The Closure Trichotomy Theorem

Deriving the Three-Role Decomposition of Admissible Closure Operations from Primitive VERSF Assumptions

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General Reader Abstract

The companion paper [VERSF–CLOS-ALG] proved that physics has three structural layers — matter (spin-½), gauge fields (spin-1), and gravity (spin-2) — corresponding to the three graded layers of a single closure algebra $\mathfrak{c} = \mathfrak{c}^{(0)} \oplus \mathfrak{c}^{(1)} \oplus \mathfrak{c}^{(2)}$. That theorem rested on five stated assumptions, the most consequential of which was the *trichotomy*: the claim that every admissible operation on closure states must do one of three things — identify a state, compare neighbouring states, or register accumulated response. The companion paper acknowledged that the trichotomy was a structural commitment rather than a derived theorem, and identified its primitive derivation as the leading open problem of the closure-algebra programme.

The present paper closes that gap. We prove that the three-role decomposition is itself a consequence of more primitive VERSF assumptions — finite distinguishability, irreversible commitment, law closure, closure–operational equivalence, compositional completeness, and pre-factual algebraic reversibility. The proof is a structural case analysis on how many distinguishability fibres an admissible operation can involve and at what differential order: the cases collapse, exhaustively, into the three roles plus their composites and derivatives. Apparent fourth-role candidates (nonlocal primitives, higher-order operations, multi-fibre couplings) are each shown to be either composite or inadmissible by the record-theoretic constraints of [VERSF–FSN] and [VERSF–LAW-REP].

This converts the closure-algebra picture from a theorem-under-five-assumptions-plus-load-bearing-step into a theorem-under-four-assumptions. Two structural commitments of [VERSF–CLOS-ALG] are discharged by the present paper: the trichotomy (Step 4 of that paper's proof, owned in §7.3(i) as a structural commitment) becomes a derived theorem here, and the locality assumption (A4) of that paper becomes a derived consequence of the record-theoretic constraints. The unconditional No-Alternative Theorem still requires resolving the continuum-limit question (§7.3(ii)) and the minimal-representation postulate at layer 0 (§7.3(iii)), but the structurally hardest commitment of the closure-algebra picture is now a theorem.

Abstract

We prove the **Closure Operation Trichotomy Theorem**: under finite distinguishability, irreversible commitment, law closure, closure–operational equivalence, compositional completeness, and pre-factual algebraic reversibility, every admissible operation on the closure-defined record algebra $\mathcal{A}$_stable belongs to exactly one of three irreducible classes — *state identity* ($\mathcal{C}^0$), *local comparison* ($\mathcal{C}^1$), or *accumulated response* ($\mathcal{C}^2$). Higher-order primitive operations are excluded: any candidate operation either reduces to a composite of the three classes via compositional completeness, fails admissibility by not corresponding to a stable observer-comparable record structure, or violates locality and is excluded by the record-theoretic constraints on observer comparability. The proof proceeds by an exhaustive structural case analysis on (a) the number of distinguishability fibres an operation involves and (b) the differential order at which it acts, with the cohomological closure results imported from [VERSF–GAUGE] §7 and Anderson 1989 handling the higher-order reduction. The result establishes the closure algebra grading

$$\mathfrak{c} = \mathfrak{c}^{(0)} \bigoplus \mathfrak{c}^{(1)} \bigoplus \mathfrak{c}^{(2)}$$

as structurally forced rather than postulated, and resolves the leading residual commitment of [VERSF–CLOS-ALG] §7.3(i) (the trichotomy) while additionally deriving the locality assumption (A4) of [VERSF–CLOS-ALG] §7.2 from record-theoretic constraints (Lemma 5.7, with horn (b) empty by (P4)). The No-Alternative Theorem of [VERSF–CLOS-ALG] now stands on four labelled structural assumptions plus the foundational VERSF primitives, with no remaining structural commitment in its proof beyond what those primitives plus the substrate-metric input of (A3) supply. The three-pillar spin hierarchy {½, 1, 2} of fundamental physics is, accordingly, two structural commitments closer to being derived rather than assumed.

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1. Introduction

1.1 What this paper does and why

The companion paper [VERSF–CLOS-ALG] proved, under five stated structural assumptions (A1–A5 in that paper), a *No-Alternative Theorem*: the closure algebra $\mathfrak{c}$ over the distinguishability substrate $\mathcal{S}$ has exactly three graded layers, corresponding to the {½, 1, 2} spin hierarchy of fundamental physics, and no independent fourth grade exists. The proof of that theorem turned, at Step 4, on the **trichotomy of admissible roles**: the claim that every admissible operation on the closure bundle $\mathcal{E} \to S$ must do one of three things — identify a closure state (role i), compare states at neighbouring substrate points (role ii), or register accumulated response or obstruction (role iii).

[VERSF–CLOS-ALG] §7.3 owned the trichotomy as the principal *structural commitment* of that theorem, distinct from the derived content. It was justified by an enumeration of three candidate operations — finite-path comparisons, bundle automorphisms, and multi-state couplings — each shown to fall within one of the three roles or reduce to composites of them. But the enumeration was not exhaustive in any formal sense: it pressure-tested the trichotomy against three plausible counterexamples without deriving the trichotomy from a more primitive characterization.

The present paper supplies that derivation. We work from six primitive VERSF assumptions — finite distinguishability, irreversible commitment, law closure, closure–operational equivalence, compositional completeness, and pre-factual algebraic reversibility — and prove that the three-role decomposition is a forced consequence of these primitives, not an additional assumption layered on top of them.

The contribution is structurally specific: it converts the No-Alternative Theorem of [VERSF–CLOS-ALG] from a five-assumption theorem (plus a load-bearing structural step) to a four-assumption theorem, by discharging *two* commitments of that paper: the trichotomy (Step 4 of CLOS-ALG's proof, owned as residual commitment §7.3(i)) becomes a derived theorem (Theorem 6.1 here), and the locality assumption (A4) of CLOS-ALG §7.2 becomes a derived consequence of the record-theoretic constraints (Lemma 5.7 here, which establishes that closure-invisible content is operationally invisible and that all admissible nonlocal observables are composites of grade-1 and grade-2 generators). The present paper does not address the other two residual commitments of [VERSF–CLOS-ALG] §7.3 — the continuum-limit gap (§7.3(ii)) and the minimal-representation postulate at layer 0 (§7.3(iii)) — which remain open.

A note on assumption accounting. A skeptical reading would object that we have replaced two structural commitments (the trichotomy and the locality assumption (A4)) with six primitives (P1)–(P6), so the total cost has gone up. We disagree: (P1)–(P6) are foundational VERSF primitives independently load-bearing across [VERSF–CHS], [VERSF–FSN], [VERSF–LAW-REP], and [VERSF–PAR-CC]. They were already in force when [CLOS-ALG] §7.2 was proved. What changes here is not the foundational assumption count; it is the assumption count *specific to the closure-algebra picture*. Two commitments — the trichotomy and (A4) — move from spin/gauge/gravity-specific to consequences of the framework's foundations: the trichotomy via Theorem 6.1 (a substantive structural result) and (A4) via Lemma 5.7 (which collapses to "every admissible nonlocal operation is a (P5)-composite of local primitives" once horn (b) is shown empty by (P4)). The reduction in the trichotomy is the structurally heavier of the two, since it was the load-bearing step rather than a labelled assumption; (A4)'s discharge is via (P4), which is foundational and uncontroversial. Both are genuine reductions in commitment-specific cost, modulo the validity of the foundational primitives.

1.2 The shape of the argument

The proof rests on a two-axis classification of admissible operations:

- **Fibre cardinality**: how many distinguishability fibres does the operation involve? One, two-neighbouring, or two-or-more-non-neighbouring.
- **Differential order**: at what order in the substrate displacement does the operation act non-trivially? Zero (point-local), one (first-order differential), two (curvature/obstruction), or three-or-more.

The case analysis on these two axes is genuinely exhaustive — every admissible operation is characterizable by some pair (fibre cardinality, differential order) — and we show that every cell of the table either falls into one of the three roles, reduces to composites of them, or is excluded by record-theoretic admissibility. The cohomological closure results imported from [VERSF–GAUGE] §7 (Henneaux–Teitelboim 1992) and Anderson 1989 handle the differential-order ≥ 3 cases; the record-theoretic constraints from [VERSF–FSN] and [VERSF–LAW-REP] handle the non-neighbouring multi-fibre cases.

The result is a closed argument with exhausted case structure, replacing the previous enumeration-of-candidates with structural completeness.

1.3 Notation and structure

We use the notation of [VERSF–CLOS-ALG]: $\mathcal{S}$ is the distinguishability substrate, $\boldsymbol{\mathcal{E}} \to \mathcal{S}$ the closure bundle, $\mathscr{H}$ the complex Hilbert space at a substrate point, $\mathbb{P}(\mathscr{H})$ the corresponding ray space, $\mathcal{A}$_stable the algebra of stable records derived in [VERSF–LAW-REP]. The closure algebra $\mathfrak{c} = \mathfrak{c}^{(0)} \oplus \mathfrak{c}^{(1)} \oplus \mathfrak{c}^{(2)}$ has the form-degree structure of [VERSF–CLOS-ALG] Definition 3.2.

§2 states the primitive assumptions. §3 defines admissible operations and the record algebra they act on. §4 develops the fibre-cardinality / differential-order classification. §5 proves the case-by-case reduction. §6 states the Closure Trichotomy Theorem and the corollary No-Alternative result. §7 makes explicit how this discharges [VERSF–CLOS-ALG] §7.3(i) and the locality assumption (A4) of [VERSF–CLOS-ALG] §7.2. §8 fixes scope and residual commitments. §9 concludes.

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2. Primitive Assumptions

We state the six primitive VERSF assumptions used in the proof. Each is established in a prior VERSF paper; we restate them here with citations and brief operational summaries, to make the present argument readable without round-tripping through the foundations.

Labelling convention. We label the present paper's primitive assumptions (P1)–(P6) — using the letter **P** for "primitive" — to keep them lexically distinct from the (A1)–(A5) assumptions of [VERSF–CLOS-ALG] §7.2, which appear in cross-references throughout the paper. Theorem 7.1 below invokes both sets in a single statement, and the lexical distinction prevents confusion. Wherever we write "(A1)–(A5) of [VERSF–CLOS-ALG]" or "(A4) of [VERSF–CLOS-ALG]," we mean that paper's labelling; (P1)–(P6) without further qualification refers to this paper's primitives.

P1. Finite Distinguishability (FD) — [VERSF–CHS]

Only finitely resolvable distinctions can be physically realised. Continuous parameters appear only as limits of finite-distinguishability structures; no observable corresponds to a distinction with infinite informational content.

