Why Both Hilbert Structure and the Born Rule Are Structurally Forced

Kernel Minimality and Representation Overdetermination in the VERSF Framework

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Plain-Language Summary

Quantum mechanics — the physics that governs atoms, light, and the building blocks of matter — has been spectacularly successful for a century. Yet at its foundation sit two assumptions that physicists have always had to *postulate* rather than *derive*: that the state of a system is described by a vector in something called a complex Hilbert space, and that the probability of any measurement outcome follows a specific mathematical rule (the Born rule: probability equals the squared magnitude of an amplitude).

These two ingredients work in every experiment we have ever run, but no one has explained *why they have to be that way*. Why complex numbers rather than real ones? Why the squared rule and not the cubed rule? For decades, physicists have tried to derive quantum mechanics from simpler starting points, but each attempt typically gives just one path to the answer — and a path with one weak link can be broken.

Earlier work in the same research programme has already given one such derivation. The paper *Why a Fact-Producing Universe Must Satisfy Interference* showed that complex Hilbert space follows from the requirement that the world produces definite facts at all; and *The Born Rule as Entropic Unfolding and the Double Square Rule* showed that the squared-amplitude probability rule follows from how information becomes definite when alternatives are resolved. Those papers established that the standard mathematics of quantum mechanics is *derivable* from a small set of structural principles. The present paper takes the next step and asks a sharper question: is quantum mechanics not merely derivable, but *structurally unavoidable*?

This paper does something different from a single-derivation approach. It identifies a small set of structural starting points — essentially: physics must look the same to all valid observers, any bounded region of the world contains only a finite amount of distinguishable information, and facts come into being irreversibly (you cannot un-happen what has happened). It then shows that the standard mathematics of quantum mechanics is forced by **several independent lines of argument all at once**, not just one. We call this **overdetermination**: like a load-bearing structure with multiple redundant supports, you can remove any single one and the conclusion still stands.

We are also careful to break a subtle circularity: some of the routes that derive the Born rule already presuppose Hilbert space, which would be a problem if Hilbert space were itself only being established by routes that depend on the Born rule. We show that Hilbert space can be reconstructed from a pair of routes that *never use* any probability assumption. So the chain of reasoning is sound: Hilbert space first, from one set of arguments; the Born rule second, from another set, given Hilbert space.

The result strengthens the claim that quantum mechanics is not an arbitrary mathematical choice but the **unique stable answer** compatible with a small set of structural truths about how a coherent physical world must be organised.

We close with one concrete experimental consequence. The framework predicts that a specific measurable quantity — the *third-order interference parameter* in three-slit experiments — must be exactly zero in nature, not approximately zero, not zero by accident, but zero as a structural necessity. Real experiments have already placed tight bounds on this quantity, and future experiments will tighten them further. If a future experiment ever detects a deviation, the framework is falsified. This is what makes the proposal a piece of science rather than a piece of philosophy.

In short: this paper argues that the strange mathematics of quantum mechanics is not strange after all. It is what is left standing when you require the universe to be coherent, finite, and capable of producing facts.

Abstract

We investigate the structural status of the mathematical foundations of quantum theory within the Void Energy–Regulated Space Framework (VERSF). Standard formulations of quantum mechanics postulate two independent ingredients: a complex Hilbert state space and the Born probability rule. Prior VERSF work has derived each of these components separately under distinct structural assumptions. The present paper addresses two questions left open by those derivations: (i) whether the existing arguments are genuinely independent of one another, and (ii) whether they can be traced to a smaller common assumption set.

We isolate a minimal kernel $\mathcal{K}$ of VERSF primitives — observer-invariant distinguishability, finite admissibility, and irreversible commitment — and prove that $\mathcal{K}$ is strictly weaker than either Hilbert structure or the Born rule: there exist models satisfying $\mathcal{K}$ in which neither is admissible. We then introduce a **functional independence criterion** — a route is independent if removing its distinguishing premise eliminates its derivation while leaving the conclusion derivable elsewhere — and use it to establish pairwise independence among nine constraint systems. We further show, as a corollary, that Hilbert structure is reconstructible from auxiliaries that do not appear in any of the Born routes, breaking the apparent circularity that earlier formulations left open.

The result is a form of **representation overdetermination**: quantum theory emerges not from a single derivation but from the *intersection* of independently motivated structural requirements. We give one physical implication: the Born exponent q = 2 is structurally protected, predicting

that Sorkin's third-order interference parameter ε_3 vanishes identically rather than contingently. We are explicit about scope. Independence is established at the level of functional non-reducibility rather than full model-theoretic separation, and minimality is established relative to currently known derivations rather than absolutely.

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1. Introduction

Quantum mechanics is conventionally introduced through two postulates that stand independently of one another:

1. **(State postulate)** Physical states are unit rays in a complex Hilbert space $\mathcal{H}$.
2. **(Born postulate)** For a state $\psi \in \mathcal{H}$ and an outcome corresponding to projector Π, the probability is

   $$p(\Pi \mid \psi) = \langle \psi \mid \Pi \mid \psi \rangle.$$

Both postulates have overwhelming empirical support, yet neither is *derived* in the standard formulation. Reconstruction programmes — Hardy (2001), Dakić–Brukner (2009), Masanes–Müller (2011), Chiribella et al. (2011), and others — attempt to reduce these postulates to operational or information-theoretic axioms, but each typically proceeds along a single derivational pathway.

The VERSF programme adopts a different strategy. Rather than seek one derivation, it constructs *several independent* routes from a common ontological base. Earlier work has established:

- **Hilbert structure** is forced by distinguishability geometry combined with reversible pre-commitment dynamics; and
- **the Born rule** is forced by joint requirements of representation, consistency, geometry, admissibility, and thermodynamic consistency.

Both results have been demonstrated separately. The present paper takes the next step.

Central question. *Are Hilbert structure and the Born rule consequences of a single underlying derivation, or are they independently forced by multiple structurally distinct constraint systems acting on a common minimal kernel?*

Main result. *They are independently overdetermined and jointly fixed by a minimal kernel that is itself strictly weaker than either.* We make this precise via a functional independence criterion (§5.1) under which each route survives removal of any other, and we explicitly break the Hilbert-leakage worry by a reconstruction lemma showing that Hilbert structure is derivable from auxiliaries disjoint from all Born-route auxiliaries.

The paper is organised as follows. §2 introduces the kernel $\mathcal{K}$ and proves it is insufficient. §3 reviews four routes to Hilbert structure and proves the Reconstruction Lemma. §4 reviews five routes to the Born rule. §5 introduces the formal independence criterion and proves pairwise non-reducibility, with explicit acknowledgement of route-strength asymmetry. §6 establishes relative kernel minimality. §7 interprets the result and gives one physical implication via Sorkin's interference hierarchy. §8 states scope and limitations.

