

# Orthogonality as Closure Incompatibility

## A VERSF Derivation of Quantum State Distinguishability

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### Plain-language Summary

A central feature of quantum theory is that two possible measurement outcomes can be perfectly distinguishable from one another — a measurement can sort them apart with certainty. The mathematics that captures this perfect distinguishability is called *orthogonality*: in the geometry of quantum states, mutually exclusive outcomes correspond to "perpendicular" directions. Textbooks introduce this as a starting assumption. But why should distinguishability take this geometric form? Why should quantum theory speak in the language of perpendicular vectors at all?

This paper offers an answer drawn from the Void Energy-Regulated Space Framework (VERSF), the broader programme of which it forms a part. In VERSF, physical reality is built up from irreversible *facts* — moments at which one possibility is selected and all alternatives are ruled out. Facts are formed from a deeper layer of *admissible configurations*: the underlying possibilities that could in principle complete into a fact. Two potential outcomes are mutually exclusive in the strongest sense when they share no admissible configuration that could complete into either of them. This relation we call *closure incompatibility*. It lives at the level of physical possibility, before any geometry has been introduced.

The central result is a bridge: under stated conditions, quantum orthogonality turns out to be exactly the mathematical reflection of closure incompatibility. The geometric property follows from the structural one, not the other way around. The Born rule — the rule giving the probabilities of measurement outcomes — then drops out as a downstream consequence rather than as an independent postulate.

The discrete  $K = 7$  cell of the VERSF interface, established elsewhere in the programme as the minimum unit capable of supporting observer-invariant facts, supplies the substrate from which the entire structure arises. The paper does not attempt a complete reconstruction of quantum theory; that work is distributed across several VERSF papers. What it does is identify the physical content of one of quantum mechanics' most basic features and show that the standard formalism, far from being a list of arbitrary postulates, is the unique faithful image of how facts distinguish themselves at the  $K = 7$  interface.

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## Abstract

This paper supplies the missing VERSF bridge between physical distinguishability and Hilbert-space orthogonality, showing that quantum orthogonality is the representation-theoretic image of **closure incompatibility** at the  $K = 7$  interface. Within the **Void Energy-Regulated Space Framework**, in which physical reality consists of irreversible commitments (*facts*) selected from admissible pre-commitment configurations, two outcomes are mutually exclusive precisely when their admissible configuration sets share no closure-completable element. We show that this set-theoretic relation is the structural origin of orthogonality in any faithful representation of admissible outcome classes.

The argument is staged as follows. We identify the structural conditions that admissible outcome classes must satisfy, distinguishing carefully between properties **derived** from VERSF primitives (orthocomplementation, atomicity, boundedness) and **structural conditions** required of any well-formed VERSF closure architecture (orthomodularity, covering law), whose fuller derivations belong to companion work in the programme. Conditional on these conditions plus the standard hypotheses of Solèr-type representation theorems, the admissible lattice is faithfully represented by the lattice of closed subspaces of a Hilbert space; the VERSF interference result fixes the underlying field as  $\mathbb{C}$ . Theorem 1—the central result of this paper—then asserts that closure incompatibility corresponds exactly to orthogonality of subspaces under any such representation. The Born rule follows as a corollary, drawing on the non-contextuality argument established in companion VERSF work rather than being re-derived here.

The discrete  $K = 7$  closure cell of the VERSF interface, established as a structural necessity by the no-go theorem on non-simplicial relational substrates, supplies the combinatorial substrate from which admissible partitions arise. The result completes one segment of the VERSF programme's reconstruction of quantum theory: it identifies the physical content of orthogonality and shows that the standard inner-product geometry is the unique faithful image of admissible closure structure under the structural conditions made explicit in the body.

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## 1. Introduction

Orthogonality is the geometric expression of distinguishability in quantum mechanics. Two states  $|\psi\rangle$  and  $|\phi\rangle$  are orthogonal when

$$\langle\psi|\phi\rangle = 0,$$

and a measurement is then said to be able to discriminate them perfectly. This formulation is operationally successful but conceptually opaque. It does not explain why distinguishability should be encoded geometrically, why mutually exclusive outcomes correspond to vanishing overlap, or what physical structure underlies the separation.

VERSF starts from a different foundation. Physical reality is taken to consist only of observer-invariant facts, formed through irreversible commitment under finite-distinguishability constraints. The relevant question is therefore not "why is the inner product zero?" but rather: *what structural conditions must hold for two potential outcomes to be mutually exclusive facts?*

This paper isolates the answer. Its distinctive contribution is the bridge between **physical distinguishability** (closure incompatibility, a structural relation on the pre-commitment configuration space) and **Hilbert-space orthogonality** (an algebraic relation between subspaces). We do not in this paper attempt a full reconstruction of the Hilbert formalism from scratch; that programme draws on several VERSF results already established and is still in progress in others. What we do attempt, and complete, is the identification of the precise physical

content of orthogonality as the representation-theoretic image of closure incompatibility, and the demonstration that under VERSF-derived structural conditions this image is essentially unique.

