Exact Scalar Perturbation Theory in VERSF: From Nonlocal Memory Dynamics to the Mukhanov–Sasaki Limit

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For the General Reader

The previous paper in this series — *Primordial Scalar Perturbations from Void Unfolding* — showed that the slight unevenness in the early universe's density, the primordial tilt, can be derived from a single foundational idea: physical reality is built from irreversible events, moments at which a distinction between possible states becomes a permanent fact. Every such event leaves a trace in a field called κ. Those traces accumulate over time, and their accumulation gently pushes the universe away from perfect uniformity. The result was a specific number — the spectral index $n_s \approx 0.972$ — derived with no free parameters, sitting within 1% of what telescopes actually measure.

That derivation had one honest gap. The argument that the memory traces don't overwhelm the calculation — that they remain a small perturbation — was supported by an inequality, a careful estimate, rather than by working out the full exact equations of motion for the memory field and showing directly that they produce only a small effect. The first paper identified this as its one remaining open item.

The present paper closes that gap. The strategy is to introduce a second field, M, whose sole job is to track the running memory of all past commitment events. M is not a new physical ingredient: it is a bookkeeping device, an auxiliary field, that encodes the same causal memory response that was previously treated as a nonlocal integral over history. We show that introducing M — with its coupling strength fixed entirely by the same physics that fixes everything else in the framework — allows us to write down an exact equation governing how quantum fluctuations in the primordial universe evolved. That equation turns out to be the celebrated Mukhanov–Sasaki equation of standard cosmology, but derived rather than assumed, and with a small calculable correction from the memory field.

Three things are established. First, the auxiliary field M can be constructed with no free parameters — its properties are entirely determined by the existing framework. Second, the exact perturbation equation, written out for the first time, is a version of the Mukhanov–Sasaki equation with a memory correction term that depends on the entire past history of the mode. Third, in the regime relevant for the cosmic microwave background, that memory correction is provably small — suppressed by roughly one part in three hundred — so the leading-order result of the first paper is not just a good approximation but the exact leading term of a well-controlled expansion. The correction goes in the right direction: it makes the predicted spectral index slightly more red, moving the theoretical value closer to the observed one.

Taken together, the two papers provide a complete derivation of one of the most precisely measured numbers in cosmology from first principles — the spectral index follows not from a chosen potential or a fitted parameter, but from the logical structure of how irreversible facts accumulate in a universe like ours.

Abstract

This is the second of two papers on primordial scalar perturbations in the Void Energy-Regulated Space Framework (VERSF). The first paper, *Primordial Scalar Perturbations from Void Unfolding* (Paper I), derived the spectral index $n_s \approx 0.972$ at leading order with no adjustable parameters, using the result that the irreversible commitment memory sector contributes only at $O(1/(N \ln N))$ to the quadratic perturbation action — a controlled bound that justified reduction to the canonical Mukhanov–Sasaki equation without deriving the full perturbation dynamics. Paper I identified this as its one remaining open item: the memory sector was characterised by an inequality rather than an explicit dynamical operator, and the exact scalar mode equation of VERSF was not written down.

The present paper closes that gap. We promote the memory sector from a suppression estimate to a fully specified dynamical theory by introducing a local auxiliary field M whose retarded Green's function exactly reproduces the VERSF memory kernel, with all coupling parameters fixed from previously established VERSF results and no new free parameters beyond those already present in the VERSF memory sector. From this localised two-field (κ, M) action we derive the exact quadratic scalar perturbation theory — including a non-trivial proof that lapse and shift constraint elimination in the two-field system introduces no additional cross-sector mixing at leading slow-roll order — and integrate out the auxiliary sector to obtain the exact scalar mode equation.

We establish three results. **Theorem 1** (local auxiliary representation): the VERSF retarded memory sector admits a local auxiliary-field construction with all parameters determined by the κ-field mass theorem and CCC architecture. **Theorem 2** (exact nonlocal mode equation): scalar perturbations are governed by the retarded nonlocal integro-differential equation

$$v_k'' + (k^2 - z''/z)\, v_k + \int_{\tau_i}^{\tau} d\tau'\, \Sigma_k(\tau, \tau')\, v_k(\tau') = 0,$$

where Σ_k is the VERSF memory self-kernel, structurally determined with no free parameters. **Theorem 3** (adiabatic MS limit): in the slow-roll unfolding regime relevant for observable CMB modes, the nonlocal term is suppressed by $O(1/(N \ln N))$ uniformly across the CMB window, and the exact equation reduces to the standard local Mukhanov–Sasaki equation at leading order. The leading-order result of Paper I,

$$n_s - 1 = -2/N_\star,$$

is thereby elevated from a result justified by a controlled bound to a theorem derived from the exact VERSF perturbation dynamics. The subleading memory correction

$$\delta n_s^{mem} = O(1/(N_\star \ln N_\star)),$$

is shown to be sign-fixed negative (enhanced red tilt), consistent with Paper I's identification of sign-fixed corrections that move n_s toward the Planck observed value 0.9649.

Taken together, the two papers provide a complete, parameter-free derivation of the primordial scalar spectrum in VERSF: Paper I establishes the structural result and the leading-order prediction; the present paper provides the exact dynamical foundation from which that result follows as a theorem.

Keywords: VERSF; primordial perturbations; scalar perturbation theory; κ-field; auxiliary-field localisation; Mukhanov–Sasaki equation; nonlocal memory kernel; commitment events

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1. Introduction

The companion paper *Primordial Scalar Perturbations from Void Unfolding* (Taylor 2026, hereafter Paper I) established a leading-order derivation of the primordial scalar spectral index in the VERSF framework:

$$n_s - 1 = -8 / \ln(\rho_{void} / \rho_{CCC}) \approx -0.028, \quad n_s \approx 0.972.$$

That derivation rested on three structural inputs: a fixed κ-field mass $m^2_\kappa = (4/3)\xi^{-2}$ (a theorem from the K = 7 minimal fact architecture), a logarithmic slow-roll displacement driven by accumulated commitment memory, and a leading-order perturbative reduction of the scalar sector to the canonical Mukhanov–Sasaki (MS) equation. The first two inputs are internal VERSF theorems. The third was established through a controlled suppression bound: the memory-sector contribution to the quadratic perturbation action is $O(1/(N \ln N))$ relative to the canonical term in the slow-roll unfolding regime, justifying the leading-order reduction.

That bound was correct and sufficient for the leading-order spectral prediction. But it left one identifiable gap: the memory sector entered the perturbation theory as an external source term characterised by a controlled inequality, rather than as an explicit dynamical operator derived from an action. The local Mukhanov–Sasaki equation was justified at leading order, but the exact scalar mode equation of VERSF — including the full memory contribution — was not written down.

The purpose of the present paper is to close that gap completely. The strategy is as follows. We represent the retarded memory response by a local auxiliary scalar field M, with coupling parameters determined explicitly from VERSF structure. We derive the exact quadratic action of the coupled (κ, M) system, paying careful attention to a non-trivial issue: whether the lapse and shift constraint equations, which in a two-field system receive contributions from both fields, introduce additional cross-sector mixing terms that would complicate the reduction. We show that they do not at leading slow-roll order. We then integrate out M exactly, obtaining a single retarded nonlocal scalar mode equation. We prove that this equation reduces to the standard local MS equation in the slow-roll adiabatic limit relevant for CMB modes, and we identify the structure and sign of the subleading memory correction to n_s.

This gives a more rigorous perturbation theory in four respects. First, the memory kernel is an explicit dynamical operator rather than a suppression estimate. Second, the exact scalar mode equation is an integro-differential equation with a structurally determined retarded kernel. Third, the suppression factor $R_{pert} \sim 1/(N \ln N)$ is recovered as the asymptotic control parameter of the kernel rather than as a heuristic extension from the homogeneous sector. Fourth, the first memory correction to the spectral tilt is sign-fixed and calculable without new parameters.

Throughout, we distinguish sharply between proven theorems, derived results, and modelled assumptions (the last carrying no free cosmological parameters).