P2. Irreversible Commitment (IC) — [VERSF–FSN]

A *fact* is the production of an irreversible distinguishability increment — a closure event that cannot, by closure dynamics alone, be undone. Uncommitted distinctions cannot serve as stable variables in macroscopic laws. The fact-production threshold is governed by the CCC quartic inequality of [VERSF–CCC].

P3. Law Closure (LC) — [VERSF–LAW-REP]

All admissible macroscopic laws are statements over $\mathcal{A}$_stable, the algebra of stable, committed, observer-comparable distinctions. Equivalently: no physical law can refer to variables outside $\mathcal{A}$_stable, and any operation that fails to act on $\mathcal{A}$_stable cannot appear in a law statement. This is the representation theorem for admissible observables.

P4. Closure–Operational Equivalence (COE) — [VERSF–FSN]

The closure equivalence classes on $\mathcal{S}$ coincide exactly with the operational distinguishability classes. There is no physical structure that distinguishes states beyond what closure can register, and no closure-equivalent states that are operationally distinguishable. This is a faithful-representation statement: closure structure exhausts the distinguishability content of the substrate.

P5. Compositional Completeness (CC) — [VERSF–PAR-CC]

Every admissible element of the structure participates in the compositional closure of the domain: any well-defined composite of admissible operations is itself admissible, and any admissible operation is expressible (modulo gauge) as a composition of operations on the basic primitives. Composites do not introduce new primitive content.

P6. Pre-Factual Algebraic Reversibility (PAR) — [VERSF–PAR-CC]

In the pre-factual domain (before fact-production), no irreducible one-way transitions exist: all operations on uncommitted closure structure are algebraically reversible, with irreversibility entering only through the fact-production threshold of (P2). This rules out primitive operations that are irreducibly directional in the pre-factual algebra.

P5 reframed: composability as a consistency requirement, not a structural choice

A reasonable referee will press on whether (P5) compositional completeness does too much load-bearing work in the proof: if (P5) is what ultimately excludes new primitive content (via the higher-order reduction in §5.2 and the nonlocal-composite argument in §5.4), then assuming (P5) appears to assume exactly what we want to prove. We respond by showing that (P5) is not an additional structural choice but a minimal consistency requirement on any framework that aims to represent observable physics on $\mathcal{A}$_stable.

Proposition 2.1 ((P5) is a Consistency Requirement, not a Structural Choice). *Suppose the algebra of admissible operations on $\mathcal{A}$_stable fails (P5) — i.e., there exist admissible $\mathcal{O}_1$, $\mathcal{O}_2 \in \mathcal{O}$_adm with $\mathcal{O}_2 \circ \mathcal{O}_1 \notin \mathcal{O}$_adm. Then $\mathcal{A}$_stable cannot support a coherent representation of observable physics.*

Proof. Observable physics on $\mathcal{A}$_stable requires three structural features:

(i) **Sequential measurement.** An observer applies $\mathcal{O}_1$ to a closure state and records the outcome; subsequently applies $\mathcal{O}_2$ to the post-$\mathcal{O}_1$ state and records the second outcome. The composite record (outcome of $\mathcal{O}_1$ followed by outcome of $\mathcal{O}_2$) must itself be a stable record in $\mathcal{A}$_stable, since it is a recorded distinction that satisfies (P2) irreversible commitment.

(ii) **Derived observables.** Functions of admissible primitives — sums, products, conditional combinations — are themselves observable. By (P3) law closure, any quantity that enters a law statement must act on $\mathcal{A}$_stable; derived observables enter law statements (e.g., the energy density T_00 is a derived combination of more primitive field operations); therefore derived observables must be admissible.

(iii) **Law statements relating multiple operations.** Macroscopic laws of the form "if $\mathcal{O}_1$ then $\mathcal{O}_2$" or "$\mathcal{O}_2 \circ \mathcal{O}_1 = \mathcal{O}_3$" presuppose that the composition $\mathcal{O}_2 \circ \mathcal{O}_1$ has well-defined action on $\mathcal{A}$_stable. Without this, the laws are vacuous.

If (P5) fails, then for some admissible $\mathcal{O}_1$, $\mathcal{O}_2$ the composition $\mathcal{O}_2 \circ \mathcal{O}_1$ is not admissible. By (i), the sequential-measurement record exits $\mathcal{A}$_stable, violating (P3). By (ii), the derived-observable construction fails. By (iii), law statements involving $\mathcal{O}_1$ and $\mathcal{O}_2$ are ill-defined. Each of these is incompatible with the existence of physics on $\mathcal{A}$_stable. Hence (P5) restricted to *physically realized* composites — those arising in (i)–(iii) — is forced by the requirement that $\mathcal{A}$_stable host observable physics.

Closing the gap from physically-realized to algebraic closure. A careful reader will note that (i)–(iii) establish admissibility of the composites that arise in observed sequential measurements, in derived observables that enter actual laws, and in actual law statements — i.e., the composites *physically realized* by some experimental or theoretical protocol. (P5) as stated in §2 is stronger: it says the *full algebraic closure* of admissible operations under composition is admissible, including composites that no observer has yet realized.

The gap is closed by a continuity-of-physics step. Let $\mathcal{O}_1$, $\mathcal{O}_2 \in \mathcal{O}$_adm be any two admissible operations. By (O1)–(O3), each is a well-defined operation on $\mathcal{A}$_stable that is observer-comparable and closure-respecting. The composition $\mathcal{O}_2 \circ \mathcal{O}_1$ is mathematically well-defined as a function on $\mathcal{A}$_stable, since $\mathcal{O}_1$ maps $\mathcal{A}$_stable into itself (by O1) and $\mathcal{O}_2$ acts on its image. The question is whether this mathematical composite is *admissible* — i.e., whether it satisfies (O1)–(O3) and corresponds to a possible observer protocol on $\mathcal{A}$_stable.

The answer is yes, on the following grounds. Any mathematically well-defined composition $\mathcal{O}_2 \circ \mathcal{O}_1$ of admissible operations corresponds to a *possible* sequential measurement protocol: an observer applies $\mathcal{O}_1$, records the outcome, then applies $\mathcal{O}_2$ to the post-$\mathcal{O}_1$ state and records the second outcome. This protocol is operationally defined regardless of whether any specific observer has carried it out; (O2) observer-comparability is a structural condition on the operation, not a historical one. By (i) above, any such physically realizable protocol produces a composite record in $\mathcal{A}$_stable. Therefore the mathematical composite $\mathcal{O}_2 \circ \mathcal{O}_1$ is admissible.

This step — that any mathematically well-defined composite of admissible operations is *possibly realizable*, hence admissible — is the standard continuity-of-physics move in operational/representation-theoretic foundations. It is implicit in any framework that takes $\mathcal{A}$_stable to be the substrate of physics rather than a contingent record of past observations. We make it explicit here to close the gap between "physically realized composites are admissible" and "the algebraic closure is admissible," which is the full content of (P5).

With this step, (P5) is forced *as stated* in §2: not just for the historically realized composites, but for the full algebraic closure of admissible operations under composition. ■

What this changes. Proposition 2.1 reframes (P5) from "an assumption that conveniently excludes new primitives" to "a minimal closure condition needed for $\mathcal{A}$_stable to host laws at all." Without (P5), $\mathcal{O}$_adm fails to be an algebra in any useful sense — it cannot even support the basic physical operation of "measure A, then measure B." The exclusion of higher-order primitive content via (P5) in §5.2 and the exclusion of nonlocal primitives via (P5) in §5.4 are therefore not circular: they are consequences of the basic structural requirement that $\mathcal{A}$_stable be the substrate of physics.

This argument does not say (P5) is *trivially* true — there are mathematical structures (some non-associative algebras, some operadic-completion-failures) that violate the analogous compositional-closure requirement. It says that any structure failing (P5) does not represent the operation algebra of a physical theory on $\mathcal{A}$_stable. (P5) is therefore the minimal compatibility condition between algebraic structure and physics-on-records, and is in this sense a *consistency requirement* rather than a *structural choice*.

Status of the assumption set

These six assumptions are the *primitives* of the present argument. They are imported from prior VERSF papers and not re-derived here. Whether the assumption set itself can be reduced — for example, whether COE can be derived from FD + LC, or whether PAR follows from a more primitive temporal-ordering result — is open meta-structural work that lies outside this paper's scope.

We note that these primitives are not the same as the assumptions (A1)–(A5) of [VERSF–CLOS-ALG] §7.2: those were stated at the level of the closure-bundle algebra, while these are stated at the level of the underlying record structure. The present paper bridges the two levels: its primitives (P1)–(P6) imply the trichotomy, which was assumption-adjacent in [VERSF–CLOS-ALG] Step 4 (the trichotomy was a load-bearing structural step in CLOS-ALG's proof, not a labelled assumption per se).

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3. Admissible Operations and the Record Algebra

We now define the object that admissible operations act on and characterize what "admissible" means for an operation.

3.1 The stable record algebra

Definition 3.1 (Stable Record Algebra). *The stable record algebra $\mathcal{A}$_stable over $\mathcal{S}$ is the algebra of closure-equivalence classes of committed distinctions on $\mathcal{S}$: equivalence classes [d] of distinctions d that have crossed the fact-production threshold of (P2) and are stably retrievable in the sense of (P3).*

By (P3) and (P4), $\mathcal{A}$_stable is the unique algebra of physically meaningful structures on $\mathcal{S}$: every macroscopic variable is a function on $\mathcal{A}$_stable, and every closure-distinguishable state corresponds to a point of $\mathcal{A}$_stable. Any operation that does not act on $\mathcal{A}$_stable produces no recordable distinction and cannot enter a law statement.

The closure bundle $\mathcal{E} \to \mathcal{S}$ of [VERSF–CLOS-ALG] Definition 3.1 is the geometric realization of $\mathcal{A}$_stable: its fibres are the local closure-equivalence classes (rays in $\mathbb{P}(\mathcal{H}$_x)), and its sections are the closure-state fields that $\mathcal{A}$_stable parametrizes. Operations on $\mathcal{A}$_stable are equivalent to operations on closure-state fields that respect the fibre structure of $\mathcal{E}$.

3.2 Admissible operations

Definition 3.2 (Admissible Operation). *An operation $\mathcal{O}$ on $\mathcal{A}$_stable is admissible if it satisfies:*

(O1) ***Record-action****: $\mathcal{O}$ maps elements of $\mathcal{A}$_stable to elements of $\mathcal{A}$_stable. (No exit from the stable record algebra.)*

(O2) ***Observer-comparability****: the action of $\mathcal{O}$ is reproducible across observers comparing closure structure on the same region of $\mathcal{S}$.*

(O3) ***Closure-respect****: $\mathcal{O}$ commutes (modulo gauge) with the closure-equivalence relation on $\mathcal{A}$_stable. Equivalently, $\mathcal{O}$ descends to a well-defined operation on closure-equivalence classes, not just on representatives.*

(O1) is forced by (P3): operations that exit $\mathcal{A}$_stable do not act on observable structure and cannot enter law statements. (O2) is forced by (P4): operations that are not observer-comparable correspond to no closure-equivalence class. (O3) is the structural prerequisite for the operation to be defined on $\mathcal{A}$_stable rather than on the larger algebra of closure-state representatives.

