The Overdetermined Born Rule

A Convergence Analysis of Five Independent Derivations within VERSF

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General-Reader Abstract

When a quantum measurement is made, the result is unpredictable in a way that does not seem to reduce to ordinary ignorance. The universe itself appears not to have decided which outcome will occur until the measurement happens. The probabilities of the different outcomes nevertheless follow a precise mathematical rule — the squared modulus of a complex-valued amplitude, known as the Born rule. In the textbook formulation this rule is simply assumed. It sits alongside the other axioms of quantum theory as an additional ingredient with no internal explanation. For nearly a century, physicists have asked whether it must really be assumed, or whether it can be derived from something more fundamental. Several derivations exist — Gleason's theorem, Deutsch–Wallace decision theory, Zurek's envariance, the operational reconstructions of Hardy and Masanes–Müller — but each replaces the Born rule with a different starting assumption whose physical status is itself contested. The puzzle has resisted a clean solution.

This paper takes a different approach. Rather than offering yet another single derivation of the Born rule, it analyses *five* derivations that have already been completed within a single physical framework — the Void Energy-Regulated Space Framework (VERSF) — and asks what the existence of five independent derivations *jointly* tells us. Each of the five proceeds from a different starting point: from the structure of mutually-exclusive outcomes, from the consistency of the universe's record of facts, from the geometry of paths between distinguishable states, from the requirement that any alternative rule fail under physical scrutiny, and from the thermodynamic cost of recording an outcome as permanent. Each is its own paper, with its own theorems and its own proof techniques. Each terminates at exactly the same probability rule: the Born rule and nothing else.

This kind of multiple convergence has a precedent in physics. The numerical value of Avogadro's constant, for instance, was not finally settled by any single experiment but by the convergence of many — Brownian motion, X-ray crystallography, blackbody radiation, oil-drop electrometry, electrochemistry — each open to its own systematic objections, all converging on the same value. By 1920 the community's confidence in N_A rested less on any single experiment than on the structural fact that *every independent route gave the same answer*. To dismiss the value would have required showing that five different sets of assumptions were jointly unsound in a way that produced the same numerical agreement — a claim with no plausible mechanism behind it.

The paper argues that the Born rule has now reached an analogous status within VERSF. It is not that any one of the five derivations is unimpeachable; it is that to undermine the rule, an alternative would have to violate not one but five independent constraints simultaneously, each derived from a different physical principle. We make this argument carefully: we isolate the small set of background assumptions all five derivations *do* share — irreversibility, finite distinguishability, reversible pre-commitment dynamics, compositional structure, and observer invariance — and show that this shared kernel is far weaker than the Born rule itself, and that each derivation closes the gap between kernel and conclusion through *different additional* primitives.

The result is a stronger form of foundational support than any single proof can supply: not one derivation but a convergence. The Born rule transitions from "postulated" to "structurally overdetermined" — within the limits we set out honestly in the technical body of the paper. Five independent paths to the same destination are harder to dismiss than five independent attempts to find one.

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Abstract

Five derivations of the Born rule $p_i = |\langle i|\psi\rangle|^2$ have now been completed within the Void Energy-Regulated Space Framework (VERSF), each starting from a distinct primitive: closure incompatibility on admissible partitions (Route I, the representation-theoretic route, via Piron–Solèr and Busch-extended Gleason); single-source consistency of the committed record (Route II); pairwise selection on path correlations in discrete informational geometry (Route III); the physical admissibility filter on alternative probability rules (Route IV); and a maximum-caliber argument on an entropic unfolding action (Route V, the thermodynamic route). We show that these five routes share no proof step and no premise beyond the common VERSF kernel, and we prove a Convergence Theorem stating that they entail the same probability measure — not approximately, but identically. We argue that this convergence constitutes a structural overdetermination of the Born rule with no parallel in the existing reconstruction literature: standard derivations (Gleason, Deutsch–Wallace, Zurek, Hardy, Masanes–Müller) each replace the Born postulate with a single alternative postulate of contested status, while the present result places the rule at the intersection of five operationally independent constraints, each of which would have to fail for the rule to be wrong. Independence is established here at the *operational* level — by inspection of distinct primitive vocabularies and proof structures — rather than at the *model-theoretic* level; we are explicit about this distinction and identify formal model-theoretic independence as an open technical task. We analyse the residual common dependencies — the irreducible kernel of VERSF assumptions on which all five routes rest — and identify the precise content of the overdetermination: not that the Born rule cannot be wrong, but that its failure modes are now comprehensively constrained, and that any alternative probability structure must violate not one but five admissibility conditions simultaneously. Within VERSF, the Born rule is no longer a free postulate but a structurally overdetermined consequence.

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1. The Argument from Overdetermination

Most reconstructions of the Born rule replace one postulate with another. Gleason's theorem trades the rule for non-contextuality of frame functions on a presupposed Hilbert lattice. Deutsch–Wallace trade it for decision-theoretic rationality axioms applied to branching worlds. Zurek's envariance trades it for environment-induced symmetries of entangled states. Hardy's reconstruction trades it for operational primitives of state tomography; Masanes–Müller for information-theoretic compositional axioms. Each derivation succeeds. None settles the foundational discomfort, because the replacement postulate is no less contested than what it replaces — and in some cases more so.

The present paper does not add a sixth derivation. It analyses the five derivations already completed within the VERSF programme and argues that their joint structure — five operationally independent routes terminating at the same probability measure — produces a category of foundational support that no individual derivation can supply.

The argument has a familiar shape in the history of physics. The numerical value of Avogadro's constant, for instance, was not established by any single experiment but by the convergence of methods that drew on independent physical principles: Brownian motion, X-ray crystallography, blackbody radiation, electrochemistry, oil-drop electrometry. Each method, considered in isolation, was open to systematic objection. Their convergence was not. To dismiss the value of N_A on the basis that one method's assumptions were unsound would have required showing that five independent sets of assumptions were *jointly* unsound in a way that produced the same numerical agreement — a claim with no plausible mechanism behind it.

The Born rule, we will argue, has now reached an analogous status within VERSF, with one specific observation that translates the abstract claim into a concrete asymmetry against the existing reconstruction literature. The historically contested premise of Gleason-style derivations — non-contextuality of probability assignments on the projection lattice — does not appear as a free postulate in any of the five routes here. It is *derived* along Route I from observer-invariant fact formation; it is *derived* along Route II from the single-source theorem $\mathcal{O} = \boldsymbol{\mathcal{F}}[\rho]$; and it does not appear at all in Routes III, IV, V. The premise that has stood as the hard kernel of the Gleason controversy for nearly seventy years is therefore replaced twice independently and routed around three times within a single overdetermined argument. This is the most concrete sign that the present convergence delivers something the existing literature does not.

The headline claim, in one line:

The Born rule is not uniquely derived once — it is uniquely recovered across mutually independent derivational logics.

The remainder of the paper is the unpacking of that line.

The paper is organised as follows. Section 2 catalogues the five routes and identifies their primitive structures, with explicit framing of Route I as the representation-theoretic bridge to standard quantum-logic reconstructions, Route IV as an elimination argument over a parametric space of alternatives rather than a constructive derivation, and Route V as a thermodynamic argument with the symmetry sub-route of *Born Rule as Entropic Unfolding* §14 explicitly *not counted* (since it duplicates Route I's closing step). Section 3 establishes the routes' *operational* independence, is candid about the gap between this and full model-theoretic independence, and addresses two structural concerns directly: the Hilbert-structure dependency across routes II, IV, V, and the closeness in content between Routes I and II's derivations of non-contextuality. Section 4 states and proves the Convergence Theorem, with explicit attention to what "the same measure" means across the five routes' native vocabularies. Section 5 analyses the common VERSF kernel — the irreducible set of assumptions shared by all five — and frames its weakness relative to the Born rule as a conjecture rather than a theorem, with a counter-example demonstration identified as the headline open task. Section 6 develops the methodological argument: what overdetermination buys, and what it does not. Section 7 compares the structure to other overdetermined results in physics, with a deliberate weakening of the Avogadro analogy that the result does not require. Section 8 sets out what would, in principle, be required to overturn the result.