We are explicit about what kind of result Theorem 1 is. Once a faithful, orthocomplementation-preserving representation  $\varphi$  is stipulated, both directions follow from the lattice axioms with little further work. The theorem's contribution is therefore interpretive rather than mathematically novel: it identifies the physical content of orthogonality as the representation-theoretic image of closure incompatibility. The mathematical labour is in §7's representation step, not in §5's bridge.

The Born rule arises as a downstream consequence. We mark its derivation as a corollary that draws on the non-contextuality argument established elsewhere in the programme, rather than re-running the full machinery here.

The  $K = 7$  closure cell of the VERSF interface supplies the underlying combinatorial substrate. With that in place, the chain runs: closure incompatibility  $\rightarrow$  admissible-lattice structure  $\rightarrow$  Hilbert representation  $\rightarrow$  orthogonality is closure incompatibility, with Born rule as corollary.

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## 2. Pre-Commitment Structure in VERSF

VERSF distinguishes carefully between two ontological levels:

- **Pre-commitment configurations** — admissible, reversible possibilities prior to fact formation.
- **Facts** — irreversible commitments that constitute physical reality.

Let  $\mathcal{A}$  denote the set of admissible configurations of a system prior to commitment. Each potential outcome corresponds to a subset

$$\mathcal{A}_i \subseteq \mathcal{A},$$

subject to three structural requirements:

1. **Observer-invariant distinguishability.** The partition must not be relative to any privileged frame; an admissible configuration belongs to a definite class regardless of who or what registers the resulting fact.
2. **Closure consistency.** Every configuration in  $\mathcal{A}_i$  satisfies the full set of interface closure constraints, so that completion to a fact is structurally permitted.
3. **Finite resolution.** No infinite refinement is permitted; admissibility is granular at the  $K = 7$  scale established in §10.

A *fact* is formed when one admissible subset  $\mathcal{A}_i$  is selected irreversibly, eliminating all incompatible alternatives from further participation.

The pre-commitment space  $\mathcal{A}$  is not itself a Hilbert space and need not be a manifold. It is a combinatorial structure of admissible interface configurations. The Hilbert formalism arises only as a representation of how its admissible partitions compose.

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### 3. Closure and Compatibility

Let  $\mathcal{C}$  denote the closure structure on  $\mathcal{A}$ : the family of conditions an admissible configuration must satisfy in order to support a complete fact. A valid outcome corresponds to a self-consistent closure pathway through  $\mathcal{C}$ .

**Definition 1 (Closure compatibility).** Two outcome sets  $\mathcal{A}_i$  and  $\mathcal{A}_j$  are *closure-compatible* if there exists at least one configuration

$$a \in \mathcal{A}_i \cap \mathcal{A}_j \subseteq \mathcal{A} \text{ (intersection taken in } \mathcal{A}\text{)}$$

that simultaneously satisfies the closure constraints for both outcomes.

**Definition 2 (Closure incompatibility).** The sets are *closure-incompatible* if no such configuration exists in the admissible space:

$$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset \text{ (intersection taken in } \mathcal{A}\text{)}.$$

It is critical that the intersection is taken inside the admissible space  $\mathcal{A}$  rather than in any ambient set of nominal possibilities. A configuration that fails admissibility cannot serve as a witness to compatibility, because it cannot complete to a fact. Closure incompatibility is therefore strictly stronger than abstract disjointness: it asserts the absence of any *physically realisable* configuration that could simultaneously underwrite both outcomes.

Two further notions will be useful:

**Definition 3 (Refinement).** A partition  $\Pi'$  refines  $\Pi$  if every cell of  $\Pi'$  is contained in some cell of  $\Pi$ .

**Definition 4 (Joint admissibility).** Two partitions  $\Pi$  and  $\Pi'$  are *jointly admissible* if there exists a common refinement  $\Pi''$  all of whose cells are admissible. Otherwise they are *complementary*.

Joint admissibility is the structural counterpart of commuting observables; complementarity, the structural counterpart of incompatible measurements.

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### 4. Structural Conditions on the Admissible Outcome Lattice

Before establishing the orthogonality–closure correspondence, we identify the algebraic structure that admissible outcome classes inherit from VERSF. We are deliberately explicit about which properties follow from VERSF primitives, which are required as structural conditions on any well-formed VERSF closure architecture (with their fuller derivations belonging to companion work), and which are imported as standard mathematical hypotheses needed to invoke a representation theorem.

This explicitness matters. The history of quantum-logic foundations contains several attempts that quietly assert the orthomodular-lattice structure rather than derive it. We do not wish to inherit that vice. The conditions below are not asserted as a quantum-logical postulate; they are itemised, with their VERSF status flagged in each case.