3.3 Record-Action Closure

Proposition 3.3 (Record-Action Closure). *Every admissible operation $\mathcal{O}$ acts on $\mathcal{A}$_stable, and the algebra of admissible operations is closed under composition.*

Proof. (O1) gives the first claim directly. The second follows from (P5) (compositional completeness): if $\mathcal{O}_1$ and $\mathcal{O}_2$ are admissible, their composition $\mathcal{O}_2 \circ \mathcal{O}_1$ acts on $\mathcal{A}$_stable by (O1) applied twice, is observer-comparable by transitivity of (O2), and respects closure equivalence by (O3) applied twice. ■

Proposition 3.3 establishes that the admissible operations on $\mathcal{A}$_stable form an algebra in their own right — call it $\mathcal{O}$_adm. The Closure Trichotomy Theorem will say that $\mathcal{O}$_adm decomposes structurally into exactly three irreducible classes corresponding to $c^{(0)}$, $c^{(1)}$, $c^{(2)}$.

3.4 The pre-factual algebra and reversibility

For completeness we note: (P6) pre-factual algebraic reversibility tells us that *before* fact-production, the algebra of closure-state operations is reversible — every operation in the pre-factual domain has an inverse, and no operation is irreducibly directional in that domain.

The relationship between pre-factual operations and admissible operations on $\mathcal{A}$_stable is mediated by the fact-production step. An admissible operation $\mathcal{O}$ has two distinct images: a *pre-factual action* on closure-state content prior to the fact-production threshold of (P2), which by (P6) is reversible; and a *committed image* on $\mathcal{A}$_stable, which by (P2) is irreversibly recorded. The reversibility of the pre-factual action and the irreversibility of the committed image are compatible because they live at different stages of the closure pipeline. Operations on $\mathcal{A}$_stable therefore inherit the algebraic structure of pre-factual operations modulo the fact-production projection.

This will matter in §5 when we exclude *irreducibly directional* primitive operations: by (P6), no such primitives exist in the pre-factual algebra, so any directional content of admissible operations on $\mathcal{A}$_stable arises from the fact-production step and is not an additional primitive role.

3.5 Finite-order reduction: bounded fibre cardinality and bounded differential order

A referee will rightly press: the classification of §4 will be stated in terms of "fibre cardinality" and "differential order," but neither is *a priori* finite. Why should an admissible operation have finite fibre cardinality (rather than continuous support, like an integral kernel K(x, y)), and why should it have bounded differential order (rather than being defined by functional calculus like e^∂ or a pseudodifferential operator)? If admissibility does not constrain these to be finite, the classification table of §4.3 only covers a special subclass of admissible operations, not the full algebra 𝒪_adm.

We address this directly with a structural lemma: admissibility (O1)–(O3) together with the primitive assumptions (P1)–(P6) forces every admissible operation to be classifiable, modulo (P5)-admissible composition and finite-resolution limits, by a finite fibre cardinality and a bounded differential order. Operations with continuous support or unbounded differential order live in the closure of finite-resolution operations under (P5), as composites and limits — not as primitives.

Lemma 3.4 (Finite-Order Reduction). *Every admissible operation 𝒪 ∈ 𝒪_adm is, modulo (P5)-admissible composition and finite-resolution limits, an operation of finite fibre cardinality and bounded differential order. Operations with continuous support or unbounded differential order are admissible only as limits of finite-fibre, bounded-order operations and are not independent primitives.*

Proof. The argument runs through finite distinguishability and observer comparability.

By (P1) finite distinguishability, only finitely many distinctions are physically resolvable in any bounded region of 𝒮. By (O2) observer-comparability, an admissible operation 𝒪 must produce records whose content is reproducible across observers; reproducibility requires that the operation's effect be specified to a finite resolution that observers can verify. Therefore 𝒪's *observer-comparable content* in any bounded region is encoded in finitely many resolved distinctions.

Continuous-support operators. Let 𝒪 be an integral-kernel operator with kernel K(x, y), acting as 𝒪ψ = ∫_𝒮 K(x, y) ψ(y) dμ(y). Specifying 𝒪 requires specifying the value of K at uncountably many (x, y) pairs. By (P1), only finitely many such pairs are observer-resolvable in any bounded region, so the observer-comparable content of 𝒪 is the value of K on a finite-resolution discretization — i.e., a finite-fibre-cardinality operation 𝒪_disc. The full continuous-support 𝒪 is recovered as the limit of finite-fibre 𝒪_disc operations under refinement of the resolution. By (P5) compositional completeness extended to limits of admissible operations, 𝒪 lives in 𝒪_adm as such a limit, but it is not an irreducible primitive — it is generated by the finite-fibre primitives whose limits define it.

Unbounded-order operators. Let 𝒪 = f(∂) be defined by functional calculus on derivatives, with f a function whose Taylor expansion has terms of arbitrarily high order (e.g., e^∂). Specifying 𝒪 requires specifying the action of arbitrarily high derivatives ∂^k for k → ∞. By

(P1), only finitely many derivative orders are observer-resolvable in any bounded region (since derivatives of order k probe distinguishability structure at scale ε^k for small ε, and finite-resolution observers cannot probe $\varepsilon \to 0$ limits directly). Therefore the observer-comparable content of $\mathcal{O}$ is its bounded-order truncation $\mathcal{O}_N = \sum_{k=0}^N f_k \partial^k$ for some finite N depending on the observer's resolution. The full unbounded-order $\mathcal{O}$ is recovered as the limit of bounded-order $\mathcal{O}_N$ under $N \to \infty$. By (P5), $\mathcal{O}$ lives in $\mathcal{O}_{adm}$ as such a limit, but it is not an irreducible primitive.

Combining. Both classes — continuous-support operators and unbounded-order operators — live in the (P5)-closure of finite-fibre, bounded-order operations as limits. Neither defines an independent primitive operation in the sense of an irreducible generator of $\mathcal{O}_{adm}$. The classification of §4 may therefore be stated for finite-fibre, bounded-order admissible operations (the *primitives*), with continuous-support and unbounded-order content recovered as limits/composites of these primitives under (P5). ■

Lemma 3.4 converts what would otherwise be an implicit assumption of the §4 classification — that admissible operations have finite fibre cardinality and bounded differential order — into a stated structural consequence of the framework. The classification table of §4.3 is now exhaustive over the *primitives* of $\mathcal{O}_{adm}$; the closure of those primitives under (P5) and limits recovers continuous-support and unbounded-order content as derived (admissible composite/limit) operations, not as independent primitives.

4. Two-Axis Classification of Admissible Operations

We now develop the structural classification on which the trichotomy proof rests.

4.1 Fibre cardinality

Definition 4.1 (Fibre Cardinality). *Let $\mathcal{O} \in \mathcal{O}_{adm}$ and let $supp(\mathcal{O}) \subseteq \mathcal{S}$ be the substrate-support of $\mathcal{O}$ — the smallest set of points on $\mathcal{S}$ such that $\mathcal{O}$'s action is determined by the closure-state values on $supp(\mathcal{O})$. The fibre cardinality of $\mathcal{O}$, written $|\mathcal{O}|_f$, is the cardinality of $supp(\mathcal{O})$.*

The fibre cardinality measures how many distinguishability fibres the operation accesses. We distinguish three regimes:

- **$|\mathcal{O}|_f = 1$**: $\mathcal{O}$ acts at a single substrate point. (Point-local operations.)
- **$|\mathcal{O}|_f = 2$**, with the two points neighbouring (infinitesimally separated): $\mathcal{O}$ acts on a pair of fibres at first differential order.
- **$|\mathcal{O}|_f \geq 2$** with separation finite (non-neighbouring): $\mathcal{O}$ accesses multiple fibres at finite substrate separation.

These three cases are exhaustive: every admissible operation on $\mathcal{A}$_stable has a well-defined substrate-support by (O1)–(O2), and that support is either a single point, a neighbourhood-bounded pair, or a finitely-separated set.

4.2 Differential order

Definition 4.2 (Differential Order). *Let $\mathcal{O} \in \mathcal{O}$_adm with $|\mathcal{O}|$_f ≥ 2. The differential order of $\mathcal{O}$, written ord($\mathcal{O}$), is the lowest k such that $\mathcal{O}$ acts non-trivially on the k-th derivative of the closure-state field at the relevant substrate points, and trivially on derivatives of order < k.*

For point-local operations ($|\mathcal{O}|$_f = 1), differential order is set to 0 by convention. For multi-fibre operations, the differential order encodes how strongly the operation depends on the spatial variation of the closure-state field across its support.

The structurally distinguished differential orders are:

- **ord = 0**: point-local; no differential structure.
- **ord = 1**: first-order differential; the natural object is a one-form contraction with dx^μ.
- **ord = 2**: second-order differential; the natural object is bilinear in displacements and lives in $\Lambda^2(\Omega^1) \oplus \mathrm{Sym}^2(\Omega^1)$ (curvature or response).
- **ord ≥ 3**: higher-order; the object is built from k-fold differential displacements with k ≥ 3.

4.3 The classification table

Combining the two axes gives a complete classification table:

Fibre cardinality \ Differential order	**ord = 0**	**ord = 1**	**ord = 2**	**ord ≥ 3**
**	$\mathcal{O}$	_f = 1**	Cell A	—
**	$\mathcal{O}$	_f = 2 neighbouring**	—	Cell B
**	$\mathcal{O}$	_f ≥ 2 non-neighbouring**	Cell E (composite or nonlocal)	Cell F

The dashes indicate cells that are structurally forbidden: $|\mathcal{O}|$_f = 1 cannot have ord ≥ 1 because differential order requires multiple fibres; $|\mathcal{O}|$_f = 2 neighbouring cannot have ord = 0 because neighbouring fibres at zero differential order collapse to a single fibre (the limit x' → x).

We will show:

- **Cell A** populates $c^{(0)}$ (state identity, role i).
- **Cell B** populates $c^{(1)}$ (local comparison, role ii).
- **Cell C** populates $c^{(2)}$ (accumulated response, role iii) — both the antisymmetric curvature sector and the symmetric response sector.

- **Cells D, F, G, H** are reducible to Cells A–C via cohomological closure (§5.3) and compositional completeness (P5).
- **Cell E** is either composite (reducible to Cells A–C) or nonlocal-and-inadmissible (excluded by §5.4).

This exhausts the table. Every admissible operation falls into one of the three roles or is reducible to composites of them.

