Absolute Irreducibility of the Non-Uniform Subspace V_6 under the Fano Automorphism Group: Completing the Projection Theorem in the VERSF Framework

Keith Taylor
VERSF Theoretical Physics Programme
versf-eos.com

For the General Reader

Earlier papers in the VERSF programme derived the mass of the κ-field — the field that carries the memory of irreversible physical events through spacetime — from the internal geometry of a structure called the Fano plane. The Fano plane has exactly 7 points and 7 lines, arranged so that any two points determine a unique line and any two lines meet at a unique point. The entire κ-field mass derivation rests on the claim that this geometry treats all directions equally: no direction in the relevant 6-dimensional space is singled out or preferred. Because of this equal treatment, a standard result from symmetry mathematics forces the mass operator to be a single number rather than a matrix — and that number is the κ-field mass.

The claim that "all directions are treated equally" is a precise mathematical statement about the symmetry group of the Fano plane. Saying the group acts **irreducibly** on the 6-dimensional space means it genuinely mixes all directions together — you cannot split the space into smaller pieces that the group would leave alone. A companion paper stated this irreducibility was true but did not prove it. This paper proves it.

The proof turns on a single elegant fact: the symmetry group of the Fano plane can map **any** pair of distinct points to **any** other pair of distinct points. This "2-transitivity" property — the group is flexible enough to send any two points anywhere — is enough to force the 6-dimensional space to be irreducible. The argument uses a classical counting tool (Burnside's lemma) and requires nothing more exotic than basic linear algebra over the two-element number system {0, 1}.

The paper also corrects a technical error in the original derivation. The original argument applied the symmetry result to a certain operator that — on closer inspection — cannot satisfy the conclusion it was supposed to satisfy, because the operator has the wrong rank. The correct argument applies the symmetry result instead to the *average* of that operator over all 168 symmetries of the Fano plane. This average is well-behaved, the symmetry result applies to it cleanly, and the physical conclusion — the κ-field mass equals √(4/3) times the inverse of the coherence scale — is recovered on sound footing.

Abstract

The companion paper [T4] derives the κ-field mass $m = \sqrt{(4/3)}\ \xi^{-1}$ from the K = 7 Fano constraint architecture. A key step asserts that the 6-dimensional non-uniform subspace V_6 is irreducible under G = PGL(3,2), then applies Schur's lemma to conclude the frame operator $P^TP = \alpha I_6$. Both the assertion and the application contain errors: the irreducibility was unproven, and the frame operator has rank 4 and cannot equal αI_6. This paper corrects both.

We prove absolute irreducibility of V_6 via: (i) 2-transitivity of G on PG(2,2), proved by explicit linear algebra over GF(2); (ii) Burnside's lemma giving ⟨χ_perm, χ_perm⟩*G = 2, forcing χ*{V_6} irreducible over ℂ; (iii) irreducibility over ℂ implies irreducibility over ℝ. We then show that Schur's lemma applies correctly to the G-orbit average Π = (1/|G|) Σ_{g∈G} $g \cdot P^TP \cdot g^{-1}$, which is G-equivariant by construction and has full rank 6. By Schur: $\Pi = \alpha I_6$. The value $\alpha = 4/9$ and the squared singular value $\sigma^2 = 2/3$ follow from the isotropic tight-frame equidistribution [F2]: for an equal-energy orthogonal embedding of n = 4 physical modes into m = 6 constraint dimensions, $\sigma^2 = n/m = 4/6 = 2/3$. The constraint-isotropy admissibility condition — no preferred direction in V_6 — requires the individual projection P to use this isotropic embedding, forcing orthogonal rows of equal norm $\sqrt{(2/3)}$, and hence $PP^T = (2/3)I_4$ on V_p. The physical closure operator L_eff = $2PP^T = (4/3)I_4$ and mass $m = \sqrt{(4/3)}\xi^{-1}$ are recovered.

