Generic interior points carry no independent variable at cell resolution: by locality (Category I), any constraint at a generic interior point depends only on the constraint fields in a finite neighbourhood, which at cell resolution are the seven variables already defined at the vertices and hub. A constraint at an intermediate location is, at one-cell resolution, a function of existing vertex/hub data after coarse-graining. No sub-cell lattice structure exists at cell resolution (by finite entropy density, Category I, and economy A4).

Total independent constraint sites: 6 + 0 + 1 + 0 = **7**. ■

Exclusion of K > 7. Any proposed eighth independent binary constraint must be localised somewhere on the hexagonal cell. By Lemma A.1, all independent constraint sites are occupied. A proposed eighth constraint therefore falls into one of three cases: (a) localised at an existing boundary vertex or hub, making it algebraically dependent on an existing variable; (b) localised at a boundary edge midpoint, making it determined by the paired vertex constraints; or (c) non-local, violating Category I locality.

Furthermore, the Nullity-1 Lemma requires nullity exactly 1. Adding an eighth independent constraint to the already-saturated 7-site system would either reduce nullity below 1 (destroying the required global gauge mode) or be algebraically redundant. Neither is consistent with stable committed closure.

$K > 7$ is therefore impossible. ■

A.7 Conclusion of Theorem A.1

$K \geq 6$ (Step 1 and §A.3), $K = 6$ is insufficient (Step 2 and §A.4), $K = 7$ is sufficient (Step 3 and §A.5), $K > 7$ is impossible (Step 4 and §A.6). Therefore $K = 7$ is the unique minimal closure count. ■

A.8 Corollaries Used in the Main Text

Corollary A.1 (Six Interface Channels). The committed hexagonal cell has exactly six independent interface channels $\{C_1,\dots,C_6\}$, corresponding to the six boundary vertices. Each is associated with one independent link variable φ_i. The hub constraint is an interior degree of freedom, not an interface channel; edge midpoints carry no independent variable. This is Result B of the main text.

Corollary A.2 (Nullity-1 Lemma). The response matrix M_ext of the committed K=7 cell has rank 6 and nullity exactly 1. The single null eigenvector is the global gauge mode. This is used in Proposition 3 and throughout §3 of the main text.

Corollary A.3 (Phase Budget). The K=7 committed cell supports at most 2^7 = 128 distinguishable binary states, providing the upper bound $N_\varphi \leq 2^K$ used in Propositions 1 and 3 of the main text.