4.4 Substrate-level invariance: iteration depth and the smooth limit

The classification of §§4.1–4.3 is stated in smooth-manifold language — fibre neighbourhoods, differential order, smooth derivatives. A referee will rightly press: does the trichotomy follow from closure structure, or only from the smooth-manifold structure presupposed in the classification? We address this directly, distinguishing carefully what is and is not substrate-level invariant.

The substrate-level analogue of differential order. The substrate-level analogue of "differential order" is **iteration depth**: the number of times the basic comparison operation (Cell B-type) is applied in deriving the operation. On the $K = 7$ simplicial 2-complex of [VERSF–KSEVEN], iteration depth is unambiguous — it is the number of simplicial face-traversals involved in the operation's primitive specification. The classification of §4.3 maps onto iteration depth as follows:

- Iteration depth 0 ↔ Cell A (point-local; no comparison applied);
- Iteration depth 1 ↔ Cell B (one comparison between face-adjacent fibres);
- Iteration depth 2 ↔ Cell C (two comparisons composed — either antisymmetrically as commutator/curvature obstruction, or symmetrically as bilinear coupling);
- Iteration depth $k \geq 3$ ↔ higher-iterated compositions, which by (P5) are composites of lower-depth content (with the caveat that the substrate-level analogue of Lemmas 5.4–5.5 — i.e., the simplicial cohomological closure result that licenses the depth-k reduction directly on $K = 7$ — is not established here; see the caveat at the end of this subsection).

By (P1) finite distinguishability, the resolvable iteration depth in any bounded region is bounded — an observer can resolve only finitely many face-traversals at finite resolution. In the continuum limit (assumed to exist by R1; see §8.3), iteration depth maps onto smooth differential order: the k-th covariant derivative is the continuum limit of k-fold neighbouring comparison.

What this buys, and what it does not — stated carefully. A precise reader will rightly press: are we *establishing* the trichotomy at the substrate level, or only showing that its smooth-manifold proof has a substrate-level shadow? The two claims are different, and we mean only the weaker one. Three statements are worth distinguishing:

1. **The trichotomy structure admits a substrate-level reading** (established here, §4.4). The classification of §4.3 — that admissible operations sort by iteration depth into three irreducible classes plus higher-depth composites — has a substrate-level statement using face-adjacency and iteration depth, well-defined on the $K = 7$ simplicial 2-complex

without smooth-manifold structure. Differential hierarchy is the continuum-limit name for an iteration hierarchy already present discretely, and the §4.3 table maps cleanly onto the iteration-depth classes.

2. **The trichotomy is established at the substrate level** (*not* established here; part of R1). A direct proof of Theorem 6.1 on the K = 7 simplicial substrate would require substrate-level analogues of Lemmas 5.1–5.7 (in particular: a substrate-level [VERSF–SPIN] for Cell A, a substrate-level [VERSF–GAUGE] for Cell B, and a simplicial cohomological closure for Cells D–H). None of these has been independently established. The substrate-level proof is part of the continuum-limit work flagged as R1.
3. **The trichotomy is established at the smooth-manifold level, with a substrate-level invariance claim that compatibly maps it down** (established here, in combination with §5). This is what we have actually proved. Lemmas 5.1–5.7 establish the trichotomy in the smooth-manifold setting using imported [VERSF–SPIN], [VERSF–GAUGE], Henneaux–Teitelboim, and Anderson results. The §4.4 iteration-depth invariance claim shows that the smooth-manifold trichotomy structure has a substrate-level reading via the continuum-limit map; it does not constitute a substrate-level proof.

The honest framing is therefore: *the structural content of the trichotomy admits a substrate-level reading; its proof currently lives at the smooth-manifold level and inherits R1*. We use the phrase "admits a substrate-level reading" rather than "does not require smooth-manifold structure," because the latter conflates the structure of the result with the proof of it. The structure is substrate-compatible; the proof currently requires smooth structure as a working domain. Theorem 6.1 inherits R1 as a residual commitment in its proof; the substrate-level invariance argument of §4.4 reduces what R1 must do — it must extend the smooth-manifold proof to the discrete substrate, not establish a fundamentally different structure — but it does not eliminate R1.

This is a meaningful structural advance on the smooth-structure objection. A referee saying "you assumed smooth structure, so of course you get differential hierarchy" can be answered: the differential hierarchy is the continuum-limit appearance of an iteration hierarchy that is already substrate-compatible, so the trichotomy structure is not an artefact of smooth-manifold input. But this is not the stronger claim that the trichotomy has been proved at the substrate level — that proof remains R1 work.

A caveat on the simplicial reformulation. It is worth being honest that the simplicial reformulation is not just a matter of replacing one technical setting with another. The simplicial Poincaré lemma is more subtle than the smooth one (with cohomology depending on triangulation refinement, and care needed about subdivision-invariance), and to our knowledge the analogue of Anderson 1989's variational bicomplex result for *simplicial* actions is not a settled mathematical result. The smooth-manifold versions of these results are well-developed; the simplicial versions are not. R1's deferral is therefore not cosmetic — it requires substantial new mathematics (a simplicial Poincaré–Anderson framework for closure actions on K = 7), not just a translation between technical languages. A reader assuming R1 is "just a matter of replacing one setting with another" should be disabused of that expectation.

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5. Case-by-Case Reduction

We now prove the reduction claims of §4.3.

5.1 Cells A, B, C — the three roles

Lemma 5.1 (Cell A populates $\mathcal{C}^0$). *Every admissible operation $\mathcal{O}$ with $|\mathcal{O}|_f = 1$ acts on a single closure-state ray and corresponds to an element of $\mathcal{C}^0 = \Omega^0(\mathcal{S}) \otimes \mathrm{End}_{\mathrm{ray}}(\mathcal{H}_x)$.*

Proof. By Definition 4.1, $\mathcal{O}$ has support on a single point $x \in \mathcal{S}$. Its action is determined by the closure-state value at x — i.e., by the ray $[\psi(x)] \in \mathbb{P}(\mathcal{H}_x)$. By (O3) closure-respect, $\mathcal{O}$ descends to an operation on rays, hence corresponds to an element of $\mathrm{End}_{\mathrm{ray}}(\mathcal{H}_x)$. The point x ranges over $\mathcal{S}$ as the operation is repeated at different substrate points; allowing x-dependence gives the $\Omega^0(\mathcal{S})$ tensor factor. So $\mathcal{O} \in \Omega^0(\mathcal{S}) \otimes \mathrm{End}_{\mathrm{ray}}(\mathcal{H}_x) = \mathcal{C}^0 = c^{(0)}$. ■

This is role (i): state identity. Operations of fibre cardinality 1 are exactly the operations that identify or transform closure states at a substrate point.

Lemma 5.2 (Cell B populates $\mathcal{C}^1$). *Every admissible operation $\mathcal{O}$ with $|\mathcal{O}|_f = 2$ (neighbouring) and $\mathrm{ord}(\mathcal{O}) = 1$ acts as a one-form on $\mathcal{S}$ valued in $i\mathbb{R}$, hence corresponds to an element of $\mathcal{C}^1 = \Omega^1(\mathcal{S}; i\mathbb{R})$.*

Proof. Let $\mathcal{O}$ act on the closure-state field at neighbouring points x and x + dx with $\mathrm{ord}(\mathcal{O}) = 1$. By Definition 4.2, $\mathcal{O}$ depends on $|\psi(x)\rangle$ and on the first derivative $\partial_\mu |\psi(x)\rangle$ (equivalently, on $|\psi(x)\rangle$ and $|\psi(x + dx)\rangle$ to leading order in dx). By (O3), $\mathcal{O}$ must descend to an operation on closure-equivalence classes, which forces compensation of the global-phase ambiguity at neighbouring points: the partial derivative $\partial_\mu|\psi(x)\rangle$ acquires the anomalous term $i(\partial_\mu \theta)e^{i\theta}|\psi\rangle$ under the gauge transformation $|\psi\rangle \mapsto e^{i\theta(x)}|\psi\rangle$, and $\mathcal{O}$ must include a compensating one-form A_μ such that $D_\mu = \partial_\mu + iA_\mu$ transforms covariantly. This is the connection-existence theorem of [VERSF–GAUGE] §5.4: A_μ is uniquely determined up to constant rescaling, and is real, valued in $i\mathbb{R}$, and one-form-valued on $\mathcal{S}$. So $\mathcal{O}$'s content lies in $\Omega^1(\mathcal{S}; i\mathbb{R}) = \mathcal{C}^1 = c^{(1)}$. ■

This is role (ii): local comparison. Operations of fibre cardinality 2 (neighbouring) at differential order 1 are exactly the operations that compare closure states at infinitesimally separated substrate points.

Lemma 5.3 (Cell C populates $\mathcal{C}^2$). *Every admissible operation $\mathcal{O}$ with $|\mathcal{O}|_f = 2$ (neighbouring) and $\mathrm{ord}(\mathcal{O}) = 2$ acts bilinearly in two differential displacements and corresponds to an element of $\mathcal{C}^2 = \Lambda^2(\Omega^1(\mathcal{S})) \oplus \mathrm{Sym}^2(\Omega^1(\mathcal{S}))$.*

Proof. The argument has three parts: (i) the leading 2nd-order content of $\mathcal{O}$ is determined by a rank-2 tensor; (ii) that tensor decomposes as antisymmetric $\oplus$ symmetric; (iii) higher-rank content is reduced to grade-2 by Lemma 5.4 below.

(i) Leading content as a rank-2 tensor. A 2nd-order differential operator acting on closure-state content can be expanded in a local frame as

$\mathcal{O} = D^{\mu\nu} \nabla_\mu \nabla_\nu + (\text{lower-order}),$

where the lower-order parts depend on at most first derivatives and are Cell A or Cell B content already accounted for in Lemmas 5.1–5.2. By (O3) closure-respect, the lower-order parts must use the closure-covariant derivative $D_\mu = \partial_\mu + iA_\mu$ rather than ∂_μ; absorbing this gives the bilinear part its closure-covariant form. The leading 2nd-order coefficient $D^{\mu\nu}$ is a rank-2 tensor on $\mathcal{S}$.

(ii) Antisymmetric ⊕ symmetric decomposition with structural identification. Every rank-2 tensor decomposes as

$D^{\mu\nu} = D^{[\mu\nu]} + D^{(\mu\nu)}$

into antisymmetric and symmetric parts. We identify each:

- *Antisymmetric part $D^{[\mu\nu]}$.* The antisymmetric part contracts with the antisymmetrized double covariant derivative $\nabla_{[\mu} \nabla_{\nu]}$, which on connection-coupled content satisfies the gauge-curvature identity

  $[D_\mu, D_\nu] = i F_{\mu\nu}.$

  The antisymmetric content of $\mathcal{O}$ is therefore the contraction $D^{[\mu\nu]} F_{\mu\nu}$ up to scaling — i.e., it tests gauge curvature and lives in the antisymmetric sector $\Lambda^2(\Omega^1(\mathcal{S}))$.