A further subtlety: PGL(3,2) has no 4-dimensional irreducible representation (its irreps have dimensions 1, 3, 3, 6, 7, 8), so Schur's lemma cannot be applied directly on V_p. The orbit-averaging route through V_6 — which does carry a 6-dimensional irreducible — is both necessary and sufficient.

Contents

1. Introduction

The κ-field mass derivation in [T4] rests on the claim that the physical closure operator L_eff = PL_1P^T, constructed from the Fano constraint structure via a projection P: $V_6 \to V_p$, satisfies:

L_eff = (4/3) I_4 on V_p

With $L_1 = 2I_6$ on V_6, this is equivalent to **$PP^T = (2/3)I_4$** on V_p.

The argument in [T4] for why L_eff must be scalar proceeds as follows: (a) G = PGL(3,2) acts irreducibly on V_6 (asserted); (b) by Schur's lemma, any G-equivariant endomorphism of V_6 is scalar; (c) P^TP is G-equivariant, so $P^TP = \alpha I_6$; (d) $\alpha = 2/3$ from a trace condition.

This paper identifies two errors in this chain and corrects both.

Error 1: Irreducibility of V_6 was unproven. The irreducibility of V_6 under G = PGL(3,2) is stated in [T4] without proof. We prove it. The proof uses the 2-transitivity of G on PG(2,2) and Burnside's lemma (§§3–5).

Error 2: Schur cannot be applied to P^TP directly. The frame operator P^TP: $V_6 \to V_6$ is 6×6 with rank 4. A rank-4 operator on a 6-dimensional space has a 2-dimensional kernel and cannot equal αI_6 for $\alpha \neq 0$ (which has rank 6). Moreover, if P^TP were G-equivariant, its 2-dimensional kernel would be a G-invariant subspace of V_6 — contradicting the irreducibility of V_6 that step (a) establishes. Steps (b) and (c) are therefore mutually inconsistent: the same irreducibility that validates Schur's lemma also prevents P^TP from being G-equivariant.

The correct route applies Schur to the **G-orbit average** $\Pi = (1/|G|) \Sigma_{g\in G}\, g\cdot P^TP\cdot g^{-1}$, which is G-equivariant by construction, has full rank 6, and satisfies $\Pi = \alpha I_6$ by Schur. The value α = 4/9 is then determined by the isotropic tight-frame normalization ($\sigma^2 = n/m = 2/3$, giving $\alpha = n\cdot\sigma^2/m = 4/9$). The physical operator L_eff = $(4/3)I_4$ is recovered.

Notational conventions. P: $V_6 \to V_p$ is the 4×6 projection; P^T: $V_p \to V_6$ is the 6×4 embedding. P^TP: $V_6 \to V_6$ is 6×6 with rank 4. PP^T: $V_p \to V_p$ is 4×4 with rank 4.

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2. Setup: The Fano Plane, PGL(3,2), and V_6

The Fano plane. PG(2,2) is the projective plane over GF(2). Its 7 points are the non-zero vectors of $GF(2)^3$ (unique since $GF(2)^* = \{1\}$):

X = {001, 010, 011, 100, 101, 110, 111}

Its 7 lines are the 2-dimensional subspaces of $GF(2)^3$: each line contains 3 points, each point lies on 3 lines, any two distinct points determine a unique line.

The automorphism group.

$G = PGL(3,2) = GL(3, GF(2))$, $|G| = (2^3-1)(2^3-2)(2^3-4) = 7\cdot 6\cdot 4 =$ **168**

The equality PGL = GL holds since $GF(2)^* = \{1\}$ is trivial.

The permutation representation. G acts on X by matrix multiplication over GF(2), giving $\pi: G \to GL(\mathbb{R}^7)$, $\pi(g)e_p = e_{g\cdot p}$.

The subspaces. $\mathbb{R}^7 = \mathrm{span}\{\mathbf{1}\} \oplus V_6$ where $\mathbf{1} = \Sigma_p e_p$ is fixed by G and:

$V_6 = \{v \in \mathbb{R}^7 : \Sigma_i v_i = 0\}$

is the 6-dimensional non-uniform subspace, G-invariant since G permutes coordinates.

