# Determining η: The Entropy Conversion Factor Between Closure and Thermodynamic Entropy in VERSF

**VERSF Theoretical Physics Program**

**For the General Reader**

Physics has two separate entropy languages. In quantum information and abstract distinguishability theory, entropy is a dimensionless number --- the logarithm of the number of distinguishable states. In thermodynamics, entropy carries units of energy per temperature, set by Boltzmann's constant k_B. Any theory that derives thermodynamics from a more fundamental distinguishability structure must bridge these two languages. The bridge is a single dimensionless number: η.

In the Void Energy-Regulated Space Framework (VERSF), physical reality emerges through *commitment* --- the irreversible selection of one outcome from competing possibilities. The framework defines an internal entropy measure S̃ on distinguishability classes of closure states. The central question this paper addresses is: **what is $\eta = \Theta_0 / (k\_B \ln 2)$**, the ratio of the physical entropy cost of one primitive commitment event to the Landauer quantum?

The paper shows that η = 1 is the unique value consistent with the framework's causal, statistical, and partition structures. This means S = k_B S̃: closure entropy and thermodynamic entropy are the same quantity in different unit systems, with k_B as the conversion factor --- not imposed externally, but forced by what thermodynamic emergence requires. As a consequence, the commitment barrier $\Phi_c = r$, and with r = 1 as the primitive consistency value, the commitment energy scale is approximately 2.5 meV --- within reach of low-temperature experiments.

Two important caveats. First, the numerical value 2.5 meV assumes the cosmological vacuum energy density as the VERSF substrate energy density ρ; this identification is a working assumption pending derivation. Second, results depending on K = 7 as the closure cell are established within the wheel-graph family but not yet proven to be unique among all connected graphs --- that enumeration is future work.

The paper is self-contained. Readers unfamiliar with VERSF can treat the framework axioms as given and follow the derivations from Section 3 onward.

Commitment capacity χ(L) determines whether irreversible commitment events are possible in a region, while the commitment barrier $\Phi_c$ specifies the minimum entropy cost required to produce each event.

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**If This Framework Is Correct: Implications**

The paper's central result is $\eta = 1$. If correct, this has implications beyond the internal consistency of the VERSF framework. The following consequences are stated conditionally --- each holds if $\eta = 1$ and the associated structural claims are correct --- and each is in principle falsifiable by the experimental test of Section 10.2. They are presented here as a guide to what would be established, not as established results.

**A Candidate Physical Mechanism for Quantum Measurement**

Standard quantum mechanics describes measurement as a postulate: at some point during interaction with an apparatus, a superposition becomes a definite outcome. The theory does not say at what point in the causal ordering this occurs, where, or at what cost. It is simply assumed.

If VERSF is correct, measurement is not a postulate but a physical process with a quantifiable entropy threshold. A superposition does not collapse because an observer looks --- it commits because the entropy amplification of the distinguishability relation has crossed the barrier $\Phi_c = 1$. Below the threshold, the system remains in a reversible superposition. Above it, an irreversible record is produced and the outcome is fixed. The transition is sharp, physical, and in principle detectable.

The framework thereby provides a candidate physical mechanism for determining at what point in the causal ordering a measurement commits, with a precise answer in terms of the energy scale $E_c \approx 2.5$ meV and the entropy amplification of the measurement interaction --- where "point in the causal ordering" refers to the emergent causal structure produced by commitment, not a pre-given clock time.

**The Arrow of Emergent Time Naturally Explained**

The laws of physics at the microscopic level are time-symmetric. Yet the macroscopic world has a definite arrow of time: records are formed in one direction, not the other. This asymmetry is one of the deepest unexplained features of physics.

If VERSF is correct, the arrow of emergent time is naturally explained as a consequence of commitment. Time does not flow independently --- it is constituted by the irreversible production of distinguishable commitment events (facts). Each commitment event creates a stable record that cannot be undone, and the ordered accumulation of such records is what we experience as the forward direction of emergent time. On this view, temporal ordering is not a background against which commitment occurs; it is produced by commitment. The commitment barrier $\Phi_c$ is therefore not just a number about measurement: it is the minimum entropy cost of one irreversible step in the causal ordering of commitment events (facts).

If $E_c \approx 2.5$ meV is correct, there is a smallest possible thermodynamic cost for the production of one unit of irreversible causal structure. The grain size of fact-production has a measurable energy scale.

**A Structural Explanation for the Landauer Bound**

Landauer's principle --- that erasing one bit of information costs at least k_B ln 2 of entropy --- is usually treated as a result of thermodynamics applied to computation. Its connection to quantum measurement has never been derived from first principles.

If VERSF is correct, the framework provides a structural explanation for why the Landauer bound appears: the minimum cost of one irreversible commitment event (fact) is k_B ln 2 because the minimum irreversible closure event is a binary partition refinement, and the entropy of a binary partition is ln 2 in closure units, converting to k_B ln 2 under the physical entropy identification. Landauer and commitment meet at the same value because they describe the same minimal event in different languages. This is not a numerical coincidence in the framework --- it follows from the binary minimality of distinguishability refinement.

**A Structural Origin for Decoherence Thresholds**

Decoherence theory explains why macroscopic objects appear classical: entanglement with environmental degrees of freedom suppresses quantum interference. But decoherence theory does not explain *why* environmental interaction produces irreversible records rather than reversible entanglement indefinitely. It assumes irreversibility; it does not derive it.

If VERSF is correct, the framework suggests a structural origin for decoherence thresholds: irreversibility is produced whenever the entropy amplification of a distinguishability relation crosses $\Phi_c = 1$. Decoherence appears effectively immediate for macroscopic objects because many environmental modes contribute simultaneously to entropy amplification, driving the interaction across the commitment threshold after only a small causal extent. Small quantum systems remain coherent because their interaction entropy remains below $\Phi_c$. The quantum--classical crossover is governed not by system size alone but by whether the entropy of the measurement interaction has crossed the commitment threshold.

This suggests characteristic decoherence signatures near $E_c \approx 2.5$ meV that would be distinguishable from standard environmental decoherence models --- a prediction that can be tested independently of the full framework.

**Thermodynamics as a Consequence of Distinguishability Geometry**

In standard physics, thermodynamics sits as a separate layer above mechanics. The second law is an empirical fact about the universe's initial conditions and the statistics of large systems --- not a consequence of the fundamental dynamical laws.

If VERSF is correct, the second law is naturally explained as a consequence of the geometry of distinguishability. Every commitment event increases the total closure entropy of the universe by at least $\tilde{\Theta}_0 = \ln 2$ in closure units. The irreversibility of thermodynamics reflects the irreversibility of commitment: entropy increases because commitment events (facts) accumulate, and they accumulate because commitment is one-directional by construction. The second law is not an empirical regularity about statistics but a theorem about the structure of irreversible distinguishability refinement.

**An Experimentally Accessible Scale**

Perhaps the most striking implication is the most concrete. The framework predicts a specific energy scale, $E_c \approx 2.5$ meV, at which primitive commitment occurs. This corresponds to a temperature of approximately 30 K --- cold, but well within the reach of standard cryogenic equipment. Josephson junction experiments, superconducting qubit systems, and precision atom interferometry all operate in this regime.

If the prediction is correct, experiments in this energy range should reveal anomalies in decoherence rates, transition probabilities, or entropy production that cannot be accounted for by standard quantum mechanics alone. The commitment barrier would manifest as a threshold --- a minimum entropy cost below which irreversible outcomes cannot be produced --- that is absent from the standard formalism.

This makes the framework falsifiable in the near term, not in the distant future of Planck-scale physics. A null result --- no anomalous threshold near 2.5 meV in carefully designed quantum coherence experiments --- would constitute strong evidence against the framework. A positive result would constitute evidence for a physical basis of quantum measurement that has been sought since the founding of quantum mechanics.

**Global Failure Modes**

The paper's main prediction is $\eta = 1$. Because the derivation is structured as a chain of named, separable results, a deviation from $\eta = 1$ does not simply refute the framework --- it points to a specific link in the chain that would need revision. Each failure mode has a precise diagnostic.

The failure modes divide into three tiers of severity. The first tier concerns whether discrete commitment structure exists at all: if decoherence is fully continuous with no detectable entropy threshold, the framework's foundational assumption of irreversible commitment events has no physical correlate and the entire programme is undermined. The second tier concerns the energy scale r: if anomalous thresholds exist but $E_c$ and $\Phi_c$ are inconsistent with the predicted relation $E_c = r \cdot \hbar c/\xi$, the commitment-capacity derivation or the Wilson sector normalization fails while the existence of discrete structure may still survive. The third tier --- the most informative failure --- is $\eta \neq 1$ with both r and $E_c$ well-defined: this means the unconditional spectral structure is intact but the physical entropy identification is wrong, pointing specifically to Route D and the Class Identification Assumption.

The following table maps each failure mode to its diagnostic and structural consequence:

| Failure mode | η diagnostic | What it would show |
|---|---|---|
| η measured > 1 via $\Phi_c \neq r$ | $\eta \neq 1$ | Closure and thermodynamic entropy are proportional but not equal; α ≠ k_B; Route D and the Class Identification Assumption fail |
| No threshold near $E_c$ in decoherence experiments | r or η unknown | The commitment barrier has no physical correlate at the predicted scale; the entire discrete-commitment |

| Failure mode | η diagnostic | What it would show |
|---|---|---|
| | | programme loses its experimental anchor |
| Decoherence fully continuous, no entropy scale | r undefined | The framework's discrete commitment structure has no physical correlate; the foundational commitment-event ontology is not instantiated in nature |
| $E_c$ and $\Phi_c$ inconsistent with $E_c = r \cdot \hbar c/\xi$ | Routes A or B | Commitment-capacity derivation or Wilson sector normalization breaks down; the quartic scaling $L^4$ or the action-budget interpretation is incorrect |
| VERSF fails to recover standard thermodynamics | Route D collapses | Class Identification Assumption is not forced; thermodynamic emergence programme is incomplete |
| Two structural features of this failure map deserve emphasis. First, the failure modes are ordered: finding a threshold (falsifying the third row) is a prerequisite for testing the remaining rows. Measuring $E_c$ gives r unconditionally; measuring $\Phi_c$ separately then gives η. Only if both are measured can Routes A--D be probed. Second, the unconditional results survive every failure mode in the table --- $\Delta\tilde{S}_{prim} = r$ and $E_c \sim r \cdot \hbar c/\xi$ hold regardless of whether $\eta = 1$ is confirmed. What fails in each case is the conditional prediction $\eta = 1$, and with it $\Phi_c = 1$ and $E_c \approx 2.5$ meV. The two-observable test of Section 10.2 is designed to read off η directly from independent measurements of $E_c$ and $\Phi_c$, and to identify which structural claim requires revision if $\eta \neq 1$. | | |

**Summary of Implications**

| Domain | Implication if η = 1 |
|---|---|
| Closure--thermodynamic bridge | S = k_B S̃ exactly; the Boltzmann constant emerges as the unique conversion factor, not imposed |

| Domain | Implication if $\eta = 1$ |
| --- | --- |
| Quantum measurement | Candidate physical mechanism: irreversible commitment occurs when entropy amplification crosses $\Phi_c = \eta r = 1$ |
| Arrow of emergent time | Temporal directionality naturally explained as consequence of commitment irreversibility; one commitment event (fact) costs k_B ln 2 |
| Landauer's principle | Structural explanation: k_B ln 2 threshold follows from binary minimality of distinguishability refinement and $\eta$ = 1 |
| Decoherence | Structural origin for quantum--classical crossover at the commitment threshold |
| Second law | Entropy increase naturally explained as theorem about distinguishability geometry |
| Experimental test | Measure $\eta = \Phi_c/r$ via two-observable test; $\eta = 1$ confirms S = k_B S̃; $\eta \neq 1$ constrains the discrepancy precisely |
| The framework does not claim to have resolved these questions | |
| definitively. It claims that $\eta$ is a single measurable number whose value | |
| determines whether all of them are addressed simultaneously --- and that | |
| four convergent structural arguments predict $\eta = 1$. | |


**Abstract**

We derive the dimensionless entropy conversion factor $\eta = \Theta_0/(k\_B \ln 2)$ --- the ratio of the physical thermodynamic entropy cost of one primitive commitment event to the Landauer quantum --- from the closure-entropy structure of the Void Energy-Regulated Space Framework (VERSF). This quantity bridges the framework's internal dimensionless entropy measure S̃ and physical thermodynamic entropy S. The derivation is structured as a sequence of named results separating unconditional from conditional claims at every step.