Let  $\mathcal{L}$  denote the collection of all admissible outcome classes  $\mathcal{A}_i \subseteq \mathcal{A}$  obtained by varying the measurement, ordered by inclusion.

### Structural conditions on $\mathcal{L}$ .

**(L1) Bounded.** The full admissible space  $\mathcal{A}$  is the maximum element (the trivial measurement, certain to occur); the empty class  $\emptyset$  is the minimum (the impossible outcome). *Status:* derived from VERSF primitives. The trivial measurement is admissible by construction; the impossible outcome is admissibility-empty by definition.

**(L2) Orthocomplementation.** For each admissible class  $\mathcal{A}_i$ , the closure-incompatible complement

$\mathcal{A}_i^\perp := \{ a \in \mathcal{A} : a \text{ is admissibly closure-incompatible with every element of } \mathcal{A}_i \}$  is itself admissible, realised by the binary measurement  $\{\mathcal{A}_i, \mathcal{A}_i^\perp\}$ . Order-reversal and the double-complement identity  $(\mathcal{A}_i^\perp)^\perp = \mathcal{A}_i$  follow from the symmetry of closure incompatibility:  $a$  is incompatible with every element of  $\mathcal{A}_i$  iff every element of  $\mathcal{A}_i$  is incompatible with  $a$ , hence the orthocomplement operation is involutive on admissibility-saturated sets. *Status:* derived from Definition 2 together with the admissibility of binary partitions, which follows from finite distinguishability.

**(L3) Atomicity.** Finite distinguishability and the  $K = 7$  discreteness of the interface guarantee the existence of minimal admissible classes (atoms): maximally fine-grained outcomes that cannot be properly refined while remaining admissible. *Status:* derived from VERSF primitives, conditional on the no-go theorem on non-simplicial substrates fixing  $K = 7$ .

**(L4) Orthomodularity.** If  $\mathcal{A}_i \subseteq \mathcal{A}_j$  then

$\mathcal{A}_j = \mathcal{A}_i \vee (\mathcal{A}_j \wedge \mathcal{A}_i^\perp)$ , i.e. any admissible outcome containing  $\mathcal{A}_i$  decomposes uniquely into  $\mathcal{A}_i$  and its admissible relative complement. *Status:* structural condition required of any well-formed VERSF closure architecture. Its motivation is that successive admissible measurements should compose without producing inadmissible residues; the fuller theorem-level derivation belongs to the **PAR/CC** paper (Pre-Factual Algebraic Reversibility and Compositional Completeness) and the **IAC** paper (Internal Admissible Closure). We note that this motivation is gestural rather than

self-contained: a fully rigorous argument distinguishing orthomodularity from weaker compositional conditions (modularity) or stronger ones (full distributivity) belongs to the PAR/CC and IAC papers and is not attempted here.

**(L5) Covering law and atomic exchange.** If  $a$  is an atom and  $\mathcal{A}_i \in \mathcal{L}$  with  $a \notin \mathcal{A}_i$ , then  $\mathcal{A}_i \vee a$  covers  $\mathcal{A}_i$  in  $\mathcal{L}$ . *Status:* structural condition required to invoke a Solèr-type representation theorem. The natural VERSF justification is that an atomic outcome is the maximal refinement compatible with  $K = 7$  closure, so any class strictly between  $\mathcal{A}_i$  and  $\mathcal{A}_i \vee a$  would violate closure granularity. We acknowledge that this is the weakest link in §4: the inference from "atoms exist" (L3) to "no admissible class sits strictly between an outcome and its join with a covering atom" requires explicit reconstruction from  $K = 7$  granularity, which the source no-go paper supports structurally but does not derive at theorem level. A complete derivation from VERSF primitives is an open task.

Conditions (L1)–(L3) are derived. Conditions (L4)–(L5) are required structural conditions whose VERSF status is in some respects already established (L4 via PAR/CC and IAC) and in others is the target of further work (L5). Throughout the rest of the paper, statements that depend on (L4) or (L5) are flagged accordingly.

The present paper does not require that (L4)–(L5) hold universally; it requires only that the admissible closure architecture of physically realised systems satisfy them in the regimes where a faithful representation exists. This is strictly weaker than universal validity and is what the representation step of §7 actually needs.

$\mathcal{L}$  is, under these conditions, an orthocomplemented, atomistic, orthomodular lattice satisfying the covering law. It is *not* distributive: complementary measurement contexts produce non-distributive joins, exactly as in standard QM.

This lattice is the precise object that any faithful representation of VERSF measurement structure must preserve.

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## 5. Orthogonality as Closure Incompatibility

We now state and prove the central theorem in its sharpened form.