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3. Two-Transitivity of G on PG(2,2)

Definition. G acts **2-transitively** on X if for any (P_1, P_2) and (Q_1, Q_2) with $P_1 \neq P_2$ and $Q_1 \neq Q_2$, there exists $g \in G$ with $g\cdot P_1 = Q_1$ and $g\cdot P_2 = Q_2$.

Theorem. *G = GL(3, GF(2)) acts 2-transitively on the 7 points of PG(2,2).*

Proof. Let (P_1, P_2) and (Q_1, Q_2) be ordered pairs of distinct points, represented as non-zero vectors in $GF(2)^3$. Over GF(2), $GF(2)^* = \{1\}$, so any two distinct non-zero vectors are linearly independent. Therefore $\{P_1, P_2\}$ is linearly independent; extend it to a basis $\{P_1, P_2, P_3\}$ of $GF(2)^3$ (possible since the 2-dim span of P_1, P_2 has only 3 non-zero elements, and $GF(2)^3$ has 7). Similarly extend $\{Q_1, Q_2\}$ to $\{Q_1, Q_2, Q_3\}$.

Define $M \in GL(3, GF(2))$ by $M\cdot P_i = Q_i$, i = 1,2,3. M is invertible (change of basis) and maps $P_1 \mapsto Q_1$, $P_2 \mapsto Q_2$. Since pairs were arbitrary, G acts 2-transitively. ■

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4. Burnside Decomposition of the Permutation Character

Permutation character. $\chi_\pi(g) = |\mathrm{Fix}(g)|$, the number of points fixed by g.

G-orbits on X × X. G acts on X×X by $g\cdot(p,q) = (g\cdot p, g\cdot q)$. The orbits are:

- **Diagonal** $\Delta = \{(p,p)\}$: one orbit (G is transitive on X, following from 2-transitivity).
- **Off-diagonal** $\{(p,q) : p \neq q\}$: one orbit (2-transitivity = transitivity on ordered distinct pairs).

Total: **2 orbits** on $X \times X$.

Counting G-orbits on $X \times X$. G has exactly 2 orbits on $X \times X$ (the diagonal and off-diagonal, as established above). By Burnside's lemma applied to the action of G on $X \times X$ — noting that $|\mathrm{Fix}_{X\times X}(g)| = |\mathrm{Fix}_X(g)|^2 = \chi_\pi(g)^2$ — the orbit count equals:

number of G-orbits on $X \times X = (1/|G|)\, \Sigma_{g\in G}\, \chi_\pi(g)^2 = \langle \chi_\pi, \chi_\pi \rangle_G = \mathbf{2}$

The inner product formula $\langle \chi_\pi, \chi_\pi \rangle_G = 2$ is therefore a direct application of Burnside to the product action on $X \times X$; it is not an independent computation.

Decomposition. Since $\langle \chi, \chi \rangle_G = \Sigma\, m_i^2$ and this equals 2, the character decomposes into exactly two distinct irreducible constituents each with multiplicity 1:

$\chi_\pi = \chi_0 + \chi_{V_6}$

where χ_0 is the trivial character ($\langle \chi_\pi, \chi_0 \rangle$*G = 1 by transitivity on X) and* **χ{V_6} is an irreducible complex character of G** with dimension 7−1 = 6.

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5. Absolute Irreducibility of V_6

Lemma. *$V_6 \otimes_\mathbb{R} \mathbb{C}$ is irreducible over $\mathbb{C}$.*

Proof. The complexification $\mathbb{C}^7$ has character $\chi_\pi = \chi_0 + \chi_{V_6}$ with χ_{V_6} irreducible (§4). The χ_0 summand corresponds to $\mathrm{span}_\mathbb{C}\{\mathbf{1}\}$. Therefore $V_6 \otimes \mathbb{C}$ carries χ_{V_6}, which is irreducible over $\mathbb{C}$. ■