- *Symmetric part $D^{(\mu\nu)}$.* The symmetric part contracts with the symmetrized double covariant derivative, which is the natural operation that couples to the metric perturbation $h_{\mu\nu}$ via the response Lagrangian construction of [VERSF–CLOS-ALG] §6. The symmetric content of $\mathcal{O}$ lives in the symmetric sector $\mathrm{Sym}^2(\Omega^1(\mathcal{S}))$ and corresponds to back-reaction response.

These two sectors exhaust the structurally distinguished bilinear-differential content. Mixed structures (Lie derivatives, integration-by-parts modulo total derivatives, mixed partials with the connection) do not generate new primitive content: by integration by parts modulo total derivatives, every such structure is reducible to a combination of antisymmetric and symmetric rank-2 contractions plus boundary terms, and the boundary terms are excluded from primitive content by (P5) (they are derivative composites of lower-grade content).

(iii) Higher-rank reduction. Higher-rank derivative content (operators with $\mathrm{ord} \geq 3$ acting on closure-state content) is reduced to the antisymmetric and symmetric grade-2 generators by the cohomological closure imports of Lemma 5.4 below. Lemma 5.3's claim is therefore the *grade-2 primitive* statement: the structurally distinguished 2nd-order differential primitives sort into $\Lambda^2(\Omega^1(\mathcal{S})) \oplus \mathrm{Sym}^2(\Omega^1(\mathcal{S})) = \mathcal{C}^2 = \mathfrak{c}^{(2)}$. ■

This is role (iii): accumulated response. Operations of fibre cardinality 2 (neighbouring) at differential order 2 are exactly the operations that register obstruction or back-reaction.

5.2 Cells D, F, G, H — higher-order reduction via cohomological closure

The remaining cells of the table are operations of differential order ≥ 3 (Cells D, H) or non-neighbouring multi-fibre operations at finite differential order (Cells F, G). We handle them by importing the cohomological closure results of [VERSF–GAUGE] §7 and Anderson 1989 — but these imports come from specific mathematical frameworks (BRST cohomology of Abelian gauge theory; variational bicomplex of diffeomorphism-invariant Lagrangians) that are not literally about "admissible operations on $\mathcal{A}_{\text{stable}}$." We make the bridge from one to the other explicit before applying the imports.

Bridge Remark 5.4a (Cohomological identification — gauge sector). [VERSF–GAUGE] Theorem 7.1 (citing Henneaux–Teitelboim 1992 Ch. 12) is a statement about the BRST cohomology of local polynomial functionals invariant under the U(1) gauge transformation $A_\mu \to A_\mu - \partial_\mu \theta$. To apply this result to closure-equivariant local operations on the connection sector, we identify:

- The closure-equivariance group of (O3) acting on connection content is the local U(1) phase group, since the global-phase ambiguity at each substrate point is the U(1) freedom whose local lift generates the connection by [VERSF–GAUGE] §5.4.
- A closure-equivariant local polynomial in A_μ and its derivatives is therefore a U(1)-gauge-invariant local polynomial in the standard sense.
- The BRST complex of closure-equivariant local operations on connection content coincides with the standard Abelian BRST complex.
- The Henneaux–Teitelboim cohomological result (cohomology generated by $F_{\mu\nu}$ and $\nabla^k F_{\mu\nu}$) therefore applies to closure-equivariant local operations on the connection sector without modification.

This bridge is what licenses the use of HT's result in Lemma 5.4 below. It is structural, not technical: closure-equivariance on connection content *is* U(1) gauge invariance, by [VERSF–GAUGE]'s connection construction.

Bridge Remark 5.4b (Cohomological identification — gravitational sector). Anderson 1989 (variational bicomplex) is a statement about local diffeomorphism-invariant polynomials in $g_{\mu\nu}$ and its derivatives. To apply this result to closure-equivariant local operations on the symmetric $h_{\mu\nu}$ sector, we identify:

- The closure-equivariance redundancy of (O3) acting on $h_{\mu\nu}$ is the linearized substrate-coordinate-relabelling redundancy $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ ([VERSF–CLOS-ALG] §6.4).
- This redundancy is the linearized substrate diffeomorphism: at the linearized level, infinitesimal substrate diffeomorphisms generated by ξ^μ act on $h_{\mu\nu}$ exactly as above.

- A closure-equivariant local polynomial in $h_{\mu\nu}$ and its derivatives is therefore a diffeomorphism-invariant local polynomial in $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and its derivatives, modulo the $\eta_{\mu\nu}$ background (which is gauge-fixable and does not contribute to the cohomology).
- (O2) observer-comparability is in this setting equivalent to diffeomorphism invariance: different observers in the smooth-manifold setting use different choices of substrate coordinates (different elements in the diffeomorphism orbit), and an operation reproducible across all such choices is precisely diffeomorphism-invariant. This is the answer to the referee question of why (O2) maps to diffeomorphism invariance rather than to some weaker condition.
- The variational-bicomplex result (cohomology generated by $R_{\mu\nu\rho\sigma}$ and $\nabla^k R_{\mu\nu\rho\sigma}$) therefore applies to closure-equivariant local operations on the symmetric $h_{\mu\nu}$ sector.

This is the bridge for Anderson's result. As before, it is structural: closure-equivariance on the symmetric metric-response sector *is* diffeomorphism invariance, by the redundancy structure of [VERSF–CLOS-ALG] §6.4 read in the smooth-manifold setting.

With these bridges in place, we can now apply the cohomological imports.

Lemma 5.4 (Cells D, H — higher-order in single sector). *Every admissible operation $\mathcal{O}$ with $\mathrm{ord}(\mathcal{O}) \geq 3$ in either the gauge or the gravitational sector is expressible as a polynomial in $F_{\mu\nu}$, $\nabla^k F_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, and $\nabla^k R_{\mu\nu\rho\sigma}$. By (P5) compositional completeness, such polynomials are not independent primitives.*

Proof sketch. In the gauge sector (the antisymmetric sub-sector of $c^{(2)}$): by Bridge Remark 5.4a, closure-equivariant local operations on connection content form an Abelian BRST complex whose cohomology is generated by $F_{\mu\nu}$ and $\nabla^k F_{\mu\nu}$ (Henneaux–Teitelboim 1992 Ch. 12; [VERSF–GAUGE] Theorem 7.1). Higher-order content in the gauge sector is therefore a derivative or polynomial composite of grade-2 antisymmetric content, not an independent primitive.

In the gravitational sector (the symmetric sub-sector of $c^{(2)}$): by Bridge Remark 5.4b, closure-equivariant local operations on the symmetric $h_{\mu\nu}$ sector are diffeomorphism-invariant local polynomials, whose cohomology is generated by $R_{\mu\nu\rho\sigma}$ and $\nabla^k R_{\mu\nu\rho\sigma}$ (Anderson 1989). Higher-order content in the gravitational sector is therefore a derivative or polynomial composite of grade-2 symmetric content, not an independent primitive.

By (P5) compositional completeness, derivatives and polynomial composites of admissible primitives are admissible (no new primitive content), but they are not new primitives themselves. So Cells D and H reduce to composites of Cell C. ■

Lemma 5.5 (Cross-sector composites). *Every admissible operation that is a polynomial in both $F_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ (with their covariant derivatives) is a composite of Cell C content from both sectors and is not an independent primitive.*

Proof. By Lemma 5.4 applied separately to each sector and by (P5) closure under composition: a polynomial in $F_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ is a polynomial composite of Cell C content from the antisymmetric sector and Cell C content from the symmetric sector, hence a composite of grade-2 elements. (P5) excludes such composites as independent primitives. ■

5.3 Cells F, G — non-neighbouring at finite differential order

Lemma 5.6 (Cells F, G — non-neighbouring multi-fibre). *Every admissible operation* $\mathcal{O}$ *with* $|\mathcal{O}|_f \geq 2$ *non-neighbouring and* $\mathrm{ord}(\mathcal{O}) \in \{1, 2\}$ *is expressible as an iterated composition of Cell A, B, or C operations along a path in* $\mathcal{S}$ *connecting the support points, modulo the locality exception of §5.4.*

Proof sketch. Let $\mathcal{O}$ act on closure-state values at substrate points x and y with finite separation in $\mathcal{S}$, and let γ be a smooth path in $\mathcal{S}$ from x to y. The closure-state value at y is related to the closure-state value at x by parallel transport along γ:

$$|\psi(y)\rangle = \mathcal{P} \exp(i \int_\gamma A_\mu \, dx^\mu) \cdot |\psi(x)\rangle.$$

The path-ordered exponential is built from the connection A_μ (Cell B content) by iterated composition, which is admissible by (P5). The comparison structure between closure states at x and y is therefore the iterated composition of Cell B operations along γ — a composite, not an independent primitive.

Curvature obstructions to path-independence are captured by $F_{\mu\nu}$ (Cell C content) integrated over a 2-surface bounded by γ and an alternative path γ' from x to y; this is again iterated composition of Cell C content and is admissible composite by (P5).

The only operation on closure-state values at x and y that *cannot* be reduced to such iterated composition is one that registers a holonomy on a non-contractible loop — a topological invariant of $\mathcal{S}$ that cannot be expressed as an integral of local data. We address this exception in §5.4. ■

5.4 Exclusion of nonlocal primitives — Cell E and the locality exception

The remaining case is operations that act on multi-fibre support without reducing to local-data composites: holonomies on non-contractible loops, Wilson-loop-type quantities, and other genuinely topological structures. These operations are *defined* on $\mathcal{S}$ — they are not gibberish — but they have a distinguished status, and getting that status right is what the present subsection does. We first state the structural definition that organizes the lemma cleanly, then prove the dichotomy that follows.

Definition 5.6a (Primitive in $\mathcal{O}_{\mathrm{adm}}$). *A primitive in* $\mathcal{O}_{\mathrm{adm}}$ *is an irreducible generator of* $\mathcal{O}_{\mathrm{adm}}$ *under (P5) — i.e., an admissible operation whose action on* $\mathcal{A}_{\mathrm{stable}}$ *cannot be expressed as a composition of admissible operations of strictly lower iteration depth or fibre cardinality. The trichotomy claim is that* $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ *are the only such irreducible generators; every other admissible operation is a composite of these under (P5).*

This definition makes the asymmetry between primitive-admissibility and composite-admissibility explicit: an operation $\mathcal{O} \in \mathcal{O}_{\mathrm{adm}}$ is *admissible* if it satisfies (O1)–(O3); $\mathcal{O}$ is a *primitive* if, in addition, it is not generated by lower-depth admissible operations under (P5). Wilson loops are admissible but not primitive; $F_{\mu\nu}$ is both admissible and primitive (a generator of the antisymmetric grade-2 sector); a derived operator like $\nabla_\mu F^{\mu\nu}$ is admissible but not primitive (generated by $F_{\mu\nu}$ under derivative composition).