**Theorem 1 (Orthogonality–closure correspondence).** *Let  $\varphi : \mathcal{L} \rightarrow \mathcal{P}(\mathcal{H})$  be any faithful representation of the admissible lattice  $\mathcal{L}$  as a sublattice of the lattice of closed subspaces of a complex inner-product space  $\mathcal{H}$  that preserves orthocomplementation. Then for any two admissible outcome classes  $\mathcal{A}_i, \mathcal{A}_j \in \mathcal{L}$ :*

*$\mathcal{A}_i$  and  $\mathcal{A}_j$  are closure-incompatible  $\Leftrightarrow \varphi(\mathcal{A}_i) \perp \varphi(\mathcal{A}_j)$ .*

Throughout, "faithful" means order-preserving, order-reflecting, and orthocomplementation-preserving. The order-reflecting requirement is what licenses the inference from  $\varphi(\mathcal{A}_i) \subseteq \varphi(\mathcal{A}_i)^\perp$  to  $\mathcal{A}_i \subseteq \mathcal{A}_i^\perp$  in the ( $\Leftarrow$ ) direction.

**Proof.**

( $\Rightarrow$ ) Suppose  $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$  in  $\mathcal{A}$ . Then  $\mathcal{A}_i \wedge \mathcal{A}_j = \emptyset$  in  $\mathcal{L}$ , the lattice minimum. Faithful order-preserving representations send the minimum to the zero subspace, so  $\varphi(\mathcal{A}_i) \cap \varphi(\mathcal{A}_j) = \{0\}$ . Closure incompatibility further implies  $\mathcal{A}_j \subseteq \mathcal{A}_i^\perp$  (every element of  $\mathcal{A}_j$  is admissibly incompatible with every element of  $\mathcal{A}_i$ ). Preservation of orthocomplementation gives  $\varphi(\mathcal{A}_i^\perp) = \varphi(\mathcal{A}_i)^\perp$ , hence  $\varphi(\mathcal{A}_j) \subseteq \varphi(\mathcal{A}_i)^\perp$ , i.e.  $\varphi(\mathcal{A}_i) \perp \varphi(\mathcal{A}_j)$ .

( $\Leftarrow$ ) Suppose  $\varphi(\mathcal{A}_i) \perp \varphi(\mathcal{A}_j)$ . Then  $\varphi(\mathcal{A}_j) \subseteq \varphi(\mathcal{A}_i)^\perp = \varphi(\mathcal{A}_i^\perp)$ . Faithfulness of  $\varphi$  then yields  $\mathcal{A}_j \subseteq \mathcal{A}_i^\perp$ . By definition of the orthocomplement in  $\mathcal{L}$ ,  $\mathcal{A}_j$  contains no element that is closure-compatible with any element of  $\mathcal{A}_i$ . Hence  $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ . ■

This is the central result of the paper. It is non-circular: the inner product is not used in the proof, only the lattice operations of intersection, complement, and join. The theorem therefore holds for any representation  $\varphi$  that exists and is faithful in the stated sense. Whether such a representation exists, and whether it is essentially unique, is the separate question addressed in §7.

**Remark (Robustness of Theorem 1).** The proof of Theorem 1 does not depend on conditions (L4)–(L5). It uses only (L1) (boundedness, supplying the lattice minimum) and (L2) (orthocomplementation, used in both the orthocomplement-preservation step and the definition of  $\mathcal{A}_i^\perp$ ), together with the stipulated faithfulness and orthocomplementation-preservation of  $\varphi$ . Conditions (L4)–(L5) enter only in the representation step of §7, where they are needed to invoke Solèr's theorem and establish that such a faithful representation exists. If (L4) or (L5) were challenged, modified, or replaced by alternative compositional axioms, the orthogonality–closure correspondence would still hold for whatever faithful, orthocomplement-preserving representation the modified architecture admitted. The bridge result is therefore robust to revisions in the representation-theoretic machinery downstream of §4 — these conditions enter only in the representation step, not in the identification of orthogonality with closure incompatibility.

The corresponding statement at the level of unit vectors is the familiar one: if  $|\psi_i\rangle \in \varphi(\mathcal{A}_i)$  and  $|\psi_j\rangle \in \varphi(\mathcal{A}_j)$  are unit vectors representing atomic outcomes, then

$$\langle \psi_i | \psi_j \rangle = 0 \Leftrightarrow \mathcal{A}_i \cap \mathcal{A}_j = \emptyset.$$

Quantum orthogonality is therefore the representation-theoretic image of closure incompatibility. It is not a geometric primitive but the formal shadow of an underlying structural impossibility at the level of admissible configurations.

## 6. Measurement, Basis Dependence, and Complementarity

Measurement, in this framework, is not a passive readout but an irreversible selection process. The system transitions from the full admissible set to one of its closure-compatible cells:

$$\mathcal{A} \rightarrow \mathcal{A}_i,$$

with all incompatible cells eliminated from further participation in fact formation. Closure incompatibility between cells of the partition guarantees two operational properties of this selection:

- **No ambiguity in fact formation.** The realised outcome belongs to exactly one cell  $\mathcal{A}_i$ ; no admissible configuration witnesses two cells at once.
- **No residual overlap.** There is no admissible configuration shared between the realised outcome and any unrealised alternative; nothing of the unrealised cells survives in the post-fact state.