**Unconditional results. Theorem A** proves that any admissible closure entropy measure satisfying monotonicity, additivity, and the null-singleton condition must take the form $\tilde{S} = \ln N$, forcing the logarithmic form from axioms; the proof uses the integer Cauchy functional equation. **Lemma B** establishes binary minimality: the smallest nontrivial irreversible partition refinement gives $1 \rightarrow 2$, so the primitive closure entropy quantum is $\boldsymbol{\tilde{\Theta}_0 = \ln 2}$ (Corollary C) --- unconditionally. **Lemma $F_1$** establishes the non-kernel requirement: any faithful spectral representation of the partition $\{s_1\} \mid \{s_2\}$ must involve at least one non-kernel mode --- trivial but necessary. **Lemma $F_2$** derives the dimension $\geq 2$ lower bound via branch exchange symmetry:

primitive commitment in VERSF has symmetry group $G = \mathbb{Z}_2$ (both outcomes are indistinguishable before commitment, by definition), and any faithful equivariant representation of $\mathbb{Z}_2$ must contain both the trivial and sign irreducible representations, forcing $\dim V \geq 2$ by standard representation theory. **Theorem D** establishes $\lambda^* = 2$ by Rayleigh--Ritz over an independently defined admissible class $A_{prim}$, and the **Spectral Lemma** gives $\Delta\tilde{S}_{prim} = r$ --- the primitive closure action cost with no residual factor. **Bridge Principle F** connects counting and spectral sides: Lemma $F_1$ plus Courant--Fischer plus the K = 7 spectrum gives the $\lambda^* = 2$ eigenspace as the unique minimal spectral realization of binary commitment --- fully rigorous. **Theorem E** proves $(1/2)\cdot\lambda^*(K) = 1 \Leftrightarrow K = 7$, with the wheel spectrum derived from first principles in the Appendix. **Theorem I** derives $\chi(L) = \rho L^4/(\hbar c)$ from the causal action budget --- the $L^4$ scaling is forced --- and the coherence scale $\xi$ from $\chi(\xi) = 1$ at the effective causal layer. Together these give the unconditional barrier formula

$$\boldsymbol{\Phi_c = \eta \cdot r,}$$

where $\eta$ is left undetermined by the unconditional tier. The unconditional structural picture is therefore: the closure framework has a well-defined primitive entropy quantum $\tilde{\Theta}_0 = \ln 2$, a well-defined spectral cost $\Delta\tilde{S}_{prim} = r$, and a well-defined barrier formula $\Phi_c = \eta r$ --- but the physical scale of the entropy, encoded in $\eta$, requires additional input.

**Determination of η.** The paper presents four convergent consistency arguments that $\eta = 1$, approaching the question from different structural directions. They are not claimed to be mutually independent proofs: Routes A, B, and C are effective-layer consistency checks that all assume the effective thermodynamic layer is already in place, and they share the common premise that this layer is standard statistical mechanics with k_B as its entropy unit. Route D is the most fundamental, deriving $\eta = 1$ from the VERSF emergence claim alone. **Route A (primitive commitment-cell matching):** the coherence cell $\chi(\xi) = 1$ uniquely identifies the minimal physical domain carrying one primitive binary event; once the effective thermodynamic layer is admitted, comparing closure entropy (ln 2) and Boltzmann entropy (k_B ln 2) on that same domain fixes α = k_B. Route A is a domain-matching argument, not an independent derivation of k_B. **Route B (action--thermal non-redundancy):** within VERSF's emergence programme, $\eta$ cannot be a free primitive parameter, because the emergence claim requires that the closure entropy scale *be* the thermodynamic entropy scale --- any $\eta \neq 1$ would mean they differ, which contradicts what emergence means; the Structural Non-Redundancy Principle makes this precise in the context of the effective layer's existing action-to-entropy map. **Route C (statistical):** the Boltzmann factor e^{−E/(k_B T)} is consistent with closure entropy only if α = k_B --- any other value forces the effective temperature to absorb a spurious factor. **Route D (partition):** the **Proposition on Uniqueness of the Record-Induced Partition** shows that within VERSF's emergence programme, the closure partition and the Boltzmann microstate partition cannot differ, forcing α = k_B via the Matching Lemma. Routes A--C are consistent with $\eta = 1$ given the effective thermodynamic layer; Route D derives it from the emergence claim. Together they constitute a multi-directional case for $\eta = 1$ whose strength rests primarily on Route D, with Routes A--C providing supporting consistency. The physical entropy identification **S = k_B S̃** is the unique normalization compatible with all four.

**Conditional results** (on the Class Identification Assumption underlying Route D). Under $\eta = 1$, the barrier formula gives $\mathbf{\Phi_c = r}$. The **Consistency Theorem** selects $r = 1$ as the primitive consistency value --- the unique value at which the unconditional spectral structure and the conditional physical normalization are mutually consistent under natural normalization. The commitment energy scale is then $\mathbf{E_c \approx 2.5}$ **meV**, placing the primitive commitment threshold in an experimentally accessible low-temperature regime. **Landauer's principle** appears as an external consistency check: the derived $\Theta_0 = k\_B \ln 2$ satisfies it, confirming that $\eta = 1$ is thermodynamically coherent, but Landauer is not used as a premise.

The central result of the paper is $\eta = 1$ --- the statement that closure entropy and thermodynamic entropy are the same quantity, expressed in different unit systems related by the Boltzmann constant.

## 1. Introduction and Motivation

The Void Energy-Regulated Space Framework (VERSF) defines an internal entropy measure $\tilde{S}$ on distinguishability classes of closure states. This measure is forced to be logarithmic in the number of distinguishable states by the axioms of the framework (Theorem A). Physical thermodynamic entropy is also logarithmic in the number of microstates, with coefficient k_B. The proportionality $S = \alpha\tilde{S}$ is structurally forced --- both entropies are logarithmic measures of distinguishability. The question is: what is $\alpha$?

This question is equivalent to asking for $\boldsymbol{\eta = \Theta_0/(k\_B \ln 2)}$ --- the ratio of the physical entropy cost of one primitive commitment event to the Landauer quantum k_B ln 2. The parameter $\eta$ is the central quantity of this paper.

- If $\eta = 1$: closure entropy and thermodynamic entropy agree exactly; $S = k\_B\, \tilde{S}$; the Boltzmann constant is the unique conversion factor; and the framework's internal entropy measure literally *is* physical entropy.
- If $\eta \neq 1$: the two entropy measures are proportional but not equal; the thermodynamic emergence programme requires additional structure to explain the discrepancy.

The paper presents four convergent consistency arguments that $\eta = 1$. They are not claimed to be mutually independent proofs; rather, they approach the question from different structural directions, and their common premise is identified explicitly. Routes A, B, and C are effective-layer consistency checks: they assume the effective thermodynamic layer is in place and show $\eta \neq 1$ is inconsistent with standard statistical mechanics at that layer. Route D is the most fundamental: it derives $\eta = 1$ from the VERSF emergence claim alone, without requiring the effective layer to be pre-established. Route D is the primary result; Routes A--C provide supporting convergence from different structural directions.

**Unconditional tier.** Theorems A, D, E, I; Lemma $F_1$; Lemma $F_2$; Bridge Principle F; Corollary C; Spectral Lemma; Proposition (barrier formula $\Phi_c = \eta r$). These establish the closure-unit entropy structure, the spectral cost $\Delta\tilde{S}_{prim} = r$, and the barrier formula $\Phi_c = \eta r$ --- with $\eta$ left as a free parameter.

**Conditional tier.** Routes A--D (Section 7.3 and Section 8), Conditional Theorem G, Conditional Corollary H, and the Consistency Theorem r = 1. These determine $\eta = 1$ and its consequences. Route D requires the thermodynamic emergence claim; Routes A--C additionally require the effective thermodynamic layer to be in place with k_B as its standard entropy unit.

**K = 7 conditional.** Theorems D, E, the Spectral Lemma, Bridge Principle F, and all downstream results (including the barrier formula and energy scale) depend on the K = 7 closure cell. Within the wheel-graph family, K = 7 is uniquely determined (Theorem J). Whether it is minimal among all connected graphs is an open spectral classification problem --- a full determination requires enumerating connected graphs on n = 4, 5, 6 vertices, which is finite but has not been carried out in this paper. All K = 7-dependent results should be understood as conditional on the wheel-family assumption until that enumeration is complete.

Within VERSF, coherence capacity governs reversible pre-commitment possibilities, commitment capacity governs the production of irreversible events, and persistence capacity governs the stability of records once formed. The stability of records after formation is governed by persistence capacity, which is not treated in this paper. This paper is concerned exclusively with commitment capacity and the commitment barrier.

**Notation convention.** Throughout this paper, tilde quantities ($\tilde{S}$, $\tilde{\Theta}_0$) denote dimensionless closure-unit quantities. Untilded quantities ($S$, $\Theta_0$, $\Phi_c$) denote corresponding physical thermodynamic quantities. They are related by $S = k\_B\ \tilde{S}$ once $\eta = 1$ is established. All unconditional results are stated and proved in closure units; the physical interpretation is applied only once $\eta$ is determined.

**Logical spine.** The closure action cost of a mode with eigenvalue $\lambda$ is

$$\Delta\tilde{S}(\lambda) := (r/2) \cdot \lambda.$$

The commitment barrier is

$$\Phi_c = \Delta S_c / (k\_B \ln 2) = \eta \cdot (r/2) \cdot \lambda^*, \quad \eta := \Theta_0 / (k\_B \ln 2).$$

The unconditional tier pins $\lambda^* = 2$ and $\Delta\tilde{S}_{prim} = r$. The conditional tier derives $\eta = 1$. Together: $\Phi_c = r = 1$ and $E_c \approx 2.5$ meV.

## 2. Closure-Entropy Framework

### 2.1 Closure-Stiffness and the Wilson Penalty

The Wilson term is the unique local quadratic entropy penalty associated with closure roughness. The entropy functional is

$$S_{ent}[\Psi] = \lambda \langle \Psi, D\_G^2\ \Psi \rangle,$$

with the corresponding Hamiltonian correction

$H\_W = (r/2) \cdot (\hbar c/\xi) \cdot (D\_G^2 \otimes \beta)$,

where $r$ is the dimensionless closure-stiffness coefficient and $\xi$ is the closure coherence scale. The prefactor 1/2 is structural --- it arises from the quadratic nature of the entropy penalty in the second variation of the closure-entropy functional --- and is independent of the cell spectrum. Independent stability arguments establish $r = O(1)$: a too-soft closure is unstable against fluctuations.

### 2.2 The K = 7 Closure Cell

The K = 7 closure cell has graph Laplacian L with spectrum

$\mathrm{Spec}(L) = \{ 0, 2, 2, 4, 4, 5, 7 \}$.

| Eigenvalue | Multiplicity | Sector |
|---|---|---|
| 0 | 1 | Constant (trivial) --- global phase |
| 2 | 2 | Minimum transport --- primitive commitment |
| 4 | 2 | Second transport |
| 5 | 1 | Near-UV |
| 7 | 1 | UV --- suppressed by H_W |
| The spectrum is derived from first principles in the Appendix. Section | | |
| 9.4 (Theorem J) establishes K = 7 as the *minimal* admissible closure | | |
| cell within the wheel-graph family --- making K a derived minimum rather | | |
| than a free parameter. | | |

## 3. Closure Entropy: Forced Logarithmic Form and Binary Minimality

### Theorem A --- Admissible Closure Entropy Form

**Theorem A (Forced Logarithmic Form).** *Let $\tilde{S} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be any function assigning a closure entropy to a system with N distinguishable states, satisfying:*

*(i)* ***Monotonicity.*** *$N' > N \Longrightarrow \tilde{S}(N') > \tilde{S}(N)$.*

*(ii)* ***Additivity.*** *$\tilde{S}(N_1 \cdot N_2) = \tilde{S}(N_1) + \tilde{S}(N_2)$ for independent subsystems.*

*(iii)* ***Null singleton.*** *$\tilde{S}(1) = 0$.*

*Then $\tilde{S}(N) = C \ln N$ for some constant $C > 0$. In closure units, $C = 1$, giving $\tilde{S}(N) = \ln N$.*

**Proof.**

*Step 1.* Define $f(N) := \tilde{S}(N)$. Condition (ii) gives $f(N_1 \cdot N_2) = f(N_1) + f(N_2)$ for positive integers $N_1, N_2$ --- the multiplicative Cauchy functional equation on $\mathbb{N}$.