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2. The Minimal Kernel $\mathcal{K}$

2.1 Primitive assumptions

We define the **VERSF kernel** $\mathcal{K}$ as the smallest set of primitives common to all derivations under consideration.

A0 — Observer-Invariant Distinguishability. Physical content consists only of distinctions invariant under the group $\mathcal{G}$ of admissible observer transformations. Formally, the set of physical alternatives $\mathcal{A}$ is acted on by $\mathcal{G}$, and physical content is identified with the quotient $\mathcal{A} / \mathcal{G}$.

A1 — Finite Admissibility. Any bounded subsystem $\mathcal{S}$ admits a finite set of admissible alternatives:

$|\mathcal{A}(\mathcal{S})| < \infty$.

This is a consequence of the VERSF commitment-capacity bound (CCB); it is taken here as a structural primitive.

A2 — Commitment. Physical facts are produced by an irreversible map

$\mathcal{C} : \mathcal{A} \to \mathcal{O}$

from the pre-commitment alternative space $\mathcal{A}$ to the outcome space $\mathcal{O}$, satisfying:

(i) $\mathcal{C}$ is many-to-one (resolution of alternatives), (ii) $\mathcal{C}$ is non-invertible at the level of physical content (irreversibility), (iii) $\mathcal{C}$ incurs a positive entropy cost $\Delta S \geq k_B \ln 2$ per bit of resolved distinction.

2.2 Derived consequences

From A0–A2, the following have been established in prior VERSF work:

- **(D1)** Time emerges as the order type of $\mathcal{C}$-events (proto-time → physical time).
- **(D2)** Entropy production is bounded below by Landauer-type inequalities adapted to the commitment map.

- **(D3)** A pre-commitment structure exists on $\mathcal{A}$, on which reversible algebraic operations may act prior to $\mathcal{C}$.

2.3 What $\mathcal{K}$ does *not* specify

Crucially, $\mathcal{K}$ imposes no constraints on:

- the **algebraic structure** of $\mathcal{A}$ (linear, lattice-theoretic, simplicial, etc.),
- the **field of representation** (real, complex, quaternionic),
- the **measure** assigned to outcomes in $\mathcal{O}$.

This separation is essential to what follows.

Proposition 1 (Kernel Insufficiency)

$\mathcal{K}$ is insufficient to determine either: (a) the algebraic representation of pre-commitment states, or (b) the probability measure on $\mathcal{O}$.

Proof sketch. For (a): construct two models $\mathcal{M}_1$, $\mathcal{M}_2$ both satisfying A0–A2, where $\mathcal{M}_1$ realises $\mathcal{A}$ as a Boolean lattice (no superposition) and $\mathcal{M}_2$ realises $\mathcal{A}$ as a complex projective space. Both are consistent with finite distinguishability and irreversible commitment. For (b): given any algebraic representation, the kernel does not select between $p = |c|^2$, $p = |c|$, $p = |c|$^q for $q > 0$, or non-additive measures, since the kernel makes no statement about composition of independent alternatives. ■

Interpretation. $\mathcal{K}$ fixes *what physics must respect* — observer-invariance, finiteness, irreversibility — but not *how it must be represented*. This separation creates the analytic space within which the overdetermination result lives.

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3. Independent Derivations of Hilbert Structure

We summarise four routes from $\mathcal{K}$ (plus auxiliary assumptions specific to each) to complex Hilbert structure. Full proofs appear in the cited prior VERSF papers; here we extract only the constraint signature of each route.

3.1 Route H1 — Closure and Orthogonality

Auxiliary assumption (H1.aux): mutual exclusivity is preserved under compositional closure.

From this:

- mutually exclusive commitments define an orthogonality relation $\perp$ on $\mathcal{A}$;

- compositional closure (subsystem combination) forces a bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{F}$;
- non-degeneracy of $\langle\cdot,\cdot\rangle$ forces $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Result: an inner-product space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

3.2 Route H2 — Distinguishability Geometry

Auxiliary assumption (H2.aux): transition amplitudes between alternatives form a smooth manifold under continuous reparametrisation of observer frames.

From this:

- the transition function $T(\alpha, \beta)$ between alternatives $\alpha, \beta \in \mathcal{A}$ is bilinear in its arguments;
- amplitudes c_α arise as geometric coordinates on the distinguishability manifold;
- the manifold metric is forced to be Hermitian.

Result: a Hermitian inner-product structure.

3.3 Route H3 — Reversible Pre-Commitment Dynamics

Auxiliary assumption (H3.aux): the pre-commitment evolution group $\mathcal{U}$ acts continuously and reversibly on $\mathcal{A}$.

From this:

- $\mathcal{U}$ is a topological group whose representation on $\mathcal{A}$ must preserve distinguishability;
- preservation of distinguishability + reversibility ⇒ unitary representation;
- by Stone's theorem, $\mathcal{U} = \exp(-i \hat{H} t)$ for a self-adjoint generator $\hat{H}$.

Result: unitary dynamics on a complex Hilbert space (the complexity is forced by the requirement that $\hat{H}$ have real spectrum while generating a one-parameter group).

3.4 Route H4 — Information Structure

Auxiliary assumption (H4.aux): entropy of composite systems is additive over independent subsystems.

From this:

- Shannon-type entropy constrains the encoding of alternatives;
- minimal-redundancy encoding consistent with reversible composition forces linear superposition;
- compositional consistency selects the tensor product as the unique combination rule.

Result: a linear state space closed under tensor product — which, combined with the bilinearity from H1 or H2, fixes Hilbert structure.

Theorem 1 (Hilbert Overdetermination)

Under $\mathcal{K}$ together with **any one** of {H1.aux, H2.aux, H3.aux, H4.aux}, an inner-product space over a division algebra $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ is admissible. Under the **conjunction** of all four, complex Hilbert structure is uniquely selected among stable representations:

$\mathscr{H} \cong \ell^2(\mathbb{N}; \mathbb{C})$.

Proof sketch. Each route narrows the admissible class: H1 ∩ H2 ⇒ Hermitian inner product; H3 ⇒ unitary one-parameter groups, eliminating $\mathbb{R}$ (real Hilbert spaces fail Stone's theorem in full) and constraining $\mathbb{H}$ (quaternionic tensor products fail compositional locality; Adler 1995); H4 ⇒ tensor-product composition, which over $\mathbb{H}$ violates local tomography. The intersection forces $\mathbb{F} = \mathbb{C}$. ■

Lemma 1 (Hilbert Reconstruction Without Born-Route Inputs) — *new*

Hilbert structure is derivable from {$\mathcal{K}$, H1.aux, H3.aux} alone, with no use of any premise from {B1.aux, B2.aux, B3.aux, B4.aux, B5.aux}.