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2. VERSF Background and the Memory Problem

2.1 The κ-Field and the Memory Kernel

VERSF associates irreversible commitment events with a scalar field κ whose effective continuum dynamics are sourced by the history of committed distinctions. In the companion paper and prior VERSF results (Taylor 2026a,b), the homogeneous κ-equation in flat FRW background takes the form

$\ddot{\kappa}_0 + 3H\,\dot{\kappa}_0 + m^2_\kappa\,\kappa_0 = J_{mem}[\kappa_{hist}]$,

where J_{mem} is a functional of the κ-history encoding the cumulative influence of past commitment events through the retarded Green's function of the damped κ-field. The late-time asymptotic of the effective memory kernel is (Taylor 2026b):

$$M_{eff}(\tau) \sim \cos(\omega_\kappa \tau + \varphi) / (m_\kappa \tau), \quad \tau \gg m_\kappa^{-1}, \quad \ldots (1)$$

with $\omega_\kappa = \sqrt{m^2_\kappa - \gamma^2_m}$. The algebraic $1/\tau$ tail is a theorem from the causal retarded propagator structure and is absent for reversible or massless sources.

2.2 The Memory Problem in Perturbation Theory

In Paper I, the perturbation sector was treated by estimating the ratio of memory-sector contributions to the canonical restoring force in the quadratic perturbation action. The estimate $R_{pert} \sim 1/(N \ln N)$ was derived from the background quasi-static balance and extended to the perturbation sector on the grounds that the kernel $K(t, t')$ carries no explicit k-dependence. That extension was correct but acknowledged as an estimate pending a rigorous derivation from the second-order perturbation equations of motion.

The present paper provides that derivation. The approach is to construct an explicit local action for the memory sector via an auxiliary field, derive the exact perturbation equations of this local two-field system, and then recover the single-field limit by integrating out the auxiliary sector. In this formulation, the kernel k-independence is not assumed but derived from the auxiliary-sector structure, and the suppression bound becomes a theorem from the adiabatic expansion of the exact retarded Green's function.

2.3 Previously Established VERSF Inputs Used Here

The following results from prior VERSF papers are used without re-derivation:

- **$m^2_\kappa = (4/3)\xi^{-2}$** (proven, K = 7 architecture theorem; Taylor 2026c,d)
- **$\xi = (\hbar c/\rho)^{1/4}$** (proven, CCC threshold condition; Taylor 2026e)
- **$m_\kappa/H_{eff} = \sqrt{32/3} \approx 3.27$** at primordial onset (derived, Paper I §3.1)
- **$\varepsilon_V = \eta_V = 1/(2N_\star)$** (derived, Paper I §4.4–4.5)
- **$\eta_H = \eta_V - \varepsilon_V = 0$** (proven theorem from quadratic V_{eff}; Paper I §4.4)
- **$N_{tot} = \frac{1}{4} \ln(\rho_{void}/\rho_{CCC}) \approx 71$** (derived, Paper I §5)
- **Late-time kernel asymptotic** (1) (derived, Taylor 2026b)

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3. Auxiliary-Field Construction: Parameters and Matching

3.1 Statement of the Construction Problem

The retarded memory response of the κ-sector is an explicitly nonlocal functional of κ-history. To bring it within a Lagrangian perturbation theory framework, we require a local field-theoretic representation whose equations of motion reproduce the target retarded kernel upon elimination of the new degree of freedom.

This is not merely a technical convenience. The local representation provides an action from which the exact quadratic perturbation theory can be derived without further ansatz, and from which the lapse/shift constraint equations of the two-field system can be obtained and solved systematically.

3.2 The Target Kernel and Its Factored Structure

The VERSF retarded memory kernel (Taylor 2026b) is

$K(\tau, \tau') = \theta(\tau - \tau')\, e^{-\gamma_m(\tau-\tau')} \sin(\omega_\kappa(\tau-\tau')) / \omega_\kappa,\ \omega_\kappa = \sqrt{m^2_\kappa - \gamma^2_m}$. ... (2)

This is exactly the retarded Green's function of a damped harmonic oscillator with mass-squared $\omega^2_M = \omega^2_\kappa$ and damping coefficient γ_m. The identification is precise: the target kernel has the same functional form as G_{ret} of an oscillator

$\partial^2_\tau M + 2\gamma_m \partial_\tau M + \omega^2_M M = \text{source}$, ... (3)

with $\omega^2_M = m^2_\kappa - \gamma^2_m$ and $G_{ret}(\tau, \tau') = \theta(\tau-\tau')\, e^{-\gamma_m(\tau-\tau')} \sin(\omega_M(\tau-\tau'))/\omega_M$.

This factored form is the basis of the auxiliary-field construction. The auxiliary field M is introduced precisely as the field whose retarded Green's function *is* the VERSF memory kernel (2). This fixes the auxiliary-sector parameters without ambiguity.

3.3 Determining the Auxiliary-Field Parameters from VERSF Structure

The three parameters entering the auxiliary sector are ω_M, γ_m, and g_m. We now show that each is fixed by previously established VERSF results.

Parameter ω_M: From the identification above, $\omega_M = \omega_\kappa = \sqrt{m^2_\kappa - \gamma^2_m}$. With $m^2_\kappa = (4/3)\xi^{-2}$, the value ω_M is determined once γ_m is fixed (see below). In the underdamped regime $\gamma_m < m_\kappa$ — which is required for the oscillatory kernel structure — $\omega_M < m_\kappa$.

Parameter γ_m: The damping rate γ_m of the κ-field enters the memory kernel through the exponential envelope in (2). In VERSF, γ_m arises from the coupling of the κ-field to the commitment bath — the ensemble of past irreversible events that constitutes the κ-field source. From the κ-field equation of motion (Taylor 2026b), the bath spectral density $J(\omega)$ determines γ_m via the standard fluctuation-dissipation structure. The precise derivation of γ_m from $J(\omega)$ is an open calculation identified in Paper I and in the epistemic table of Section 11 below. However, the key structural point is this: *the auxiliary-field construction does not introduce γ_m as a free parameter*. It inherits γ_m from the κ-field damping that is already present in the proven VERSF memory kernel (2). Whatever value γ_m takes within the underdamped range, the auxiliary-field theory with that value of γ_m reproduces the VERSF kernel exactly. The kernel structure — and hence Theorem 1 below — holds for all $\gamma_m \in (0, m_\kappa)$; the specific numerical value affects only the phase $\omega_\kappa = \sqrt{m^2_\kappa - \gamma^2_m}$ and thereby the subleading correction to n_s in Section 10.

Parameter g_m: The coupling g_m sets the amplitude of the memory forcing on the κ-background. From the quasi-static balance in Paper I (equation ⟨20⟩), the memory forcing amplitude is $\lambda_m \rho_0/m^2_\kappa$, where λ_m is the linear response coefficient. The coupling g_m is identified with λ_m: in the auxiliary-field formulation, integrating out M at the background level gives an effective source $g^2_m \kappa/\omega^2_M$ for the background κ-equation, which matches the VERSF memory source when $g^2_m/\omega^2_M = \lambda_m$. This fixes g_m in terms of λ_m and ω_M, both of which are VERSF-internal quantities. No new free parameter is introduced.

3.4 The Localised Action

With parameters fixed as above, the localised action is

$$S[g_{\mu\nu}, \kappa, M] = \int d^4x \sqrt{-g} \, [M^2_{Pl}/2 \cdot R - \tfrac{1}{2} g^{\mu\nu} \partial_\mu\kappa \, \partial_\nu\kappa - \tfrac{1}{2} m^2_\kappa \kappa^2 + \tfrac{1}{2} g^{\mu\nu} \partial_\mu M \, \partial_\nu M - \tfrac{1}{2} \omega^2_M M^2 + g_m \kappa M] + S_{damp}[\kappa, M]. \quad ... (4)$$

Several points require comment.

Kinetic sign: The M kinetic term has a standard positive sign $+\tfrac{1}{2}(\partial M)^2$. The retarded Green's function of the M-sector — a damped harmonic oscillator with positive kinetic term, mass ω_M, and retarded boundary conditions — is exactly

$$G_{ret}(\tau, \tau') = \theta(\tau-\tau') \, e^{-\gamma_m(\tau-\tau')} \sin(\omega_M(\tau-\tau')) / \omega_M,$$

which is the sine form. The VERSF target kernel (2) has precisely this sine form. No sign gymnastics is required: the standard positive-kinetic damped oscillator reproduces the target kernel directly.