*Step 2.* Condition (i) gives strict monotonicity of f on $\mathbb{N}$. By the integer version of the Cauchy functional equation (Aczél 1966, §2.1), any monotone solution on positive integers is $f(N) = C \log N$ for some C

0 and any base. Condition (iii) gives $f(1) = C \log 1 = 0$, satisfied automatically. Monotonicity forces $C > 0$. Setting $C = 1$ in closure units gives $\tilde{S}(N) = \ln N$. **Remark.** No extension to real arguments is required --- the integer domain suffices. The logarithmic form is forced: any alternative ($N - 1$, $\sqrt{N}$, etc.) violates additivity or monotonicity.

**Lemma B --- Binary Minimality**

**Lemma B (Binary Minimality).** *Under finite distinguishability (VERSF axiom A1), the smallest nontrivial irreversible refinement step increases partition cardinality by exactly 1. Starting from a single indistinguishable class ($N = 1$), the smallest nontrivial refinement produces $N = 2$.*

**Proof.** A partition refinement producing a new stable record must separate at least one previously indistinguishable pair into distinct classes --- otherwise no new distinguishable fact is produced and the event is trivial. Any nontrivial refinement therefore increases cardinality by at least 1. An increase of exactly 1 is achievable (split any one pair), so the minimum is 1. For the primitive case $N = 1$, the minimum nontrivial refinement produces $N = 2$, which is the binary partition $\{s_1\} \mid \{s_2\}$. **Corollary C --- Primitive Entropy Quantum**

**Corollary C.** *The entropy change of the smallest nontrivial irreversible closure event is*

$\tilde{\Theta}_0 = \ln 2$ (in closure units).

**Proof.** By Lemma B, cardinality changes from $N = 1$ to $N = 2$. By Theorem A: $\tilde{\Theta}_0 = \tilde{S}(2) - \tilde{S}(1) = \ln 2 - 0 = \ln 2$. This result is unconditional within VERSF: it depends only on Theorems A and B, with no reference to k_B or thermodynamics.

## 4. Primitive Commitment: Admissible Class and Spectral Theorem

### 4.1 The Admissible Primitive Class

Define the **admissible primitive class** $A_{prim}$ as the set of closure configurations satisfying:

**Condition 1 --- Non-triviality.** $\Psi \neq 0$ and $\Psi \perp \ker(L)$.

**Condition 2 --- Irreducibility.** $\Psi$ cannot be written as $\Psi = \Psi_1 + \Psi_2$ with both $\Psi_i \neq 0$, $\Psi_i \perp \ker(L)$, and each achieving a Rayleigh quotient strictly less than that of $\Psi$:

$\langle\Psi_i, L\,\Psi_i\rangle / \langle\Psi_i, \Psi_i\rangle < \langle\Psi, L\,\Psi\rangle / \langle\Psi, \Psi\rangle$, i = 1, 2.

The class $A_{prim}$ makes **no reference to any specific eigenvalue**. Define the variational minimum

$\lambda^* := \min \{ \langle\Psi, L\,\Psi\rangle / \langle\Psi, \Psi\rangle : \Psi \in A_{prim} \}$.

**4.2 Theorem D --- Spectral Minimum**

**Theorem D (Spectral Minimum).** *For the K = 7 closure cell with spectrum {0, 2, 2, 4, 4, 5, 7},* $\lambda = 2$. The minimum is achieved by configurations in the two-dimensional eigenspace at eigenvalue 2. **Proof.**

*Step 1.* All $\Psi \in A_{prim}$ are nonzero and orthogonal to ker(L) by Condition 1. By Courant--Fischer--Weyl, $\min\{ R(\Psi) : \Psi \perp \ker \} = \mu_1$, the first nonzero eigenvalue. Since any eigenvector at $\mu_1$ satisfies Condition 2 (Step 2), the constrained minimum over $A_{prim}$ equals $\mu_1$.

*Step 2.* Let $\Psi^*$ be an eigenvector at $\mu_1$. Suppose for contradiction $\Psi = \Psi_1 + \Psi_2$ with $\Psi_i \neq 0$, $\Psi_i \perp \ker$, and $R(\Psi_i) < \mu_1$. But each $\Psi_i \perp \ker(L)$, so Courant--Fischer gives $R(\Psi_i) \geq \mu_1$ --- contradicting $R(\Psi_i) < \mu_1$. Therefore $\Psi^*$ is irreducible and $\Psi^* \in A_{prim}$.

*Step 3.* By explicit computation of the K = 7 spectrum (Appendix), $\mu_1 = 2$. **4.3 Spectral Lemma --- Primitive Closure Cost**

**Spectral Lemma.** *The dimensionless closure action cost of the primitive commitment mode is*

$\Delta\tilde{S}_{prim} := (r/2) \cdot \lambda^* = (r/2) \cdot 2 = r$.

*The Wilson prefactor 1/2 and the cell eigenvalue* $\lambda = 2$ cancel exactly; no residual numerical factor remains. **Proof.** Direct substitution of $\lambda^* = 2$ from Theorem D. Both inputs are independently fixed: 1/2 from the quadratic entropy penalty structure, 2 from the K = 7 combinatorics. Theorem E below proves this exact cancellation is unique to K = 7.

**5. Bridge Principle: Connecting Spectral Structure and Binary Refinement**

The previous sections establish two independent minimality results:

- **Entropy minimality.** Lemma B: the smallest irreversible refinement is binary ($1 \to 2$).
- **Spectral minimality.** Theorem D: the smallest nontrivial admissible spectral sector has eigenvalue $\lambda^* = 2$ with multiplicity 2.

These results arise from different structures --- combinatorial distinguishability and graph spectral theory. The bridge connecting them requires two steps, each captured by its own lemma. **Lemma $F_1$** establishes the trivial but necessary lower bound: any faithful spectral representation of a binary refinement must involve at least one non-kernel mode. **Lemma $F_2$** establishes the stronger claim via representation theory: primitive binary commitment has symmetry group $G = \mathbb{Z}_2$ (derived from the VERSF definition of "primitive"), and any faithful equivariant representation of $\mathbb{Z}_2$ must contain both irreducible representations --- the trivial irrep and the sign irrep --- giving dimension at least 2. Bridge Principle F then combines these with the K = 7 spectrum via Courant--Fischer.

**Lemma $F_1$ --- Branch-Distinguishing Mode**

**Lemma $F_1$ (Branch-Distinguishing Mode).** *Any faithful spectral representation of the partition $\{s_1\} \mid \{s_2\}$ relative to the indiscrete partition $\{s_1, s_2\}$ requires at least one non-kernel mode.*

**Proof.** The kernel of L consists of constant functions --- functions that assign the same value to every vertex. A constant function cannot distinguish $s_1$ from $s_2$: it assigns the same amplitude to both, encoding no information about which branch was selected. Therefore any representation that faithfully encodes the partition $\{s_1\} \mid \{s_2\}$ --- meaning it assigns distinguishable values to $s_1$ and $s_2$ --- must have support outside ker(L), i.e., must involve at least one non-kernel (non-constant) mode. **Remark.** Lemma $F_1$ is trivial in the sense that it merely says: a constant function is useless for distinguishing things. Its role is to anchor the argument rigorously at the first step, establishing that the realizing sector lies strictly outside the kernel. The non-trivial content comes from Lemma $F_2$.

**Lemma $F_2$ --- Equivariant Realization Requires Multiplicity ≥ 2**

**Setup: the symmetry group of primitive binary commitment.**

In VERSF, a primitive commitment event is defined as the smallest irreversible refinement of a distinguishability partition starting from a single indistinguishable class $\{s_1, s_2\}$. Prior to commitment, the two states are by definition indistinguishable --- the framework's axioms assign no structural property distinguishing the roles of $s_1$ and $s_2$ before commitment occurs. The commitment event produces the distinction; it does not exploit a pre-existing asymmetry. Therefore the physical system has a symmetry group

$$G = \mathbb{Z}_2 = \{e, \sigma\},$$

where σ is the branch exchange $\sigma: s_1 \leftrightarrow s_2$, and e is the identity. This is not imposed: it follows directly from the definition of "primitive" in VERSF.

**Lemma $F_2$ (Equivariant Multiplicity Lower Bound).** *Let V be a spectral sector of L that provides a faithful representation of primitive binary commitment. Faithfulness requires:*

*(i) V contains a $\mathbb{Z}_2$-invariant mode that distinguishes the committed state from the trivial (kernel) state --- i.e., there exists $f_{occ} \in V_+$ with $f_{occ}(s_1) = f_{occ}(s_2) \neq 0$.*

*Physical justification for condition (i): the fact that "a commitment occurred" is a branch-exchange invariant statement --- it is true regardless of which branch was selected, and* *swapping $s_1 \leftrightarrow s_2$ does not change whether the refinement happened. Therefore the mode encoding commitment occurrence must lie in $V_+$, the symmetric subspace. Condition (i) formalises this requirement.*

*(ii) V contains a mode that distinguishes $s_1$ from $s_2$ --- i.e., there exists $f_{br} \in V$ with $f_{br}(s_1) \neq f_{br}(s_2)$, so V encodes branch identity.*

*V is equivariant under $G = \mathbb{Z}_2$: $f \in V \Longrightarrow \sigma \cdot f \in V$.*

*Under these conditions,*

*$\dim V \geq 2$.*

**Proof.**

*Step 1 --- Irreducible representations of $\mathbb{Z}_2$.* $G = \mathbb{Z}_2$ has exactly two irreps over $\mathbb{R}$: the trivial irrep $\rho_+$ ($\sigma$ acts as $+1$) and the sign irrep $\rho_-$ ($\sigma$ acts as $-1$). Every real equivariant representation decomposes as $V \cong V_+ \oplus V_-$.

*Step 2 --- Equivariant decomposition of V.* Since V is equivariant, it decomposes as $V \cong V_+ \oplus V_-$ where $V_+ = \{f \in V : \sigma \cdot f = f\}$ and $V_- = \{f \in V : \sigma \cdot f = -f\}$.

*Step 3 --- Both summands are non-trivial.*

**$V_+ \neq \{0\}$:** Condition (i) directly requires a non-trivial element $f_{occ} \in V_+$. Therefore $V_+ \neq \{0\}$.

**$V_- \neq \{0\}$:** By condition (ii), there exists $f_{br} \in V$ with $f_{br}(s_1) \neq f_{br}(s_2)$. Decompose $f_{br} = f_+ + f_-$ where $f_+ = (f_{br} + \sigma \cdot f_{br})/2 \in V_+$ and $f_- = (f_{br} - \sigma \cdot f_{br})/2 \in V_-$. Since $f_{br}(s_1) \neq f_{br}(s_2)$, we have $f_{br} \neq \sigma \cdot f_{br}$, so $f_- \neq 0$. Therefore $V_- \neq \{0\}$.

*Step 4 --- Conclusion.* Since $V_+ \neq \{0\}$ and $V_- \neq \{0\}$:

$\dim V = \dim V_+ + \dim V_- \geq 1 + 1 = 2$. **Remark.** The gap that a pure sign sector $V = V_-$ might satisfy all conditions is closed by condition (i): a purely antisymmetric sector cannot contain a $\mathbb{Z}_2$-invariant (symmetric) mode, so it fails condition (i). Conversely, a purely symmetric sector $V = V_+$ cannot contain a branch-distinguishing mode (a symmetric function takes the same value at $s_1$ and $s_2$), so it fails condition (ii). Therefore any sector satisfying both conditions must contain both $V_+$ and $V_-$ as non-trivial summands.

**Graph-Theoretic Perspective**

The argument of Lemma $F_2$ has a natural graph-cut interpretation that connects directly to the spectral structure of the closure cell.

In a connected graph, the kernel of the Laplacian consists of constant functions --- there is one such mode (the global constant), corresponding to the zero eigenvalue. A binary refinement corresponds to a **minimal nontrivial cut** of the graph: a partition of vertices into two non-empty sets corresponding to the two outcome branches.

**Cut perspective.** A single oriented cut function --- assigning +1 to vertices on one side and −1 to the other --- is antisymmetric under branch exchange and encodes which side each vertex belongs to (branch identity). But a cut function alone does not encode the occurrence of commitment: it does not distinguish the committed state from any other two-sided assignment. For the representation to separately encode *that* a commitment occurred (distinguishing the post-commitment state from the pre-commitment trivial state), a symmetric mode is required. This is exactly the $V_+/V_-$ decomposition forced by Lemma $F_2$.

In $W_7$, the perimeter Fourier modes at eigenvalue $\lambda^* = 2$ come in symmetric/antisymmetric pairs under the dihedral symmetry of the cycle --- precisely the $\mathbb{Z}_2$ symmetry exchanging the two primitive branches. The two-dimensional $\lambda^* = 2$ eigenspace decomposes as one copy of $\rho_+$ and one copy of $\rho_-$, exactly saturating the lower bound of Lemma $F_2$. The graph structure of $W_7$ therefore realises the multiplicity lower bound exactly: the first nonzero perimeter modes of a hexagonal cycle provide the minimal equivariant representation of $\mathbb{Z}_2$ action on the two primitive branches.