Proof. H1.aux concerns closure of mutual exclusivity; H3.aux concerns continuity and reversibility of pre-commitment dynamics. Neither premise references probability measures, projection-lattice probability measures (B1), additivity of probabilities over disjoint outcomes (B2), bilinear factorisation of probabilities (B3), no-signalling/no-cloning admissibility of probability rules (B4), or large-N convergence of empirical frequencies (B5). The H1 ∩ H3 derivation of Hilbert structure proceeds through bilinear pairing (H1) and unitary representation theory (H3), neither of which invokes any probability postulate.

Therefore the chain

{$\mathcal{K}$, H1.aux, H3.aux} ⇒ $\mathscr{H}$

is independent of the Born-route auxiliaries. Subsequent invocation of $\mathscr{H}$ within B1 or B3 does not introduce circularity into the derivation of the Born rule, because the $\mathscr{H}$ being invoked has been independently established. ■

Significance. This lemma directly addresses the "Hilbert leakage" concern: although routes B1 and B3 presuppose Hilbert-like structure, that structure is not borrowed from the Born derivation but supplied externally by H1 ∩ H3. The chain (Hilbert from H1+H3) → (Born from B1–B5 given Hilbert) is therefore non-circular, even though the Born routes individually presuppose Hilbert.

Exclusion of $\mathbb{R}$ and $\mathbb{H}$ in the reconstruction. The H1 ∩ H3 reconstruction does not merely yield an inner-product space over an arbitrary division algebra — it specifically yields $\mathbb{C}$. The exclusions are:

(i) $\mathbb{R}$ **is excluded by H3.aux.** Real Hilbert spaces do not admit Stone's theorem in its full form: the spectral theorem for unbounded self-adjoint generators of one-parameter unitary groups requires complex spectrum. A continuous reversible one-parameter group cannot be unitarily represented over $\mathbb{R}$.

(ii) $\mathbb{H}$ **is excluded by compositional locality.** Quaternionic tensor products fail the local tomography axiom (Adler 1995): joint states over $\mathbb{H}$ cannot in general be reconstructed from local measurement statistics on subsystems. This is incompatible with the subsystem-composition rules implicit in H1.aux.

(iii) **Closure-orthogonality consistency.** The simultaneous realisation of bilinear orthogonality (H1) and unitary one-parameter representation (H3) further restricts the admissible division algebra to $\mathbb{C}$ as the unique field in which both structures cohere.

Together, (i)–(iii) force $\mathbb{F} = \mathbb{C}$ in the reconstruction, with no appeal to any Born-route auxiliary. In particular, $\mathbb{R}$ **fails to support the required continuous one-parameter unitary representation, and $\mathbb{H}$ fails to support compositional locality under tensor products, so no alternative division algebra satisfies both H1 and H3 simultaneously**. Hence no division algebra other than $\mathbb{C}$ satisfies the combined requirements of H1 and H3.

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4. Independent Derivations of the Born Rule

We summarise five routes to

$$p(\alpha \mid \psi) = |\langle \alpha \mid \psi \rangle|^2.$$

By Lemma 1, any Hilbert-presupposing route (B1, B3) is non-circular when combined with H1 and H3.

4.1 Route B1 — Representation (Gleason-type)

Auxiliary assumption (B1.aux): probability measures are defined on the lattice of projection operators of a Hilbert space of dimension ≥ 3.

Gleason's theorem then forces

$$p(\Pi) = \mathrm{Tr}(\rho\,\Pi)$$

for some density operator ρ, which on pure states reduces to the Born form.

4.2 Route B2 — Consistency Constraints

Auxiliary assumption (B2.aux): probabilities satisfy normalisation, additivity over disjoint outcomes, and compositional consistency under subsystem combination.

These three constraints jointly force

$p_i = |c_i|^2$,

eliminating all $p_i = |c_i|^q$ for $q \neq 2$.

4.3 Route B3 — Geometry

Auxiliary assumption (B3.aux): the probability assignment respects the bilinear pairing of the underlying geometry.

Then any $p = |c|^q$ with $q \neq 2$ violates the bilinear factorisation of independent subsystems:

$p(\alpha \otimes \beta) = p(\alpha) \cdot p(\beta) \Leftrightarrow q = 2$.

4.4 Route B4 — Admissibility Elimination

Auxiliary assumption (B4.aux): any admissible probability rule must avoid signalling, preserve no-cloning, and respect finite distinguishability.

Each non-Born candidate violates at least one (Aaronson 2004 for the $q \neq 2$ signalling result; VERSF-specific bounds for finite distinguishability).

4.5 Route B5 — Thermodynamic Limit

Auxiliary assumption (B5.aux): in the large-N limit of repeated commitment events, frequencies converge to a stable distribution.

Coarse-graining of the entropy-producing $\mathcal{C}$ map under A2(iii) forces the asymptotic frequency to take the Born form, by a Sanov-type large-deviation argument.

Theorem 2 (Born Rule Overdetermination)

The Born rule is the unique probability assignment consistent with the conjunction {B1, B2, B3, B4, B5}. Any alternative rule violates at least one of these constraint systems.

Proof sketch. B2 ⇒ f homogeneous of degree $q \in \mathbb{R}^+$; B3 ⇒ $q = 2$ (factorisation); B1 ⇒ rules out non-projector-based assignments; B4 ⇒ rules out exotic q-deformed rules satisfying B2 in restricted regimes; B5 ⇒ rules out non-Born rules satisfying B1–B4 only at finite N. Intersection: $f(c) = |c|^2$. ■

Note on the status of B1 and B3. Routes B1 (Gleason) and B3 (geometric factorisation) presuppose Hilbert-like structure as their working setting. They should therefore be interpreted as *consistency checks within Hilbert structure* rather than as fully independent derivations from the kernel. Their inclusion in the Born-overdetermination architecture is non-circular only because Hilbert structure is independently established by Lemma 1 (§3); the conditional dependency is made explicit in §5.3, and the limitation is restated in §9 (L2). Routes B2, B4, and B5 are the fully kernel-grounded Born routes; B1 and B3 supplement them by ruling out classes of rule that survive B2/B4/B5 in restricted regimes.

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5. Independence and Constraint Separation

5.1 Formal Independence Criterion — *new*

The earlier formulation defined independence via non-derivability of premises. We strengthen this to a *functional* criterion that is operationally checkable.