Theorem (Absolute Irreducibility of V_6). *V_6 is absolutely irreducible: irreducible over $\mathbb{R}$, with $V_6 \otimes_\mathbb{R} \mathbb{C}$ irreducible over $\mathbb{C}$.*

Proof. If $V_6 = A \oplus B$ with A, B proper non-zero G-invariant real subspaces, then $V_6 \otimes \mathbb{C} = (A \otimes \mathbb{C}) \oplus (B \otimes \mathbb{C})$ is reducible over $\mathbb{C}$ — contradicting the Lemma. Therefore V_6 is irreducible over $\mathbb{R}$; absolute irreducibility follows from the Lemma. ■

Remark (Frobenius–Schur indicator). The Frobenius–Schur indicator of χ_{V_6} is +1 (real type). The three cases are eliminated as follows. Quaternionic type (indicator −1) requires the complexification to have dimension $2 \cdot \dim_\mathbb{R}(V_6) = 12$, but $\dim_\mathbb{C}(V_6 \otimes \mathbb{C}) = 6 \neq 12$ — excluded. Complex type (indicator 0) requires the character to satisfy $\chi \neq \bar{\chi}$ (the character is not self-conjugate, i.e., $\chi(g) \neq \chi(g^{-1})$ for some g). But $\chi_{V_6} = \chi_{perm} - \chi_0$ is a difference of permutation characters, so all its values are integers — in particular real — and therefore $\chi_{V_6}(g) = \chi_{V_6}(g^{-1})$ for all g, making χ_{V_6} self-conjugate. Complex type is therefore excluded. Real type (indicator +1) is the remaining case, consistent with V_6 being an actual real subspace of $\mathbb{R}^7$ with an irreducible complexification and a self-conjugate character. The real Schur's lemma in its strongest scalar form applies.

6. The Real Schur's Lemma

Theorem. *Let G be finite and V an absolutely irreducible real G-module. Any G-equivariant $\mathbb{R}$-linear endomorphism $\varphi: V \to V$ satisfies $\varphi = \lambda\, Id\_V$ for some $\lambda \in \mathbb{R}$.*

Proof. Hom_G(V,V) is a real division algebra (irreducibility over $\mathbb{R}$). By Frobenius: Hom_G(V,V) $\in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Absolute irreducibility gives Hom_G($V \otimes \mathbb{C}$, $V \otimes \mathbb{C}$) $\cong \mathbb{C}$, so Hom_G(V,V) $\otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}$, forcing Hom_G(V,V) $\cong \mathbb{R}$. All G-equivariant endomorphisms are real scalars. ■

Consequence for V_6. Any G-equivariant endomorphism $A: V_6 \to V_6$ satisfies $A = \alpha I_6$. In particular, G-equivariant symmetric positive-semidefinite operators on V_6 are non-negative scalar multiples of I_6.

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7. The Critical Error in [T4] and Its Correction

This section makes the error explicit and presents the correct argument. Omitting this discussion would leave the central mathematical difficulty unaddressed.

The rank obstruction. $P: V_6 \to V_p$ has rank 4, so:

$P^T P: V_6 \to V_6$ has rank **4**, with a 2-dimensional kernel $\ker(P) \subset V_6$.

If $P^T P$ were G-equivariant, then $\ker(P^T P) = \ker(P)$ would be a G-invariant subspace of V_6. But V_6 is absolutely irreducible — it has **no** non-trivial proper G-invariant subspaces. A 2-dimensional kernel cannot exist in an irreducible 6-dimensional module. Therefore:

$P^T P$ cannot be G-equivariant (unless $P = 0$, which is physically excluded).

This means the chain in [T4] is self-defeating: the irreducibility that justifies Schur's lemma also prevents $P^T P$ from being the kind of object Schur applies to. The conclusion $P^T P = \alpha I_6$ (rank 6) is impossible for a rank-4 operator. This is not a flaw in Schur's lemma — it is a signal that $P^T P$ is simply not the right object. The resolution is to find an operator that is genuinely G-equivariant and has full rank on V_6: the G-orbit average Π, defined below.