2. The Five Routes

We label the five derivations I–V. Each is given here in compressed form; the full statements appear in the cited companion papers.

Route I — Closure-theoretic representation

Framing. Route I is, deliberately, *the representation-theoretic route*: it builds on the standard quantum-logic reconstruction pipeline of Piron–Solèr followed by Busch-extended Gleason, with VERSF supplying the physical grounding for the lattice structure and the contested premise of non-contextuality. It plays a dual role within the present argument that is worth making explicit:

1. **Bridge.** Route I is the bridge between the standard quantum-logic reconstruction tradition and the four genuinely VERSF-internal derivations (II–V). It tests whether the VERSF framework reproduces the standard pipeline's conclusion when run through the standard pipeline's machinery, with the contested premise (non-contextuality) replaced by a derived one.
2. **Consistency check.** Equivalently, Route I answers the question: *"Does VERSF, when fed through the standard quantum-logic reconstruction, produce the Born rule rather than a deformed Born rule?"* This is a different and smaller question than "does VERSF derive the Born rule from closure structure?" — and we are open about that. Route I's answer to the first question is yes, conditional on the audit items listed below.

We include Route I not because it is the most novel of the five (it is the closest to a standard reconstruction) but because *both* roles are useful within an overdetermination argument. If Routes II–V agreed with each other but disagreed with the standard pipeline, the convergence would be parochial to VERSF. The fact that Route I — running through the same machinery as Gleason's theorem, with a contested premise replaced by a derived one and several audit items honestly flagged — terminates at the same Born rule strengthens the convergence rather than diluting it. A reader who, on the basis of Route I's audit backlog, prefers to count it as a consistency check rather than as a fifth independent derivation can make that choice; the resulting four-route claim ("four VERSF-internal derivations plus one consistency check with the standard pipeline") is a defensible reading of the same material. We retain the five-route count in this paper because Route I does, in fact, terminate at the Born rule and does so via machinery distinct from II–V — but the bridge/consistency-check reading is acknowledged as legitimate.

Primitive structure. Admissible configuration space $\mathcal{A}$ with closure relation $\mathcal{C}$; admissible outcome classes ordered by inclusion; admissible partitions $\Pi = \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ satisfying closure consistency, finite distinguishability, and observer invariance.

Central theorems.

- *Lattice structure (Proposition 1).* The collection $\mathscr{L}$ of admissible outcome classes, ordered by inclusion, forms an orthocomplemented, atomistic, orthomodular lattice. Atomicity is realised concretely by the discreteness of the $K = 7$ closure cell. The orthomodularity condition is motivated in the source paper through the requirement that successive admissible measurements compose without producing inadmissible residues; we note that this motivation is gestural and a fully self-contained argument distinguishing orthomodularity from weaker (modularity) or stronger (distributivity) compositional conditions remains an open task within the Route I paper itself.

- *Closure–orthogonality correspondence (Theorem 1).* For any faithful representation $\varphi : \mathscr{L} \to \mathcal{P}(\mathscr{H})$ — where "faithful" means order-preserving, order-reflecting, and orthocomplementation-preserving — closure-incompatibility $\mathcal{A}_i \cap\_\mathcal{C}\ \mathcal{A}_j = \emptyset$ is equivalent to subspace orthogonality $\varphi(\mathcal{A}_i) \perp \varphi(\mathcal{A}_j)$. We note that this theorem's contribution is *interpretive* rather than mathematically novel: once φ is stipulated to be faithful and orthocomplementation-preserving, both directions follow from the lattice axioms with no further work. The theorem's value lies in the identification of orthogonality with closure incompatibility, not in any non-trivial representation-theoretic content.
- *Representation theorem (Theorem 2).* Conditional on the Piron–Solèr conditions (covering law, irreducibility, and an *infinite* orthonormal-style sequence of atoms in the relevant sectors), $\mathscr{L}$ is isomorphic to the lattice of closed subspaces of a Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$; the companion VERSF interference result selects $\mathbb{C}$. The representation is unique up to unitary equivalence. The covering-law and infinite-sequence conditions are non-trivial — the former requires arguing from K = 7 granularity that no admissible class sits strictly between an outcome and its join with a covering atom; the latter requires VERSF to deliver an *infinite* atomic sequence under successive admissible measurements, not merely an arbitrarily large finite one. Both conditions are flagged as audit items under F1.
- *Non-contextuality as a derived condition.* The Route I paper argues that non-contextuality of probability assignments — the standard contested premise of Gleason's theorem — follows from observer-invariant fact formation: the admissible weight of an outcome $\mathcal{A}_i$ is held to be a property of $\mathcal{A}_i$ and its closure structure, not of the partition in which it is grouped. We note that the gap between observer-invariance (the *fact* $\mathcal{A}_i$ is observer-independent) and full Gleason non-contextuality (the *probability* $p(\Pi_i)$ is partition-independent) is wider than the prose admits: one could in principle imagine a framework where the fact is observer-invariant but its admissibility weight depends on which other cells appear in the partition, without obviously violating observer-invariance. Closing this gap requires an additional locality-of-weight assumption that the Route I paper does not make fully explicit. Within the overdetermination, this matters less than it might because Route II derives non-contextuality from independent grounds (the single-source theorem on ρ); see §3.1 and §6.3.
- *Born rule via Busch-extended Gleason.* Conditional on the representation theorem and the non-contextuality derivation surviving the audit above, Gleason's theorem applied to admissible projectors yields $p(P) = \mathrm{Tr}(\rho P)$ for $\dim \mathscr{H} \geq 3$; the Busch–Caves–Fuchs–Schack extension to POVMs (natural in VERSF, since coarse-grained admissible measurements are POVM-valued) closes the d = 2 case unconditionally. For pure states, $p_i = |\langle e_i|\psi\rangle|^2$.

Source. *Orthogonality as Closure Incompatibility: A VERSF Derivation of Quantum State Distinguishability and the Born Rule* (Taylor 2025).

Route II — Single-source consistency weighting

Primitive structure. Committed record density $\rho(x, t)$; single-source theorem $\mathcal{O} = \mathcal{F}[\rho]$; admissible extension set $\mathcal{C}(\rho, \{|i\rangle\})$ of candidate commitments; consistency weighting W as the measure of invariance of admissible observables under admissible extensions.

Central theorems.

- *Disjoint-sector factorisation (Proposition 1).* For $\rho = \rho_A \oplus \rho_B$, every admissible W satisfies $W(C_{(i,j)}) = W^A(C_i) \cdot W^B(C_j)$.
- *Consistency-weighting theorem (Theorem 1).* W satisfying non-negativity, non-contextuality, unitary invariance, and compositional multiplicativity takes the form $W_i(|\psi\rangle) = \kappa|\langle i|\psi\rangle|^2$, giving $p_i = |c_i|^2$.
- *No-go theorem (Theorem 2).* No alternative admissible probability structure exists: any such candidate either depends on data outside ρ (forbidden by the single-source theorem) or violates one of the four admissibility conditions.
- *Gibbs corollary.* $p_i = \exp(-\Delta I_i)$ with extension-information $\Delta I_i = -\ln|c_i|^2$.

Source. *Probability as Consistency: Eliminating the Born Rule as an Independent Postulate* (Taylor 2025).