M as a dynamical field with no observable propagating mode: M is a dynamical field in the localised representation — it has a kinetic term and propagating modes in principle. However, its mass $\omega_M \sim m_\kappa \sim 3.27$ H satisfies $\omega_M \gg H$ throughout the slow-roll unfolding epoch, which means M does not contribute an independent propagating degree of freedom to the observable scalar spectrum. The reduction to a single effective scalar degree of freedom follows from integrating out the heavy field M in the regime $\omega_M \gg H$ — the standard EFT argument for decoupling a massive mode from low-energy observables. Additionally, M is adiabatically slaved to κ in the slow-roll regime ($\dot{M}_0 = O(\varepsilon_V H M_0)$, established in §4), so it carries no independent initial conditions and introduces no independent adiabatic clock. The single-clock structure of §5.2 follows from both of these properties and is independent of any kinematic sign choice.

Relation between sine kernel and late-time cosine form: The retarded propagator G_{ret} has the sine structure at the level of the kernel between two points. The late-time cosine form

$M\_eff(\tau) \sim \cos(\omega\_\kappa\, \tau + \varphi)/(m\_\kappa\, \tau)$ in (1) arises upon convolution of G_ret with a spatially extended commitment source, which introduces the phase shift φ (Taylor 2026b, Appendix B). The two forms are consistent: the auxiliary-field construction reproduces the memory sector at the level of G_ret, and the late-time cosine asymptotics of M_eff emerge through source convolution exactly as in the original derivation.

Status of action (4) and S_damp[κ, M]: Dissipation cannot be represented in a closed-system Lagrangian. Action (4) is therefore shorthand for the system action within the Caldeira-Leggett in-in path integral: M is coupled to an explicit bath of harmonic oscillators, which is then traced out. The equations of motion (6)–(7) below are the effective equations of the bath-traced theory, not the variation of S as written. The variation of the closed-system action (4) gives a conservative M-equation; the damping term 2γ_m in (7) arises from the bath trace. This distinction matters for the interpretation of action (4) but does not affect any of the theorems: what enters the perturbation theory is the effective equation of motion (3), which has the damped-oscillator form regardless of whether it is derived from a Caldeira-Leggett bath or treated as a phenomenological input. The microscopic origin of S_damp is listed as an open calculation in Section 11 and carries no adjustable cosmological parameter.

3.5 Theorem 1: Local Auxiliary Representation

Theorem 1 (local auxiliary-field construction). *The retarded memory sector of the VERSF κ-field admits a local auxiliary-field representation with action (4), where the parameters ω_M, γ_m, g_m are fixed by previously established VERSF results. No new free cosmological parameters are introduced; the remaining open coefficient — the precise value of γ_m — is fixed once the bath spectral density is determined within the VERSF framework. Integrating out the auxiliary field M from this action yields an effective bilocal retarded κ-action whose retarded kernel is*

$$K\_eff(\tau, \tau') = g^2\_m \cdot \theta(\tau-\tau')\, e^{-\gamma\_m(\tau-\tau')} \sin(\omega\_M(\tau-\tau')) / \omega\_M, \; ... \; (5)$$

which reproduces the VERSF memory kernel (2) exactly upon the identification $g^2\_m = \lambda\_m\, \omega\_M$.

Proof: Solve the M-equation of motion (3) with retarded boundary conditions:

$$M(x) = g\_m \int d^4x' \sqrt{(-g(x'))}\, G\_ret(x, x')\, \kappa(x'),$$

where G_ret is the retarded Green's function of the damped oscillator (3). Substituting back into the M-sector of action (4) and performing the functional integration over M yields the effective bilocal term

$$S\_eff^{mem} = ½\, g^2\_m \int d^4x\, d^4x' \sqrt{(-g(x))} \sqrt{(-g(x'))}\, \kappa(x)\, G\_ret(x, x')\, \kappa(x').$$

The bilocal kernel is $K\_eff(\tau, \tau') = g^2\_m\, G\_ret(\tau, \tau')$, which is equation (5). The retarded structure $\theta(\tau-\tau')$ is inherited directly from G_ret. Matching (5) to the VERSF kernel (2) requires $g^2\_m = \lambda\_m\, \omega\_M$ with $\omega\_M = \omega\_\kappa$. We verify this identification is satisfied by VERSF-internal quantities: g_m is defined as the κ-M coupling amplitude; λ_m is the linear response coefficient from Paper I ⟨20⟩; $\omega_M = \omega_\kappa$ is the damped oscillation frequency fixed by m_κ and γ_m. The relation $g^2_m = \lambda_m \omega_M$ is therefore a consistency condition between VERSF-internal definitions, not an additional constraint imposed from outside. It is satisfied by construction when g_m is identified as the square root of $\lambda_m \omega_M$, both of which are fixed by VERSF structure. No new free parameter is introduced. ■

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4. Background Equations in the Localised (κ, M) Theory

We specialise to spatially flat FRW:

$ds^2 = -dt^2 + a^2(t)\, \delta_{ij}\, dx^i\, dx^j$,

with homogeneous background fields $\kappa_0(t)$, $M_0(t)$. The background field equations from action (4) are

$\ddot{\kappa}_0 + 3H\, \dot{\kappa}_0 + m^2_\kappa\, \kappa_0 = g_m\, M_0$, ... (6) $\ddot{M}_0 + (3H + 2\gamma_m)\, \dot{M}_0 + \omega^2_M\, M_0 = g_m\, \kappa_0$. ... (7)

Adiabatic slow-roll limit. In the slow-roll unfolding regime, M_0 tracks κ_0 quasi-statically. Setting $\dot{M}_0 \approx 0$ and $\ddot{M}_0 \approx 0$ in (7):

$M_0 \approx g_m\, \kappa_0 / \omega^2_M$.

Substituting into (6):

$\ddot{\kappa}_0 + 3H\, \dot{\kappa}_0 + (m^2_\kappa - g^2_m/\omega^2_M)\, \kappa_0 = 0$.

This reproduces the effective κ-equation of Paper I with the memory-corrected mass. The correction $g^2_m/\omega^2_M = \lambda_m/\omega_M$ is of order $R \sim 1/(N \ln N)$ relative to m^2_κ in the slow-roll regime (consistent with Paper I §4.3), confirming that the adiabatic approximation is internally self-consistent.

Logarithmic memory build-up. For the cumulative background memory integral, the localised theory reproduces the logarithmic growth of Paper I §3.4. In the adiabatic limit, $M_0(t)$ encodes the running average of κ-history weighted by the exponentially damped kernel, and the net displacement δs_{QS} grows as $(\rho_0/m_\kappa) \ln N_{fwd}$ as shown there. The localised formulation is therefore dynamically equivalent to the nonlocal background treatment of Paper I.

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5. Scalar Perturbations and Gauge Structure

5.1 Perturbation Variables

We expand both fields:

$\kappa = \kappa_0(t) + \delta\kappa(t, \mathbf{x})$, $M = M_0(t) + \delta M(t, \mathbf{x})$.

The metric is written in ADM form with scalar perturbations parametrised by lapse δN, shift ∂i *B, and spatial curvature ψ in spatially flat gauge (*$\delta g\{ij\} = 0$ in the spatial sector). In spatially flat gauge, all scalar fluctuations reside in $\delta\kappa$ and δM; the curvature perturbation is the gauge-invariant combination

$R = -(H/\dot{\kappa}_0)\, \delta\kappa + \ldots$, ... (MS.1)

at leading slow-roll order, where the ellipsis denotes subdominant terms from the M-sector (shown to be negligible in §6.3).

5.2 Single-Clock Structure

Single-clock closure is the condition that only one scalar degree of freedom contributes independent propagating modes to the primordial spectrum. A critical question for the two-field system (κ, M) is whether this condition holds — i.e., whether M contributes an independent propagating scalar at observable energy scales.

It does not, for two independent reasons.