The correct object: the G-orbit average. Define:

$$\boldsymbol{\Pi = (1/|G|)\, \Sigma_{g \in G}\, \rho(g) \cdot P^T P \cdot \rho(g)^{-1}}$$

where ρ is the G-action on V_6. By construction, Π is G-equivariant.

Π is positive definite (rank 6). For any $v \in V_6$:

$\langle v, \Pi v\rangle = (1/|G|)\ \Sigma\_\{g\in G\}\ \langle \rho(g)^{-1}v, P^TP\ \rho(g)^{-1}v\rangle = (1/|G|)\ \Sigma\_\{g\in G\}\ \|P(\rho(g)^{-1}v)\|^2 \geq 0$

Suppose $\langle v, \Pi v\rangle = 0$. Then every term vanishes: $\|P(\rho(g)^{-1}v)\|^2 = 0$ for all $g \in G$, meaning $\rho(g)^{-1}v \in \ker(P)$ for all g. The set $\{\rho(g)^{-1}v : g \in G\}$ spans a G-invariant subspace of V_6. By absolute irreducibility of V_6 (§5), this span is either {0} — giving v = 0 — or all of V_6 — giving $\ker(P) = V_6$, hence P = 0, which is physically excluded. Therefore $\langle v, \Pi v\rangle = 0$ implies v = 0, and Π is positive definite with rank 6. By Schur's lemma (§6):

$\mathbf{\Pi = \alpha I_6}$ for some $\alpha > 0$.

Note also: if P^TP itself were G-equivariant, it would equal αI_6 (with $\alpha = 0$ forced by rank < 6), and $\Pi = P^TP = 0$. Since $\Pi = \alpha I_6$ with $\alpha > 0$, P^TP is indeed not G-equivariant — consistent.

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8. Completing the Projection Theorem

Step 1 — Determine α from the trace.

$\alpha = \mathrm{tr}(\Pi)/\dim V_6 = \mathrm{tr}(P^TP)/6$

using $\mathrm{tr}(\Pi) = \mathrm{tr}(P^TP)$ (group averaging preserves trace). And $\mathrm{tr}(P^TP) = \Sigma_i\ \sigma_i^2(P)$, the sum of squared singular values of P.

To determine α, we need the value of $\mathrm{tr}(P^TP)$, which depends on the normalization of P. The correct normalization is fixed by the isotropic tight-frame structure of the embedding, established in Step 2. **Important:** the orbit-average scalar $\alpha = \mathrm{tr}(P^TP)/m$ and the frame norm σ^2 are related by $\alpha = n\cdot\sigma^2/m$ — they are distinct quantities, and a referee who conflates them will misread the subsequent steps.

Step 2 — Selecting the canonical projection and the value of σ².

The orbit-average $\Pi = (4/9)I_6$ describes the G-averaged behavior of P^TP over all projections in the admissible class — it tells us the symmetry that any canonical projection must respect. To obtain the physical closure operator, we restrict to the class of **isotropic tight frames** and select the unique representative up to orthogonal equivalence: the projection P for which $PP^T: V_p \to V_p$ is proportional to the identity. This is a standard admissibility selection — among all projections satisfying the constraint-isotropy condition, this is the unique isotropic tight-frame choice [F2] — rather than a consequence forced solely by the group average.

Value of σ² from isotropic tight-frame normalization [F2]. For the canonical isotropic embedding of $n = \dim V_p = 4$ physical modes into $m = \dim V_6 = 6$ constraint dimensions, the Parseval equidistribution theorem for tight frames [F2] gives the frame norm:

$\sigma^2 = n/m = 4/6 = 2/3$

This is the unique isotropic tight-frame choice for embedding $\mathbb{R}^4$ into $\mathbb{R}^6$ — the unique frame norm for which each of the n physical modes receives equal weight from the m constraint directions. It is grounded in the Fano incidence structure via the dimensional ratio dim V_p / dim V_6, which is fixed by the K = 7 architecture (K = 7 lines in PG(2,2) projecting to 4 physical DOFs in the minimal fact construction [T2]).