Route III — Pairwise selection on path correlations

Primitive structure. Discrete metric space $(\mathcal{S}, d)$ of distinguishable microstates; isometry group G of reversible transformations; geometric phase $\theta : \text{Paths}(\mathcal{S}) \to \mathbb{R}/2\pi\mathbb{Z}$ from holonomy of G; irreversible selection acting on path-correlation structures.

Central theorems.

- *Bilinearity (Theorem 4.1, Part II).* Selection on correlations forces probability to take the bilinear form $P(A) = \Sigma_{\{P, P' \in R_A\}} W(P, P')$.
- *Rank-one structure (Theorem 5.2, Part II).* Factorisation forces W to be rank-one: $W(P, P') = \varphi(P)\, \varphi(P')^*$.
- *Phase form (Theorem 5.3, Part II).* $\varphi(P) = e^{i\theta(P)}$ is the unique form compatible with multiplicativity and geometric dependence.
- *Born rule.* Substituting yields $P(A) = |\Sigma_{\{P \in R_A\}} e^{i\theta(P)}|^2 = |\psi_A|^2$.
- *Quadratic uniqueness (Theorem 5.4, Part II).* No exponent $p \neq 2$ satisfies factorisation and unitary normalisation jointly.

Source. *The Double Square Rule: A Derivation of Quantum Probability from Discrete Informational Geometry* (Taylor 2025, Part II).

Route IV — Physical admissibility filter

Framing. Route IV is, by design, an *elimination* argument rather than a constructive derivation. It does not build the Born rule from positive primitives in the way Routes I, II, III do; instead, it characterises the structurally relevant neighborhood of candidate probability rules — phase structures, selection orders (k-linear), path-weight schemes, harmonic representations (n in $p_i \propto |c|^n$), exponents (p in $p_i = \|\psi_i\|^p$), and so on — and shows that every member of this neighborhood except the Born rule fails at least one admissibility condition (TPB-consistency, BCB compatibility, normalisation preservation under reversible evolution, factorisation). This is the neighborhood of alternatives that have actually been considered in physics, including those proposed in the existing reconstruction literature; it is not "all logically possible probability rules." Route IV's contribution to the convergence is therefore *negative-valent*: it shows that the alternatives a working physicist would actually entertain are systematically excluded by VERSF admissibility. A referee may legitimately note that this is a different epistemic activity from the constructive routes, and that any alternative far outside the considered neighborhood remains in principle possible. We agree, and treat Route IV as a complement to the constructive routes rather than a stand-alone proof.

Primitive structure. Tick-Per-Bit (TPB) time advancement; Bit Conservation and Balance (BCB); finite relational distinguishability; temporal extensibility; admissibility as the requirement of stable existence, measurability, and consistent composition.

Central theorems.

- *Continuous holonomy (Theorem 2.1, Companion).* Finite holonomy is incompatible with TPB + temporal extensibility; the minimal admissible holonomy is U(1).
- *Higher-order impossibility (Theorems 4.1–4.2, Companion).* Any k-linear selection kernel with $k > 2$ either reduces to a product of bilinear kernels or fails normalisation preservation under unitary evolution (TPB-violation), with positivity failure as a secondary obstruction.
- *Quadratic uniqueness from TPB-consistency (Theorem 8.1, Companion).* Among rules of the form $p_i = \|\psi_i\|^p$, only $p = 2$ is unitarily normalisation-preserving and hence TPB-consistent.
- *Admissibility filter.* Every alternative to the Born structural assumptions (no phase, individual-path selection, higher-order kernels, non-uniform path weights, higher harmonics, non-factorising probability, non-quadratic exponent) fails at least one admissibility condition.

Source. *Physical Necessity of Quantum Probability Structure* (Taylor 2025).

Route V — Maximum-caliber entropic unfolding

Framing. Route V is *the thermodynamic route*. It builds the Born rule from a maximum-caliber inference on an entropic unfolding action, recovering the rule in the iso-entropic limit and predicting small calculable deviations otherwise. It carries an interpretive commitment to maximum-caliber as a rational-inference principle that some readers will resist, and we are open about this. Sub-route Vb in earlier drafts of this paper — the frame-function-and-symmetry argument of §14 of *Born Rule as Entropic Unfolding* — is **not counted** toward the present convergence claim, because that sub-route closes via Busch-extended Gleason on a setup that is essentially Route I's closing step. To count it as independent of Route I would double-count the same proof. We mention the symmetry sub-route in the source paper for completeness, but the overdetermination argument here uses only the thermodynamic derivation.

Primitive structure. Reversible unitary kinematics on Hilbert space $\mathscr{H}$; entropy export $\Delta S_i \geq 0$ required to stabilise outcome i as a temporal record; maximum-caliber inference on outcome paths under fixed expected unfolding action.

Central theorems.

- *Uniqueness of the unfolding action (Theorem 3.0).* The action functional satisfying extensivity on products, operational regularity, iso-entropic neutrality, and thermodynamic additivity is uniquely $M_i = -\ln a_i + \lambda\Delta S_i$.
- *Gibbs-biased law (Theorem 1, Part I).* Maximum-caliber extremisation gives $P_i = (a_i$ e^{$-\lambda\Delta S_i$}) / $Z(\lambda)$.
- *Born limit (Corollary 1, Part I).* For ΔS_i = const, $P_i = a_i = |c_i|^2$.

Vulnerability note. Route V is the most exposed of the five to interpretive objection, because it relies on MaxCal inference and a notion of "entropy export to stabilise outcomes." A reader who rejects MaxCal as a primitive may reject Route V entirely. The four-route convergence on routes I, II, III, IV would then survive without it. We include Route V because the thermodynamic argument is independently developed and recovers the same Born form via genuinely distinct machinery; we acknowledge that its primitive set is the most contested.

Source. *Born Rule as Entropic Unfolding* (Taylor 2025, Part I).

Compact comparison

	Route I	**Route II**	**Route III**	**Route IV**	**Route V**
Primitive	closure relation $\mathcal{C}$	record density ρ	discrete metric d	admissibility filter	unfolding action M_i
Probability is	partition lattice measure	extension consistency	path-pair correlation	unique survivor	Gibbs-biased flux
Key theorem	Piron–Solèr representation	Consistency-weighting	Rank-one kernel	TPB-consistency	Action uniqueness
Proof technique	Orthomodular lattice → Piron–Solèr → Busch-extended Gleason	Cauchy multiplicative + additive	Mercer + factorisation	Elimination	Lagrange extremisation
Result	p_i =	⟨e_i	ψ⟩	²	p_i =

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3. Independence

Convergence on a common conclusion is uninformative if the routes secretly share a premise that does the work. Establishing independence — or, more precisely, the appropriate operational form of it — is therefore the central technical task of an overdetermination argument.

We claim:

Independence claim. The five routes are *pairwise operationally independent given their distinct primitive vocabularies*: no route's conclusion appears as a premise of any other route, and no route's central theorem can be derived from another's premises without reintroducing that other route's primitive structure.

We are deliberate about the qualifier "operationally." Full model-theoretic independence — exhibiting, for each pair (i, j), an explicit model in which Route i's premises hold but Route j's central theorem fails — is a substantial technical task that we leave open. The pairwise inspections below do not prove model-theoretic independence; they establish that no derivation in one route's vocabulary translates directly into a proof in another's vocabulary without re-running an independent argument. This is the operationally relevant condition for the convergence-based reasoning of §§4–6: it is what rules out the worst case in which two "independent" derivations are merely the same proof in different notation.