Mass decoupling. The M-field has mass $\omega_M \sim m_\kappa \sim 3.27\, H$ in the slow-roll unfolding regime. Since $\omega_M \gg H$, M is too heavy to contribute low-energy propagating modes to the observable scalar spectrum — it decouples from the infrared in the standard EFT sense. The reduction to a single effective scalar degree of freedom follows from integrating out the heavy field M in the regime $\omega_M \gg H$, which is satisfied throughout the unfolding epoch. This is exactly the EFT decoupling argument: modes with mass $\gg H$ do not participate in horizon-crossing amplification and do not appear in the superhorizon power spectrum.

Adiabatic slaving. As established in §4, M_0 tracks κ_0 quasi-statically in the slow-roll regime: $\dot{M}_0 = O(\varepsilon_V H M_0)$. The M-sector has no independent initial conditions and therefore introduces no independent adiabatic clock. The curvature perturbation R is sourced entirely by $\delta\kappa$ at leading slow-roll order, with the M-sector contributing only at $O(\varepsilon_V)$ through the lapse/shift mixing (established in §6.2).

After integrating out M in Section 8, only κ remains as a propagating scalar in the exact mode equation. The single-field description is therefore exact at leading slow-roll order, not approximate. This is consistent with the single-clock closure demonstrated in Paper I §6.2 by three independent arguments; the present formulation makes the decoupling mechanism transparent at the Lagrangian level.

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6. Exact Second-Order Action: Lapse/Shift Elimination in the Two-Field System

6.1 The Key Technical Issue

In a single-field theory, eliminating the lapse δN and shift B from the quadratic action is standard and yields the canonical Mukhanov-Sasaki action. In a two-field theory, both fields source the lapse and shift constraints, and the elimination can introduce cross-sector mixing terms of the form $\delta N \cdot \delta M$, $B \cdot \delta M$, or derivatives thereof. These mixing terms would modify the quadratic action in a way that goes beyond the simple sum $S^{(2)}_\kappa + S^{(2)}_M + S^{(2)}_{mix}$ written in the draft companion paper.

We now address this explicitly.

6.2 The Hamiltonian and Momentum Constraints

The lapse and shift in the perturbed metric enter the action through the Einstein equations. At linear order in perturbations, the Hamiltonian constraint (from varying with respect to δN) and the momentum constraint (from varying with respect to B) are:

Hamiltonian constraint: $2M^2_{Pl} [3H \dot\psi + (k^2/a^2) \psi] = \dot\kappa_0 \delta\dot\kappa + m^2_\kappa \kappa_0 \delta\kappa - g_m M_0 \delta M$ + (M-sector energy terms)

Momentum constraint: $2M^2_{Pl} (H \delta N + \dot\psi) = \dot\kappa_0 \delta\kappa$ + (M-sector momentum terms)

The M-sector contributions to these constraints take the schematic form:

[M-sector contribution to H-constraint] $\sim \dot M_0 \delta\dot M + \omega^2_M M_0 \delta M - g_m \kappa_0 \delta M$ [M-sector contribution to mom-constraint] $\sim \dot M_0 \delta M$

Now apply the adiabatic slow-roll condition. In the slow-roll unfolding regime, $\dot M_0 \approx 0$ (adiabatic tracking). Therefore:

- The M-sector contribution to the momentum constraint is $O(\dot M_0) = O(\varepsilon_V H M_0)$ — suppressed by the slow-roll parameter $\varepsilon_V \sim 1/(2N\bigstar)$ relative to the κ-sector contribution.
- The M-sector contribution to the Hamiltonian constraint involves $\omega^2_M M_0 \delta M$ and $-g_m \kappa_0 \delta M$. In the adiabatic limit, these cancel against each other at leading order (from the M background equation (7) with $\dot M_0 \approx 0$: $\omega^2_M M_0 = g_m \kappa_0$). The residual is the $\ddot M_0$ correction. In the slow-roll regime, $\ddot M_0$ can be estimated from differentiating the adiabatic balance $\omega^2_M M_0 \approx g_m \kappa_0$: $\ddot M_0 \approx (g_m/\omega^2_M) \ddot\kappa_0 \sim (g_m/\omega^2_M) \varepsilon_V H^2 \kappa_0 \sim \varepsilon_V H^2 M_0$. The residual Hamiltonian constraint contribution is therefore $O(\ddot M_0/\omega^2_M) \times \delta M \sim O(\varepsilon_V H^2/\omega^2_M) \times M_0 \delta M \sim O(\varepsilon_V/11) \times$ (κ-sector term), which is suppressed relative to $O(\varepsilon_V)$ by the additional factor of $H^2/\omega^2_M \sim 1/11$.

Theorem (lapse/shift elimination at leading slow-roll order). *In the slow-roll unfolding regime ($\varepsilon_V \sim 1/(2N★) \ll 1$), the Hamiltonian and momentum constraints sourced by the auxiliary field M contribute to the κ-sector quadratic action only at $O(\varepsilon_V)$ relative to the leading canonical term. At leading slow-roll order, the lapse and shift solutions are determined by the κ-sector alone, and the standard lapse/shift elimination of the single-field theory carries over to the two-field system with corrections suppressed by $\varepsilon_V \sim 1/(2N★)$.*

This is the non-trivial technical result of this section. It holds specifically because M is an adiabatically slaved response field ($\dot{M}_0 \approx 0$ in slow-roll) and its constraint contributions therefore cancel or are suppressed. It would not hold for a second independently dynamical scalar with $\dot{M}_0 \sim H M_0$.

6.3 The Quadratic Action

After lapse/shift elimination, the quadratic action to leading slow-roll order is:

$$S^{(2)} = S^{(2)}_\kappa + S^{(2)}_M + S^{(2)}_{mix}, \quad \ldots \text{(MS.2)}$$

where:

κ-sector (canonical):

$$S^{(2)}_\kappa = \tfrac{1}{2} \int d\tau\, d^3x\, [(v')^2 - (\nabla v)^2 + (z''/z)\, v^2],\ v \equiv zR,\ z \equiv a\, \dot{\kappa}_0/H_{eff} \ \ldots \text{(MS.3)}$$

M-sector:

$$S^{(2)}_M = \tfrac{1}{2} \int d\tau\, d^3x\, [+(\delta M')^2 - (\nabla \delta M)^2 - a^2 \omega^2_M (\delta M)^2] + S^{(2)}_{damp} \ \ldots \text{(MS.4)}$$

The M-sector action has the standard positive-kinetic form, consistent with the full action (4). M does not contribute propagating modes to the physical spectrum below the Hubble scale because $\omega_M \gg H$ throughout the unfolding epoch (§5.2); its role in the perturbation theory is as a heavy mediator that is integrated out in Section 8 to yield the nonlocal self-kernel Σ_k.

Mixing sector:

$$S^{(2)}_{mix} = g_m \int d\tau\, d^3x\, a^2\, \delta\kappa\, \delta M + O(\varepsilon_V) \ \ldots \text{(MS.5)}$$

The $O(\varepsilon_V)$ correction is the residual from the lapse/shift elimination and is suppressed by $1/(2N★) \sim 0.7$–1% relative to the leading mixing term. It is retained for the subleading correction in Section 10 and negligible for the leading-order spectral-index result.

Proposition (exact quadratic structure). *The non-trivial content of the quadratic action (MS.2)–(MS.5) is the absence, at leading slow-roll order, of cross-metric-field terms of the form $\delta N \cdot \delta M$ or $B \cdot \delta M$. This absence follows from the adiabatic constraint cancellation established in §6.2 and is not a general feature of two-field systems — it holds specifically because the M-sector is a response field with $\dot{M}_0/H M_0 = O(\varepsilon_V)$.*

7. Exact Coupled Mode Equations

Notation. Equations (6)–(7) in Section 4 are in cosmic time (dots denote ∂_t, $H = \dot{a}/a$). The perturbation equations below are in conformal time (primes denote ∂_τ, $\mathcal{H} = a'/a = aH$). The damping coefficient $2\gamma_m$ in the cosmic-time M-equation (7) becomes $2a\gamma_m$ in conformal time, since γ_m has dimensions of inverse time and the conformal-time equation acquires an extra factor of a at each time derivative. This factor is carried explicitly in (7.2) below.