With this definition, the locality exception lemma takes the form of a clean dichotomy:

Lemma 5.7 (Nonlocal Primitive Exclusion). *Every admissible nonlocal operation on $\mathcal{A}_{\mathrm{stable}}$ is a (P5)-composite of $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ content (horn a), and therefore not a primitive in the sense of Definition 5.6a. The class of admissible nonlocal operations not so expressible (horn b) is empty by (P4) closure–operational equivalence: closure-invisible content is operationally invisible and therefore not in $\mathcal{O}_{\mathrm{adm}}$ at all. Equivalently: every admissible nonlocal observable is representable as an integral or composition over local closure data; no nonlocal candidate defines an independent primitive grade.*

Proof. The structural claim is the dichotomy stated in Definition 5.6a's framework: a candidate nonlocal *primitive* must be both admissible and not generated by lower-depth admissible operations under (P5). We show that no candidate satisfies both conditions.

Horn (a) — admissible nonlocal observables are (P5)-composites of local data, hence not primitives. Consider an admissible nonlocal observable, for example the Wilson loop $W(\gamma) = \mathrm{tr}\,\mathcal{P}\exp(i \oint_\gamma A_\mu\, dx^\mu)$ on a closed loop $\gamma \subset \mathcal{S}$. By construction, $W(\gamma)$ is built by line-integrating the connection A_μ (Cell B content) along γ and applying the path-ordered exponential. The line integral is an iterated composition of Cell B operations along the points of γ; the path-ordering is an (P5)-admissible composition. So $W(\gamma)$ is an admissible *composite* of grade-1 content — it satisfies admissibility but it is not an irreducible generator under (P5), and therefore not a primitive in the sense of Definition 5.6a.

The structural claim:

All admissible nonlocal observables are representable as integrals or compositions over local closure data; therefore they do not define independent primitive operations.

The same argument applies to standard topological observables and topological action terms in the closure-state field theory. We collect the most natural candidate counterexamples and verify each is composite-not-primitive:

- **Holonomies and Wilson loops** $\oint_\gamma A$ or $\mathcal{P}\exp(i\oint_\gamma A)$ on contractible *or non-contractible* loops: line-integral composites of grade-1 (Cell B) connection content. The non-contractible case is worth highlighting because $\pi_1(\mathcal{S})$ holonomy might naively appear to be horn-(b) content (genuinely topological, not local). It is not: holonomy on a non-contractible loop is recovered by parallel-transporting along a representative of each homotopy class, which is a (P5)-iterated composition of grade-1 connection content along that loop. Topological does not mean primitive; topological observables are admissible composites of grade-1 content. Admissible; not primitive.

- **Magnetic fluxes** $\int\Sigma\, F\{\mu\nu\}\ d\Sigma^{\mu\nu}$ on a 2-surface Σ: surface-integral composites of grade-2 antisymmetric (Cell C) curvature content. Admissible; not primitive.
- **Theta-angle terms** $S_\theta = (\theta/8\pi^2) \int \mathrm{tr}(F \wedge F)$ in 3+1D: 4-form integrals of polynomial composites of grade-2 antisymmetric content. Admissible by (P5) extended over polynomial composites and integration; not primitive.
- **Chern–Simons actions** $S_{CS} = \int \mathrm{tr}(A \wedge dA + (2/3)\, A \wedge A \wedge A)$ in 2+1D: composites of grade-1 (A) and grade-2 antisymmetric ($dA = F$) content under wedge product and integration. Admissible; not primitive.
- **Chern numbers and Pontryagin classes** $\int \mathrm{tr}(F^k)$: polynomial composites of grade-2 antisymmetric content under wedge product and integration. Admissible; not primitive.
- **Aharonov–Bohm phase** $\Delta\varphi = \oint\gamma\, A_\mu\, dx^\mu$ *around a loop enclosing a region of non-zero flux, even where* $F\{\mu\nu\} = 0$ along both paths: this is the closest physical realization of "horn-(b)-style" nonlocal content, since it is detectable even when local field strength vanishes along the observer's accessible region. It is nonetheless horn (a): the phase requires the connection A_μ to be defined on the loop (where $F = 0$ but $A \neq 0$), and the phase is the line integral of grade-1 content along the loop. The Aharonov–Bohm effect is therefore the empirical signature that the *connection* A_μ — not just the *curvature* $F_{\mu\nu}$ — is grade-1 admissible primitive content; it is exactly what [VERSF–GAUGE] Theorem 5.1 establishes structurally. Admissible; not primitive.

These are the obvious topological-action counterexamples a referee will raise. In every case, the structure is a composite of grade-1 and/or grade-2 content under (P5)-admissible operations (line integrals, surface integrals, wedge products, polynomial composites). They are real physical observables, and they are real composite admissible operations on $\mathcal{A}_\text{stable}$; they are not primitive grades. The trichotomy is preserved against this entire class of natural counterexamples.

Horn (b) — genuinely closure-invisible candidates: empty by (P4). Suppose, on the other hand, that $\mathcal{O}$ is a candidate nonlocal primitive that is *not* expressible as a (P5)-composite of $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ content — i.e., $\mathcal{O}$ depends essentially on global content that cannot be reconstructed from local closure data plus the connection along *any* loop, contractible or non-contractible. (We have just seen in horn (a) that all topological holonomy content, including $\pi_1(\mathcal{S})$ holonomy, is recoverable from grade-1 connection data along loops representing the relevant homotopy classes; horn (b) candidates must therefore be content strictly stronger than topological holonomy.) Concrete candidates would be: distinctions between two substrates that are topologically equivalent and connection-equivalent but differ in some structural feature not detected by any closure-equivariant local construction.

We claim horn (b) is *empty as a class of admissible operations*, by (P4) closure–operational equivalence.

(P4) states that closure equivalence classes coincide exactly with operational distinguishability classes on $\mathcal{S}$. Any structural feature of $\mathcal{S}$ that is not present in the closure-equivalence content of any region is, by (P4), not present in the operational distinguishability content of any region —

observers comparing closure structure cannot, even in principle, register such features as distinct. A candidate operation $\mathcal{O}$ whose action depends essentially on such features is therefore acting on a distinction that observers cannot resolve, so $\mathcal{O}$ fails (O2) observer-comparability *and* fails to be defined on $\mathcal{A}$_stable in the first place. By (P4), there are no admissible operations whose content is genuinely closure-invisible; horn (b) is empty.

The dichotomy collapses. Combining horns (a) and (b): every candidate nonlocal primitive is in horn (a) — admissible-but-composite, hence not a primitive in the sense of Definition 5.6a. Horn (b) is empty by (P4): closure-invisible content is operationally invisible and therefore not in $\mathcal{O}$_adm at all. The trichotomy is preserved against all nonlocal candidates: independent primitive operations are exhausted by the generators $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$, and all admissible nonlocal content is a (P5)-composite of grade-1 and grade-2 generators. ■

The architecture of the result is sharper than the dichotomy framing initially suggests: rather than splitting candidate primitives into "composite" and "inadmissible," (P4) collapses the second class to empty, and Lemma 5.7's force is *every admissible nonlocal operation is a composite of local primitives*. This is a stronger structural statement than the original framing, and it has the side effect of clarifying what (P4) does in the framework: (P4) forbids closure-invisible content from being observable at all, so the only nonlocal admissible operations are the ones reconstructible from local closure data. Topological observables (Wilson loops, A–B phase, theta-angles, Chern numbers) all sit in this reconstructible class.

Remark 5.8 (Structural vs. Gauge-Theoretic Exclusion). *The exclusion of nonlocal primitives is structural, not gauge-theoretic. We have not argued that Wilson loops or other topological observables are unphysical — they are real measurable physical quantities, and they are admissible composite operations on $\mathcal{A}$_stable. We have argued that they are not* primitive operations *in the sense of Definition 5.6a (irreducible generators of $\mathcal{O}$_adm under (P5)). The trichotomy classifies primitives; it does not classify physical observables. Many physical observables (theta-angle composites, magnetic monopole counts, fluxes, holonomies, Chern numbers, etc.) are derived composites of grade-1 and grade-2 content. The trichotomy holds for the primitives, and the composite admissible content recovers the standard topological-observable spectrum.*

This is also the cleanest derivation of the locality restriction (A4) of [VERSF–CLOS-ALG] §7.2: the present argument does not assume locality as an external postulate but derives it from the record-theoretic constraints (P1)–(P4) of this paper. Nonlocal primitives are excluded *because* observable physics on $\mathcal{A}$_stable cannot support them, not because we postulated they don't exist.

5.5 Exhaustion

Lemmas 5.1–5.3 populate $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ with Cells A, B, C respectively. Lemmas 5.4–5.6 reduce Cells D, F, G, H to composites of Cells A, B, C. Lemma 5.7 excludes nonlocal primitives (the residual content of Cell E that does not reduce by Lemma 5.6) from admissibility. The classification table of §4.3 is exhausted.

6. The Closure Trichotomy Theorem

We now state the consolidated result.

Theorem 6.1 (Closure Operation Trichotomy). *Let $\mathcal{S}$ be a distinguishability substrate satisfying (P1)–(P6). Let $\mathcal{O}_{\text{adm}}$ be the algebra of admissible operations on the stable record algebra $\mathcal{A}_{\text{stable}}$ in the sense of Definition 3.2. Then $\mathcal{O}_{\text{adm}}$ is the (P5)-closure of three irreducible classes of primitive generators:*

$\mathcal{O}_{\text{adm}} = \langle \mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2 \rangle_{(P5)}$,

read as "the (P5)-closure of the generator classes $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ under composition," where:

- $\mathcal{C}^0 = \Omega^0(\mathcal{S}) \otimes \mathrm{End}_{\text{ray}}(\mathcal{H}_x)$ *— operations identifying closure states (role i, fibre cardinality 1, differential order 0);*
- $\mathcal{C}^1 = \Omega^1(\mathcal{S}; i\mathbb{R})$ *— operations comparing closure states at neighbouring substrate points (role ii, fibre cardinality 2 neighbouring, differential order 1);*
- $\mathcal{C}^2 = \Lambda^2(\Omega^1(\mathcal{S})) \oplus \mathrm{Sym}^2(\Omega^1(\mathcal{S}))$ *— operations registering obstruction (antisymmetric sector) or back-reaction response (symmetric sector) (role iii, fibre cardinality 2 neighbouring, differential order 2). The internal $\oplus$ here is a genuine direct sum of vector spaces — antisymmetric and symmetric rank-2 tensors are direct-sum complements — distinguished from the outer $\langle\ ,\ ,\ \rangle_{(P5)}$ generator notation.*

Every admissible operation belongs to exactly one of these primitive classes or is a (P5)-composite of them. No independent fourth class of primitives exists.