A serious referee may legitimately press on this point. We acknowledge it openly: the operational independence claim is supported by inspection of proof structures and primitive vocabularies, not by formal counter-models. Strengthening it to a model-theoretic theorem is a target for future work and is listed explicitly under F1 in §8.

To support the operational claim we examine each pair.

I vs. II. Route I works with admissible partitions of a configuration space and a closure relation $\mathcal{C}$; Route II works with a record density ρ and an extension set on it. The notion of "closure pathway" in Route I has no analogue in Route II's primitive vocabulary, and the single-source theorem $\mathcal{O} = \boldsymbol{\mathcal{F}}[\rho]$ used as an axiom in Route II plays no role in Route I's representation theorem. Both routes reach non-contextuality of the probability measure but from genuinely different primitive bases: Route I derives non-contextuality from observer-invariant fact formation on admissible cells, while Route II derives it from the single-source theorem's prohibition on hidden contextual data. The two routes can be derived in either order without circularity.

I vs. III. Route I derives Hilbert structure from an orthomodular lattice of admissible outcome classes via the Piron–Solèr representation theorem, then applies Busch-extended Gleason on the resulting projection lattice. Route III derives Hilbert structure from path holonomy in ($\mathcal{S}$, d) via Mercer decomposition of a positive-semidefinite correlation kernel, with rank-one structure forced by factorisation. The two routes draw on entirely different mathematical traditions — quantum-logic representation theorems on one side, kernel decomposition theorems on the other — and neither construction's apparatus is available within the other's vocabulary. Route I's atoms (minimal admissible cells) and Route III's individual paths play structurally different roles:

atoms are objects in the lattice $\mathscr{L}$, while paths are arguments of the kernel W(P, P′). Neither construction presupposes the other.

I vs. IV. Route I assumes the existence of admissible partitions and proves a representation theorem about them; Route IV assumes TPB and proves that finite holonomy is impossible. These results address different layers: Route IV concerns the holonomy structure of reversible dynamics; Route I concerns the algebraic structure of measurement contexts. Route IV's continuous-holonomy theorem is closer to a *prerequisite* for Route III than for Route I.

I vs. V. Route I produces the Born rule from closure structure with no thermodynamic input; Route V produces it from a thermodynamic action with no closure-theoretic input. Route V's iso-entropic limit recovers the Born rule, but the Born rule is not assumed in deriving the unfolding action.

II vs. III. Route II's Proposition 1 (disjoint-sector factorisation) and Route III's Theorem 5.2 (rank-one kernel) both produce multiplicative structure, but from different inputs: Proposition 1 from disjointness of ρ-sectors, Theorem 5.2 from factorisation on tensor-product path spaces. The intermediate functional equations (Cauchy multiplicative in II, eigenfunction multiplicativity in III) are formally similar but logically distinct: each holds in its own primitive vocabulary and neither implies the other without additional translation work.

II vs. IV. Route II's no-go theorem (Theorem 2) and Route IV's elimination table both establish that no alternative probability rule survives admissibility. But the admissibility conditions differ: Route II uses single-source structure to forbid hidden data; Route IV uses TPB-consistency to forbid normalisation leakage. Either could in principle survive without the other.

II vs. V. Route II's Corollary 1 produces a Gibbs-form $p_i = \exp(-\Delta I_i)$ that resembles Route V's $p_i \propto \exp(-\lambda\Delta S_i)$, but the underlying functionals are distinct: ΔI_i in Route II is the extension-information cost on ρ (a structural quantity), whereas ΔS_i in Route V is the thermodynamic entropy exported in stabilising outcome i (an operational quantity). The two coincide only at the level of the iso-entropic limit, where both reduce to the Born form. Their derivational machinery — Cauchy additivity in Route II, MaxCal extremisation in Route V — is distinct.

III vs. IV. Route IV is the closest of the five to a partial prerequisite for Route III: continuous holonomy must exist before path phases can be defined. But Route IV does not assume pairwise selection; it derives the *prohibition* of higher-order selection from TPB-consistency, in parallel with Route III's positive derivation of bilinearity from selection on correlations. The two arguments converge on bilinearity from opposite directions.

III vs. V. Already analysed in the companion papers (Section 9B of Part II). The pairwise-correlation structure (III) and the iso-entropic Gibbs structure (V) coincide in the diagonal/iso-entropic limit but rest on entirely distinct primitives: discrete distinguishability geometry vs. thermodynamic action.

IV vs. V. Both routes invoke admissibility, but in different senses: Route IV through TPB-consistency on holonomy structure, Route V through operational regularity on the unfolding action. Neither's proof steps are present in the other.

A fuller dependency diagram is given in Figure 1 (notional).

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          [common VERSF kernel: §5]
          /      |      |      |      \
      I        II     III    IV      V
      |        |      |      |       |
      v        v      v      v       v
             pᵢ = |⟨i|ψ⟩|²
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Each arrow into the conclusion is independent of each other arrow. No horizontal arrows exist between I–V at the level of *derived theorems*; the only horizontal connections are between *primitives* via the common kernel of §5.

We do not claim the routes share *no* assumptions whatever — they are all VERSF derivations and all rest on the irreducible VERSF kernel discussed in §5. We claim that *beyond that kernel*, none of the five derivations presupposes another.

3.1 Two structural concerns about independence

A careful reader will press on two further points that the pairwise inspections above do not fully address. We take them in turn.

The Hilbert-structure dependency. Route I constructs the complex Hilbert space $\mathscr{H}$ from scratch — Piron–Solèr representation theorem applied to the orthomodular lattice $\mathscr{L}$, with $\mathbb{C}$ selected by the companion VERSF interference paper. Route III also constructs Hilbert space from scratch — discrete metric → isometry group → phase → amplitudes → inner product → Hilbert space, in §6 of *The Double Square Rule*. Routes II, IV, and V, by contrast, *invoke* Hilbert structure at the start: Route II opens with "(P1) Pre-commitment states are rays in a complex Hilbert space $\mathscr{H}$"; Route IV's holonomy lives in U(1) ⊂ $\mathbb{C}$^d; Route V begins from "reversible unitary kinematics on Hilbert space $\mathscr{H}$." Each of II, IV, V cites prior VERSF work — most centrally, the interference paper — as the source of this structure.

This means a referee can reasonably observe that *the Hilbert-space construction is not independently re-derived along Routes II, IV, V; it is imported from a common background source*. If that source is one of the five routes, the independence claim collapses for those routes. The interference paper is not one of the five routes — it is background structural work — so the worst case does not arise. But the framing must be honest:

The convergence presented here is conditional on a body of background VERSF work, including the interference paper that selects $\mathbb{C}$ and the no-go theorem on non-simplicial substrates that fixes K = 7. Two of the five routes (I, III) reconstruct Hilbert structure within their own derivations, providing an internal check that this background work is not arbitrary. Three of the five routes (II, IV, V) import Hilbert structure from background. The convergence is therefore *not* "five independent constructions of Hilbert space all yielding the same Born rule"; it is *two independent constructions of Hilbert space plus three derivations on a Hilbert space that is conditional on the interference companion result*.

This is a genuine narrowing of the claim. We retain it because the two independent Hilbert constructions (I, III) are themselves a substantive overdetermination of Hilbert structure, and because Routes II, IV, V derive their probability rules through machinery distinct from the Hilbert construction itself. A future strengthening would be to re-run each of Routes II, IV, V with Hilbert structure independently sourced from within their own primitive vocabularies — an open task we flag explicitly under F1 in §8.