**Bridge Principle F --- Minimal Spectral Realization**

**Bridge Principle F (Minimal Spectral Realization of Binary Refinement).** *The* $\lambda = 2$ eigenspace of the $K = 7$ closure cell is the unique lowest-roughness spectral sector capable of realizing primitive binary refinement in a manner satisfying Lemma $F_1$ and Lemma $F_2$. **Proof.**

*Step 1 --- Non-kernel requirement (Lemma $F_1$).* Any realizing sector must lie outside ker(L). This eliminates the $\lambda = 0$ sector entirely.

*Step 2 --- Multiplicity lower bound (Lemma $F_2$).* Any realizing sector that faithfully and symmetrically represents primitive binary commitment must have dimension at least 2. This rules out any one-dimensional eigenspace.

*Step 3 --- Minimal roughness (Courant--Fischer).* Among all spectral sectors outside the kernel, the lowest roughness is achieved at the first nonzero eigenvalue. For $K = 7$ this is $\lambda^* = 2$, with eigenspace multiplicity exactly 2. The $\lambda^* = 2$ sector simultaneously satisfies all requirements: it lies outside the kernel (Step 1), it has dimension $2 \geq 2$ (Step 2), and it achieves the minimum admissible roughness (Step 3).

*Step 4 --- Exclusion of higher sectors.* Any sector with $\lambda > 2$ has closure action cost $\Delta\tilde{S} = (r/2)\lambda > r$, exceeding the primitive closure cost and violating minimality of primitive commitment.

Therefore the $\lambda^* = 2$ eigenspace is the unique lowest-roughness sector satisfying all four requirements. **Logical structure of the bridge.**

[Lemma $F_1$: non-kernel requirement --- trivial, rigorous]

- [Lemma $F_2$: G = $\mathbb{Z}_2$ equivariance forces dim V ≥ 2 via irrep decomposition --- rigorous representation theory]
- [K = 7 spectral structure: λ* = 2 with multiplicity exactly 2, saturating the lower bound]
- [Courant--Fischer: λ* = 2 is the lowest admissible roughness outside the kernel]

The chain is: (1) primitive commitment exists in a system with symmetry group G = $\mathbb{Z}_2$, derived from VERSF's definition of "primitive"; (2) faithful equivariant representations of $\mathbb{Z}_2$ decompose into both irreps and have dimension ≥ 2; (3) the K = 7 spectral structure provides a sector of exactly dimension 2 at the minimum roughness, saturating the bound. The argument is now anchored in standard representation theory of finite groups rather than geometric intuition.

**Consequence.** The two minimality results of the previous sections are not coincidental: $\tilde{\Theta}_0 = \ln 2$ (Corollary C) and λ* = 2 (Theorem D) both describe the same primitive closure event. The direction of implication runs from the combinatorial structure of binary distinguishability and the $\mathbb{Z}_2$ symmetry of primitive commitment to the spectral multiplicity, not the reverse.

**6. Wheel Uniqueness --- Theorem E**

**Theorem E (Uniqueness of Primitive Spectral Normalization).** *Let W_K denote the wheel graph on K vertices total: one central hub vertex connected to a cycle of K − 1 perimeter vertices. Let λ*(K) be the smallest nonzero eigenvalue of the unweighted combinatorial Laplacian of W_K. Then (1/2) · λ*(K) = 1 if and only if K = 7.

**Proof.** The Laplacian of W_K block-decomposes into hub and perimeter sectors (derivation in Appendix). The perimeter modes satisfy

$\lambda_m(K) = 3 - 2\cos(2\pi m/(K-1))$, m = 1, ..., K−2,

and the hub mode has eigenvalue K. For K ≥ 4 the hub eigenvalue K exceeds all small-m perimeter modes, so the smallest nonzero eigenvalue is the m = 1 perimeter mode:

$\lambda^*(K) = 3 - 2\cos(2\pi/(K-1))$.

*Verification for K = 7:* The cycle has K − 1 = 6 vertices, giving $\lambda_m = 3 - 2\cos(\pi m/3)$ for m = 1, ..., 5:

| m | cos(πm/3) | $\lambda_m$ |
|---|---|---|
| 1, 5 | 1/2 | 2 |
| 2, 4 | −1/2 | 4 |
| 3 | −1 | 5 |
| hub | --- | 7 |

This recovers Spec($W_7$) = {0, 2, 2, 4, 4, 5, 7}. ✓

The condition $(1/2)\cdot\lambda^*(K) = 1$ becomes $\cos(2\pi/(K-1)) = 1/2$. The general solution is $2\pi/(K-1) = \pm\pi/3 + 2\pi n$. For $K \geq 4$, the unique integer solution is $2\pi/(K-1) = \pi/3$, giving $K - 1 = 6$, hence **K = 7**. For the full wheel-cell family:

$(1/2) \cdot \lambda^*(K) > 1$ for $K < 7$ (over-committed) $(1/2) \cdot \lambda^*(K) = 1$ for $K = 7$ (exact normalization) $(1/2) \cdot \lambda^*(K) < 1$ for $K > 7$ (under-committed)

$K = 7$ is the unique crossing point. The spectral cancellation $\Delta\tilde{S}_{prim} = r$ is a structural property of $K = 7$ alone.

## 7. The Barrier Formula

### 7.1 Proposition --- Barrier Formula (Unconditional)

Define the entropy quantum $\Theta_0$ as the physical entropy cost of one primitive commitment event, and let $\eta := \Theta_0/(k\_B \ln 2)$. The physical entropy cost of the primitive mode is

$\Delta S_c = (\Theta_0/\tilde{\Theta}_0) \cdot \Delta\tilde{S}_{prim} = (\Theta_0/\ln 2) \cdot r$.

**Proposition (Barrier Formula).**

$\boldsymbol{\Phi_c = \eta r}$. [unconditional]

**Proof.** $\Phi_c := \Delta S_c/(k\_B \ln 2) = \Theta_0 r/(k\_B \ln 2) = \eta r$. This expression is fully general. The parameter $\eta$ encodes the currently undetermined relationship between closure entropy and thermodynamic entropy; it is fixed only once the physical entropy identification is granted.

### 7.2 Falsification

The derivation weakens to $\Phi_c = \eta r$ with $\eta \neq 1$ if any of the following fails:

- **Theorem A fails** (closure entropy is not logarithmic): $\tilde{\Theta}_0$ changes, $\eta$ acquires a correction factor.
- **Lemma B fails** (primitive events are not binary-minimal): $\tilde{\Theta}_0 = \ln N$ for $N > 2$, giving $\eta = \ln N / \ln 2 > 1$.
- **Theorem G fails** (VERSF classes do not coincide with Boltzmann microstates): $\alpha \neq k\_B$ and $\Phi_c = \alpha r/k\_B$.

Each failure mode predicts specific deviations $\eta \neq 1$ distinguishable in principle through independent measurements of $E_c$ and $\Phi_c$.

### 7.3 Convergent Routes to η = 1

The barrier formula $\Phi_c = \eta r$ is unconditional, but $\eta$ itself has so far been left as a free parameter encoding the unknown relationship between closure entropy and physical entropy. This subsection presents four convergent routes that select $\eta = 1$. They approach $\eta$ from different structural directions --- causal geometry, the emergence constraint, statistical mechanics, and partition uniqueness --- and use different inputs from the framework. They are not mutually independent proofs; their common premise and logical hierarchy are set out below.

**Logical hierarchy of the routes.** The routes differ in what they prove and what they assume:

- **Route C** (statistical) is the primary statistical determination: it shows $\eta \neq 1$ produces an inconsistency in the Boltzmann factor at the effective thermodynamic layer.
- **Route D** (partition) is the emergence-necessity argument: it shows that if thermodynamics genuinely emerges from closure, the class identification --- and hence $\eta = 1$ --- is unavoidable.
- **Route A** (commitment-cell matching) does not fix k_B from scratch; it proves that the coherence cell is the unique minimal physical carrier of one primitive bit, reducing the $\eta$ question from an open conversion ambiguity to a single coefficient-comparison on one and the same physical domain.
- **Route B** (non-redundancy) does not infer k_B by dimensional analogy; it proves that the effective thermal layer already contains a unique action-to-entropy conversion, and $\eta \neq 1$ would insert a second, structurally redundant conversion factor with no independent origin in the framework.

Routes A and B do not independently define the Boltzmann coefficient from scratch. Rather, they show that once the effective causal and thermal layers are in place, any value $\eta \neq 1$ introduces a mismatch or redundancy that is absent elsewhere in the framework. Routes C and D make the stronger claim that $\eta \neq 1$ is outright inconsistent with the statistical and partition structures.

**Route A --- Primitive Causal Cell Matching**

Theorem I establishes the commitment-capacity invariant $\chi(L) = \rho L^4/(\hbar c)$ at the effective causal layer, and defines the coherence scale $\xi$ by $\chi(\xi) = 1$. The coherence cell is, by construction, the smallest causal domain in which the action-budget-limited capacity permits the production of one primitive commitment event. No smaller domain can produce a complete irreversible commitment event.

**Step 1 --- The coherence cell is the smallest domain whose capacity bound permits the production of one primitive commitment event.** The primitive commitment event is binary (Lemma B), so its closure entropy is $\tilde{\Theta}_0 = \ln 2$ (Corollary C). The coherence cell corresponds to the minimal domain at which one primitive commitment event becomes possible: $\chi(\xi) = 1$ means the action-budget-limited capacity upper bound first reaches one, not that exactly one event is guaranteed to occur. The closure entropy associated with one such event is ln 2, determined by binary minimality.

**Step 2 --- The coherence cell therefore corresponds to one primitive thermodynamic commitment event.** Suppose thermodynamics genuinely emerges from closure dynamics (the VERSF emergence hypothesis). Then the thermodynamic entropy associated with the same coherence cell must be consistent with the closure entropy of that cell. Since the closure entropy of the cell is ln 2, the thermodynamic entropy must be of the form

$$\Theta_0 = \alpha \cdot \ln 2$$

for some universal conversion factor α. Route A does not fix α independently of the effective thermodynamic layer. What it establishes is a matching condition: the primitive closure bit and the primitive thermodynamic bit are necessarily evaluated on the same physical domain --- the coherence cell --- so the η problem collapses from an open conversion ambiguity to a single coefficient comparison on one and the same well-defined domain.

**Step 3 --- Once the effective thermodynamic reading is admitted on the coherence cell, the coefficient is fixed.** The coherence cell supports one binary commitment event with two distinguishable outcomes. Once the effective thermodynamic layer is in place and Boltzmann entropy is admitted as the entropy measure on that layer, the Boltzmann entropy of the same two-state cell is S_B = k_B ln 2. Since both closure entropy ($\tilde{\Theta}_0$ = ln 2) and Boltzmann entropy (S_B = k_B ln 2) are computed over the same two-outcome physical domain, their ratio fixes the conversion factor: α = k_B. This gives $\Theta_0$ = k_B ln 2 and η = 1.

The admission of Boltzmann entropy on the coherence cell is not an additional assumption beyond the effective thermodynamic layer --- it is what the effective thermodynamic layer *means* applied to that specific domain. Route A's role is to ensure the domain is the right one: the minimal domain at which the action-budget-limited capacity permits the production of one primitive commitment event.

**Route A conclusion:** The coherence cell χ(ξ) = 1 is the unique minimal domain at which the action-budget-limited capacity permits the production of one primitive irreversible commitment event. Route A proves that the closure and thermodynamic entropy readings of that cell must refer to the same physical domain, reducing the η problem to a single coefficient comparison. Once the effective thermodynamic layer is admitted on that domain, the comparison gives α = k_B, hence $\Theta_0$ = k_B ln 2 and η = 1. Route A is a strong reduction argument, not an independent derivation of k_B from scratch.

**Route B --- The Emergence Constraint on η**

Within VERSF's emergence programme, η is not a free primitive parameter in the way that, say, the fine-structure constant is a free parameter of QED. The distinction is important: QED makes no claim to derive electromagnetism from a more fundamental combinatorial structure, so α ≈ 1/137 is legitimately measured rather than derived. VERSF, by contrast, claims that thermodynamics *emerges* from closure dynamics. The meaning of this claim is that the thermodynamic entropy scale is not an independent input but is the closure entropy scale --- they are the same thing. A value η ≠ 1 would mean they are not the same thing: the closure entropy of a primitive event would be ln 2, but its thermodynamic entropy would be η · k_B ln 2, with η ≠ 1 encoding a genuine discrepancy between the two scales. That discrepancy is exactly what the emergence claim says cannot exist.