Passing to Fourier space, varying the action (MS.2)–(MS.5) with respect to $\delta\kappa_k$ and δM_k yields the coupled second-order system:

$\delta\kappa_k'' + 2\mathcal{H}\, \delta\kappa_k' + (k^2 + a^2 m^2_\kappa + \Delta_\kappa)\, \delta\kappa_k = g_m a^2\, \delta M_k + C_k$, ... (7.1) $\delta M_k'' + (2\mathcal{H} + 2a\gamma_m)\, \delta M_k' + (k^2 + a^2 \omega^2_M + \Delta_M)\, \delta M_k = g_m a^2\, \delta\kappa_k + D_k$. ... (7.2)

Several points:

- Equation (7.2) has positive damping coefficient $(2\mathcal{H} + 2a\gamma_m)$ throughout, since both $\mathcal{H} > 0$ and $\gamma_m > 0$. There is no anti-damping instability in the auxiliary sector.
- Δ_κ and Δ_M collect background-induced mass corrections from the slow-roll expansion of the FRW metric perturbations. At leading slow-roll order, $\Delta_\kappa \sim O(\varepsilon_V)\, m^2_\kappa$ and $\Delta_M \sim O(\varepsilon_V)\, \omega^2_M$.
- C_k and D_k are residual mixing terms from lapse/shift elimination, of order $O(\varepsilon_V)$ as established in §6.2.

8. Elimination of the Auxiliary Sector and the Exact Nonlocal Mode Equation

8.1 Retarded Green's Function of the Auxiliary Sector

Define $G_k(\tau, \tau')$ as the retarded Green's function of the auxiliary-sector perturbation operator from (7.2):

$[\partial^2_\tau + (2\mathcal{H} + 2a\gamma_m)\, \partial_\tau + k^2 + a^2 \omega^2_M]\, G_k(\tau, \tau') = \delta(\tau - \tau')$, $G_k(\tau, \tau') = 0$ for $\tau' > \tau$. ... (8.1)

The operator on the left has positive-definite mass term $a^2\omega^2_M$ and positive damping $(2\mathcal{H} + 2a\gamma_m)$, so the auxiliary sector is stable throughout. The retarded condition $G_k = 0$ for $\tau' > \tau$ follows from the causal structure of the auxiliary-sector equation and ensures that the memory term in the final mode equation is strictly past-dependent.

The late-time asymptotic of G_k in the slow-roll FRW background is:

$G_k(\tau, \tau') \sim \theta(\tau-\tau')\, F_k(\tau, \tau')\, e^{-\gamma_m(\tau-\tau')} \sin(\omega_M(\tau-\tau')) / \omega_M$, $m_\kappa(\tau-\tau') \gg 1$, ... (8.2)

where $F_k(\tau, \tau')$ is a slowly varying envelope arising from the FRW background. In the slow-roll regime, F_k is well approximated by the scale factor ratio: $F_k(\tau, \tau') \approx a(\tau')/a(\tau) = e^{-H(\tau-\tau')} \approx 1 - H(\tau-\tau') + O((H/m_\kappa)^2(\tau-\tau')^2)$. Over one kernel oscillation period $\Delta\tau \sim \pi/\omega_M \sim \pi/m_\kappa$, the fractional variation of F_k is of order $H/m_\kappa \sim 0.31$. This means F_k introduces a correction to the leading kernel of the same order as the H/m_κ correction already identified in §9.1: it enters the adiabatic expansion at the same order as the subleading term in (9.1) and does not modify the leading-order suppression. At horizon crossing, F_k is evaluated at a specific τ and contributes an O(1) prefactor to the integrated kernel weight in §9.2; it introduces no new parameters.

k-dependence of G_k. An important structural point: the asymptotic form (8.2) is independent of k at leading order. The k^2 term in the auxiliary-sector operator (8.1) is subdominant relative to the mass term $a^2\omega^2_M$ for the modes of interest ($k^2 \sim a^2H^2 \sim a^2\omega^2_M/11$ at primordial onset), so the short-distance structure of G_k is set by the mass and damping, not by k. The kernel is therefore approximately k-independent for the CMB window modes, justifying the uniform-in-k suppression established in Paper I §6.4 — now as a derived consequence of the auxiliary-sector structure rather than an assumption. The caveat on the asymptotic nature of this k-independence argument is given in §9.1.

8.2 Retarded Solution for the Auxiliary Perturbation

From (7.2) with retarded boundary conditions:

$\delta M_k(\tau) = g_m \int_{\tau_i}^{\tau} d\tau'\, G_k(\tau, \tau')\, a^2(\tau')\, \delta\kappa_k(\tau')$ + [D_k contributions]. ... (8.3)

The D_k contribution is $O(\varepsilon_V)$ and enters only the subleading correction.

8.3 The Exact Nonlocal κ-Equation

Substituting (8.3) into (7.1):

$\delta\kappa_k'' + 2H\, \delta\kappa_k' + (k^2 + a^2 m^2_\kappa + \Delta_\kappa)\, \delta\kappa_k - g^2_m a^2(\tau) \int_{\tau_i}^{\tau} d\tau'\, G_k(\tau,\tau')\, a^2(\tau')\, \delta\kappa_k(\tau') = \tilde{C}_k$, ... (8.4)

where $\tilde{C}_k$ collects $O(\varepsilon_V)$ residuals. The memory integral enters with a minus sign: the coupling g_m on the source side of (7.1) and the $+g_m$ in the retarded solution (8.3) combine to give a net $-g^2_m$ term when the self-consistent condition is imposed. To resolve any apparent tension with the background picture: at the background level, memory drives the cumulative displacement of κ away from equilibrium over many e-folds; at the perturbative level, the retarded response of the auxiliary sector enters as a restoring self-interaction around the slowly evolving equilibrium configuration. These are two distinct roles acting on two distinct timescales — the background displacement builds logarithmically over the full unfolding epoch, while the perturbative self-interaction acts instantaneously at each e-fold around the current equilibrium. They are consistent, not competing. After canonical normalisation $v_k = z R_k$ and passing to conformal time, the mode equation becomes:

Theorem 2 (Exact Nonlocal Scalar Mode Equation).

The scalar perturbation sector of the VERSF κ-field, with memory response encoded exactly by the auxiliary field M, is governed by the retarded nonlocal integro-differential equation

$$v_k'' + (k^2 - z''/z)\, v_k + \int_{\tau_i}^{\tau} d\tau'\, \Sigma_k(\tau, \tau')\, v_k(\tau') = 0, \quad \ldots (8.5)$$

where the retarded memory self-kernel is

$$\Sigma_k(\tau, \tau') = -g^2_m\, N(\tau, \tau')\, G_k(\tau, \tau'),\quad N(\tau, \tau') = a^2(\tau)\, a^2(\tau') / (z(\tau)\, z(\tau')), \quad \ldots (8.6)$$

with G_k retarded ($\Sigma_k = 0$ for $\tau' > \tau$) and $G_k > 0$ (positive for $\Delta\tau \in (0, \pi/\omega_M)$), so that $\Sigma_k < 0$ on the dominant first half-cycle. All parameters are determined by previously established VERSF results.

Proof: Equation (8.4) is derived by substituting the retarded solution (8.3) into (7.1) and rearranging. The sign of the nonlocal term is $-g^2_m$: the source coupling $g_m M_0$ on the RHS of (7.1) combined with the retarded solution $\delta M_k = +g_m \int G_k\, \delta\kappa_k\, d\tau'$ gives $+g^2_m$ on the RHS; moving to the LHS gives $-g^2_m$. The canonical normalisation $v_k = z R_k$ with $z = a \dot{\kappa}_0/H_{eff}$ introduces the pump term z''/z. The normalisation factor $N(\tau, \tau')$ arises from converting between $\delta\kappa_k$ and v_k at times τ and τ'. The retarded structure of Σ_k is inherited from G_k. All parameters — g_m, ω_M, γ_m, m_κ — are fixed by Theorem 1 and the VERSF structural inputs of §2.3. ■

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9. Slow-Roll Reduction and the Mukhanov–Sasaki Limit

9.1 Domain of the Adiabatic Expansion

The reduction of (8.5) to the local Mukhanov–Sasaki equation proceeds by expanding $v_k(\tau')$ around τ inside the memory integral. This expansion is valid when v_k varies slowly over the kernel support. We must specify the k-domain in which this holds.

Superhorizon modes ($k \ll aH$): v_k is frozen on superhorizon scales ($v_k \sim$ const), so $v_k(\tau') \approx v_k(\tau)$ throughout the kernel support. The adiabatic expansion is exact at leading order.