Routes I and II both involve non-contextuality — are they two views of one constraint? A skeptical referee may read Route I's *argument* for non-contextuality (from observer-invariant fact formation on admissible cells) and Route II's *derivation* of non-contextuality (from the single-source theorem $\mathcal{O} = \boldsymbol{\mathcal{F}}[\rho]$) as two formulations of the same underlying VERSF commitment — namely, that hidden contextual data is forbidden. We must concede a real point about Route I's *argument*: the gap between observer-invariance (the *fact* $\mathcal{A}_i$ is observer-independent) and full Gleason non-contextuality (the *probability* $p(\Pi_i)$ is partition-independent) is wider than the Route I source paper makes fully explicit, requiring an additional locality-of-weight assumption that is not obviously a consequence of observer-invariance alone. Route II's derivation via the single-source theorem $\mathcal{O} = \boldsymbol{\mathcal{F}}[\rho]$ does not have this gap: if probabilities are functionals of ρ alone, partition-independence follows immediately.

But the *independence* of Route I and Route II as derivations is a separate question, and the answer is that the two routes do not merely use different formalisations — they operate at different ontological layers on different physical objects:

- The closure relation $\mathcal{C}$ (Route I) is a relation on *configurations of admissible spaces prior to any commitment count*. It belongs to the pre-commitment layer of VERSF: it characterises which configurations are compatible with which closure pathways before any fact has been produced. It has no temporal or spatial extension; it is a relational structure on a combinatorial space.
- The record density ρ (Route II) is a *scalar field on space-time encoding cumulative commitment events*. It belongs to the post-commitment layer: it is the actual physical distribution of facts that the universe has produced, indexed by location and time. It has temporal and spatial extension; it is a field on a space-time manifold.

These are not different formalisations of one constraint. They are different physical objects living at different layers of the VERSF ontology — the pre-commitment combinatorial layer ($\mathcal{C}$) and the post-commitment field-theoretic layer (ρ). Route I derives orthogonality from compatibility relations on pre-commitment configurations; Route II derives the consistency weighting from invariance properties of admissible observables on the post-commitment record. The proof machinery is correspondingly distinct: Route I uses orthomodular-lattice representation theorems; Route II uses Cauchy multiplicative and additive functional equations on disjoint-sector decompositions of ρ.

A referee who insists these are "two views of one constraint" would have to argue either (a) that $\mathcal{C}$ and ρ are not genuinely distinct physical objects but two formalisations of one underlying object, which we reject — they live at different layers and refer to different things — or (b) that the constraint each route closes (Route I's faithfulness on partition lattices vs. Route II's compatibility with the single-source theorem) is a single constraint in disguise, which would require an explicit argument we have not seen. We commit, on these grounds, to counting Routes I and II as logically independent for the purposes of the convergence argument.

We make these concessions to Route I's source-paper non-contextuality argument openly, because honesty about the gap there strengthens rather than weakens the independence claim: Route II's cleanly grounded derivation does not inherit Route I's lacuna, so even if Route I's non-contextuality argument were entirely set aside, Route II would still terminate at the Born rule independently.

3.2 Summary

Independence holds at the level of primitives and proof machinery; it remains open at the level of model construction.

The pairwise inspections of §3 establish operational independence — distinct primitive vocabularies, distinct proof techniques, no derivation in one route's language that translates into another's without re-running an independent argument. What they do not establish is model-theoretic independence — exhibiting, for each pair, a model where one route's premises hold but the other's central theorem fails. The first is what most of the convergence-based argument of §§4–6 needs; the second is what would close F1 in §8.

We note an asymmetry that a careful referee will press on: §6's individual arguments lean on operational and model-theoretic independence to different degrees. §6.2 (cross-validation at the proof-step level) and §6.4 (explanatory unification) need only operational independence — they observe that the routes' intermediate steps and physical readings cohere, which is established by inspection. §6.1 (robustness against premise failure) and §6.5 (falsification structure) lean harder on something closer to model-theoretic independence — they assert that a failed premise in one route does not propagate to the others, and that overturning the rule requires multiple admissibility violations, which strictly speaking requires that the premises and admissibility conditions not be hidden duplicates. Operational independence is consistent with this but does not by itself guarantee it. Strengthening to model-theoretic independence (the F1 program) is therefore particularly consequential for §6.1 and §6.5; under the current operational reading, those two arguments are partially conjectural, which we acknowledge openly. The remainder of the paper proceeds on the operational reading, with the second flagged honestly as open work.

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4. Convergence

4.1 What "the same measure" means

Before stating the convergence theorem, we make explicit a step that the result tacitly assumes: that the five routes' conclusions denote the *same* probability measure rather than five different measures that happen to share a notational form. Each route's conclusion is stated in its native vocabulary: Route I gives $\mathrm{Tr}(\rho P)$ on a Piron–Solèr-derived $\mathbb{C}$-Hilbert space; Route II gives $|c_i|^2$ where c_i are extension-set coefficients; Route III gives $|\psi\_A|^2$ where ψ_A is a path-amplitude sum; Route IV gives $\|\psi_i\|^2$ as the unique TPB-consistent exponent rule; Route V gives $|c_i|^2$ in the iso-entropic Gibbs limit. To say these are the same measure, the c_i in II, the path-sum ψ_A in III, the ψ_i in IV, and the c_i in V must all map to Route I's eigenstate decomposition under a common Hilbert-space structure. Concretely: the c_i in Route II's vocabulary correspond to Route I's amplitudes $\langle e_i|\psi\rangle$ under the identification fixed by the background VERSF interference result; the path-amplitude sum ψ_A in Route III corresponds to Route I's eigenvector under the Mercer-decomposition isomorphism developed in *The Double Square Rule* §6; the ψ_i in Route IV corresponds to a basis component in Route I's Hilbert space under the U(1)-holonomy reading; and Route V's c_i are by construction Route I's amplitudes since Route V opens with Hilbert kinematics as A1. Routes I and III construct the common Hilbert-space structure internally; Routes II, IV, V import it from background VERSF work, as discussed in §3.1. Under that common structure — and conditional on the background work surviving its own scrutiny — the five conclusions denote the same measure. The convergence theorem below is to be read as making the convergence claim *under this identification*, not in a vocabulary-free sense.

4.2 The convergence theorem

We now state the central result.

Theorem 1 (Convergence). *Within their respective primitive frameworks, Routes I–V each entail the conclusion*

$$p_i = |\langle i|\psi\rangle|^2$$

as the unique admissible probability assignment on outcomes of a measurement event in finite-dimensional pre-commitment Hilbert space.

Proof. By inspection of the cited companion papers. Specifically:

(I) Theorem 2 (Piron–Solèr representation) of *Orthogonality as Closure Incompatibility* yields the Hilbert-space representation of admissible outcome classes; the Busch-extended Gleason theorem applied to the resulting projection lattice yields $p(P) = \mathrm{Tr}(\rho P)$, giving $p_i = |\langle e_i|\psi\rangle|^2$ for pure states.

(II) Theorem 1 of *Probability as Consistency* yields $p_i = |c_i|^2$ as the unique consistency weighting; Theorem 2 strengthens to a no-go statement.

(III) The derivation in §5.5 of *The Double Square Rule* yields $P(A) = |\psi\_A|^2$; Theorem 5.4 establishes the uniqueness of the exponent.

(IV) Theorem 8.1 of *Physical Necessity of Quantum Probability Structure* yields the quadratic exponent as the unique TPB-consistent option; the §9 elimination table closes the structural alternatives.

(V) Corollary 1 of *Born Rule as Entropic Unfolding* yields $P_i = |c_i|^2$ in the iso-entropic limit. (As discussed in §2, the symmetry sub-route of §14 of that paper is not counted toward the convergence here.)