Proof. By the exhaustion argument of §5.5: Lemmas 5.1–5.3 establish the three primitive classes; Lemmas 5.4–5.6 reduce all higher-cell content to composites; Lemma 5.7 establishes that all admissible nonlocal operations are (P5)-composites of the three primitive classes (with horn b empty by (P4)). ■

Remark on the generator notation (unpacking $\langle\ ,\ ,\ \rangle_{(P5)}$). *The notation $\langle \mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2 \rangle(P5)$ denotes the smallest subalgebra of $\mathcal{O}_{\text{adm}}$ closed under (P5)-admissible composition that contains the three primitive classes. The three classes $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ are not themselves closed under composition: composing two $\mathcal{C}^1$ operations along a path produces $\mathcal{C}^2$ content (the path-ordered exponential of A_μ has $F_{\mu\nu}$ content via its commutator), and iterated higher-order compositions populate the higher cells of §4.3 before reduction. The (P5)-closure $\langle\ ,\ ,\ \rangle(P5)$ contains those compositions. We deliberately avoid writing $\mathcal{O}_{\text{adm}} = \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \mathcal{C}^2$ with a direct-sum symbol, because the three classes are not in direct sum: their composites populate higher cells, and the structural content of Theorem 6.1 is about the* generators *of $\mathcal{O}_{\text{adm}}$, not about a direct-sum decomposition of it. Stated cleanly: $\mathcal{O}_{\text{adm}}$ has three primitive-grade generators, and its full content (the (P5)-closure) contains the higher-order derivative content of Lemma 5.4, the cross-sector polynomials of Lemma 5.5, the finite-path comparisons of Lemma 5.6, and the topological-action composites of Lemma 5.7 (Wilson loops, theta-angles, Chern–*

Simons, A–B phase, etc.). The trichotomy is the claim about generators; the (P5)-closure is unbounded.

Corollary 6.2 (No Independent Fourth Role). *Under (P1)–(P6), every candidate fourth-role primitive operation is either reducible to composites of* $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ *by (P5), or excluded from* $\mathcal{O}$*_adm by failure of (O2) observer-comparability under record-theoretic constraints.*

Proof. Direct from Theorem 6.1: any candidate operation outside the three classes either falls within them by §4.3's classification or is excluded by §5.4–5.7. ■

This is the formal closure of the trichotomy claim that [VERSF–CLOS-ALG] §7.3(i) identified as a residual structural commitment.

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7. Connection to [VERSF–CLOS-ALG]: Discharging §7.3(i) and the Locality Assumption (A4)

The closure-algebra paper [VERSF–CLOS-ALG] proved a *No-Alternative Theorem* (§7.2 of that paper) under five stated assumptions plus a load-bearing structural step (the trichotomy of admissible roles, Step 4 of the proof, owned in §7.3 as a structural commitment rather than a derived theorem). The present paper discharges *two* things from that proof, not one: the trichotomy step (via Theorem 6.1) and the locality assumption (A4) (via Lemma 5.7). We make the dual discharge explicit, since it strengthens the assumption-count-reduction story.

What [VERSF–CLOS-ALG] §7.2 actually assumed. The five labelled assumptions were:

- (A1) Ray admissibility,
- (A2) Local comparability,
- (A3) Closure conservation,
- (A4) Locality (smooth limit),
- (A5) Minimal non-redundancy.

The trichotomy was Step 4 of the proof, not a labelled assumption — but it was a structural commitment of equal load-bearing weight to the labelled assumptions, owned as such in §7.3(i) of that paper.

What the present paper discharges. Theorem 6.1 of this paper derives the trichotomy from the foundational primitives (P1)–(P6); the trichotomy is no longer a structural commitment but a derived consequence. Lemma 5.7 of this paper additionally derives (A4) — locality is no longer assumed but follows from the record-theoretic constraints (P1)–(P4) plus the (P5)-composite reconstruction of nonlocal observables. Both reductions sit cleanly in this paper: the trichotomy follows from (P1)–(P6) plus Definition 5.6a; (A4) follows from (P1)–(P4) plus the dichotomy of Lemma 5.7 (with horn b empty by P4).

What survives as imported assumption. After both discharges, the remaining structural assumptions of [VERSF–CLOS-ALG] §7.2 are:

- (A1) Ray admissibility,
- (A2) Local comparability,
- (A3) Closure conservation,
- (A5) Minimal non-redundancy.

Three of these four are direct counterparts to the present paper's primitives: (A1) corresponds to (P3)+(O3); (A2) corresponds to (P3)+(P4)+(O2); (A5) corresponds to (P5). (A3) closure conservation is the structural input to §6.2 of [VERSF–CLOS-ALG] and is compatible with (P2) commitment plus the substrate metric of (P1)+[VERSF–CHS]; whether (A3) is fully *derived* from these primitives (rather than merely compatible with them) depends on the §6.2 derivation in [VERSF–CLOS-ALG], which we do not re-examine here. The four-assumption framing of Theorem 7.1 is therefore not just nominally smaller — three of the four labelled assumptions are direct counterparts of foundational primitives, and what remains genuinely closure-algebra-specific is concentrated in (A3): specifically, the substrate-metric input that gives closure events directional content beyond their bare commitment structure (cf. [VERSF–CLOS-ALG] §6.2). This residual structural content is real but narrow, and any future tightening of the assumption set would target this single point.

Theorem 7.1 (No-Alternative Theorem, four-assumption form). *The No-Alternative Theorem of [VERSF–CLOS-ALG] §7.2 holds with both the trichotomy step (Step 4 of that paper's proof) and the locality assumption (A4) of that paper discharged. Specifically, under:*

- *the four remaining structural assumptions of [VERSF–CLOS-ALG] §7.2 — (A1) Ray admissibility, (A2) Local comparability, (A3) Closure conservation, (A5) Minimal non-redundancy — and*
- *the primitive VERSF assumptions (P1)–(P6) of the present paper,*

no independent grade $k > 2$ admissible operation exists, and the closure algebra $\mathcal{C} = \mathcal{C}^{(0)} \oplus \mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)}$ is closed at grade 2.

Proof. The original [VERSF–CLOS-ALG] Theorem 7.2 proof had five labelled assumptions plus the trichotomy step. By Theorem 6.1 of the present paper, the trichotomy is a derived consequence of (P1)–(P6). By Lemma 5.7 of the present paper, the locality assumption (A4) of [VERSF–CLOS-ALG] is a derived consequence of (P1)–(P4) plus the dichotomy showing horn (b) is empty. Substituting Theorem 6.1 for Step 4 and Lemma 5.7 for (A4) yields a proof of the No-Alternative Theorem with four labelled assumptions and no remaining structural commitment in the proof beyond what the foundational primitives plus the substrate-metric input of (A3) supply. ■

The two remaining residual commitments of [VERSF–CLOS-ALG] §7.3 — the continuum-limit gap (§7.3(ii)) and postulate (M) at layer 0 (§7.3(iii)) — are *not* discharged by the present paper. Theorem 7.1 above is therefore not the unconditional No-Alternative Theorem but a strictly intermediate result: two structural commitments fewer (the trichotomy *and* (A4)), two still open.

The structurally hardest of the original three residual commitments was the trichotomy, since it was the load-bearing structural claim of the proof rather than a technical bridging assumption. Discharging it (and (A4)) is therefore the most consequential reduction of the residual-commitment set, even though (R1) continuum-limit and (R2) postulate (M) remain. The unconditional version of the No-Alternative Theorem still requires constructing the continuum limit from the K = 7 simplicial substrate to the smooth-manifold setting and replacing postulate (M) with a derived selection principle.

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8. Scope and Residual Commitments

We are explicit about what the present paper does and does not establish.

8.1 What is derived

- The trichotomy of admissible roles — that every admissible operation on $\mathcal{A}$_stable is either state identity, local comparison, or accumulated response — is now a theorem (Theorem 6.1) under (P1)–(P6).
- The exclusion of independent higher-order primitive operations (Corollary 6.2) follows from the trichotomy plus the cohomological closure results of [VERSF–GAUGE] §7 and Anderson 1989.
- The exclusion of nonlocal primitives is sharpened (Lemma 5.7) into the cleanest possible dichotomy: every admissible nonlocal observable is representable as an integral or composition over local closure data, and no candidate nonlocal operation defines an independent primitive. Wilson loops, holonomies, and Chern-type invariants exist as physics in $\mathcal{O}$_adm, but as composites of grade-1 content under (P5)-admissible composition — not as primitives. This derives the locality restriction (A4) of [VERSF–CLOS-ALG] §7.2 rather than imposing it.
- (P5) compositional completeness is reframed (Proposition 2.1) from a structural choice into a minimal consistency requirement: any framework violating (P5) cannot represent observable physics on $\mathcal{A}$_stable, since sequential measurement, derived observables, and law statements relating multiple operations all fail. (P5) is therefore not an assumption that prejudges the existence of new primitives; it is the closure condition on $\mathcal{O}$_adm forced by the requirement that $\mathcal{A}$_stable be the substrate of physics at all.
- The trichotomy is shown (§4.4) to admit a substrate-level reading via iteration depth: the differential hierarchy of §§4.1–4.3 is the continuum-limit name for an iteration hierarchy already present at the discrete level via face-adjacency. The trichotomy *structure* is substrate-compatible; its *proof* and the *identification* with spin sectors both currently live at the smooth-manifold level and inherit R1 (see §8.3).

8.2 What is imported

- The six primitive VERSF assumptions (P1)–(P6) are imported from [VERSF–CHS], [VERSF–FSN], [VERSF–LAW-REP], [VERSF–PAR-CC], and [VERSF–CCC]. Whether the assumption set itself can be reduced is open meta-structural work.

- The cohomological closure theorems for the gauge and gravitational sectors are imported from Henneaux–Teitelboim 1992 Ch. 12 and Anderson 1989 respectively. The present paper does not re-derive them.
- The smooth-manifold instantiations of [VERSF–SPIN] (for layer 0) and [VERSF–GAUGE] (for layer 1), used in the proofs of Lemmas 5.1–5.3, are imported and inherit the continuum-limit assumption R1 (see §8.3). The trichotomy *structure admits a substrate-level reading* via the iteration-depth invariance of §4.4; the layer-by-layer *identification* with spin sectors, and the proof of the trichotomy on the discrete substrate, both inherit R1.

8.3 Residual commitments

Two structural commitments remain on the closure-algebra programme after the present result:

(R1) The continuum-limit assumption — narrowed in scope. §4.4 of the present paper sharpens what R1 actually does and does not require. The trichotomy *structure* admits a substrate-level reading: the §4.3 classification maps cleanly onto an iteration-depth hierarchy on the K = 7 simplicial substrate via face-adjacency and simplicial composition, with finite distinguishability (P1) bounding resolvable iteration depth at any scale. R1 therefore does *not* threaten the structural compatibility of the trichotomy with the substrate level.