Modes near horizon crossing ($k \sim aH$): v_k varies on the Hubble timescale H^{-1}. Since the kernel oscillates on the shorter timescale $m_\kappa^{-1} = (3.27\, H)^{-1}$, the amplitude v_k changes by a fraction $\Delta v_k/v_k \sim H/m_\kappa \approx 0.31$ over one kernel oscillation. The adiabatic expansion

$$v_k(\tau') = v_k(\tau) - (\tau-\tau')\, v_k'(\tau) + \tfrac{1}{2}(\tau-\tau')^2\, v_k''(\tau) + \ldots$$

converges with the leading correction $O(H/m_\kappa) \sim 0.31$, and the first term in the expansion gives a fractional accuracy of ~ 31% for the instantaneous kernel value. However, this first-order inaccuracy in the integrand does not translate to a 31% inaccuracy in the integrated kernel contribution to (8.5). The kernel is oscillatory, so the leading error term — proportional to $(\tau-\tau')$ $v_k'(\tau)$ times a damped oscillatory kernel — integrates to a small value. Specifically, the integral $\int (\tau-\tau') \cos(\omega_\kappa(\tau-\tau'))/(m_\kappa(\tau-\tau')) e^{-\gamma_m(\tau-\tau')} d\tau'$ is convergent and is suppressed relative to the leading term by an additional factor of $\omega_\kappa^{-1} \sim m_\kappa^{-1} \sim H$, giving an overall correction to Σ_k of order $O((H/m_\kappa)^2) \sim 0.09$. This is beyond the leading-order result and enters only the subleading correction of Section 10.

Subhorizon modes ($k \gg aH$): Modes deep inside the horizon oscillate rapidly with frequency k. In principle, if $k \sim \omega_\kappa$ there could be a resonance between the mode and the kernel. However, at the primordial onset $\omega_\kappa \sim m_\kappa \sim 3.27$ H, so the resonance condition $k \sim \omega_\kappa$ corresponds to $k \sim aH$ — i.e., horizon-crossing, already covered above. For $k \gg aH$, mode and kernel oscillate at incommensurate frequencies and the integral in (8.5) is further suppressed by phase cancellation. Subhorizon modes are therefore not adversely affected by the memory kernel; the adiabatic expansion overestimates the kernel contribution for these modes.

Limit of the asymptotic k-independence argument: The conclusion that Σ_k is k-independent rests on the late-time asymptotic form of G_k in (8.2), which is derived in the regime $m_\kappa(\tau-\tau') \gg 1$. The full Green's function at finite separation includes k-dependent transient terms that are not captured by the asymptotic expansion. A complete proof of k-independence across all regimes — including the transition from subhorizon to superhorizon and the short-separation behaviour of G_k — requires explicit evaluation of the Green's function beyond the asymptotic expansion and is identified as an open calculation. For the CMB-window modes treated here, the asymptotic argument is sufficient: those modes satisfy $m_\kappa(\tau-\tau') \gg 1$ across the bulk of the kernel support at horizon crossing, and the transient contributions are suppressed by the same damping envelope.

Conclusion on domain: The adiabatic expansion leading to Theorem 3 is valid uniformly across the CMB-relevant window, with the leading memory correction $O(1/(N \ln N))$ established below, and the next correction $O((H/m_\kappa)^2 /(N \ln N)) \sim 0.09/(N \ln N)$ — an order of magnitude smaller. The expansion is controlled not by H/m_κ itself, but by the oscillatory structure of the kernel, which suppresses subleading terms by an additional factor of $(H/m_\kappa)^2$, yielding an effective expansion parameter $(H/m_\kappa)^2 \sim 0.09$. The mechanism is explicit: the leading and subleading integrated kernel weights involve the standard damped-oscillator integrals

$$\int_0^\infty \sin(\omega u) e^{-\gamma u} du = \omega/(\omega^2+\gamma^2), \int_0^\infty u \sin(\omega u) e^{-\gamma u} du = 2\omega\gamma/(\omega^2+\gamma^2)^2,$$

so the ratio of subleading to leading integrated weight is $2\gamma_m/(\omega^2_M + \gamma^2_m)$. This ratio has dimensions of time (inverse frequency); setting $\omega_M \sim 3.27H$ and $\gamma_m \sim H$ gives $2\gamma_m/(\omega^2_M + \gamma^2_m) \sim 2H/(10.7H^2 + H^2) \sim 0.17 H^{-1}$. The dimensionless suppression of the subleading term in the adiabatic expansion (9.1) arises upon multiplication by H from the $v_k'(\tau)$ prefactor in (9.1): the subleading term is proportional to $v_k'(\tau) \times$ [integral ratio], and $v_k'(\tau) \sim H v_k(\tau)$ near horizon crossing, so the net dimensionless suppression is $2H\gamma_m/(\omega^2_M + \gamma^2_m) \sim 2(H/m_\kappa)^2 \sim 0.18$. It is this dimensionless ratio — not H/m_κ alone — that governs the convergence of the adiabatic expansion. The expansion is therefore well-controlled even though H/m_κ is $O(1)$ rather than $\ll 1$.

9.2 The Adiabatic Expansion

For modes in the CMB window, expand v_k inside the memory integral:

$$\int_{\tau_i}^{\tau} d\tau' \, \Sigma_k(\tau, \tau') \, v_k(\tau') = v_k(\tau) \int_{\tau_i}^{\tau} d\tau' \, \Sigma_k(\tau, \tau') - v_k'(\tau) \int_{\tau_i}^{\tau} d\tau' \, (\tau-\tau') \, \Sigma_k(\tau, \tau') + \dots \quad \dots (9.1)$$

Leading term: The integral $\int \Sigma_k(\tau, \tau') \, d\tau'$ is the integrated kernel weight. From (8.2) and (8.6):

$$\int_{\tau_i}^{\tau} d\tau' \, \Sigma_k(\tau, \tau') \sim -g^2_m \, (m_\kappa)^{-1} \int F_k \, e^{-\gamma_m \Delta\tau} \sin(\omega_M \Delta\tau)/\omega_M \, d\Delta\tau \times N.$$

For $\tau-\tau_i \gg \gamma_m^{-1}$ (i.e., after the initial transient), this converges to a constant of order $g^2_m/(m_\kappa \, \omega_M \, \gamma_m) \times N$. Using $g^2_m = \lambda_m \, \omega_M$ and $N \sim a^2(\tau)/z^2(\tau)$:

$$\int \Sigma_k \, d\tau' \sim \lambda_m/(m_\kappa \, \gamma_m) \times (\text{background slow-roll factors}) \sim O(R_{pert} \times H^2_{eff}).$$

The ratio R_{pert} = (leading memory term)/$(k^2 - z''/z) \sim 1/(N \ln N)$ is the same suppression derived in Paper I §4.3 and §6.4, now obtained from the exact auxiliary-field structure rather than from a heuristic extension.

Subleading term (first correction): The term proportional to $v_k'(\tau)$ involves $\int(\tau-\tau') \, \Sigma_k \, d\tau'$, which is suppressed by an additional factor of $(H/m_\kappa)^2 \sim 0.09$ relative to the leading term, as argued in §9.1.

9.3 Theorem 3: Local MS Limit

Theorem 3 (Mukhanov–Sasaki reduction). *In the slow-roll unfolding regime relevant for observable primordial scalar modes (modes crossing the VERSF Hubble radius during the unfolding epoch), the exact nonlocal mode equation (8.5) reduces to*

$$v_k'' + (k^2 - z''/z) \, v_k = O(1/(N \ln N)) \times v_k. \quad \dots (9.2)$$

Equivalently, the local Mukhanov–Sasaki equation

$$v_k'' + (k^2 - z''/z) \, v_k = 0 \quad \dots (9.3)$$

is the leading-order adiabatic limit of the exact VERSF scalar perturbation theory, valid uniformly for modes in the CMB window.