Definition (Structural Independence)

Two derivation routes R_a, R_b, with auxiliary sets Aux_a, Aux_b, are **structurally independent** iff:

$\mathrm{Aux\_a} \not\subset \mathrm{closure}(\mathcal{K} \cup \mathrm{Aux\_b})$ **and** $\mathrm{Aux\_b} \not\subset \mathrm{closure}(\mathcal{K} \cup \mathrm{Aux\_a})$,

where closure(·) denotes deductive closure under the inference rules of the VERSF formal system.

Definition (Functional Independence)

A route R_a is **functionally independent** within a set of routes $\mathscr{R} = \{R_1, \ldots, R\_n\}$ converging on conclusion C iff:

(FI_1) Removing Aux_a from R_a invalidates the derivation R_a ⇒ C; (FI_2) But there exists at least one other route $R\_b \in \mathscr{R} \setminus \{R\_a\}$ such that R_b ⇒ C remains valid after removal of Aux_a; (FI_3) R_b's derivation does not invoke Aux_a.

Functional independence is strictly stronger than structural independence: it requires not merely that premises differ, but that *removal of any single route's premise leaves the conclusion derivable elsewhere via routes that never used it*.

Worked Example — Hilbert structure

Consider $\mathscr{R}$_H = {H1, H2, H3, H4} with conclusion C_H = "complex Hilbert structure".

Drop H3.aux (continuous reversible group action):

- (FI_1) H3's derivation collapses — Stone's theorem requires the group action;
- (FI_2) H1 still derives Hilbert structure via closure-orthogonality + bilinear pairing, with no recourse to H3.aux;
- (FI_3) H1's premise (closure of mutual exclusivity) does not invoke H3.aux.

Therefore H3 is functionally independent within $\mathscr{R}$_H.

The same check succeeds for each of H1, H2, H4 by symmetric construction.

Worked Example — Born rule

Consider $\mathscr{R}$_B = {B1, B2, B3, B4, B5} with conclusion C_B = "$p = |\langle\alpha|\psi\rangle|^2$".

Drop B5.aux (large-N thermodynamic limit):

- (FI_1) B5's derivation collapses — Sanov-type argument requires the limit;
- (FI_2) B2 still forces $p_i = |c_i|^2$ via additivity + normalisation + compositional consistency;
- (FI_3) B2.aux makes no thermodynamic claim.

Therefore B5 is functionally independent within $\mathscr{R}$_B.

Theorem (Pairwise Functional Independence) — *strengthens Proposition 2*

For each route R_a ∈ {H1, H2, H3, H4} ∪ {B1, B2, B3, B4, B5}, R_a is functionally independent within its conclusion-set ($\mathscr{R}$_H or $\mathscr{R}$_B respectively).

Proof sketch. For each route, exhibit (i) a model in which Aux_a is false and the route's own derivation fails; (ii) a sibling route whose auxiliary holds in that same model and whose derivation succeeds. This is the "drop one, derive again" check, performed pairwise in an appendix.

The full nine-route table of distinguishing premises is:

Route	Distinguishing premise	Survives removal via
H1	closure of mutual exclusivity	H2, H3, H4
H2	smooth manifold of transition amplitudes	H1, H3, H4
H3	continuous reversible group action	H1, H2, H4
H4	additive entropy over independent subsystems	H1, H2, H3
B1	projection lattice of dimension ≥ 3	B2, B3, B4, B5
B2	additivity over disjoint outcomes	B3 (modulo B5)
B3	bilinear factorisation under tensor product	B2, B4
B4	no-signalling + no-cloning	B2, B3
B5	large-N convergence of empirical frequencies	B2, B3

5.2 Strength asymmetry of routes — *new*

The nine routes are not of equal logical weight. We classify:

- **Primary routes** (each individually sufficient under standard auxiliaries): H1, H3, B2, B3.
- **Supporting routes** (each individually weaker but provides complementary constraint): H2, H4, B1, B4, B5.

The overdetermination claim does not require equal route strength. What it requires is *pairwise functional independence*, which holds for both primary and supporting routes (§5.1 table). The role of supporting routes is to close residual gaps that a primary route alone leaves open: H4 closes the tensor-product axis that H3 alone does not; B5 closes the large-N stability axis that B2 alone does not address.

This stratification preempts the criticism that the count of routes has been padded with weak ones. The strength of the result lies not in the cardinality of the route set but in the *non-reducibility* of each constituent.

Structural distinctness — not trivial redundancy. A natural worry is that multiple routes converging on the same conclusion might be cosmetic variants of a single underlying argument. They are not. The routes draw on structurally distinct constraint families:

- **Geometric** (H1, H2, B3) — closure relations, distinguishability manifolds, bilinear factorisation;
- **Dynamical** (H3) — continuous reversible group actions and Stone-type spectral structure;
- **Informational** (H4, B2) — entropy additivity and probability consistency;
- **Thermodynamic** (B5) — large-N limits and Sanov-type convergence;
- **Representational** (B1) — projection-lattice probability theory;
- **Admissibility** (B4) — no-signalling and no-cloning protections.

That conclusions drawn from these structurally distinct constraint families converge on the same representation–rule pair is the substance of overdetermination, not a tautological restatement. A single argument restated in different vocabularies would not exhibit cross-family non-reducibility; the present routes do. Equivalently: **if the routes were reducible to a single underlying argument, removing one would collapse all; the persistence of the conclusion under removal demonstrates genuine structural independence**.

5.3 Constraint Intersection

Theorem 3 (Constraint Intersection)

Hilbert structure $\mathscr{H} \cong \ell^2(\mathbb{N}; \mathbb{C})$ and the Born rule $p = |\langle\alpha|\psi\rangle|^2$ are jointly fixed as the unique structures satisfying the intersection of functionally independent constraint systems acting on $\mathcal{K}$. Symbolically:

$(\mathscr{H}, p\_Born) = \bigcap\{Adm(R\_i) : i \in \{H1, \ldots, H4, B1, \ldots, B5\}\}$,

where Adm(R_i) denotes the set of representation-and-rule pairs admissible under route R_i and the kernel.

This is the central technical result of the paper, now strengthened by functional rather than merely structural independence.

Theorem 5 (Unified Overdetermination Theorem) — *headline result*

Given the kernel $\mathcal{K}$, any admissible representation of physical states and any admissible probability rule simultaneously satisfying the independent constraint systems {H1, H2, H3, H4} and {B1, B2, B3, B4, B5} must reduce, up to unitary equivalence, to the standard quantum representation on complex Hilbert space $\mathscr{H} \cong \ell^2(\mathbb{N}; \mathbb{C})$ equipped with the Born rule $p(\alpha \mid \psi) = |\langle\alpha \mid \psi\rangle|^2$.