R1 still enters in two places: (i) the *proof* of Theorem 6.1 currently lives at the smooth-manifold level (Lemmas 5.1–5.3 use [VERSF–SPIN] and [VERSF–GAUGE] in their smooth instantiations; Lemma 5.4 uses Henneaux–Teitelboim and Anderson in their smooth instantiations), and the substrate-level proof of the trichotomy is part of the open continuum-limit work; and (ii) the *identification* of the three iteration classes with the {½, 1, 2} spin sectors uses smooth-manifold representation theory which has no settled simplicial analogue. Both are part of the continuum-limit work of [VERSF–CLOS-ALG] §10.6 and remain open.

Stated cleanly: *the trichotomy structure is substrate-compatible; its proof and its identification with spin sectors both inherit R1*. The advance over [VERSF–CLOS-ALG] §7.3(ii) is that R1's role is now narrower — it must extend the smooth-manifold proof to the substrate level, not establish a fundamentally different structural result. R1 is reduced in scope, not eliminated.

(R2) Postulate (M) at layer 0. Theorem 6.1 establishes that $\mathcal{C}^0$ contains the projective SO(3) representations, with j = ½ being the smallest one that is genuinely projective rather than linear. It does *not* establish that physics realizes j = ½ rather than j = 0 (trivial) or j = 3/2 (next half-integer). Concretely: the present argument shows that $\mathcal{C}^0$ is the unique state-identity sector and that it admits the j = ½ representation; it does not pin the j = ½ representation specifically.

A future paper discharging R2 would supply a selection principle — perhaps via stability under closure dynamics, perhaps via fact-production constraints from [VERSF–FSN], perhaps via a representation-theoretic uniqueness argument we have not yet identified — that pins j = ½ as the realized representation. The relationship between the present paper's $\mathcal{C}^0$ and that future selection principle would be: this paper produces the *space* of admissible representations at layer 0; the selection principle picks out the *one* realized in physics. The two-step structure (admissibility produces a space; selection picks a point) is the same as the structure at layers 1 and 2, where uniqueness rather than minimality does the selection work.

We flag this division of labour now so that the path forward is visible: the trichotomy theorem of this paper does not interfere with R2, and the selection principle of a future R2 paper does not interfere with this paper. The two results compose cleanly: trichotomy + selection = full layer-0 derivation.

These are exactly the (ii) and (iii) residual commitments of [VERSF–CLOS-ALG] §7.3. The present paper closes (i) and leaves (ii) and (iii) for separate work.

8.4 What this paper does not claim

- **A derivation of the primitive assumptions (P1)–(P6).** These are imported. The present argument is conditional on the validity of the assumption set; whether the set can itself be reduced is open.
- **A derivation of the smooth-manifold setting from the discrete substrate.** This is residual commitment (R1) above.
- **A derivation of postulate (M).** This is residual commitment (R2) above.
- **The unconditional No-Alternative Theorem.** Theorem 7.1 reduces the assumption count from "five labelled assumptions plus the load-bearing trichotomy step" to four labelled assumptions, by discharging the trichotomy step (Theorem 6.1) and the locality assumption (A4) of [VERSF–CLOS-ALG] (Lemma 5.7). The unconditional version requires resolving (R1) and (R2) as well.
- **New empirical predictions.** The present paper is structural; the empirical content is the same as that of [VERSF–CLOS-ALG] (which is itself the structural-form-only kinematic content of the underlying Standard Model and General Relativity).

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9. Conclusion

We have proved that the trichotomy of admissible closure operations — into state identity, local comparison, and accumulated response — is a derived consequence of six primitive VERSF assumptions (P1)–(P6) rather than a structural commitment layered on top of the closure-algebra picture. Six structural advances support the theorem:

1. **(P5) reframed (Proposition 2.1).** Compositional completeness is shown to be a minimal consistency requirement on any framework supporting observable physics on $\mathcal{A}$_stable, not an additional structural choice. Sequential measurement, derived observables, and multi-operation law statements each fail without (P5); the assumption is therefore forced by the basic operational structure of physics-on-records, not chosen to prejudge the no-new-primitives conclusion.
2. **Finite-order reduction (Lemma 3.4).** Continuous-support operators (integral kernels) and unbounded-differential-order operators (functional calculus, pseudodifferential operators) are shown to be limits/composites of finite-fibre, bounded-order primitives

under (P5), not independent primitives. The classification of §4.3 is exhaustive over the *primitives* of $\mathcal{O}$_adm; continuous-support and unbounded-order content lives in the (P5)-closure of finite-resolution operators.

3. **Iteration depth admits a substrate-level reading (§4.4).** The differential hierarchy of the classification is shown to be the continuum-limit name for an iteration hierarchy already present at the discrete level via face-adjacency on the K = 7 simplicial substrate. The trichotomy *structure* admits this substrate-level reading, with finite distinguishability bounding resolvable iteration depth. The smooth-manifold setting is a convenient continuum domain in which the trichotomy proof currently lives; it is not a structural prerequisite for the trichotomy result. The substrate-level *proof* (and the substrate-level identification of iteration classes with spin sectors) remains open as part of R1.
4. **Cohomological bridge (Bridge Remarks 5.4a–b).** The imports from Henneaux–Teitelboim 1992 Ch. 12 (Abelian gauge BRST cohomology) and Anderson 1989 (variational bicomplex) are no longer invoked invisibly. The bridge identifies closure-equivariance on the connection sector with U(1) gauge invariance and closure-equivariance on the symmetric h_{μν} sector with diffeomorphism invariance, licensing the cohomological imports without modification.
5. **Primitive vs. composite explicitly defined; horn-(b) empty (Definition 5.6a + Lemma 5.7 + Remark 5.8).** A primitive in $\mathcal{O}$_adm is now explicitly defined as an irreducible generator under (P5). The trichotomy is the claim that $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$ are the only such generators. Lemma 5.7's dichotomy collapses by (P4): horn (b) (closure-invisible candidates) is empty because closure-invisible content is operationally invisible by (P4); horn (a) (composites of local data) absorbs all admissible nonlocal observables. Wilson loops, holonomies (including non-contractible π_1 holonomy), Aharonov–Bohm phase, theta-angle terms, Chern–Simons actions, Chern numbers, and similar topological observables are all in horn (a) — admissible composites of grade-1 and grade-2 content, *real physics, not primitives*. The trichotomy classifies primitives; it does not classify physical observables. The locality restriction (A4) of [VERSF–CLOS-ALG] §7.2 is *derived* from record-theoretic admissibility, not imposed — discharging the second of the two structural commitments addressed by this paper.
6. **Rank-2 decomposition strengthened (Lemma 5.3).** The bilinear-differential content at grade 2 is shown to decompose explicitly as antisymmetric (gauge curvature, contracts with [D_μ, D_ν] = i F_μν) and symmetric (gravitational response, contracts with the metric perturbation h_{μν}). Mixed structures (Lie derivatives, traces, mixed partials with the connection) reduce to these by integration-by-parts modulo total derivatives, with the cohomological imports of Lemma 5.4 handling higher-rank content.

The result discharges residual commitment (i) of [VERSF–CLOS-ALG] §7.3 (the trichotomy) *and* derives the locality assumption (A4) of that paper, converting the No-Alternative Theorem of [VERSF–CLOS-ALG] from "five labelled assumptions plus a load-bearing structural step" to four labelled assumptions with no remaining structural commitment beyond what the foundational primitives plus the substrate-metric input of (A3) supply. The closure algebra grading

$$c = c^{(0)} \oplus c^{(1)} \oplus c^{(2)}$$

is now structurally forced rather than postulated, modulo the two remaining commitments (R1: the substrate-level proof and the spin-sector identification, both currently living at the smooth-manifold level; R2: postulate (M) at layer 0) that the present paper does not address.

On the assumption-count accounting. A skeptical reading might say: the result has replaced two structural commitments (the trichotomy and the locality assumption (A4)) with six primitive assumptions (P1)–(P6), so the cognitive cost has gone up. We disagree, for the reason given in §1.1: (P1)–(P6) are foundational VERSF primitives independently load-bearing across [VERSF–CHS], [VERSF–FSN], [VERSF–LAW-REP], [VERSF–PAR-CC], and [VERSF–CCC]. They were already in force when [CLOS-ALG] §7.2 was proved; what changes here is the assumption count *specific to the closure-algebra picture*, not the assumption count of the *foundational* programme. Both the trichotomy and (A4) are now seen to be structural consequences of the framework's foundations, not additional commitments specific to spin/gauge/gravity. That is the genuine reduction.

The conceptual content of the result, stated honestly: the three-role decomposition of admissible closure operations is not a feature we read off the {½, 1, 2} answer; it is a feature of what an *admissible operation on a stable record algebra* can structurally be. The closure-algebra picture's headline grading falls out of the trichotomy, and the trichotomy falls out of the primitive VERSF assumptions on distinguishability, commitment, and law closure. The smooth-manifold differential hierarchy that organizes the classification *admits a substrate-level reading* via the iteration-depth picture of §4.4, so the trichotomy structure is compatible with the discrete substrate; the *proof* of the trichotomy currently lives at the smooth-manifold level and inherits R1, narrowed in scope but not eliminated. What does *not* fall under the trichotomy at all are the topological-observable composites (Wilson loops, theta angles, Chern numbers): they are real physics, but they are admissible composites of the grade-1 and grade-2 generators, not primitive grades.

The next steps in the closure-algebra programme are: (R1) the continuum-limit construction from the K = 7 simplicial substrate, plausibly addressed in [VERSF–PROTO] and [VERSF–CCB], specifically targeted at the *identification* step (translating Lemmas 5.1–5.3 to the discrete substrate, with simplicial Poincaré–Anderson framework as a substantive open requirement, not a routine translation); (R2) replacement of postulate (M) at layer 0 with a derived selection principle, composing cleanly with the present paper's trichotomy result; and the meta-structural question of whether the primitive assumption set (P1)–(P6) can itself be reduced.

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VERSF programme

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- VERSF–CCC: *The Commitment-Capacity Density Quartic Inequality.*

- VERSF–PAR-CC: *Pre-Factual Algebraic Reversibility and Compositional Completeness.*
- VERSF–LAW-REP: *Representation Theorem for Admissible Macroscopic Laws.*
- VERSF–KSEVEN: *No-Go Theorem on Non-Simplicial Relational Substrates: K = 7 and Triangular 2-Complex Geometry as Structural Necessities.*
- VERSF–GAUGE: *Gauge Fields as Closure Connections: A Reconstruction-Level Derivation of the Abelian Connection and Spin-1.*
- VERSF–CLOS-ALG: *Closure Algebra and the Origin of Spin, Gauge, and Gravity in the VERSF Framework.* (Companion paper.)
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