**More precisely.** At the effective statistical layer, quantum statistical mechanics generates dimensionless thermal weighting through

$E \cdot \tau\_th / \hbar = E / (k\_B T)$,

where $\tau\_th = \hbar/(k\_B T)$ is the thermal interval. The closure framework produces a primitive event with entropy cost $\tilde{\Theta}_0 = \ln 2$. The dimensionless statistical weight of this event under the identification $S = \alpha\tilde{S}$ is $\Theta_0/k\_B = (\alpha/k\_B) \cdot \ln 2$. For this to equal the standard dimensionless closure entropy ln 2 --- which is what emergence requires, since the closure and thermodynamic descriptions must assign the same dimensionless weight to the same event --- $\alpha = k\_B$ is forced.

Any $\eta \neq 1$ would insert a factor $(\alpha/k\_B) \neq 1$ into the statistical weight of the primitive commitment event. This factor cannot be attributed to any structural sector of the framework that is independent of the identification itself: it is not fixed by the closure spectrum (which determines $\Delta\tilde{S}_{prim} = r$, not $\eta$), nor by the commitment-capacity structure (which determines $\xi$), nor by the Boltzmann factor already established at the effective layer. Within the emergence programme, the only consistent interpretation of such a factor is that the closure and thermodynamic entropy scales genuinely differ --- which is precisely what emergence claims to preclude.

**Remark on the scope of this argument.** Route B does not prove $\eta = 1$ as a theorem about all possible theories; it argues that within VERSF's emergence programme, $\eta \neq 1$ is in tension with what emergence means. A theory that does not claim thermodynamic emergence --- one that treats thermodynamics as a separate and independent layer --- could freely have $\eta \neq 1$ as a measured parameter. VERSF's claim is stronger, and Route B holds it to that claim.

**Route B conclusion:** Within VERSF's thermodynamic emergence programme, $\eta$ cannot be a free parameter --- it must equal 1 for closure entropy and thermodynamic entropy to be the same scale, which is what emergence requires. The effective-layer argument using $\tau\_th = \hbar/(k\_B T)$ makes this precise: $\eta \neq 1$ assigns different dimensionless weights to the same physical event in the closure and thermodynamic descriptions, violating the emergence identification.

**Route C --- Statistical Consistency**

At the effective thermodynamic layer, the probability of a physical microstate with energy E at temperature T is governed by the Boltzmann factor:

$P(E) \propto e^{-E/(k\_B T)}$.

For the identification $S = \alpha\tilde{S}$ to reproduce this factor correctly, we need the exponent $E/(k\_B T)$ to be expressible purely in terms of closure entropy and physical energy without any additional dimensionless correction. The Boltzmann weight requires:

$E/(k\_B T) = \alpha\tilde{S}/k\_B$.

For this to equal the standard thermodynamic exponent, we need $\alpha = k_B$. Any other value of $\alpha$ introduces a spurious factor of $\alpha/k_B$ into the exponent. Since temperature T is defined independently through the equipartition structure of the effective causal layer --- it is not a free parameter that can absorb this factor without changing the physics --- the factor $\alpha/k_B \neq 1$ would represent a genuine discrepancy between the temperature emerging from closure dynamics and the temperature in the Boltzmann distribution.

A cleaner way to see this: the Boltzmann weight $e^{-E/(k_B T)}$ assigns dimensionless thermal weight $E/(k_B T)$ to a state. Under $S = \alpha\tilde{S}$, the closure entropy of that state is $\tilde{S} = S/\alpha$, giving dimensionless thermal weight $\tilde{S} \cdot (\alpha/k_B) = S/k_B$. For this to equal $E/(k_B T)$ --- the standard thermodynamic weight --- we need $S = E/T$, which is the standard thermodynamic relation, and the weight $S/k_B = E/(k_B T)$ holds for all states only when $\alpha = k_B$ exactly.

**Route C: $\eta = 1$ is the unique value consistent with the standard Boltzmann factor at the effective statistical layer.** Any deviation $\alpha \neq k_B$ forces either a redefinition of temperature or a mismatch between the closure-derived and Boltzmann-derived statistical weights of the same physical state.

**Convergence and Logical Structure**

| Route | What it proves | Logical status |
|---|---|---|
| **A** --- Primitive commitment-cell matching | Coherence cell is the unique minimal carrier of one primitive bit. Once the effective thermodynamic layer is admitted on that cell, comparing closure entropy (ln 2) and Boltzmann entropy ($k_B$ ln 2) on the same domain fixes $\alpha = k_B$. | Effective-layer consistency argument; requires admission of effective thermodynamic layer |
| **B** --- Emergence constraint on $\eta$ | Within VERSF's emergence programme, $\eta$ cannot be a free parameter. The emergence claim requires that the closure entropy scale *be* the thermodynamic entropy scale --- any $\eta \neq 1$ means they differ. The relation $\tau_{th} = \hbar/(k_B T)$ makes this precise. | Effective-layer consistency argument; applies within the emergence programme's own requirements |
| **C** --- Statistical Boltzmann consistency | $\eta \neq 1$ breaks the Boltzmann factor $e^{-E/(k_B T)}$, forcing a spurious factor into the effective temperature. The standard Boltzmann weight requires $\alpha = k_B$ exactly. | Effective-layer consistency argument; conditional on independently defined temperature |
| **D** --- Partition identification | Thermodynamic emergence forces the closure partition and Boltzmann microstate partition to coincide. The Matching Lemma then fixes $\alpha = k_B$ without further interpretational choice. | Conditional on thermodynamic emergence programme; primary route (Section 8) |

All four routes select $\eta = 1$. Routes C and D show $\eta \neq 1$ is outright inconsistent with the statistical and partition structures respectively. Routes A and B are effective-layer consistency arguments --- they show $\eta \neq 1$ introduces a mismatch or a structurally redundant factor absent elsewhere in the framework. None of the four routes derives $k_B$ from pure closure combinatorics without reference to the effective layer; each operates within a regime where some thermodynamic structure is already in place, and shows that $\eta \neq 1$ is incompatible with that structure.

**The paper does not match closure entropy to thermodynamics by assumption. It shows that $\eta = 1$ is the unique value simultaneously consistent with four convergent structural requirements --- primitive commitment-cell matching, the emergence constraint on $\eta$, Boltzmann factor consistency, and partition uniqueness --- each approached from a distinct structural direction within the effective and emergent thermodynamic regime.** The physical entropy identification $S = k_B \tilde{S}$ is the unique normalization compatible with all four.

## 8. Physical Entropy Identification

### 8.1 Preamble --- The Partition Route to $\eta = 1$

Section 7.3 established that three effective-layer consistency arguments --- primitive commitment-cell matching (Route A), action--thermal non-redundancy (Route B), and statistical Boltzmann consistency (Route C) --- all select $\eta = 1$ within the emergent thermodynamic regime. This section provides the fourth route: the partition identification argument (Route D). It shows that if the VERSF thermodynamic emergence programme is correct, the closure distinguishability partition and the Boltzmann microstate partition cannot be different partitions of the same physical system. The identification $S = k_B \tilde{S}$ then follows from the uniqueness of the record-induced partition rather than from an interpretational choice. Route D is the only route that does not depend on the effective thermodynamic layer already being in place --- it operates at the level of the emergence claim itself.

### 8.2 Irreversibility as the Common Origin

Both closure entropy and thermodynamic entropy are measures of irreversible distinguishability, and this common origin motivates their identification.

In the VERSF framework, a primitive commitment event transforms a reversible distinguishability relation into an irreversible stable record. Once the event occurs, the system's state space is partitioned into distinguishable classes corresponding to the recorded outcomes. Closure entropy $\tilde{S}(N) = \ln N$ counts the logarithm of the number of such classes.

In statistical mechanics, thermodynamic entropy measures exactly the same kind of quantity: the logarithm of the number of microscopic configurations compatible with a macroscopic record:

$S_B(N) = k_B \ln N$.

Both expressions therefore count irreversible distinguishability. This structural parallel motivates asking whether they count the *same* distinguishable states --- and if so, what follows.

### 8.3 The Conditional Step --- Class Identification

**The physical entropy identification rests on the following conditional claim.**

**Class Identification Assumption.** VERSF distinguishability classes coincide with the microstate classes of statistical mechanics: two states are VERSF-distinguishable after a commitment event if and only if they belong to different Boltzmann microstates.

If this identification holds, then for any system of size N:

$\tilde{S}(N) = \ln N$ (VERSF closure entropy, Theorem A) $S_B(N) = k_B \ln N$ (Boltzmann entropy)

are computed over the *same* set of N states. Since both are the unique admissible logarithmic entropy on that shared state structure up to a positive constant (Theorem A), proportionality is forced and comparison with the Boltzmann formula fixes that constant:

$\alpha = k_B$, hence $\mathbf{S = k_B \tilde{S}}$.

**Matching Lemma.** *Suppose the Class Identification Assumption holds. Then $\alpha = k_B$ is uniquely determined by comparison of $\tilde{S}$ and $S_B$ over the shared state structure.* **What the Matching Lemma does not establish.** The lemma does not derive $k_B$ from closure combinatorics alone. $k_B$ is a measured constant entering through the Boltzmann formula. The lemma shows that *if* the class identification holds then $\alpha = k_B$ follows structurally --- but the identification itself is the conditional step. The real achievement of the unconditional tier is deriving the *form* $\tilde{S} = \ln N$ from axioms; the *scale* is fixed once the physical interpretation is granted.

### 8.4 Why the Class Identification Is Structurally Constrained

The Class Identification Assumption states that VERSF distinguishability classes coincide with Boltzmann microstate classes. Rather than treating this as a free interpretational choice, this section shows that it is structurally constrained: any theory in which thermodynamics genuinely emerges from the production of irreversible commitment events has at most one candidate partition, and the closure partition is it.

**Proposition (Uniqueness of the Record-Induced Partition).** *Any theory satisfying the following two conditions:*

*(a) Facts are irreversible: once a record is produced, the corresponding partition of state space is stable and cannot be reversed by the dynamics.*

*(b) Entropy counts distinguishable outcomes: the entropy of a physical state is the logarithm of the number of state-space configurations indistinguishable from it, relative to a fixed irreversible record.*

*must define a unique partition of state space into equivalence classes of mutually indistinguishable configurations, relative to any given irreversible record.*

**Proof.** Condition (b) requires that entropy be computed by counting indistinguishable configurations relative to a fixed record. For this count to be well-defined and reproducible given the record, the partition into equivalence classes must be unique given that record --- if two different partitions were both admissible given the same record, entropy would be multi-valued on the same physical state after the same record has been produced, which violates the assumption that entropy is a function of the physical state and the record. Condition (a) ensures the record and hence the partition is stable once produced. **Caveat: coarse-graining and the macroscopic description.** In statistical mechanics, different choices of macroscopic description (different coarse-grainings) yield different entropy values for the same microscopic state --- the Gibbs entropy, Boltzmann entropy, and von Neumann entropy are not identical, and none is uniquely determined by the microscopic state alone. This is not a contradiction with the Proposition, because the Proposition establishes uniqueness *given* a fixed macroscopic description (i.e., given a fixed set of irreversible records), not across all possible descriptions.

The Proposition therefore establishes: for any fixed macroscopic description of a physical system, the VERSF closure partition and the Boltzmann microstate partition are both the unique admissible partitions for that description. If both are descriptions of the same physical system at the same level of macroscopic resolution, they cannot differ.

The additional step required to identify the VERSF closure partition with the Boltzmann microstate partition --- and hence to conclude that both use the same level of macroscopic description --- is precisely the Class Identification Assumption. The Proposition shows this assumption is not arbitrary: it is required if VERSF's closure dynamics genuinely describe the same physical records as statistical mechanics does. But it does not prove the assumption from the VERSF axioms alone; it shows that the assumption is *forced* by the emergence claim rather than being optional.

**Consequence for the Class Identification.** Statistical mechanics defines a microstate partition relative to the macroscopic observables of thermodynamics. VERSF defines a distinguishability partition relative to commitment events. The Proposition shows that if both partitions describe the same physical irreversible records at the same level of macroscopic resolution, they cannot be different partitions. Therefore:

Either VERSF's records and thermodynamic records describe different macroscopic resolutions of the same system (in which case thermodynamics does not emerge from closure in the intended sense), or the partitions coincide.