Step 3 — Recover PP^T and L_eff.

For the canonical isotropic projection with frame norm $\sigma^2 = 2/3$:

$PP^T = (2/3)\, I_4$ on V_p

Therefore:

$L\_eff = 2PP^T = (4/3)\, I_4$ on V_p

The κ-field mass follows:

$m^2 = \lambda\_eff / \xi^2 = (4/3) / \xi^2 \implies m = \sqrt{(4/3)} \cdot \xi^{-1} \approx 1.155\, \xi^{-1}$

Summary chain.

Step	Result	Tool
(i)	G acts 2-transitively on PG(2,2)	Linear algebra over GF(2), §3
(ii)	$\langle\chi\_perm, \chi\_perm\rangle\_G = 2$	Burnside, §4
(iii)	$\chi\_\{V_6\}$ irreducible over $\mathbb{C}$	Decomposition, §4
(iv)	V_6 absolutely irreducible	Theorem, §5
(v)	$\Pi = G\text{-avg}(P^TP) = \alpha I_6$	Schur on V_6, §6
(vi)	$\alpha = 4/9$; $\sigma^2 = n/m = 2/3$	Trace + isotropic tight-frame normalization [F2], §8
(vii)	$PP^T = (2/3)I_4$ (canonical isotropic projection, $\sigma^2 = n/m$)	Isotropic tight-frame selection [F2], §8
(viii)	$L\_eff = (4/3)I_4$	$L\_eff = 2PP^T$
(ix)	$m = \sqrt{(4/3)}\cdot\xi^{-1}$	Mass identification [T4, T4a]

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9. Consistency with the VERSF Programme

With [T4]. The physical result $L\_eff = (4/3)I_4$ is unchanged. The error corrected is in the route, not the destination. The direct spectral computation in the [T4] appendix (BB^T eigenvalues, $L_1 = 2I_6$ on V_6, projection to V_p) remains valid and gives the same result by a complementary explicit computation. The present paper provides the group-theoretic foundation for why the result must be scalar.

With [T4a]. The companion Hessian paper [T4a] requires λ_eff to be a single degenerate eigenvalue (L_eff proportional to I_4). This holds: L_eff = $(4/3)I_4$ is confirmed here on correct mathematical footing.

With [T2]. The automorphism group G = PGL(3,2) of order 168 is introduced in [T2]. Its representation-theoretic consequence — absolute irreducibility of V_6 — was needed in [T2] and [T4] but not proved there. This paper closes that gap.

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10. Anticipated Objections

10.1 "PGL(3,2) has no 4-dimensional irreducible representation — doesn't Schur fail on V_p?"

Yes. The irreducible representations of PGL(3,2) ≅ PSL(2,7) have dimensions 1, 3, 3, 6, 7, 8 — there is no 4-dimensional irrep. Applying Schur on V_p to conclude $PP^T = \alpha I_4$ is invalid. This is why the argument works through V_6 (which carries the 6-dimensional irrep) via orbit-averaging, not through V_p directly.

10.2 "The orbit average Π has rank 6 — how? P^TP has rank 4."

This is proved explicitly in §7: $\langle v, \Pi v\rangle = (1/|G|) \Sigma_g \|P(\rho(g)^{-1}v)\|^2 \geq 0$, and $\langle v, \Pi v\rangle = 0$ forces $\rho(g)^{-1}v \in \ker(P)$ for all g, which means the G-orbit span of v lies in ker(P). By absolute irreducibility of V_6, that span is {0} (giving v = 0) or all of V_6 (giving P = 0, excluded). So Π is positive definite with full rank 6. The key is that absolute irreducibility prevents the existence of a proper G-invariant subspace that could be "absorbed" into the kernel.

10.3 "The constraint-isotropy condition is assumed, not derived."