Each result is established in its own primitive vocabulary; under the common-Hilbert-structure identification of §4.1, their joint conclusion is the Born rule. ■

Observation (joint sharpness). *The five routes converge not on a one-parameter family of measures but on a single measure. No route admits a non-trivial deformation of the Born rule consistent with its own admissibility conditions.*

This is a meta-observation rather than a freestanding theorem: each route's central uniqueness result (I: Theorems 1, 2 + Busch-extended Gleason; II: Theorems 1, 2; III: Theorems 5.2–5.4; IV: Theorems 4.1–4.2 + 8.1; V: Theorem 3.0 + iso-entropic limit) already excludes deformations within its own framework as a structural matter, not as an approximation. The Observation collects these results and notes that they exclude *the same* deformations.

The strength of these statements deserves emphasis. It is not that the five routes give *approximately* the same probability rule, with small calculable corrections between them. It is that each route, run to completion in its own vocabulary, terminates at exactly $p_i = |\langle i|\psi\rangle|^2$ — the same functional form, the same exponent, the same coefficient. Convergence is exact, not perturbative.

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5. The Common VERSF Kernel

If the five routes share *some* assumptions, an honest overdetermination argument must isolate them. We identify the common kernel as follows.

Kernel item K1 — Irreversible commitment. Physical reality contains irreversible commitment events; the fact-set is non-trivially generated. Routes I, II, III, IV, V all rest on this. Without it, none of "admissible partition," "committed record," "irreversible selection," "TPB tick," or "entropy export" has content.

Kernel item K2 — Finite distinguishability at scale. At any operational resolution $\varepsilon > 0$, only finitely many states are ε-distinguishable. This appears explicitly as BCB in Route IV and as the discrete metric in Route III; it is implicit in Routes I, II, V via the requirement that admissible partitions, extension sets, and outcome bases be finite or countable.

Kernel item K3 — Reversible pre-commitment dynamics. Between commitment events, evolution is reversible. This is invoked as unitary invariance in Routes I, II, V, as isometry preservation in Route III, and as the holonomy structure in Route IV.

Kernel item K4 — Compositional admissibility. Independent subsystems compose multiplicatively; joint observables on disjoint sectors factorise. Each route invokes this in its own form (closure-compositional factorisation in I; disjoint-sector factorisation in II; tensor factorisation in III; A9 in IV; product-state extensivity in V).

Kernel item K5 — Observer invariance. Admissible structure is determined by the physics, not by external labelling or perspective. Each route invokes a version (observer invariance in I; non-contextuality in II; relabeling invariance in III, IV; basis-label invariance in V).

This is the irreducible kernel. We claim, but do not prove, that the kernel does not by itself determine the probability structure — each route requires additional, non-overlapping primitives to close the derivation, and it is precisely the agreement across these distinct closures that constitutes the overdetermination. We state this as a conjecture rather than a theorem because a referee may reasonably press on a real concern:

Concern. K3 + K4 + K5 — reversible pre-commitment dynamics, compositional admissibility, observer invariance — is uncomfortably close to "unitary evolution on a tensor-product Hilbert space with non-contextual probability assignments." If K1–K5 already entails this much structure, the gap that the five additional primitives close is correspondingly narrower, and the assertion that the kernel is "much weaker than any route's conclusion" becomes harder to sustain.

We accept the force of the concern. Our position is that K1–K5 *as stated* does not entail the Born rule, but that this position requires explicit support that we have not yet provided in full. What would settle it definitively is a **counter-example demonstration**: a model satisfying K1–K5 in which the Born rule fails. Constructing such a model — for instance, by exhibiting a generalised probabilistic theory with irreversibility, finite distinguishability, reversible dynamics, compositional admissibility, and observer invariance, but with a non-quadratic probability rule — is an open task we flag explicitly.

In the absence of such a demonstration, we offer the following plausibility argument: Routes II, IV, and V each invoke K3, K4, K5 (in their VERSF-specific forms) but require *additional* structural primitives — single-source theorem $\mathcal{O} = \boldsymbol{\mathcal{F}}[\rho]$ (II), TPB and BCB (IV), unfolding action and MaxCal (V) — to close the gap to $p_i = |c_i|^2$. If K1–K5 alone sufficed, these additional primitives would be redundant; the fact that the proofs in II, IV, V cannot be run on K1–K5 alone is evidence that the additional primitives are genuinely doing work. We note further that these additional primitives are *substantively* different in content, not just in formulation: the single-source theorem is a statement about scalar fields on space-time, TPB is a statement about the temporal structure of irreversible commitments, and MaxCal is an inference principle. They make claims about distinct objects in distinct ontological categories. This makes it harder to argue that they are all secretly redundant given K1–K5: a single hidden consequence of K1–K5 would have to simultaneously be a scalar-field theorem, a temporal-tick principle, and an inference rule. This is suggestive, not conclusive: it shows that the *proof techniques* used in II, IV, V need more than K1–K5, but it does not rule out a different proof technique that uses only K1–K5.

The literature on generalised probabilistic theories (Hardy 2001; Barrett 2007; Janotta–Hinrichsen 2014) provides one constructive scaffolding: post-quantum theories such as the Pawłowski et al. information-causality framework satisfy compositional admissibility and observer-invariance but produce non-Born statistics, and could plausibly be adapted to satisfy K1–K5 in their VERSF-specific forms. Whether such a model can be made to satisfy K3 in a form genuinely consistent with VERSF's "reversible pre-commitment dynamics" — as opposed to a weaker local-reversibility condition — is the load-bearing question for any such construction. A worked counter-example along these lines, even a conditional one ("if K3 is relaxed to local-reversibility-only, the following non-Born model satisfies K1, K2, K4, K5"), would be a significant strengthening of the present argument.

The honest formulation of the kernel claim is therefore:

Conjecture (kernel weakness). K1–K5 does not entail $p_i = |c_i|^2$. Equivalently, there exists a model satisfying K1–K5 with a non-Born probability rule.

The structural content of the overdetermination, conditional on this conjecture, is:

Five operationally independent extensions of the common kernel K1–K5, each adding a different primitive set, each running its own proof, all terminate at the same probability measure.

If the conjecture fails — that is, if K1–K5 alone secretly entails the Born rule — then the five routes are five proofs of a kernel-level theorem, which is still a substantive result but a different one from the overdetermination claim. **The overdetermination claim is strongest in the regime where the kernel does not fix the probability structure; resolving the kernel-weakness conjecture is therefore the central structural question for the programme.** We regard it as the headline open question that would most strengthen or weaken the present argument depending on its resolution.

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6. What Overdetermination Buys

Overdetermination is not a proof of correctness. It is, however, a stronger form of structural support than any individual derivation can supply, and the precise content of that support is worth stating.

6.1 Robustness against premise failure. Suppose, for argument, that one of Routes I–V rests on a premise that turns out to be physically false. The Born rule does not collapse. It continues to be derivable from the four remaining routes, each of which proceeds in its own vocabulary without invoking the failed premise. This is the most direct epistemic benefit: the rule's foundational status does not stand or fall with any single argument.

6.2 Cross-validation at the proof-step level. Each route's intermediate steps can be checked against the others' conclusions. For instance, Route II's Proposition 1 (multiplicativity from disjoint-sector structure) and Route III's Theorem 5.2 (multiplicativity from factorisation on path products) both derive the same multiplicative structure. If they did *not* — if Route II's multiplicative kernel and Route III's were inequivalent — at least one would have to be wrong. Their agreement is itself a non-trivial check.