Proof: From (9.1), the nonlocal term in (8.5) equals $v_k(\tau) \times$ (integrated kernel weight) + $O((H/m_\kappa)^2)$ corrections. The integrated kernel weight is $O(R_{pert} \times H^2_{eff})$ from §9.2, with $R_{pert} \sim 1/(N \ln N)$ from the quasi-static force balance of Paper I. Dividing by the leading term $k^2 \sim H^2$ near horizon crossing:

$$\left|\int \Sigma_k \, d\tau' \, v_k\right| / \left|(k^2 - z''/z) \, v_k\right| \sim R_{pert} \sim 1/(N \ln N).$$

For $N \sim 60$–70: $1/(N \ln N) \sim 3 \times 10^{-3}$. The nonlocal term is therefore negligible at leading order, and the local MS equation (9.3) governs the mode dynamics. ■

The leading-order spectral index result of Paper I follows immediately:

$$n_s - 1 = -2/N_\star.$$

This is now not a result imported from standard inflation but a derived consequence of the exact VERSF perturbation theory in its adiabatic slow-roll limit. The Mukhanov–Sasaki equation is not fundamental in VERSF; it is the effective low-energy description obtained after integrating out the memory-response sector of a nonlocal, retarded dynamical system — valid precisely when $\omega_M \gg H$, which holds throughout the unfolding epoch.

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10. Sign and Structure of the Memory Correction to n_s

10.1 The Effective Pump-Term Correction

The nonlocal term in (8.5), at leading order in the adiabatic expansion (9.1), acts as a correction to the effective pump term z''/z:

$$z''/z \to z''/z - \delta\Pi_k(\tau),$$

where

$$\delta\Pi_k(\tau) = \int_{\tau_i}^{\tau} d\tau' \, \Sigma_k(\tau, \tau'). \quad \text{... (10.1)}$$

This shifts the index ν of the power-law mode solution:

$$\nu = \nu_0 + \delta\nu_{mem}, \quad \nu_0 = 3/2 + 1/N_\star, \quad \text{... (10.2)}$$

and hence shifts the spectral index:

$$n_s - 1 = 3 - 2\nu = -2/N_\star - 2\delta\nu_{mem} \equiv -2/N_\star + \delta n_s^{mem}. \quad \text{... (10.3)}$$

10.2 Sign Determination

We establish the sign of $\delta\Pi_k$ and hence of δn_s^{mem} through the following chain.

Step 1: Sign of G_k. The retarded Green's function $G_k(\tau, \tau')$ of a damped harmonic oscillator with positive mass ω_M satisfies $G_k(\tau, \tau') = \theta(\tau-\tau')\, e^{-\gamma_m(\tau-\tau')} \sin(\omega_M(\tau-\tau'))/\omega_M$. For $\tau-\tau' \in (0, \pi/\omega_M)$, the sine factor is positive, so $G_k > 0$ on the first half-cycle.

Step 2: Sign of Σ_k. From (8.6), $\Sigma_k = -g^2_m N G_k$ with $N > 0$ (it is a ratio of scale-factor products). Since $G_k > 0$ on the dominant first half-cycle, $\Sigma_k < 0$ on that half-cycle.

Step 3: Sign of $\delta\Pi_k$. The integrated kernel weight $\delta\Pi_k = \int \Sigma_k \, d\tau'$ is a sum over all half-cycles. The exponential damping $e^{-\gamma_m(\tau-\tau')}$ weights earlier (smaller $\tau-\tau'$) contributions more heavily, so the first (negative) half-cycle dominates over the second (positive) half-cycle. Therefore:

$$\delta\Pi_k < 0. \quad \text{... (10.4)}$$

Step 4: Effect on effective pump. In the adiabatic limit, the exact mode equation (8.5) becomes

$$v_k'' + (k^2 - z''/z + \delta\Pi_k) v_k = 0,$$

so the effective pump term is $z_{eff}''/z_{eff} = z''/z - \delta\Pi_k$. With $\delta\Pi_k < 0$:

$$z_{eff}''/z_{eff} = z''/z - \delta\Pi_k = z''/z + |\delta\Pi_k| > z''/z.$$

The effective pump *increases* relative to the unperturbed value.

Step 5: Sign of $\delta\nu_{mem}$. The Bessel index ν satisfies $\nu^2 - \frac{1}{4} = (z_{eff}''/z_{eff}) \tau^2$ at leading order. Since $z_{eff}''/z_{eff} > z''/z$:

$$\nu = \nu_0 + \delta\nu_{mem}, \quad \delta\nu_{mem} > 0.$$

Step 6: Sign of δn_s^{mem}. Since $n_s - 1 = 3 - 2\nu$:

$$\delta n_s^{mem} = -2\delta\nu_{mem} < 0.$$

This is a **redward** correction — the memory kernel increases $|n_s - 1|$, moving n_s toward the Planck observed value 0.9649. The sign is fixed by the damping asymmetry of G_k and carries no ambiguity. To state it plainly: the negative sign of Σ_k implies a positive correction to the effective pump term $z_{eff}''/z_{eff} = z''/z - \delta\Pi_k$, which increases ν and therefore enhances the red tilt. ■

10.3 Magnitude and Theorem 3 Completion

Theorem 3 (complete statement). *The exact scalar perturbation theory of the VERSF κ-sector predicts*

$$n_s - 1 = -2/N_\star + \delta n_s^{mem}, \quad \delta n_s^{mem} = -C/(N_\star \ln N_\star) \times (1 + O(H/m_\kappa)), \quad \text{... (10.5)}$$

where $C > 0$ is a positive coefficient determined by the integrated kernel weight $\delta\Pi_k$, computable once the bath spectral density is specified within the VERSF framework — which fixes γ_m and thereby the normalisation of G_k. No new free cosmological parameter is *introduced. The sign $\delta n_s^{mem} < 0$ (enhanced red tilt) is fixed by the damping asymmetry of G_k.*

The explicit evaluation of C requires the normalisation of the auxiliary-sector Green's function at horizon crossing, which depends on the bath spectral density determining γ_m. No new free cosmological parameters are introduced; the coefficient C is fixed once the bath spectral density is determined within the VERSF framework. This is the remaining open calculation noted in §11.

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11. Epistemic Status: Proven, Derived, and Modelled

Summary Table

Item	Status	Affects
CCC threshold $\xi = (\hbar c/\rho)^{1/4}$	**PROVEN** (prior)	m_κ, ω_M
$K = 7 \rightarrow m^2_\kappa = (4/3)\xi^{-2}$	**PROVEN** (prior)	All mass scales
$V_{eff} = \frac{1}{2}m^2_\kappa(\delta s)^2 \rightarrow \eta_V = \varepsilon_V$	**PROVEN** (prior)	$n_s - 1 = -4\varepsilon_V$
Late-time kernel asymptotic $M_{eff}(\tau)$	**PROVEN** (prior, Taylor 2026b)	Kernel structure
Auxiliary-field construction (Theorem 1)	**PROVEN** (§3.5) — parameters fixed from VERSF, no new free params beyond VERSF memory sector	Exact mode equation
Lapse/shift elimination at leading SR order (§6.2)	**PROVEN** (§6.2) — uses $\dot{M}_0 = O(\varepsilon_V)$ and adiabatic cancellation	Quadratic action validity
Exact nonlocal mode equation (Theorem 2)	**PROVEN** (§8.3) — derived from action by auxiliary elimination	Closes perturbation theory
k-uniformity of kernel suppression (§8.1, §9.1)	**DERIVED** — from auxiliary-sector G_k structure	Theorem 3 domain
MS reduction Theorem 3 (§9.3)	**PROVEN** — from adiabatic expansion of exact mode equation	$n_s = -2/N\bigstar$ at leading order
Sign of δn_s^{mem} (§10.2)	**DERIVED** — from damping asymmetry of G_k	Subleading correction direction
$n_s \approx 0.972$ (leading order)	**RESULT** (Paper I) — 1.7σ from observation	—
Exact coefficient C in δn_s^{mem}	**OPEN** — requires γ_m from bath spectral density $J(\omega)$; no new free param	Numerical value of correction
Sign of $O((H/m_\kappa)^2)$ correction	**OPEN** — computable from 2nd-order adiabatic expansion; small (< 9% of leading memory correction)	Subleading-to-subleading

Item	Status	Affects
Tensor sourcing from fold geometry	**OPEN** (prior) — $r = 16\varepsilon\_V$ is upper bound; fold suppression required by BICEP/Keck	Exact value of r
γ_m from bath spectral density	**OPEN** (prior, Taylor 2026b) — does not affect Theorems 1–3 structurally	Coefficient C; precise ω_κ

What Is and Is Not Modelled

The one genuinely modelled element in the present paper is the functional form of S_damp[κ, M]. We have specified that the M-equation of motion has the form (3) with damping coefficient γ_m; the precise microscopic Lagrangian that produces this dissipative equation of motion is a modelling input. However:

1. This modelling input carries no free cosmological parameter: γ_m is already present in the proven VERSF memory kernel (2) from prior work (Taylor 2026b).
2. Theorems 1, 2, and 3 hold for all $\gamma\_m \in (0, m\_\kappa)$ (underdamped regime). The kernel structure, the exact mode equation, and the adiabatic MS limit are independent of the specific value of γ_m.
3. The only γ_m-dependent quantity in the observable predictions is the coefficient C of the subleading correction — which is an open calculation that does not affect the leading-order spectral index.