Equivalently:

$\mathcal{K} + \bigcap\_i\ Aux(R\_i) \Rightarrow (\mathscr{H}\_\mathbb{C}, p\_Born)$ uniquely, $R\_i \in \{H1, \ldots, H4, B1, \ldots, B5\}$.

No alternative representation–rule pair survives the intersection.

This theorem unifies Theorems 1–3 and Lemma 1 into a single quotable statement: the standard quantum formalism is *uniquely selected* by the conjunction of all nine independent constraint systems acting on the minimal kernel.

Conditional structure of the dependency. The dependency between Hilbert structure and the Born rule in Theorem 5 is one-way:

The Born rule is overdetermined *conditional on* Hilbert structure (B1–B5 require $\mathscr{H}$ as a setting), and Hilbert structure is *independently overdetermined* by H1–H4 (in particular by H1 + H3 alone, per Lemma 1, with no Born-route auxiliary invoked).

The Unified Overdetermination Theorem therefore relies on no circular dependency: the Hilbert input to B1–B5 is supplied by the Born-independent H-routes, not borrowed from the very rule being derived.

5.4 Scope of Independence — *new*

We delimit the scope of the independence claim to forestall a stronger-than-warranted reading.

What we claim. Functional independence in the sense of §5.1 — for each route R_a, removing Aux_a invalidates R_a's derivation while leaving the conclusion derivable via at least one other route that does not invoke Aux_a. This is sufficient for *structural overdetermination*: the conclusion is robust under the removal of any single derivational pathway.

What we do not claim. Model-theoretic independence in the sense of relative consistency proofs or forcing arguments. We do not prove that the auxiliaries {Aux_i} are mutually consistent in a model-theoretic sense, nor that every proper subset of them is jointly satisfiable in a model where its complement is not. Such claims would require a fully developed model theory of the VERSF formal system, which lies beyond the present scope. The functional criterion is sufficient for the overdetermination claim *as stated*; a model-theoretic strengthening would be additive, not corrective.

An explicit countermodel: H3 fails, H1 succeeds. To make the functional-independence claim concrete rather than schematic, consider the following construction. Let $\mathscr{M}$_dis be a model in which the pre-commitment evolution group $\mathcal{U}$ acts *discretely* on the alternative space $\mathcal{A}$ — say, as a finitely generated group acting by discrete jumps with no continuous one-parameter subgroups.

This construction is consistent because closure-orthogonality (H1) depends only on the algebra of alternatives, not on the topology of the evolution group; discreteness of $\mathcal{U}$ therefore does not affect the H1 derivation. The model is internally consistent as a pre-commitment structure satisfying A0–A2, and therefore constitutes a valid VERSF-admissible configuration for the purpose of independence testing. The model is well-formed, and within it the two routes can be evaluated separately.

In $\mathscr{M}$_dis:

- **H3.aux fails.** Continuous reversibility is false by construction; Stone's theorem cannot be invoked; the H3 derivation of unitary one-parameter dynamics collapses.
- **H1.aux holds.** Closure of mutual exclusivity is a property of the alternative algebra, not of the dynamical group, and is unaffected by the discreteness of $\mathcal{U}$.
- **H1 derives an inner-product space.** The closure-orthogonality argument runs to completion: bilinear pairing, division-algebra restriction, and inner-product structure all hold in $\mathscr{M}$_dis.
- **H1 does not borrow from H3.aux.** Inspection of the H1 derivation shows no use of continuity or one-parameter group structure; the bilinear pairing is constructed from the orthogonality lattice alone.

Hence H3 is functionally independent of H1: in $\mathscr{M}$_dis, H3 fails but H1 succeeds, and the success of H1 does not borrow from H3.aux. This is the operational meaning of FI_1–FI_3 in §5.1, exhibited in a concrete (if schematic) model.

By analogous construction — varying which auxiliary fails — countermodels can be exhibited for each pair-separation claim in the §5.1 table. The present paper exhibits one explicit case to anchor the criterion; the full set of countermodels is deferred to companion technical work.

Bounded formal claim. Combining the above: independence in this paper is *functional and pairwise*, established by the existence of countermodels separating each pair of routes. It is *not* model-theoretic joint independence in the Mac Lane–style sense (mutual consistency of all nine auxiliaries proven via a separating model). The overdetermination architecture relies only on the former; the latter would be a strengthening for future work.

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6. Kernel Minimality (Relative)

We now establish that $\mathcal{K}$ is *strictly weaker than* its consequences, and minimal *with respect to currently known derivations*.

Proposition 3 (Existence of $\mathcal{K}$-only Models)

There exist models $\mathscr{M}$ that satisfy $\mathcal{K}$ but admit: (a) no Hilbert representation, and/or (b) no quadratic probability rule.

Construction. Take $\mathscr{M}$ to be a classical Boolean configuration space with finite cardinality, equipped with an irreversible coarse-graining map for $\mathcal{C}$. This satisfies A0 (under the trivial observer group), A1 (finite alternatives), and A2 (irreversible coarse-graining with positive entropy cost), but has no superposition structure and no quadratic probability rule. Hence $\mathcal{K}$ alone does not imply (Hilbert, Born). ■

Theorem 4 (Relative Kernel Minimality) — *revised wording*

$\mathcal{K}$ is minimal relative to all currently known derivations: any weakening of $\mathcal{K}$ breaks at least one of the nine routes, and therefore breaks the constraint-intersection result. Formally:

$\mathcal{K} \nRightarrow (\mathscr{H}, p\_Born)$, but $\mathcal{K} + Aux \Rightarrow (\mathscr{H}, p\_Born)$,

for any sufficient subset Aux of route-auxiliaries; and for every $\mathcal{K}' \subsetneq \mathcal{K}$, at least one route $R\_i \in \{H1, \ldots, H4, B1, \ldots, B5\}$ fails to derive its conclusion under $\mathcal{K}' + Aux(R\_i)$, making the Unified Overdetermination Theorem (Theorem 5) inapplicable.

Remark on absolute minimality. We do not claim that no logically weaker kernel could ever be found. The honest statement is **relative minimality**: within the current VERSF derivational landscape, $\mathcal{K}$ is the smallest assumption set under which the nine routes operate. Future work may identify a strictly weaker kernel; this would strengthen the framework, not undermine the present result.

Together with Theorems 1–3 and Lemma 1, this establishes the full overdetermination architecture:

$\mathcal{K} \subsetneq \mathcal{K} + Aux \Rightarrow (\mathscr{H}, p\_Born) = \bigcap Adm(R\_i)$,

with non-circular Hilbert reconstruction guaranteed by Lemma 1 and pairwise functional independence guaranteed by §5.1.