6.3 Reduction of primitive privilege. Some object is, of necessity, privileged across all five routes — namely the Hilbert vector $|\psi\rangle$, which appears in every route's conclusion and (as discussed in §3.1 and §4.1) is the common Hilbert-space referent under which the five route-specific objects are identified. What is *not* privileged is which structural feature of $|\psi\rangle$ is taken as fundamental: the partition-lattice atom (I), the extension-set coefficient (II), the path-amplitude sum (III), the TPB-survivor norm (IV), or the alignment-readiness amplitude in the Gibbs-biased flux (V). Each route grounds the rule in a different structural reading, and each grounding succeeds. This significantly reduces the rhetorical move "but you've assumed X is the fundamental structural feature"; if X were dropped, four other readings survive. The same reduction applies to *contested philosophical premises*: non-contextuality, historically the disputed move at the heart of Gleason-style derivations, is not posited in any of the five routes — it is derived along Route I from observer-invariant fact formation and along Route II from the single-source theorem, and it does not appear at all in Routes III, IV, V, which build the rule from path correlations, admissibility filtering, and entropic inference respectively. The overdetermination thereby replaces contested *structural-feature* assumptions and *philosophical* assumptions with derived ones, while leaving the Hilbert vector itself as the common substrate that all routes terminate on.

6.4 Explanatory unification. The five routes turn out to be different perspectives on the same underlying object. Probability appears as: a partition measure (I); an extension-information cost (II); a path-pair correlation (III); the unique admissibility survivor (IV); a Gibbs-weighted flux (V). That all five descriptions coincide is itself a substantive result about the structure of VERSF: it tells us that "probability" is the same kind of object whether approached from closure, record, path, admissibility, or thermodynamics.

6.5 Falsification structure. To overturn the Born rule within VERSF, an alternative probability measure would have to violate not one but five independent admissibility conditions: closure-incompatibility / faithful representation (I); single-source consistency on disjoint sectors (II); pairwise selection with rank-one factorisation (III); TPB-consistent normalisation (IV); and entropy-flux extensivity (V). Any single failure of admissibility would leave four routes intact; coherent failure across all five would require a coordinated mechanism for which there is no candidate.

6.6 Methodological implications. What overdetermination does *not* buy is escape from the common kernel K1–K5. If irreversibility itself, or finite distinguishability, or reversible pre-commitment dynamics turned out to be ill-founded, all five routes would fail jointly — because they share that scaffolding. The status of the Born rule within VERSF is therefore *conditional on K1–K5* but *independent of any particular extension of K1–K5 beyond that kernel*. That is the precise content of the overdetermination.

7. Comparison to Other Overdetermined Results

The structure of "multiple independent routes converging on a single conclusion" is not new in physics. Three historical cases are instructive.

Avogadro's constant. By the early twentieth century, N_A had been determined by methods drawing on Brownian motion (Perrin), X-ray crystallography, blackbody radiation (Planck), oil-drop electrometry (Millikan), and electrochemistry (Faraday). Each method was open to specific systematic objections; their numerical convergence was not. The community's confidence in N_A by 1920 rested less on any single experiment than on the joint structure of the determinations.

The speed of light. c has been measured by aberration of starlight, Fizeau's toothed-wheel method, Foucault's rotating-mirror method, microwave cavity resonance, and laser interferometry. Each method depends on different physical principles; their convergence is what fixes c as a determinate quantity rather than an artefact of any one technique.

The fine-structure constant α. α has been determined from the anomalous magnetic moment of the electron (g-2), the quantum Hall effect, atom interferometry on rubidium and caesium, and helium spectroscopy. The sub-ppb agreement across these methods is widely regarded as the strongest available test of QED.

The Born rule's situation within VERSF is structurally analogous to these historical cases, but the analogy is weaker than it first appears, and we should be honest about the direction in which it cuts. In the Avogadro/c/α cases, empirical noise makes convergence non-trivial: five different methods using different physical principles could plausibly have given five different numbers; that they didn't is informative precisely because the methods could have failed independently. In the present case, theoretical consistency *within a single framework* makes convergence partially expected: if VERSF is internally consistent, multiple proofs of the same theorem within VERSF are not deeply surprising. The historical analogy is therefore not as strong as a numerical-convergence argument would be. The strongest framing of what the present convergence does establish is:

The convergence is informative to the extent that the five routes draw on operationally independent extensions of the common kernel K1–K5.

This puts the epistemic weight back on §3, where it belongs. If the routes share more than the bare kernel, the convergence is partially expected and correspondingly less informative; if they are operationally independent in the sense of §3, the convergence is informative — though never as informative as empirical convergence between methods using independent physical principles would be.

A second feature, peculiar to the present case, deserves explicit notice. The standard Gleason reconstruction reaches the Born rule by trading the rule for the non-contextuality of frame functions on a presupposed Hilbert lattice. Non-contextuality has remained the contested philosophical move at the heart of that derivation: critics who reject Gleason typically reject non-contextuality, not the algebra. Within the present overdetermination, this move loses its single-point-of-failure status. Route I derives non-contextuality from observer-invariant fact formation; Route II derives it from the single-source theorem $\mathcal{O} = \boldsymbol{\mathcal{F}}[\rho]$; and Routes III, IV, V do not invoke it at all, building the rule from path correlations, TPB-consistent admissibility, and entropic inference. The contested premise is therefore replaced — twice independently — and routed around — three times — within a single overdetermined argument. This is not a feature any individual reconstruction in the existing literature provides.

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8. What Would Falsify the Result

We list the conditions under which the overdetermination would be undone.

F1 — Demonstration that two or more routes secretly share a premise beyond the common kernel K1–K5. This would reduce the effective number of independent routes. We have argued in §3 that this is not the case, but the argument is at the level of inspection rather than formal proof; a more rigorous independence proof would strengthen the result. A complete examination of premises beyond K1–K5 would have to address, in particular:

- *Route I's lattice-theoretic motivations*. The orthomodularity of $\mathscr{L}$ is motivated in the Route I paper through an appeal to compositional residue-freeness for successive admissible measurements; this motivation is gestural rather than self-contained, and a fully rigorous argument distinguishing orthomodularity from weaker (modularity) or stronger (full distributivity) compositional conditions remains open. The covering law is motivated through K = 7 granularity; the inference from "atoms exist" to "no admissible class sits strictly between an outcome and its join with a covering atom" requires explicit reconstruction.
- *Route I's Piron–Solèr conditions more broadly*: irreducibility, and especially the requirement of an *infinite* orthonormal-style sequence of atoms in the relevant sectors. This last is non-trivial in finite-dimensional settings and may require a direct-limit construction or an alternative representation theorem. The Route I paper currently establishes this in one sentence; full justification is open.
- *Route I's non-contextuality gap*. The inference from observer-invariant fact formation to Gleason-style probability non-contextuality requires an additional locality-of-weight assumption (that admissibility weight does not depend on which other cells appear in the partition) that the Route I paper does not make fully explicit. Either this assumption needs to be stated and defended, or non-contextuality should be flagged as a sketched-rather-than-proven step in Route I.
- *Route I's reliance on the companion VERSF interference result to fix the field as* $\mathbb{C}$; the same interference result silently underwriting the Hilbert structure imported by Routes II, IV, V (§3.1).
- *Route III's path-space regularity assumptions* (Mercer kernel boundedness, measurability).

- *Route IV's incremental-boundedness and total-boundedness assumptions* on the operational topology of holonomies (A6, A7 in the Companion).
- *Route V's MaxCal interpretive premise*.

Each item is addressed within its own paper to varying degrees of completeness, but a formal independence proof — exhibiting, for each pair (i, j), an explicit model in which Route i's premises hold but Route j's central theorem fails — would need to verify that none of these premises silently re-imports another route's structure. We acknowledge that the F1 program is non-trivial precisely because each route relies on technical premises whose interactions with the other routes' premises have not been audited: this is the most consequential outstanding task for future technical work.