S_damp[M] therefore plays the role it was assigned in Section 11 of the draft companion paper — a modelled input — but its scope is now precisely delimited: it affects only the coefficient of the subleading correction, not the structural results.

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12. Discussion

12.1 What the Paper Accomplishes

The companion paper Paper I demonstrated that the primordial spectral index follows structurally from the VERSF κ-field sector at leading order, with a controlled bound establishing that the memory contribution to the perturbation action is negligible at leading order. The present paper strengthens this in three ways.

First, the controlled bound is replaced by an exact nonlocal mode equation with a structurally determined kernel. The local Mukhanov–Sasaki equation is no longer justified by a suppression estimate but derived as the adiabatic limit of the exact perturbation theory.

Second, the lapse/shift constraint elimination in the two-field (κ, M) system is shown explicitly to produce no additional cross-sector mixing at leading slow-roll order. This had to be checked specifically for the two-field system; the result depends on the adiabatic nature of the M-sector ($\dot{M}_0 = O(\varepsilon\_V)$) and would fail for an independently dynamical second field.

Third, the sign of the subleading memory correction to n_s is fixed and shown to be redward, consistent with the companion paper's identification of sign-fixed corrections that move n_s toward the Planck observed value.

12.2 Relation to Standard Inflation

The local Mukhanov–Sasaki equation derived here is formally identical to the standard inflationary result. The physics, however, is structurally different in the following sense: in standard inflation, the MS equation is imposed as the perturbation equation of an assumed scalar field. Here it is derived as the leading-order limit of an exact nonlocal retarded integro-differential equation whose kernel is determined by the causal structure of commitment propagation. The fact that the MS equation emerges from this distinct starting point is a non-trivial structural result, not a coincidence.

The subleading memory correction δn_s^{mem} is the first term in the VERSF perturbation theory that has no analogue in standard slow-roll inflation. Its computation will provide a genuine VERSF-specific correction to the spectral index that can in principle be distinguished from standard inflationary slow-roll corrections.

12.3 Remaining Gaps

The most significant remaining gap in the VERSF cosmological programme is the tensor perturbation sector. The fold-geometry contribution to tensor modes has not been derived from the full VERSF perturbation theory, and the conditional prediction $r \sim 16\varepsilon_V \approx 0.11$ must be regarded as an upper bound. The current observational limit is $r < 0.036$ (BICEP/Keck Array 2022), which gives a ratio $r_{pred}/r_{limit} \approx 3.1$ — a roughly 3σ tension if the fold-geometry tensor suppression is absent. This is the sharpest current observational challenge to the VERSF cosmological prediction. The fold suppression of tensor modes is not merely an optional refinement but is required by observational consistency; its derivation from the VERSF fold-geometry perturbation theory is an urgent open calculation.

The second remaining gap is the VERSF Friedmann equation, which is needed to determine $N_\star$ precisely and thereby fix the subleading correction (a) of Paper I. The computation of the coefficient C in (10.5) is a third open item, dependent on γ_m from the bath spectral density. All three are open calculations within a fully specified framework and introduce no new free parameters beyond those already present in the VERSF memory sector.

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13. Conclusion

We have constructed the complete scalar perturbation theory of the VERSF κ-field sector and derived three theorems.

Theorem 1 establishes that the VERSF retarded memory sector admits an exact local auxiliary-field representation with parameters fixed from the κ-field mass theorem and CCC architecture. No new free cosmological parameters are introduced.

Theorem 2 establishes that the exact scalar perturbation dynamics of the VERSF κ-sector are governed by a retarded nonlocal Mukhanov–Sasaki-type equation with a structurally determined memory self-kernel Σ_k. This equation is derived from the localised action by auxiliary-field elimination, after proving that lapse/shift constraint elimination in the two-field system introduces no additional cross-sector mixing at leading slow-roll order.

Theorem 3 establishes that in the slow-roll unfolding regime relevant for CMB modes, the nonlocal term in the exact mode equation is suppressed by $O(1/(N \ln N))$, and the local Mukhanov–Sasaki equation emerges as the leading-order adiabatic limit of the exact VERSF perturbation theory. The k-uniformity of this suppression across the CMB window is established from the auxiliary-sector Green's function structure.

The leading-order spectral index $n_s - 1 = -2/N_\star$ of Paper I is therefore not a result imported from standard inflation but a theorem derived from the exact VERSF scalar perturbation theory in its adiabatic slow-roll limit. More precisely: the Mukhanov–Sasaki equation is not fundamental in VERSF. It is the effective low-energy description obtained after integrating out the memory-response sector of a nonlocal, retarded dynamical system — the equation that emerges when $\omega_M \gg H$, as holds throughout the entire unfolding epoch.

The complete spectral-index prediction is:

$$n_s - 1 = -2/N_\star - C/(N_\star \ln N_\star) + O((H/m_\kappa)^2/(N_\star \ln N_\star)),$$

where $C > 0$ is a positive coefficient computable from the auxiliary-sector Green's function without new free parameters, and the sign of the subleading correction is fixed and redward — consistent with the identification in Paper I of sign-fixed corrections that close the 1.7σ gap between the leading-order prediction $n_s \approx 0.972$ and the Planck observed value $n_s = 0.9649$.

The perturbation sector of VERSF is now fully closed at leading order. The framework no longer argues that the memory sector is small enough to ignore at leading order. It derives the exact nonlocal dynamics, proves when and why the familiar local equation emerges, and fixes the sign and order of magnitude of the correction that will sharpen the agreement between prediction and observation once the VERSF Friedmann equation and the bath spectral density are determined.

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References

[1] Taylor, K. (2026). *Primordial Scalar Perturbations from Void Unfolding: A Parameter-Free Derivation of the Spectral Index in the VERSF Framework* [Paper I]. VERSF Programme.

[2] Taylor, K. (2026a). *Derivation of the κ-Field Mass from Minimal Fact Architecture*. VERSF Programme.

[3] Taylor, K. (2026b). *The Memory Kernel from First Principles: A Dynamic Derivation of K(τ) in VERSF*. VERSF Programme.

[4] Taylor, K. (2026c). *From Necessary Facts to Physical Structure*. VERSF Programme.

[5] Taylor, K. (2026d). *A No-Go Theorem for Non-Simplicial Relational Substrates — K = 7*. VERSF Programme.

[6] Taylor, K. (2026e). *Causal–Coherence Compatibility and the Fact-Production Threshold*. VERSF Programme.

[7] Taylor, K. (2026f). *Fact-Momentum: Commitment-Capacity Field Dynamics*. VERSF Programme.

[8] Mukhanov, V. & Chibisov, G.V. (1981). JETP Lett. 33, 532.

[9] Liddle, A.R. & Lyth, D.H. (2000). *Cosmological Inflation and Large-Scale Structure*. Cambridge University Press.

[10] Planck Collaboration (2018). Planck 2018 results. X. Constraints on inflation. A&A 641, A10.

[11] BICEP/Keck Collaboration (2022). BK18 constraint on tensor-to-scalar ratio. Phys. Rev. Lett. 127, 151301.

[12] Caldeira, A.O. & Leggett, A.J. (1983). Quantum tunnelling in a dissipative system. Ann. Phys. 149, 374. [For S_damp bath-coupling formalism.]