This dispenses with "wavefunction collapse" as a primitive dynamical process. Collapse is replaced by the elimination of closure-incompatible alternatives once an irreversible commitment has occurred. The post-fact state is, by construction, supported on the realised cell alone. The *timing* of fact formation is not assumed: it is set by the conditions under which an admissible cell becomes the unique closure-completable channel, a topic treated in the companion proto-time and commitment-capacity papers.

A measurement is the choice of an admissible partition  $\Pi$ . Different choices of  $\Pi$  are different *closure bases* — different partitionings of  $\mathcal{A}$  into admissible alternatives. Two consequences follow.

First, orthogonality is **basis-relative**: it expresses incompatibility *with respect to* a chosen closure partition. Two outcome classes orthogonal under one partition need not be orthogonal under another, because the admissible refinements available may differ. Second, what in standard QM is described as a measurement disturbing the system is, structurally, the redefinition of the closure-compatibility relation when the partition changes; the pre-commitment configuration is not perturbed by an external act, but a different family of admissible cells is brought to bear on it.

Complementarity in the Bohrian sense is, on this account, the failure of joint admissibility (Definition 4): two partitions are complementary precisely when no admissible refinement contains both. This is a structural statement about  $\mathcal{L}$ , not a metaphysical claim about observers.

## 7. Hilbert Representation Under VERSF Structural Conditions

Theorem 1 is conditional on the existence of a faithful, orthocomplementation-preserving representation  $\varphi : \mathcal{L} \rightarrow \mathcal{P}(\mathcal{H})$ . We now indicate the route by which such a representation is

obtained, marking clearly which steps are theorems of standard mathematics and which are structural conditions whose detailed VERSF derivation belongs to companion work in the programme.

## 7.1 The shape of the result

Under the structural conditions identified in §4, supplemented by the standard hypotheses of Solèr-type representation theorems,  $\mathcal{L}$  admits a faithful representation as the lattice of closed subspaces of an inner-product space  $\mathcal{H}$  over one of the three associative real division algebras:  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . The VERSF interference result selects  $\mathbb{C}$  uniquely. The combination yields a representation that is essentially unique up to unitary equivalence.

The result is therefore stated cleanly as a *conditional*:

**Representation claim.** *Under the structural conditions (L1)–(L5) of §4, plus the standard Solèr hypotheses, plus the field selection established by the VERSF interference paper, the admissible outcome lattice  $\mathcal{L}$  is faithfully represented (uniquely up to unitary equivalence) by the lattice of closed subspaces of a separable Hilbert space  $\mathcal{H}$  over  $\mathbb{C}$ .*

We do not in this paper attempt a fully theorem-level derivation of all the Solèr hypotheses from VERSF first principles. Some are established (orthocomplementation, atomicity, boundedness via §4); some are made plausible structurally (orthomodularity via PAR/CC, the covering law via  $K = 7$  granularity); and some — notably the existence of an infinite orthonormal-style sequence of atoms in the relevant sectors — are imported from the standard mathematical literature as conditions that any well-formed admissible outcome lattice on a sufficiently rich VERSF system must satisfy.

## 7.2 Required hypotheses

The standard Solèr machinery requires, in addition to (L1)–(L5):

- **Irreducibility** — no decomposition of  $\mathcal{L}$  into independent superselection sectors. Reducible cases reproduce the standard direct-sum decomposition over superselection rules and do not affect the present argument.
- **An infinite orthonormal sequence of atoms** in the relevant sectors, used by Solèr's theorem to fix the underlying field as one of  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . We note that this is non-trivial: VERSF must deliver an actually infinite atomic sequence under successive admissible measurements on extended interface regions, not merely an arbitrarily large finite one. The structural argument is that  $K = 7$  patches admit unbounded refinement under successive admissible measurements; a fully rigorous form of this argument, possibly via a direct-limit construction, is an open task.

Both are standard in the quantum-logic literature. In VERSF the irreducibility condition corresponds to a single physical system not factoring into independent admissibility sectors.

## 7.3 Why the field is $\mathbb{C}$

Solèr's theorem leaves the underlying field as  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . The companion VERSF paper *Why a Fact-Producing Universe Must Satisfy Interference* selects  $\mathbb{C}$  uniquely; the  $\mathbb{R}$  exclusion is VERSF-internal (the interference paper's main argument), while the  $\mathbb{H}$  exclusion is a standard mathematical result, not a VERSF-internal derivation.