F2 — Discovery of an alternative probability measure compatible with all routes' admissibility conditions. Each route's uniqueness theorem (Route I: Theorems 1, 2 + Busch-extended Gleason; II: Theorems 1, 2 of *Consistency*; III: Theorem 5.4 of *Double Square*; IV: Theorem 8.1 of *Physical Necessity*; V: Theorem 3.0 of *Entropic Unfolding*) excludes alternatives within its respective vocabulary. Such a discovery would require a flaw in at least one uniqueness proof. We note that Route IV's exclusion is parametric (over the family of structurally nearby alternatives) rather than exhaustive (over all logically possible probability rules); see §2 Route IV framing.

F3 — Empirical observation of Born-rule violations not attributable to apparatus asymmetry. Route V predicts small, calculable, structured deviations in apparatus with outcome-dependent entropy costs. Empirical deviations *of this form* are consistent with the framework. Empirical deviations of any *other* form — for instance, intrinsic violations independent of entropy export — would falsify Route V and require re-examination of the others. We note that since Route V is the most exposed of the routes (interpretive commitment to MaxCal), its falsification by F3 would not by itself overturn the four-route convergence on I, II, III, IV.

F4 — Failure of the common kernel K1–K5. If irreversibility, finite distinguishability, reversible pre-commitment dynamics, compositional admissibility, or observer invariance turn out to be ill-founded, all five routes fail jointly. The Born rule's status within VERSF is conditional on this kernel. Outside the kernel, the present argument says nothing.

F5 — Resolution of the kernel-weakness conjecture against us. As stated in §5, we conjecture that K1–K5 alone does not entail the Born rule. If this conjecture is shown to fail — if K1–K5 secretly entails $p_i = |c_i|^2$ without the additional primitives invoked by the five routes — then the convergence claim collapses into "five proofs of a kernel-level theorem," which is a substantive but different result. Establishing the conjecture (by exhibiting a model satisfying K1–K5 with a non-Born probability rule) would correspondingly *strengthen* the present argument; refuting it would require us to redescribe the result. Note the contrast with F4: F4 worries that the kernel is too *weak* to support the routes (one of K1–K5 is itself ill-founded); F5 worries that the kernel is too *strong* (K1–K5 secretly entails the Born rule on its own). The two failure modes are nearly opposite, and resolving F5 in either direction is informative — confirming the conjecture strengthens the convergence claim, refuting it forces a redescription, and either outcome is genuine progress.

F6 — Failure to extend to continuous spectra. All five routes operate in finite-dimensional or countable-spectrum settings. The continuous-spectrum case (continuous projection-valued measures, observables with continuous eigenvalues such as position and momentum) is not addressed within the present argument. Each route's source paper flags this as an extension target. If the extension fails for one or more routes — for instance, if the discrete pairwise-correlation argument of Route III does not survive a continuous-limit treatment — the convergence claim is correspondingly restricted to discrete-spectrum measurements. We treat this as a scope limitation rather than a falsifier of what we have shown, but note it for completeness. Among the five routes, Route II is the natural candidate for cleanest continuous extension: ρ is already a continuous scalar field on space-time, and the consistency-weighting argument generalises naturally to continuous extension sets via integration. Route III is the hardest, since its primitive structure (discrete metric, isometry group, holonomy phases) is intrinsically combinatorial; a continuous-limit argument there would require careful reconstruction. Route IV's elimination table inherits this: TPB-consistency in continuous spectra would require a generalisation of normalisation preservation that does not currently exist in the source paper.

The structure of falsifiability is itself part of the overdetermination's content. Standard reconstructions of the Born rule fail under F1–F2 alone; the present argument additionally requires F3, F4, F5, or F6. The space of empirical observations and theoretical discoveries that could undermine it is correspondingly smaller, but the F1 audit and the F5 conjecture remain as honest open targets.

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9. Conclusion

The Born rule has been derived along five operationally independent routes within VERSF: closure-theoretic representation (I, the representation-theoretic route via Piron–Solèr and Busch-extended Gleason); single-source consistency on the committed record (II); pairwise selection on path correlations in discrete informational geometry (III); admissibility filtering over a parametric space of alternatives (IV, an elimination argument); and maximum-caliber inference on an entropic unfolding action (V, the most exposed route given its MaxCal commitment). The five routes share no proof step beyond an irreducible common kernel — irreversibility, finite distinguishability, reversible pre-commitment dynamics, compositional admissibility, and observer invariance — which we conjecture, but do not prove, to be strictly weaker than the Born rule itself.

Within the limits flagged at length in §§3.1, 5, and 8 — operational rather than model-theoretic independence; Hilbert structure constructed independently along Routes I and III but imported along Routes II, IV, V; the kernel-weakness claim explicit as a conjecture rather than a theorem; the parametric scope of Route IV's elimination; the interpretive commitment of Route V; the silence on continuous spectra — we have argued that this convergence has a specific epistemic content: the rule's foundational status is robust against the failure of any one route's premises; its proof steps are cross-validated across vocabularies; the historically contested Gleason premise of non-contextuality is replaced twice independently and routed around three times; and the rule's negation requires not one but five admissibility violations, against several of which a candidate alternative would have to succeed simultaneously.

The Born rule within VERSF is therefore not "derived" in the singular sense in which a theorem is derived from axioms. It is *operationally overdetermined*: forced, under the limits stated above, by closure structure, record consistency, path-correlation geometry, admissibility filtering, and entropic inference. Whether the operational overdetermination strengthens to model-theoretic overdetermination — the F1 program — is the most consequential outstanding technical task. Whether the kernel-weakness conjecture holds is the most consequential outstanding structural one.

Even in its present form, this is a stronger form of foundational support than any individual derivation in the existing literature carries, and we offer it as a methodological prototype: a worked case in which a postulated quantum-mechanical axiom transitions from "assumed" to "operationally overdetermined" through accumulated independent derivation, with explicit honesty about which strengthenings remain open. Whether other quantum postulates — the kinematical postulate of complex Hilbert space, the dynamical postulate of unitary evolution, the projection postulate of measurement — admit analogous overdetermination is a question for the ongoing programme.

Within the VERSF framework, the Born rule is no longer a free postulate. It is a structurally overdetermined consequence, in the operationally precise sense developed above. The unique recovery of the rule across five mutually independent derivational logics — closure incompatibility, single-source consistency, path-correlation geometry, admissibility filtering, and entropic inference — is the form of foundational support the present paper aims to deliver, with the limitations stated honestly throughout.

Companion papers

- **Route I** — *Orthogonality as Closure Incompatibility: A VERSF Derivation of Quantum State Distinguishability and the Born Rule.*
- **Route II** — *Probability as Consistency: Eliminating the Born Rule as an Independent Postulate. A Single-Source Derivation in the VERSF Framework.*
- **Route III** — *The Double Square Rule: A Derivation of Quantum Probability from Discrete Informational Geometry* (Part II).
- **Route IV** — *Physical Necessity of Quantum Probability Structure: A Companion to Born Rule as Entropic Unfolding and The Double Square Rule.*
- **Route V** — *Born Rule as Entropic Unfolding* (Part I).

Background structural papers

- *VERSF Constraint and Lagrangian (BCB)* — action-principle face of the framework.
- *Commitment-Capacity Density (CCC)* — quartic threshold for fact production.
- *Pre-Factual Algebraic Reversibility and Compositional Completeness (PAR/CC)* — the algebra underwriting compositional admissibility.
- *Internal Admissible Closure (IAC)* — the closure conditions underwriting non-contextuality.
- *No-Go on Non-Simplicial Relational Substrates* — the K = 7 microscopic substrate.
- *Why a Fact-Producing Universe Must Satisfy Interference* — derivation of complex Hilbert space from admissibility.