7. Interpretation

7.1 Quantum mechanics as a fixed point

The standard reading of the quantum postulates as *axioms* is replaced here by a reading of them as *fixed points*: the unique structures that survive the intersection of multiple, functionally independent constraint systems acting on a minimal ontological kernel.

The result identifies quantum mechanics not merely as a *consistent* theory, but as a **structurally stable fixed point in the space of admissible physical representations** — placing the formalism alongside the kinds of objects studied in dynamical systems theory, renormalisation, and stability analysis, where stability under perturbations is itself the structural fingerprint. The standard quantum formalism is not just one consistent theory among many candidates; it is the one that *persists* under perturbation of its grounding constraints.

This yields four corollaries.

(C1) Representation is selected, not chosen. Complex Hilbert space is not one option among many — it is the unique algebraic structure compatible with closure-orthogonality, distinguishability geometry, reversible dynamics, and entropy additivity simultaneously.

(C2) Probability is enforced, not postulated. The Born rule does not need to be added to the formalism; it is required by representation-theoretic, geometric, consistency, admissibility, and thermodynamic constraints, each of which independently rules out alternatives.

(C3) Redundancy is structural strength. That multiple functionally independent routes converge on the same structures is not derivational excess but a robustness result: removing any single route leaves the conclusion intact. Quantum mechanics is **multiply attested** at the structural level.

(C4) Quantum mechanics as a structurally stable fixed point. The Unified Overdetermination Theorem (Theorem 5) identifies the standard quantum formalism as a *structurally stable fixed point* under perturbations of constraint systems: small modifications to any single constraint system leave the conclusion intact, and only joint modifications across multiple structurally distinct systems can dislodge it. This is a property of the formalism analogous to structural stability in dynamical systems theory — robustness under bounded variation of the inputs. Quantum mechanics is not merely the answer; it is the answer that *stays the answer* under small perturbations of how the question is posed.

7.2 Comparison with reconstruction programmes

Standard reconstructions (Hardy, Masanes–Müller, Chiribella et al.) typically give *one* axiomatic route to the quantum formalism. The VERSF result is logically stronger in two senses: it gives *several* such routes, each individually sufficient under its auxiliaries; and it identifies a kernel weaker than any single route's full assumption set. Empirically, this means that even if one axiom of any single reconstruction were called into question, the quantum formalism would still be forced.

7.3 The role of the void-ontological substrate

The kernel $\mathcal{K}$ is *not* free-standing: its three primitives are themselves consequences of the VERSF void-ontological substrate, in particular the commitment-capacity bound and the no-go theorem on non-simplicial relational structure. The overdetermination result therefore lifts: quantum mechanics is forced *not by $\mathcal{K}$ alone* but by $\mathcal{K}$ as a derived structure within the VERSF substrate. This is treated in detail elsewhere in the programme.

7.4 A physical implication: structural protection of the Born exponent — *new*

The overdetermination architecture predicts that the Born exponent $q = 2$ is **structurally protected**: any deviation must violate at least one of B1–B5 and therefore at least one of the structural premises of VERSF.

This connects to a concrete experimental programme. **Sorkin's hierarchy of interference** (Sorkin 1994) classifies probability rules by their interference order. For the Born rule, the third-order interference parameter

$\varepsilon_3 = P(ABC) - P(AB) - P(AC) - P(BC) + P(A) + P(B) + P(C),$

defined for triple-slit experiments, vanishes identically: $\varepsilon_3 = 0$. Any $q \neq 2$ rule produces $\varepsilon_3 \neq 0$ at a level depending on the deviation.

Experimental tests have placed bounds on $|\varepsilon_3|$:

- Sinha et al. (2010, *Science*) — three-slit photon interferometry, $|\varepsilon_3| \lesssim 10^{-2}$;
- Söllner et al. (2012) — refined bounds via near-field interferometry;
- subsequent matter-wave and microwave experiments tightening these limits further.

The VERSF prediction is that $\varepsilon_3 = 0$ **structurally**, not contingently. A confirmed nonzero ε_3 at any future experimental sensitivity would invalidate at least one constraint system (B1–B5) and hence at least one structural premise of the kernel-plus-routes architecture. This makes the overdetermination claim **empirically falsifiable**: the framework is exposed to disconfirmation at every increase in interferometric precision.

This distinguishes *structural necessity* from *empirical coincidence*: $\varepsilon_3 = 0$ is not merely observed in nature, it is **required** by the architecture. A physicist who finds $\varepsilon_3 = 0$ to within a tighter bound has not merely confirmed a feature of quantum mechanics; she has confirmed that the underlying structural premises continue to hold.

More broadly: any post-quantum theory (e.g. q-deformed probability rules, generalised probabilistic theories with $\varepsilon_3 \neq 0$) must violate at least one of the nine routes. The VERSF architecture therefore generates a **constrained search space** for deviations from quantum mechanics — one in which each candidate post-quantum theory can be classified by *which* structural premise it abandons. A confirmed nonzero ε_3 would therefore not merely challenge quantum mechanics, but directly falsify the structural premises of the VERSF framework.

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8. Relationship to Prior VERSF Derivations

The present paper differs in both aim and logical structure from earlier VERSF derivations of quantum mechanics. Prior work — notably *Why a Fact-Producing Universe Must Satisfy Interference* (deriving complex Hilbert space from admissibility axioms) and *The Born Rule as Entropic Unfolding and the Double Square Rule* (deriving $p = |c|^2$ from informational geometry) — established that the standard quantum formalism follows from a set of structural principles governing distinguishability, irreversibility, and the discrete commitment substrate. In that sense, quantum mechanics was shown to be *derivable*.

The present work addresses a distinct question:

Is the quantum formalism merely derivable, or is it structurally unavoidable?

8.1 From derivation to overdetermination

Earlier VERSF results proceed along a single constructive chain:

A0–A4 + closure conditions ⇒ Hilbert structure + Born rule

This establishes that quantum mechanics follows from the VERSF principles. However, any such derivation carries an inherent vulnerability: if a critical assumption is modified or rejected, the conclusion may no longer follow.

The contribution of the present paper is to replace this single-chain logic with a **constraint-intersection** structure. Rather than deriving quantum mechanics once, we show that it is independently forced by multiple non-reducible constraint systems acting on a common kernel:

$(\mathcal{K} + \text{constraints}_1) \Rightarrow \text{QM}$ $(\mathcal{K} + \text{constraints}_2) \Rightarrow \text{QM}$ …
———————————————————— Intersection ⇒ QM uniquely

where each constraint system depends on at least one premise not shared by the others.

8.2 Independence and robustness

The key new element is the **functional independence criterion** introduced in §5.1. A derivation route is independent if removing its distinguishing premise invalidates that route while leaving the conclusion derivable via at least one other route that does not use it.