- Real coefficients fail to support reversible pre-commitment composition (no continuous one-parameter group of admissible transformations connecting orthogonal atoms via interference).
- Quaternionic coefficients over-determine the joint structure of independent subsystems (the natural tensor-product composition fails for non-abelian scalars).
- Complex coefficients are the unique choice compatible with both reversible pre-commitment dynamics and admissible composition of independent subsystems.

The combination of Solèr's theorem with the interference result therefore yields the representation as essentially unique under the structural conditions identified above.

## 7.4 What this paper does and does not claim

The present paper does *not* claim a complete, theorem-level derivation of the Hilbert formalism from VERSF first principles; that programme is distributed across several VERSF papers and is in some respects still in progress. What this paper does claim, and establish under explicit structural conditions, is the **bridge** result: that under any such representation, orthogonality is exactly the image of closure incompatibility (Theorem 1 in §5). The bridge is the distinctive contribution of this paper. The Hilbert representation is the framework within which the bridge is stated, not the result for which it is offered.

The existence of such representations is not asserted as a primitive but as the content of the representation claim itself; the present result concerns what follows once such a representation is admitted. There is consequently no circle: §5 establishes a conditional (if  $\varphi$  exists with the stated properties, then orthogonality coincides with closure incompatibility), and §7 establishes the antecedent under explicitly identified structural conditions.

More generally, the present paper isolates the representation-theoretic bridge independently of the full reconstruction. Even if the Hilbert representation were modified — for instance by replacing (L4) or (L5) with alternative compositional conditions, by passing to a coarser representation in a non-Solèr framework, or by sharpening the representation in a direction not yet anticipated — the identification of orthogonality with closure incompatibility would remain intact, so long as the representation preserved the orthocomplementation structure VERSF supplies in (L2). Theorem 1 is therefore a structural fact about VERSF, not about the specific Hilbert-space target. The paper's central claim survives any plausible revision of the representation step.

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## 8. Born Rule as Corollary

Once the Hilbert representation is in place, the Born rule follows by the non-contextuality argument developed in the companion paper *Born Rule as Entropic Unfolding* and the associated Gleason-type machinery. We do not re-derive that result here, since it is the principal subject of those papers. The route is sketched for completeness.

A probability assignment over admissible projectors is a function  $p : \mathcal{P}(\mathcal{H}) \rightarrow [0, 1]$  such that, for every complete admissible partition,

$$\sum_i p(P_i) = 1 \text{ whenever } \sum_i P_i = \mathbb{1}.$$

The crucial condition is **non-contextuality**:  $p(P)$  depends only on the projector  $P$ , not on the complete partition in which  $P$  appears. In Gleason's original treatment this condition is imposed as an axiom and has often been viewed as philosophically loaded. In VERSF it is a consequence of observer-invariant fact formation. A possible fact corresponds to a definite admissible cell  $\mathcal{A}_i$ ; its admissibility, and the admissible measure assigned to it, depend only on  $\mathcal{A}_i$  and on the closure structure  $\mathcal{C}$ , not on which other admissible cells happen to be grouped with it in a particular complete measurement. Two complete partitions sharing the projector  $P_i$  assign  $\mathcal{A}_i$  the same admissibility weight, on pain of violating observer invariance.

We acknowledge that the gap between observer-invariance (the fact  $\mathcal{A}_i$  is observer-independent) and full Gleason non-contextuality (the probability  $p(P_i)$  is partition-independent) is wider than the prose admits. Closing it strictly requires an additional locality-of-weight assumption — that the admissibility weight of  $\mathcal{A}_i$  does not depend on which other admissible cells appear alongside it in the partition — that does not follow from observer-invariance alone. The companion paper *Probability as Consistency* supplies a more rigorous derivation of non-contextuality from the single-source theorem  $\mathcal{O} = \mathcal{F}[\rho]$ ; the present sketch should be read as motivational, with the *Consistency* paper as the load-bearing argument.

Under this VERSF-grounded non-contextuality condition, Gleason's theorem (for  $\dim \mathcal{H} \geq 3$ ) and its Busch–Caves–Fuchs–Schack POVM extension (closing the  $d = 2$  case) yield the unique probability assignment

$$p(P) = \text{Tr}(\rho P),$$

with the familiar specialisation  $p_i = |\langle e_i | \psi \rangle|^2$  for pure states and one-dimensional outcome projectors. This is the **Born rule**.

The detail of these steps — the precise statement of non-contextuality from observer-invariant fact formation, the discrete entropic-unfolding derivation, the POVM extension — is the subject of the companion VERSF papers. For present purposes the corollary is that, given the bridge result of §5 and the representation framework of §7, the Born rule does not require any additional VERSF input beyond what is already in place in the programme.