The Class Identification Assumption is the claim that the two partitions describe the same records at the same resolution --- which is exactly what the thermodynamic emergence programme asserts.

### 8.4.1 Three Structural Lines as Independent Confirmation

Given the uniqueness argument above, the three structural lines below serve as independent confirmations that the closure partition and the Boltzmann partition are indeed the same object --

- approaching the identification from different structural directions rather than establishing it from scratch.

**Line 1 --- Binary Irreducibility.** Lemma B establishes that the smallest nontrivial irreversible refinement of a distinguishability partition has cardinality change $1 \to 2$. The smallest irreversible closure event therefore produces exactly one binary distinction --- one bit. This is precisely the structure of a Boltzmann microstate transition: the minimal physical record separating two previously indistinguishable configurations. The binary minimality of VERSF primitive commitment matches the binary minimality of thermodynamic bit formation, which is consistent with --- and required by --- the partitions being identical.

**Line 2 --- Commitment-Capacity Threshold.** Theorem I (Section 9.1) derives the commitment-capacity invariant $\chi(L) = \rho L^4/(\hbar c)$ from the causal action budget. The coherence scale $\xi$ defined by $\chi(\xi) = 1$ is the smallest causal domain whose capacity *bound* permits one primitive commitment event --- the scale at which the upper bound on irreversible events first reaches one, not necessarily where exactly one event is realised. Because primitive commitment is binary (Line 1), this threshold corresponds to one bit of irreversible distinguishability at the bound level. If the closure partition and the Boltzmann partition are identical, this one bit corresponds to one thermodynamic bit, and the associated entropy at the threshold must be $k_B \ln 2$. The commitment-capacity structure therefore independently requires one-bit threshold behaviour at scale $\xi$ --- consistent with the identification and with the derived value $\Theta_0 = k_B \ln 2$.

**Line 3 --- Thermodynamic Emergence.** VERSF proposes that thermodynamics is not fundamental but emerges from the irreversible production of irreversible commitment events (closure facts). The Uniqueness Proposition above shows that if this emergence is genuine, the closure partition and the Boltzmann partition must coincide. Line 3 is therefore not an independent structural line so much as the direct statement of what the Uniqueness Proposition implies: the emergence claim entails the identification. It is listed separately because it can also be approached from the global programme level --- asking whether the VERSF emergence programme is internally consistent as a whole --- rather than only through the local uniqueness argument.

These three confirmations originate from distinct structures --- combinatorial minimality, causal geometry, and the global emergence hypothesis --- and each independently supports the identification. Together with the Uniqueness Proposition, they constitute a strong and multi-layered case that the Class Identification Assumption is not a leap but a structural necessity under the emergence hypothesis.

### 8.5 Conditional Theorem G --- Physical Entropy Identification

**Conditional Theorem G.** *If the Class Identification Assumption holds --- that is, if VERSF distinguishability classes coincide with the microstate classes of statistical mechanics --- then*

$S = k_B \tilde{S}$, $\Theta_0 = k_B \ln 2$, $\eta = 1$.

**Proof.** Theorem A gives $\tilde{S}(N) = \ln N$. Under the class identification, the Matching Lemma gives $\alpha = k\_B$, so $S = k\_B \tilde{S}$. Applying to Corollary C: $\Theta_0 = k\_B \cdot \tilde{\Theta}_0 = k\_B \ln 2$. Then $\eta = \Theta_0/(k\_B \ln 2) = 1$. The Class Identification Assumption is the sole conditional step. The three structural lines of Section 8.4 motivate it; the Matching Lemma shows that once it is granted, $\alpha = k\_B$ follows without further choice.

**8.6 Landauer's Principle as an External Consistency Check**

Landauer's principle states that the minimal thermodynamic entropy cost of erasing one bit of information is $k\_B \ln 2$.

The present derivation arrives at $\Theta_0 = k\_B \ln 2$ by a different route: structural counting of primitive commitment events (Theorem A + Lemma B) gives $\tilde{\Theta}_0 = \ln 2$ unconditionally, and the Matching Lemma converts this to $k\_B \ln 2$ once the class identification is granted. The direction of reasoning is:

Closure axioms → $\tilde{\Theta}_0 = \ln 2$ → [Class Identification] → $\Theta_0 = k\_B \ln 2$ → *consistent with Landauer*.

Not:

Landauer → $\Theta_0 = k\_B \ln 2$ → $\alpha = k\_B$.

The second direction is circular: it assumes that the primitive commitment event is thermodynamically minimal in the Landauer sense, which is equivalent to assuming the class identification itself. The first direction is the correct one, with the Class Identification Assumption stated explicitly.

Landauer therefore appears as an *external consistency check*: an independent thermodynamic constraint that the derived value satisfies. If the derivation had produced $\Theta_0 \neq k\_B \ln 2$, disagreement with Landauer would constitute evidence against the class identification. Agreement instead constitutes confirmation.

**8.7 Conditional Corollary H**

**Conditional Corollary H** *(conditional on Theorem G).*

$\Phi_c = r$.

**Proof.** Theorem G gives $\eta = 1$. Substituting into the Proposition: $\Phi_c = 1 \cdot r = r$. **8.8 Summary of the Identification**

| Component | Status |
|---|---|
| $S = \alpha\tilde{S}$ for some $\alpha > 0$ | **Unconditional** --- structurally forced by shared admissibility conditions |

| Component | Status |
| --- | --- |
| Class Identification Assumption | **Conditional** --- physical interpretation claim; sole conditional step |
| $\alpha = k\_B$ | **Conditional consequence** --- Matching Lemma given class identification |
| $\Theta_0 = k\_B \ln 2$ | **Conditional** --- follows from $\tilde{\Theta}_0 = \ln 2$ and $\alpha = k\_B$ |
| $\eta = 1$ | **Conditional** |
| $\Phi_c = r$ | **Conditional** --- Corollary H |
| Landauer agreement | **External consistency check** --- not a premise |
| Three structural lines (binary irreducibility, commitment-capacity, thermodynamic emergence) | **Supporting motivation** for the Class Identification Assumption |

## 9. The Commitment-Capacity Chain and Consistency Theorem

### 9.1 Theorem I --- Quartic Commitment-Capacity Invariant

**Ontological status of time-language in this section.** VERSF holds that physical time is not a primitive ingredient of the underlying ontology but emerges from the irreversible production of distinguishable facts. Theorem I is an *effective-layer* result: it is stated and proved within the regime where causal structure and relativistic propagation bounds have already emerged from the commitment dynamics. Terms such as "causal propagation interval," "action budget," and "propagation speed *c*" are used here as properties of the effective causal layer, not as primitive notions of the pre-commitment substrate. The fundamental quantities in the underlying ontology remain irreversible commitment events, distinguishability classes, and the action scale $\hbar$. Wherever time-adjacent language appears below, it refers to causal extent or action budget at the effective layer --- not to a primitive time coordinate assumed before commitment.

**Theorem I (Quartic Commitment-Capacity Invariant).** *Let a causal domain of size L contain energy density ρ. Within the effective causal layer of VERSF, the action-limited distinguishability capacity --- the upper bound on the number of irreversible commitment events associated with a causal domain of size L --- scales as $\rho L^4/(\hbar c)$, up to order-one factors. The natural dimensionless commitment-capacity invariant is*

$$\chi(L) = \rho L^4 / (\hbar c).$$

*The quartic power $L^4$ is forced by the structure of the causal action budget.*

**Proof.**

*Step 1.* Total energy in a domain of size L: $E\_L \sim \rho L^3$.

*Step 2.* At the effective causal layer, relativistic propagation defines a causal propagation interval $\tau_L = L/c$ --- the effective causal extent associated with traversal across the domain. The available action budget over this causal extent is:

$$A_L = E_L \cdot \tau_L = \rho L^3 \cdot (L/c) = \rho L^4/c.$$

This is a property of the effective causal layer, not a primitive time interval of the underlying ontology.

*Step 3.* The Margolus--Levitin theorem, applied at the effective layer, bounds the action-limited distinguishability capacity: a system with energy E can support at most $E/(\pi\hbar/2)$ orthogonal state transitions per propagation interval. Expressed in terms of the total action budget rather than a transition rate, this gives an upper bound on the number of distinct commitment events that can be produced within the causal domain:

$$N_{max} \lesssim A_L / \hbar = \rho L^4 / (\hbar c),$$ (up to order-one factors from the Margolus--Levitin prefactor).

The coherence scale $\xi$ defined by $\chi(\xi) = 1$ is the domain size at which this capacity *bound* reaches the threshold corresponding to one primitive commitment event --- not necessarily where exactly one event is realised.

*Step 4.* The quartic scaling is forced by the action budget structure, not by energy content alone. The key point is that commitment capacity is bounded by the total action $A_L$, and action is the product of energy and causal extent --- not energy alone. Doubling the energy while halving the causal extent leaves the action, and hence the capacity bound, unchanged. The relevant scaling table:

| Quantity | Scaling |
|---|---|
| Energy in domain | $L^3$ |
| Causal propagation interval (effective layer) | $L$ |
| Action budget = energy × causal extent | $L^4$ |
| Commitment capacity bound ∝ action/$\hbar$ | $L^4$ |

No other exponent is consistent with commitment capacity proportional to the causal action budget. Define $\chi(L) := \rho L^4/(\hbar c)$.

**Remark on conditionality and ontological status.** Theorem I is unconditional within VERSF in the sense that it does not require the entropy identification $S = k_B \tilde{S}$. Neither $\chi(L)$ nor $\xi$ depends on that identification. However, the theorem does operate at the effective causal layer: it uses the emergent propagation speed *c*, the causal domain size L, and the action scale $\hbar$. These are well-defined once the effective causal structure has emerged from the underlying commitment dynamics. The theorem is therefore an effective-layer result, and all time-adjacent language in it --- "causal propagation interval,"

| Quantity | Scaling |
| --- | --- |

"causal extent," "action budget" --- should be read accordingly.

### 9.2 Corollary --- The Coherence Scale

**Corollary.** *The primitive commitment scale $\xi$ satisfies $\chi(\xi) = 1$, giving*

$\xi = (\hbar c / \rho)^{(1/4)}$.

**Proof.** Set $\chi(\xi) = \rho\xi^4/(\hbar c) = 1$ and solve for $\xi$. **On the energy density $\rho$ and the numerical estimate.** The formula $\xi = (\hbar c/\rho)^{(1/4)}$ requires a value for the energy density $\rho$ of the underlying medium. In VERSF, $\rho$ is the energy density of the void substrate --- the medium from which causal structure emerges. This is not yet derived from first principles within the framework; it is an input that requires independent determination.

In the numerical estimate used here, $\rho$ is taken to be the observed cosmological vacuum energy density, $\rho_\Lambda = \rho_c \cdot \Omega_\Lambda \approx (2.25 \times 10^{-3}\ \text{eV})^4/(\hbar c)^3 \approx 3.4 \times 10^{-30}\ \text{g/cm}^3$, which gives:

$\xi = (\hbar c/\rho_\Lambda)^{(1/4)} \approx (\hbar c / (6.7 \times 10^{-10}\ \text{J/m}^3))^{(1/4)} \approx 8 \times 10^{-5}\ \text{m}$.

With this estimate, the Wilson sector gives:

$E_c \sim r \cdot \hbar c/\xi \approx r \times 2.5\ \text{meV}$.

**Important caveats.** This numerical estimate carries significant uncertainty:

1. The identification of $\rho$ with the cosmological vacuum energy density is a working assumption, not a derived result. Other physically motivated choices --- the QCD vacuum energy density ($\sim 10^{-4}\ \text{GeV}^4/(\hbar c)^3$), the electroweak scale, or the Planck density --- differ by many orders of magnitude and would yield very different values of $\xi$ and $E_c$.
2. The calculation $\xi \approx 8 \times 10^{-5}$ m uses $\rho_\Lambda$ as stated; readers can reproduce it by substituting $\rho_\Lambda$ into $\xi = (\hbar c/\rho)^{(1/4)}$.
3. The value $E_c \approx 2.5$ meV should therefore be understood as an order-of-magnitude estimate conditional on $\rho = \rho_\Lambda$, not a precise prediction. The form $E_c = r \cdot \hbar c/\xi$ is unconditional given the commitment-capacity derivation; the numerical value is contingent on the correct identification of $\rho$.

A proper derivation of $\rho$ from within the VERSF framework --- connecting the void substrate energy density to observable quantities --- is a priority for companion work. Until that derivation is available, the 2.5 meV figure should be read as indicative of the energy regime rather than as a precise prediction.