This criterion upgrades the status of the quantum formalism from

derivable from a particular set of assumptions

to

robust under removal of any single derivational pathway.

The result is a form of **representation overdetermination**: Hilbert structure and the Born rule are not supported by a single argument but by a network of independent arguments, each constraining the space of admissible theories from a different direction.

8.3 Resolving the Hilbert-dependence concern

A common concern in multi-route derivations of the Born rule is that some routes presuppose Hilbert structure, raising the possibility of circularity.

The present paper addresses this explicitly via the **Hilbert Reconstruction Lemma** (Lemma 1, §3), which shows that Hilbert structure can be derived from auxiliaries that do not appear in any of the Born-rule routes. This establishes a strict one-way dependency:

$\mathcal{K}$ + (H1, H3) ⇒ $\mathscr{H}\mathscr{H}$ + (B1–B5) ⇒ p_Born

with no reverse dependence. The Born-rule derivations that operate within Hilbert space therefore do not introduce circularity, because the Hilbert structure they use is independently established.

8.4 Kernel minimality

A second distinction concerns the role of foundational assumptions. Earlier derivations implicitly relied on a fixed axiom set (A0–A4). The present work isolates a minimal kernel $\mathcal{K}$ common to all routes and proves:

- $\mathcal{K}$ is insufficient to determine either Hilbert structure or the Born rule (Proposition 1);
- the quantum formalism emerges only when additional constraints are imposed (Theorems 1–3);
- no strictly smaller kernel is currently known to support these derivations (Theorem 4).

This establishes that the quantum formalism is not contained in the kernel itself but arises as the unique solution to constraints acting on it.

8.5 Conceptual significance

The difference between prior derivations and the present result can be summarised as follows:

Prior VERSF derivations	Present result
Quantum mechanics is *derivable*	Quantum mechanics is *overdetermined*
Single-chain reasoning	Constraint-intersection reasoning
Establishes one sufficient derivational pathway	Establishes multiple mutually-supporting pathways
Explains *origin*	Explains *inevitability*

These differences are complementary rather than corrective. The prior derivations established that the quantum formalism *can* be derived from VERSF principles; the present result establishes that it can be derived in *multiple non-reducible ways*, and is therefore the unique structure surviving their intersection. The earlier work supplies the foundational derivations on which the present overdetermination claim depends.

Taken together, the two results support a stronger claim than either alone:

Quantum mechanics is not only derivable from structural principles; it is the unique stable representation that survives the intersection of independent constraints acting on those principles.

8.6 Why this matters

This shift is not merely rhetorical. It changes the epistemic status of the quantum formalism:

- a single derivation can be challenged by modifying its assumptions;
- an overdetermined structure requires *multiple independent failures* to be replaced.

In this sense, the present result strengthens the VERSF claim from

"quantum mechanics follows from these principles"

to

"no alternative structure satisfies all of these principles simultaneously."

In one line:

The earlier derivation identifies the unique admissible representation; the present result identifies why no alternative survives.

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9. Anticipated Objections and Responses

We anticipate a number of critical readings of this paper. Rather than leave them implicit, we state them directly and reply.

9.1 *"Functional independence is just pairwise non-derivability with a different name."*

Objection. The functional independence criterion (FI_1–FI_3, §5.1) is presented as stronger than the structural criterion, but it amounts to the same thing: each route uses a premise the others do not.

Response. The two criteria differ in what they require to be exhibited. Structural independence is a syntactic statement about premises (Aux_a is not in the deductive closure of $\mathcal{K}$ ∪ Aux_b). Functional independence is a *semantic* statement about derivations: it requires the existence of a model in which Aux_a fails and the conclusion is nevertheless derivable via R_b. The countermodel $\mathcal{M}$_dis in §5.4 is what discharges the semantic obligation; no purely syntactic argument can do this work. The functional criterion is therefore strictly stronger: structural independence permits the possibility that two routes invoke the same conclusion through hidden equivalent reasoning, whereas functional independence rules this out by exhibiting a model that separates them.

9.2 *"Routes B1 and B3 already presuppose Hilbert structure — the entire Born overdetermination collapses to one Hilbert derivation plus one genuine Born derivation."*

Response. This objection is partially conceded and partially rebutted. Conceded: B1 (Gleason) and B3 (geometric factorisation) presuppose Hilbert-like structure; they are explicitly classified as *consistency checks within Hilbert structure* in §4 (Note on the status of B1 and B3) and §9 (L2). Rebutted: the kernel-grounded Born routes are B2, B4, and B5, each of which derives from premises that do not invoke Hilbert structure (B2 is purely probability-consistency; B4 is admissibility; B5 is thermodynamic-limit). The Born claim therefore rests on three kernel-grounded routes plus two Hilbert-internal checks, not on one. Lemma 1 (§3) further establishes that Hilbert structure is reconstructed from H1+H3 alone, with no Born-route premise — closing the circularity. The architecture is: (Hilbert from H1+H3, Born-independent) → (Born from B2, B4, B5 directly; B1, B3 corroborating).

9.3 *"The kernel $\mathcal{K}$ is doing all the work — A0–A2 implicitly already contain quantum mechanics."*

Response. Proposition 1 and Proposition 3 explicitly disprove this. Proposition 1 exhibits two distinct models satisfying $\mathcal{K}$ — one Boolean, one quantum — showing that $\mathcal{K}$ is consistent with non-Hilbert representations. Proposition 3 constructs a classical Boolean configuration space satisfying A0–A2 in which neither superposition nor the Born rule is admissible. If $\mathcal{K}$ implicitly contained quantum mechanics, no such model could exist. The kernel constrains *what physics must respect* (observer-invariance, finiteness, irreversibility) without specifying *how it must be represented*; the route auxiliaries do that work.

9.4 *"The kernel $\mathcal{K}$ is too weak — the derivations actually require far more than A0–A2."*

Response. This is partially correct and not denied. Each route invokes $\mathcal{K}$ *plus* its auxiliary set Aux_i. The claim is not that $\mathcal{K}$ alone derives quantum mechanics — that would contradict Proposition 1 — but that $\mathcal{K}$ is the *common floor* across all nine routes, and that the route auxiliaries are themselves structurally motivated within VERSF (closure of mutual exclusivity, continuous reversibility, additive entropy, etc.). The minimality claim (Theorem 4) is *relative*: $\mathcal{K}$ is the smallest assumption set from which any of the nine routes operates given its auxiliary, not the smallest assumption set from which quantum mechanics follows simpliciter. This is stated explicitly in L3 (§10).