## 9. Probability and Admissible Measure

The Born rule, on the VERSF reading, has a direct interpretation as an admissible measure ratio. The squared modulus

$$P_i = |\langle \psi | \psi_i \rangle|^2$$

is the ratio of the admissible measure of the cell compatible with both the pre-commitment state  $|\psi\rangle$  and the outcome  $i$  to the admissible measure of the cell associated with  $|\psi\rangle$  alone:

$$P_i = \mu(\mathcal{A}_i \cap \mathcal{A}_\psi) / \mu(\mathcal{A}_\psi),$$

where  $\mu$  is the admissibility measure on  $\mathcal{A}$  induced by the closure constraints (developed in the **commitment-capacity density** paper). Closure incompatibility ensures non-overlapping partitions and additivity of these ratios.

Two structural features of the Born rule are explained on this reading without further postulate:

- The exponent **2** in  $|\langle \psi | \psi_i \rangle|^2$  traces back to the requirement that the admissible measure on  $\mathcal{A}$  transform sesquilinearly under unitary basis changes — the only transformation law compatible simultaneously with linearity and observer invariance over  $\mathbb{C}$ . (This is an interpretive gloss on the squared modulus, not an independent derivation of it. The full derivation belongs to *Born Rule as Entropic Unfolding* and is referenced as a corollary in §8.)
- The **non-negativity** and **normalisation** of probabilities trace back to the fact that admissible measures on  $\mathcal{A}$  are non-negative and that  $\sum_i \mu(\mathcal{A}_i) = \mu(\mathcal{A})$  for any complete admissible partition.

The Hilbert structure is the linear, observer-invariant representation of admissible partition geometry; the squared modulus is what that representation makes of the underlying admissible measure when projected onto a chosen basis.

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## 10. The $K = 7$ Closure Substrate

The VERSF  $K = 7$  closure structure provides the discrete substrate underwriting the structural conditions of §4 and the representation framework of §7.

At the interface level, each elementary admissible fact is constrained by **seven closure conditions**,

$$K = 7,$$

corresponding to six boundary closure constraints (one per hexagonal edge of the interface cell) plus one global consistency constraint that binds them — the decomposition established in the companion no-go paper, where the global constraint is shown to be independent of the six boundary ones rather than derivable from them. These seven constraints define the minimum complete closure cell capable of supporting observer-invariant fact formation. The companion no-go theorem on non-simplicial relational substrates establishes  $K = 7$  and triangular 2-complex geometry as structural necessities, not stipulations: any reduction below  $K = 7$  fails to support observer-invariant fact closure, and any non-simplicial substrate fails to support the irreversibility of commitment.

The  $K = 7$  cell supplies the primitive closure architecture from which admissible partitions arise. Each admissible outcome corresponds to a complete  $K = 7$ -consistent closure pathway. Two outcomes are orthogonal precisely when no single  $K = 7$  closure completion can realise both:

$\langle e_i | e_j \rangle = 0 \Leftrightarrow$  no shared  $K = 7$  closure completion exists.

The Hilbert basis is the linear representation of mutually exclusive  $K = 7$ -closure channels. Atomicity of  $\mathcal{L}$  (condition (L3)) is realised concretely by the discreteness of the  $K = 7$  cell: atomic outcomes correspond to maximally fine-grained  $K = 7$ -closure channels, which cannot be further refined without violating closure. The covering law (condition (L5)) reflects the same granularity, since no class strictly between an atom and its join with another atom can support a complete  $K = 7$  closure.

This connects the abstract lattice-theoretic argument of §§4–7 to the concrete combinatorial substrate established elsewhere in the VERSF programme. The bridge result of §5 then has a directly physical reading: orthogonal outcomes are outcomes whose  $K = 7$  closure completions cannot share an admissible configuration.

## 11. Physical Interpretation

Quantum orthogonality, on this account, is therefore:

- **not** a geometric primitive,
- **not** an abstract Hilbert-space axiom,
- **not** a feature imposed for predictive convenience.

It is, instead:

*the representation-theoretic image of closure incompatibility at the  $K = 7$  interface.*

Distinguishability is grounded in three concrete VERSF principles — finite distinguishability, closure consistency, and irreversible commitment — and the geometric apparatus of quantum mechanics emerges as the unique faithful representation of these principles, conditional on the structural conditions made explicit in §4 and §7.

Within this picture: superposition is unresolved cell-membership; unitary evolution is admissible relabelling of partitions; measurement is the irreversible selection of one closure-compatible cell; orthogonality is closure incompatibility; and the inner product is the representation of admissible overlap. None of these elements is independent. Each is traced, by an explicit chain of reasoning, back to the closure architecture of the  $K = 7$  interface cell.

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## 12. Implications and Connections

The result has several consequences for foundational quantum theory and for the VERSF programme.

For foundational quantum theory, orthogonality is removed as a primitive assumption and replaced by an explicit structural relation on admissible configurations. Measurement is grounded in physical structure rather than dynamical postulate. The connection to thermodynamic irreversibility is direct: the asymmetry of measurement is the asymmetry of fact formation.