### 9.3 Theorem J --- Minimal Admissible Wheel Cell

The five spectral conditions characterising an admissible closure cell are:

1. **One-dimensional kernel:** dim ker(L) = 1.
2. **Primitive eigenvalue exactly 2:** $\lambda_1 = 2$.
3. **Twofold degeneracy:** mult($\lambda_1$) = 2.
4. **Spectral gap:** $\lambda_2 > \lambda_1$.
5. **Hub-compatible structure:** the graph admits a distinguished hub vertex.

**Theorem J (Wheel Minimality).** *Within the wheel-graph family W_K, K = 7 is the unique cell satisfying all five admissibility conditions. It is therefore the minimal admissible wheel cell.*

**Proof.** Conditions 1 and 5 are satisfied by all W_K by construction. Condition 2 requires $\lambda^*(K) = 2$, which by the Appendix formula $\lambda^*(K) = 3 - 2\cos(2\pi/(K-1))$ holds if and only if K = 7. Conditions 3 and 4 are verified by inspection of the K = 7 spectrum: mult(2) = 2 and the next eigenvalue is 4 > 2. No other W_K satisfies condition 2. Therefore K = 7 is the unique minimal admissible wheel cell. **Open problem.** Theorem J does not establish that $W_7$ is minimal among *all* connected graphs. Three questions remain:

1. Must the closure cell belong to the wheel family? (Motivated by closure-flow derivation but not derived from axioms alone.)
2. Are there non-wheel graphs on fewer than 7 vertices satisfying all five conditions? (A finite spectral classification problem --- tractable by enumeration of connected graphs on n = 4, 5, 6 vertices.)
3. Is the spectrum {0, 2, 2, 4, 4, 5, 7} uniquely realised by $W_7$? (Requires cospectral graph analysis.)

If no connected graph on n < 7 vertices satisfies all five conditions, Theorem J elevates to a full uniqueness theorem without the wheel-family restriction.

**Conditional status of K = 7-dependent results.** Theorems D, E, the Spectral Lemma, Bridge Principle F, and all results that depend on them --- including the barrier formula $\Phi_c = \eta r$, the spectral cancellation $\Delta\tilde{S}_{prim} = r$, and the energy scale $E_c \sim r \cdot \hbar c/\xi$ --- are conditional on the K = 7 closure cell being the correct minimal admissible cell. Within the wheel-graph family this is established by Theorem J. Beyond the wheel family it remains open. Until the finite enumeration of connected graphs on n = 4, 5, 6 vertices is completed, all these downstream results should be understood as holding *subject to the wheel-family assumption*.

### 9.4 Consistency Theorem --- r = 1

**Consistency Theorem.** *Suppose:*

*(i)* $\Delta\tilde{S}_{prim} = r$ *(Spectral Lemma, unconditional).*

*(ii)* $\tilde{\Theta}_0 = \ln 2$ *(Corollary C, unconditional).*

*(iii) The dimensionless driving variable in closure units is normalized as* $\tilde{\Phi} := \Delta\tilde{S}/\tilde{\Theta}_0$ *--- measured in units of the primitive closure entropy quantum. Note:* $\tilde{\Phi}$ *is the closure-unit driving variable, distinct from the physical commitment barrier* $\Phi_c = \Delta S_c/(k\_B \ln 2)$*.*

*(iv) The entropy identification $S = k\_B \tilde{S}$ holds (Theorem G). Under this identification, $\Delta S_c = k\_B \cdot \Delta\tilde{S}$, so:*

*$\Phi_c = \Delta S_c/(k\_B \ln 2) = k\_B \cdot \Delta\tilde{S}/(k\_B \ln 2) = \Delta\tilde{S}/\ln 2 = \Delta\tilde{S}/\tilde{\Theta}_0 = \tilde{\Phi}$.*

*Therefore $\Phi_c = \tilde{\Phi}$ under the entropy identification.*

*Then $r = 1$.*

**Proof.** Condition (iii) fixes $\tilde{\Phi}_c = \tilde{\Theta}_0/\tilde{\Theta}_0 = 1$ by definition of the normalization. Condition (iv) promotes this to $\Phi_c = 1$. Corollary H gives $\Phi_c = r$. Therefore $r = 1$. **Remark on logical structure.** Condition (iii) is a normalization choice --- it sets the scale of the driving variable in units of the primitive event itself. It is natural but it is a *choice*, not an independent constraint. Given that choice, $r = 1$ follows from condition (iv) alone: the entropy identification is the sole load-bearing conditional step. Conditions (i) and (ii) supply the unconditional spectral and counting results that make the physical result meaningful, but they do not independently constrain $r$.

The paper does not derive $r = 1$ from dynamics alone. Rather, $r = 1$ is the unique value at which the unconditional spectral structure ($\Delta\tilde{S}_{prim} = r$) and the conditional physical normalization ($\Phi_c = 1$ under the entropy identification and natural normalization) are mutually consistent. It is therefore a **primitive consistency value** --- expressing agreement between the structural and physical tiers at exactly $r = 1$ --- not a prediction independently derived from three converging constraints.

## 10. Theoretical Predictions and Experimental Outlook

### 10.1 Predictions at Two Strengths

**Unconditional.** The spectral identity $\Delta\tilde{S}_{prim} = r$ and the energy scale form $E_c \sim r \cdot \hbar c/\xi$ are established by the combinatorial and commitment-capacity structure. A measurement of an anomalous decoherence threshold energy would constrain $r$ independently of the entropy identification.

**Conditional** (on Theorem G and the wheel-family assumption). $\eta = 1$, $\Phi_c = r$, and $r = 1$ as the primitive consistency value, giving:

$E_c \approx 2.5$ meV (order-of-magnitude, conditional on $\rho = \rho\_\Lambda$), $\Phi_c \approx 1$.

These are theoretical predictions, not unconditional values. The numerical value of $E_c$ carries the caveat of Section 9.2 on $\rho$.

### 10.2 The Two-Observable Test --- Measuring η Directly

The framework's central prediction is $\eta = 1$. The two-observable test provides the theoretical structure for measuring $\eta$ from independent observations of $E_c$ and $\Phi_c$.

From the energy scale relation:

$E_c = r \cdot \hbar c/\xi$, hence $r = E_c \cdot \xi / (\hbar c)$.

From the barrier formula $\Phi_c = \eta \cdot r$, substituting:

$\boldsymbol{\eta = \Phi_c \cdot \hbar c / (E_c \cdot \xi)}$.

Observable 1: $E_c$ [measures $r$ unconditionally] Observable 2: $\Phi_c$ [measures $\eta \cdot r$ unconditionally] $\boldsymbol{\eta = \Phi_c \cdot \hbar c / (E_c \cdot \xi)}$ [direct readout]

If $\eta = 1$ and $r = 1$: $E_c \approx 2.5$ meV, $\Phi_c = 1$ --- the full conditional prediction confirmed. If $\eta \neq 1$: the closure and thermodynamic entropy scales differ; Routes A--D identify which structural claim requires revision. If $\eta = 1$ but $r \neq 1$: the entropy identification holds but the spectral stiffness deviates from the primitive consistency value.

**Operationalization note.** The formula above is the theoretical structure of the test. $\Phi_c$ is defined as $\Delta S_c/(k\_B \ln 2)$ --- a dimensionless entropy amplification. Measuring it independently of $E_c$ requires a way to quantify the entropy cost of a transition, not just its energy scale. One natural approach: $\Phi_c$ would appear as the ratio of transition rates above and below the predicted commitment threshold, since transitions that have accumulated entropy amplification above $\Phi_c = 1$ should become irreversible while those below remain reversible. In a quantum coherence experiment, this could manifest as a sharp asymmetry in decoherence probability as a function of interaction entropy --- observable through the ratio of coherence loss to entropy production in a controlled interaction. A second approach: entropy production measurements in superconducting qubit reset operations, where the ratio of dissipated heat to k_B T ln 2 gives a direct measure of the Landauer factor and could be compared against the predicted threshold $\Phi_c = \eta \cdot r$. These are qualitative sketches. A concrete experimental protocol specifying the quantum system, interaction Hamiltonian, measurement basis, and statistical precision required for a definitive test is future work and the subject of a companion paper in preparation.

**Most robust handle.** Measuring $E_c$ gives $r$ unconditionally --- fully interpretable within the spectral and commitment-capacity tiers, with no entropy identification needed.

**Accessible scales.** The millielectronvolt range (if $\rho = \rho\_\Lambda$) corresponds to thermal energies at ~30 K. Josephson junction physics, superconducting qubit systems, and precision atom interferometry all operate in this regime and can in principle probe anomalous entropy thresholds near this scale.

### 10.3 Physical Intuition for K = 7

The spectrum {0, 2, 2, 4, 4, 5, 7} supports four distinct physical roles:

| Eigenvalue | Multiplicity | Role |
|---|---|---|
| 0 | 1 | Constant sector --- null/indistinguishable state |

| Eigenvalue | Multiplicity | Role |
|---|---|---|
| 2 | 2 | Primitive transport --- two binary commitment outcomes |
| 4, 5 | 3 | Stabilizing modes --- buffer between primitive and UV sectors |
| 7 | 1 | UV closure mode --- suppressed by H_W |

Counting: 1 + 2 + 3 + 1 = 7. The four distinct roles require a minimum of seven independent spectral degrees of freedom. This is not a proof --- Theorem J does the rigorous work --- but it makes the result intuitive: K = 7 is the smallest cell that can simultaneously hold a null state, two binary commitment channels, a stabilizing buffer, and a UV closure mode without collapsing any sector.

### 10.4 Unified Derivation Chain

VERSF axioms: finite distinguishability, irreversible commitment

↓

Theorem A: $\tilde{S}(N) = \ln N$ [logarithmic form forced by axioms]

↓

Lemma B: 1 → 2 binary minimality

↓

Corollary C: $\tilde{\Theta}_0 = \ln 2$ [primitive entropy quantum, closure units]

↓

Theorem I: $\chi(L) = \rho L^4/(\hbar c)$ [causal action budget; $L^4$ forced]

↓

Corollary: $\chi(\xi) = 1 \Longrightarrow \xi = (\hbar c/\rho)$^(1/4) [capacity bound scale]

↓

Theorem J: K = 7 minimal admissible wheel cell

↓

Theorem D: $\lambda^* = 2$ [Rayleigh--Ritz over $A_{prim}$]

↓

Spectral Lemma: $\Delta\tilde{S}_{prim} = r$ [exact cancellation: $(r/2)\cdot 2 = r$]

↓

Theorem E: $(1/2)\cdot\lambda^*(K) = 1 \Leftrightarrow K = 7$ [cancellation unique to K=7]

↓

Bridge Principle F: Lemma $F_1$ + Lemma $F_2$ + Courant--Fischer + K=7 spectrum

$\Longrightarrow \lambda^* = 2$ eigenspace is unique minimal spectral realization

[$F_1$: non-kernel; $F_2$: dim≥2 from branch exchange symmetry

(symmetry derived from VERSF definition of primitive commitment)]

↓

Proposition: $\Phi_c = \eta r$ [unconditional barrier formula; η free parameter]

↓

=== DETERMINATION OF η ===

Route A: $\chi(\xi)=1$, Boltzmann entropy at coherence cell → $\eta = 1$

[effective-layer consistency; reduces η to one-domain comparison]

Route B: emergence programme requires closure and thermo entropy scales

to coincide; $\eta \neq 1$ contradicts emergence → $\eta = 1$

[emergence-constraint argument]

Route C: Boltzmann factor consistency at effective stat-mech layer → $\eta = 1$

[effective-layer consistency]

Route D: Uniqueness of Record-Induced Partition [Section 8.4]

- Class Identification Assumption
- Matching Lemma → $\alpha$ = k_B → $\eta = 1$

[primary route; requires emergence hypothesis]

Convergence: All four routes select $\eta = 1$ convergently from distinct

structural directions; A--C are effective-layer consistency

checks; D derives from the emergence claim directly

==========================================================================
====

↓

Corollary H: $\Phi_c = r$

↓

Consistency: r = 1 [primitive consistency value; entropy identification

is sole load-bearing conditional step]

↓

Final: $\Phi_c = r = 1$, $E_c \approx 2.5$ meV [conditional]

**11. Summary: Logical Structure**

**11.1 The Full Theorem Chain**

# -- UNCONDITIONAL

Theorem A: $\tilde{S}(N) = \ln N$ [forced by monotonicity + additivity + null singleton;

integer Cauchy equation; no real extension needed]

Lemma B: Smallest nontrivial irreversible refinement: 1 → 2

Corollary C: $\tilde{\Theta}_0 = \ln 2$ [primitive entropy quantum, closure units]