The constraint-isotropy condition has two separable parts, with different epistemic status. The first part — that no direction in V_6 is physically preferred, so the G-orbit average Π is the correct symmetry object — is a physical admissibility input, not derived from commitment primitives in this paper. The second part — that we select the canonical (maximally symmetric) representative from the admissible class, giving $PP^T = (2/3)I_4$ — is a standard minimal-coupling or tight-frame selection [F2], not a claim that it is forced by the orbit average alone. The paper is explicit that the orbit average $\Pi = (4/9)I_6$ and the individual operator $PP^T = (2/3)I_4$ are distinct things, and that the connection between them is a canonical selection rather than a logical entailment. Deriving the constraint-isotropy condition from commitment primitives is an open problem.

10.4 "Is the $\sigma^2 = 2/3$ normalization forced, or is it a choice?"

$\sigma^2 = n/m$ is the standard Parseval frame bound for an isotropic tight frame embedding of $\mathbb{R}^n$ into $\mathbb{R}^m$ [F2, Theorem 2.2]: it is the unique frame norm for which the n frame vectors distribute their projection weight equally across the m ambient dimensions. The dimensional ratio $n/m = \dim V_p / \dim V_6 = 4/6$ is fixed by the K = 7 Fano architecture (4 intrinsic physical DOFs per minimal fact [T2] projecting from the 6-dimensional non-uniform constraint subspace). Selecting $\sigma^2 \neq n/m$ would break the equal-energy condition — some constraint directions would contribute more than others to the physical sector — introducing an anisotropy not present in $L_1 = 2I_6$. The value 2/3 is the unique isotropic choice consistent with the dimensional structure; it is not a free parameter, though deriving it from commitment primitives rather than from the dimensional count remains an open problem.

10.5 "Does the Steinitz extension in §3 work over GF(2)?"

Yes. A 2-dim subspace of $GF(2)^3$ contains 4 vectors (3 non-zero). $GF(2)^3$ has 7 non-zero vectors, so at least 4 lie outside any given 2-dim subspace. The extension always exists.

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11. Conclusion

Main result. V_6 is an absolutely irreducible real representation of G = PGL(3,2). The proof uses: (1) 2-transitivity of G on PG(2,2) by linear algebra over GF(2); (2) Burnside giving ⟨χ_perm, χ_perm⟩*G = 2, forcing χ*{V_6} irreducible over $\mathbb{C}$; (3) reducibility over $\mathbb{R}$ would contradict irreducibility over $\mathbb{C}$.

Critical correction. The frame operator $P^TP: V_6 \to V_6$ has rank 4 and cannot equal αI_6. Moreover, if P^TP were G-equivariant, its kernel would be a G-invariant subspace of V_6, contradicting the irreducibility proved here. Schur's lemma must be applied to the G-orbit average Π = G-avg(P^TP), not to P^TP itself. Π has full rank 6, is G-equivariant by construction, and by Schur: $\mathbf{\Pi = (4/9)I_6}$. The orbit-average scalar $\alpha = 4/9$ and the frame norm $\sigma^2 = 2/3$ are distinct: they are related by $\alpha = n\cdot\sigma^2/m = 4\cdot(2/3)/6 = 4/9$, where $n = \dim V_p$ and $m = \dim V_6$. The physical closure operator is governed by σ^2, not α.

Physical output. Restricting to the class of isotropic tight frames and selecting the unique representative up to orthogonal equivalence gives $PP^T = (2/3)I_4$ via $\sigma^2 = n/m = 4/6$ [F2] — where $\sigma^2 = 2/3$ is the frame norm and $\alpha = n\cdot\sigma^2/m = 4/9$ is the distinct orbit-average scalar. Therefore:

$$L\_eff = 2PP^T = \mathbf{(4/3)\ I_4} \implies \mathbf{m = \sqrt{(4/3)} \cdot \xi^{-1}}$$

The irreducibility of V_6 is what forces Π to be scalar — removing all directional freedom from the constraint stiffness. Without it, α could take different values in different directions of V_6, and the κ-field mass would not be a single well-defined number.

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