For the VERSF programme, this paper supplies the bridge between physical distinguishability and Hilbert-space orthogonality — the hinge on which several adjacent VERSF results turn. In particular:

- The **interference paper** (*Why a Fact-Producing Universe Must Satisfy Interference*) establishes the necessity of the complex field for the coefficients in the Hilbert representation (used in §7.3).
- The **commitment-capacity density paper** supplies the underlying admissibility measure  $\mu$  used in §9.
- The **no-go theorem on non-simplicial substrates** justifies the  $K = 7$  closure cell of §10 and underwrites condition (L3) of §4.
- The **Born Rule as Entropic Unfolding** paper supplies the corollary derivation referenced in §8.
- The **PAR/CC** paper (Pre-Factual Algebraic Reversibility and Compositional Completeness) and the **IAC** paper (Internal Admissible Closure) underwrite the orthomodularity and compositional conditions in §4.
- The **proto-time** paper supplies the timing structure under which fact formation occurs, complementing the structural account given here.

Several lines of further work follow naturally. A fully theorem-level derivation of conditions (L4) and (L5) from VERSF primitives, sharpening the conditional statements of §4 and §7. A treatment of continuous spectra, extending the lattice-theoretic argument from atomistic  $\mathcal{L}$  to direct integrals of admissible cells. The connection to algebraic QFT through the von Neumann algebras generated by admissible projectors associated with localised  $K = 7$  patches. The relation to relational quantum mechanics, which shares some of the structural moves but differs in grounding observer invariance in fact formation rather than relational reference. Each is a target for subsequent VERSF papers in the programme.

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## 13. Conclusion

We have identified the physical content of orthogonality in quantum mechanics. It is the representation-theoretic image of closure incompatibility at the  $K = 7$  interface: two outcomes are orthogonal in any faithful Hilbert representation of the admissible outcome lattice  $\mathcal{L}$  if and only if their admissible configuration sets share no closure-completable element.

The argument has been staged carefully. We isolated the structural conditions on  $\mathcal{L}$  — distinguishing those derived from VERSF primitives (boundedness, orthocomplementation, atomicity) from those required as structural conditions on any well-formed VERSF closure architecture (orthomodularity, the covering law) — and made the dependence of the Hilbert representation on those conditions and on the standard Solèr hypotheses fully explicit. The bridge result (Theorem 1) was proved without circularity, using only lattice-theoretic relations on admissible configurations. The Born rule was placed as a corollary that draws on the non-contextuality argument established in companion VERSF work, rather than re-derived here.

The  $K = 7$  closure cell of the VERSF interface, established as a structural necessity by the no-go theorem on non-simplicial relational substrates, supplies the discrete substrate from which admissible partitions arise and atomic outcomes are realised concretely.

This completes the bridge between physical distinguishability and Hilbert-space orthogonality and provides a physically grounded interpretation of quantum states. The standard quantum formalism is no longer a list of postulates: it is the unique faithful image of admissible closure structure at the  $K = 7$  interface, conditional on structural conditions whose VERSF status has been made explicit throughout.

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### Notation summary

Symbol	Meaning
$\mathcal{A}$	total admissible (pre-commitment) configuration space
$\mathcal{A}_i$	admissible configurations supporting outcome $i$
$\mathcal{A}_i^\perp$	closure-incompatible complement of $\mathcal{A}_i$
$\mathcal{C}$	closure structure on $\mathcal{A}$
$\Pi = \{ \mathcal{A}_i \}$	admissible partition (a measurement)
$\mathcal{L}$	lattice of admissible outcome classes ordered by inclusion
$\mathcal{H}$	complex Hilbert space representing $\mathcal{L}$
$\mathcal{P}(\mathcal{H})$	lattice of orthogonal projectors / closed subspaces of $\mathcal{H}$
$\varphi : \mathcal{L} \rightarrow \mathcal{P}(\mathcal{H})$	faithful VERSF representation map
$P_i$	admissible projector onto outcome $i$

Symbol	Meaning
$ e_i\rangle$	orthonormal representative of admissible class $\mathcal{A}_i$
$ \psi\rangle = \sum_i c_i  e_i\rangle$	unresolved pre-commitment state
$\rho$	density operator
$K = 7$	minimum complete closure cell at the VERSF interface
$\mu$	admissibility measure on $\mathcal{A}$

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## Key external references

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## Companion VERSF papers

- *Why a Fact-Producing Universe Must Satisfy Interference* (field selection in §7.3).
- *No-Go Theorem for Non-Simplicial Relational Substrates* ( $K = 7$  substrate in §10).
- *Commitment-Capacity Density* (admissibility measure  $\mu$  in §9).
- *Born Rule as Entropic Unfolding* (Born rule corollary in §8).
- *Probability as Consistency* (load-bearing non-contextuality argument referenced in §8).
- *Pre-Factual Algebraic Reversibility and Compositional Completeness* (orthomodularity in §4).
- *Internal Admissible Closure* (compositional conditions in §4).