Theorem I: $\chi(L) = \rho L^4/(\hbar c)$ [$L^4$ forced by causal action budget]

Corollary: $\xi = (\hbar c/\rho)^{(1/4)}$ [capacity bound scale; not exact capacity]

Theorem J: K = 7 minimal admissible wheel cell

[five spectral conditions; unique in wheel family;

broader uniqueness is open graph-spectral problem]

Theorem D: $\lambda^* = 2$ [Rayleigh--Ritz over $A_{prim}$; class defined

without reference to any eigenvalue]

Spectral Lemma: $\Delta\tilde{S}_{prim} = r$ [exact cancellation $(r/2)\cdot\lambda^* = r$]

Theorem E: $(1/2)\cdot\lambda^*(K) = 1 \Leftrightarrow K = 7$ [spectrum derived in Appendix]

Lemma $F_1$: Any faithful representation of $\{s_1\}|\{s_2\}$ requires ≥1 non-kernel mode

[trivial; anchors non-kernel requirement]

↓

Lemma $F_2$: $G = \mathbb{Z}_2$ symmetry of primitive commitment (derived from VERSF

definition); faithful equivariant representation decomposes

as $\rho_+ \oplus \rho_-$; therefore dim $V \geq 2$

[standard $\mathbb{Z}_2$ representation theory; G derived, not assumed]

↓

Bridge Principle F: Lemma $F_1$ + Lemma $F_2$ + Courant--Fischer + K=7 spectrum

$\Longrightarrow \lambda^* = 2$ eigenspace is unique minimal spectral realization

[K=7 saturates dim=2 bound with minimal roughness]

Proposition: $\Phi_c = \eta r$ [unconditional barrier formula]

Proportionality: $S = \alpha\tilde{S}$ [structurally forced; α undetermined unconditionally]

# -- η DETERMINATION --- FOUR CONVERGENT ROUTES

Route A (commitment-capacity): $\chi(\xi)=1$ + Boltzmann entropy at coherence cell

→ $\eta = 1$ [Section 7.3]

Route B (action--entropy): $\hbar$ quantum of action + thermal time scale $\hbar/k_B T$

→ $\eta = 1$ [Section 7.3]

Route C (statistical): Boltzmann factor $e^{-E/k_B T}$ consistency

→ $\eta = 1$ [Section 7.3]

Route D (partition): Uniqueness of Record-Induced Partition +

Class Identification + Matching Lemma

→ $\eta = 1$ [Section 8]

All four routes converge: $\eta = 1$, $\Theta_0 = k_B \ln 2$

# -- CONDITIONAL ON PHYSICAL ENTROPY IDENTIFICATION

Theorem G: $S = k_B \tilde{S}$, $\Theta_0 = k_B \ln 2$, $\eta = 1$

[forced by Routes A--D jointly; Route D requires

thermodynamic emergence claim]

Corollary H: $\Phi_c = r$

Consistency Thm: $r = 1$ [primitive consistency value; not derived from dynamics alone;

unique value at which unconditional spectral structure and conditional

physical normalization are mutually consistent under natural normalization]

Landauer $k_B \ln 2$ → external consistency check [Section 8.6]

# -- INDEPENDENT PHYSICAL PREDICTION

$E_c \approx r \times 2.5$ meV [unconditional in r; r = 1 conditional]

### 11.2 Status of Each Result

| Result | Status |
|---|---|
| $\tilde{S}(N) = \ln N$ --- forced logarithmic form | **Unconditional** --- Theorem A |
| $1 \rightarrow 2$ binary minimality | **Unconditional** --- Lemma B |
| $\tilde{\Theta}_0 = \ln 2$ in closure units | **Unconditional** --- Corollary C |
| $\chi(L) = \rho L^4/(\hbar c)$ --- $L^4$ forced by causal action budget | **Unconditional** --- Theorem I |
| $\xi = (\hbar c/\rho)^{(1/4)}$ from $\chi(\xi) = 1$ | **Unconditional** --- Corollary (commitment-capacity threshold scale) |
| K = 7 minimal admissible wheel cell | **Unconditional** --- Theorem J (wheel family); broader uniqueness open |
| $A_{prim}$ defined without reference to $\lambda^*$ | **Unconditional** --- Section 4.1 |
| $\lambda^* = 2$ by Rayleigh--Ritz over $A_{prim}$ | **Unconditional** --- Theorem D |
| $\Delta\tilde{S}_{prim} = r$ --- exact cancellation | **Unconditional** --- Spectral Lemma |
| $(1/2)\cdot\lambda^*(K) = 1 \Leftrightarrow K = 7$ | **Unconditional** --- Theorem E (spectrum in Appendix) |
| Lemma $F_1$: any faithful spectral representation of $\{s_1\}\|\{s_2\}$ requires at least one non-kernel mode | **Unconditional** --- trivial but necessary; anchors the non-kernel requirement rigorously |
| Lemma $F_2$: faithful equivariant representation of primitive binary commitment under $G = \mathbb{Z}_2$ has $\dim \geq 2$ ($\mathbb{Z}_2$ has two irreps $\rho_+$ and $\rho_-$; faithfulness requires both; symmetry group derived from VERSF definition of primitive commitment) | **Unconditional** --- standard representation theory of $\mathbb{Z}_2$; $G = \mathbb{Z}_2$ derived from VERSF axioms, not assumed |
| Bridge Principle F: $\lambda^* = 2$ is the unique minimal spectral realization of binary commitment | **Unconditional** --- follows from Lemma $F_1$ + Lemma $F_2$ + Courant--Fischer + K = 7 spectrum |
| $S = \alpha\tilde{S}$ for some $\alpha > 0$ | **Unconditional structural result** --- Section 8.1 |
| $E_c \sim r \cdot \hbar c/\xi$ (form) | **Unconditional in r** --- Wilson sector + Theorem I |
| $\Phi_c = \eta r$ | **Unconditional** --- Proposition |
| $\eta = 1$ via Route A (primitive commitment-cell matching: coherence cell is unique carrier of one primitive bit; Boltzmann counting on same cell fixes $\alpha = k_B$) | **Effective-layer consistency argument** --- reduces $\eta$ ambiguity to single coefficient comparison on one domain; requires admission of effective thermodynamic layer on that domain [Section 7.3] |

| Result | Status |
|---|---|
| $\eta = 1$ via Route B (emergence constraint: within VERSF's emergence programme $\eta$ cannot be a free parameter; $\eta \neq 1$ means the two entropy scales differ, contradicting emergence; $\tau_{th} = \hbar/(k_B T)$ makes this precise) | **Effective-layer consistency argument** --- argues from within the emergence programme's own requirements [Section 7.3] |
| $\eta = 1$ via Route C (statistical Boltzmann consistency: $\eta \neq 1$ breaks Boltzmann factor, forces spurious factor into effective temperature) | **Effective-layer consistency argument** --- conditional on existence of effective statistical layer with independently defined temperature [Section 7.3] |
| $\eta = 1$ via Route D (partition identification: thermodynamic emergence forces class identification; Matching Lemma fixes $\alpha = k_B$) | **Conditional** --- requires thermodynamic emergence claim [Section 8] |
| Class Identification Assumption (VERSF classes = Boltzmann microstate classes) | **Conditional** --- sole conditional step; physical interpretation claim [Section 8.3] |
| $\alpha = k_B$ | **Conditional consequence** --- Matching Lemma given class identification [Section 8.3] |
| $\Theta_0 = k_B \ln 2$, $\eta = 1$ | **Conditional** --- Theorem G [Section 8.5] |
| Landauer agreement | **External consistency check** --- not a premise [Section 8.6] |
| Three structural lines (binary irreducibility, commitment-capacity, thermodynamic emergence) | **Supporting motivation** for Class Identification Assumption [Section 8.4] |
| $\Phi_c = r$ | **Conditional** --- Corollary H [Section 8.7] |
| r = 1 --- primitive consistency value | **Conditional** --- Consistency Theorem; entropy identification is sole load-bearing step |
| $E_c \approx 2.5$ meV | **Conditional** --- requires r = 1 |
| **11.3 Unified Statement** | |

**The central result:**

$\eta = 1$ [derived from four convergent structural routes, with Route D as primary; Routes A--C are effective-layer consistency checks; Route D requires thermodynamic emergence claim]

This means: $S = k_B \tilde{S}$. Closure entropy and thermodynamic entropy are the same quantity expressed in different unit systems. The Boltzmann constant is not imposed on the theory --- it emerges as the unique conversion factor forced by the requirement that the framework's causal, quantum-action, statistical, and partition structures be mutually consistent.

**Unconditional consequences of the spectral and causal structure:**

$\Delta\tilde{S}_{prim} = r$, $\Phi_c = \eta r$, $E_c \sim r \cdot \hbar c/\xi$

**Conditional consequences** (Routes A--D, Class Identification Assumption for Route D):

$\eta = 1 \rightarrow \Phi_c = r \rightarrow$ [r = 1, Consistency Theorem] $\rightarrow$ **$\Phi_c = 1$, $E_c \approx 2.5$ meV**

**Experimental readout:**

$\eta = \Phi_c / r$, $r = E_c \cdot \xi / (\hbar c)$

Measuring $E_c$ gives r unconditionally. Measuring $\Phi_c$ then gives $\eta$. If $\eta = 1$: the closure--thermodynamic bridge is confirmed. If $\eta \neq 1$: one of Routes A--D fails, and the deviation in $\eta$ specifies which structural claim requires revision. Full uniqueness of K = 7 beyond the wheel family remains an open graph-spectral problem.

**Appendix: Wheel Graph Laplacian Spectrum --- Derivation from First Principles**

The wheel graph W_K has K vertices: one hub vertex h connected to every vertex of a (K−1)-cycle C_{K-1}. Each perimeter vertex has degree 3 (two cycle neighbours and the hub); the hub has degree K − 1.

**Perimeter equations.** For perimeter vertex i:

$3f_i - f_{i-1} - f_{i+1} - f\_h = \lambda f_i$.

**Hub equation.**

$(K-1)f\_h - \Sigma_i f_i = \lambda f\_h$.

**Fourier decoupling on the cycle.** The (K−1)-cycle admits discrete Fourier modes

$f_i$^(m) = e^(2πi·m·i/(K−1)), m = 0, 1, ..., K−2.

For m ≠ 0, the mode satisfies $\Sigma_i f_i$^(m) = 0, so the hub equation gives f_h = 0. The perimeter equation then reduces to

[3 − 2cos(2πm/(K−1))] $f_i$^(m) = λ $f_i$^(m),

yielding **perimeter eigenvalues**

$\lambda_m = 3 - 2\cos(2\pi m/(K-1))$, m = 1, ..., K−2.

**The m = 0 sector.** For the constant mode all perimeter values are equal ($f_i = a$). Two eigenvectors span this sector:

*Zero mode.* Set $f_h = a$, $f_i = a$ for all i. Hub equation: $(K-1)a - (K-1)a = 0 \rightarrow \lambda = 0$. Perimeter equation: $3a - a - a - a = 0 \rightarrow \lambda = 0$. ✓

*Hub-breathing mode.* Set $f_h = (K-1)b$, $f_i = -b$. Hub equation: $(K-1)(K-1)b - (K-1)(-b) = \lambda(K-1)b \rightarrow (K-1)b + b = \lambda b \rightarrow$ **$\lambda$ = K**. Perimeter equation: $3(-b) - (-b) - (-b) - (K-1)b = \lambda(-b) \rightarrow -b - (K-1)b = \lambda(-b) \rightarrow$ **$\lambda$ = K**. ✓

**Complete spectrum:**

$\mathrm{Spec}(W_K) = \{0\} \cup \{3 - 2\cos(2\pi m/(K-1)) : m = 1, ..., K-2\} \cup \{K\}$.

**For K = 7** (cycle of length 6, $\lambda_m = 3 - 2\cos(\pi m/3)$):

| m | cos(πm/3) | $\lambda_m$ |
|---|---|---|
| 1, 5 | 1/2 | 2 |
| 2, 4 | −1/2 | 4 |
| 3 | −1 | 5 |
| hub | --- | 7 |
| zero | --- | 0 |

**$\mathrm{Spec}(W_7) = \{0, 2, 2, 4, 4, 5, 7\}$.** ✓

**Spectral rigidity note.** This spectrum is a graph isomorphism invariant for the unweighted combinatorial Laplacian. If edge weights are allowed to vary, eigenvalues move continuously --- the present derivation uses the unweighted model throughout. Any shift in $\lambda$ requires a change in combinatorial structure, meaning $\Phi_c = \eta r$ cannot be continuously tuned within a fixed graph isomorphism class.