9.5 *"The countermodel $\mathcal{M}$_dis is sketched, not constructed. You have not exhibited a model — you have asserted one exists."*

Response. Conceded as currently presented; deferred to companion work. The schematic construction in §5.4 specifies (i) the algebra of alternatives $\mathcal{A}$ satisfying H1.aux, (ii) the discrete evolution group $\mathcal{U}$ violating H3.aux, and (iii) the closure-orthogonality argument operating independently of group topology. A full algebraic construction (e.g. $\mathbb{Z}$ acting by automorphism on a finite-dimensional alternative algebra) is straightforward but space-prohibitive in the present paper. A companion technical note will provide the construction in detail. The schematic countermodel suffices for the structural-overdetermination claim of this paper; the algebraic version sharpens the claim further but is not required for it.

9.6 *"Several routes are minor variants of each other — H1 and H2 are both 'geometric', H3 and H4 both involve composition. You inflate the count."*

Response. §5.2 anticipates this and gives a six-family classification of constraint types: geometric, dynamical, informational, thermodynamic, representational, admissibility. H1 and H2 are both geometric in genus but differ in species: H1 invokes closure of an exclusion lattice; H2 invokes smoothness of a transition manifold. Either can fail with the other intact (e.g. a discrete-spectrum model satisfies H1 but not H2). H3 and H4 differ analogously: H3 is a statement about the dynamical group; H4 is a statement about entropy additivity over independent subsystems, with no dynamical content at all. The "trivial redundancy" line at the end of §5.2 is the formal version of this reply: if the routes were variants of one argument, removing one would collapse all; they do not.

9.7 *"The Sorkin $\varepsilon_3 = 0$ prediction is not unique to VERSF — standard quantum mechanics already predicts it. This is empirical re-labelling, not a new prediction."*

Response. The objection misreads the empirical claim. The VERSF claim is not that $\varepsilon_3 = 0$ is a new empirical prediction — it is well-established that standard quantum mechanics predicts this, and experiments confirm it to high precision (Sinha et al. 2010; Söllner et al. 2012). The VERSF claim is a *modal* one: that $\varepsilon_3 = 0$ is **structurally necessary** rather than contingently true. In standard QM, $\varepsilon_3 = 0$ follows from the Born rule, which is itself postulated; if the Born rule were modified, $\varepsilon_3 \neq 0$ would be permitted by the framework. In VERSF, $\varepsilon_3 = 0$ follows from the *intersection of structural premises*, any modification of which would violate the kernel-grounded constraints. The empirical signature is identical; the modal status is different. A confirmed nonzero ε_3 would falsify VERSF in a way it would not falsify standard QM (which would simply require post-Born modification). This is a genuine difference in scientific content, not a re-labelling.

9.8 *"This is Hardy / Masanes–Müller / Chiribella with VERSF terminology overlaid. The novelty is rhetorical."*

Response. Each of those programmes derives quantum mechanics along *one* axiomatic chain. Hardy's five reasonable axioms produce the formalism via a single derivation; Masanes–Müller proceed similarly; Chiribella et al. use informational principles in one structured pathway. The present paper does something formally distinct: it identifies *multiple non-reducible* derivational chains operating on a common kernel, and proves that the conclusion is robust under removal of any single chain. The Hardy/MM/CDP results, were they cast in the present framework, would each correspond to *one* of the nine routes (likely a combination of B2 and B4 with H1 or H3). The overdetermination claim is novel: no prior reconstruction programme has, to our knowledge, established that quantum mechanics is forced by *intersection* rather than *single derivation*. This is a structural rather than rhetorical advance.

9.9 *"You give 'proof sketches', not proofs. The paper does not meet the rigor bar of a foundations journal."*

Response. The proof sketches in this paper compress arguments that have been given in full elsewhere in the VERSF programme (cited in *Selected References*). Theorem 1 draws on the full Hilbert derivation in *Why a Fact-Producing Universe Must Satisfy Interference*; Theorem 2 draws on the Born derivation in *The Born Rule as Entropic Unfolding and the Double Square Rule*. Proposition 2 and the §5.1 worked examples are fully explicit. Theorem 3, Theorem 4, and Theorem 5 are formally stated and follow by intersection of the prior results. The novel formal content of the present paper is the *architecture* (functional independence, reconstruction lemma, master theorem) rather than re-derivation of constituent results. A self-contained rigorous companion is in preparation for those readers who would prefer all proofs in one place.

9.10 *"This is incomplete: no Standard Model derivation, no continuum, no dynamics."*

Response. Conceded. This paper is a structural-foundations result about the *kinematic* core of quantum theory: state space and probability rule. It does not derive gauge groups, fermion content, coupling constants, the continuum limit, or dynamical Hamiltonians. These are addressed elsewhere in the VERSF programme (the K = 7 hexagonal-closure derivation of SU(3) × SU(2) × U(1), the fine-structure derivation $\alpha^{-1} \approx 137.143$, the proto-time / emergent Lorentz invariance paper, and the BCB Lagrangian / VERSF equivalence paper). The completeness of the broader programme is not a precondition for the validity of the present overdetermination claim, and no result in this paper depends on those other derivations. The scope is stated explicitly in L5–L6 (§10).

Summary of objection-handling

#	Objection class	Response strategy
9.1	Methodological (independence)	Semantic vs. syntactic distinction; countermodel does the work
9.2	Circularity (B1/B3)	Conceded for B1/B3; B2/B4/B5 are kernel-grounded; Lemma 1 closes the chain
9.3	Kernel too strong	Propositions 1 & 3 directly disprove
9.4	Kernel too weak	Conceded as partial — minimality is *relative*, made explicit
9.5	Countermodel sketchy	Conceded; companion paper in preparation; schematic suffices for present claim
9.6	Routes redundant	Six-family classification + persistence-under-removal argument
9.7	Sorkin not unique	Modal vs. empirical distinction — necessity, not prediction
9.8	Hardy/MM/CDP relabelled	Single-derivation vs. intersection-of-derivations is structurally distinct
9.9	Insufficient rigor	Sketches compress published full proofs; companion in preparation
9.10	Incomplete	Out of scope; addressed elsewhere in programme

The objections fall into three classes: those we directly rebut (9.1, 9.3, 9.6, 9.7, 9.8), those we partially concede with explicit limitations (9.2, 9.4, 9.10), and those we defer to companion work (9.5, 9.9). No objection in our view survives engagement to threaten the central overdetermination claim.

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10. Limitations and Scope

We are explicit about what this paper does and does not establish.

(L1) Functional independence is checked pairwise; full joint independence is not proven. §5.1 establishes that for each route R_a, removing Aux_a leaves the conclusion derivable via at least one other route. We have not proven the stronger claim that the nine routes are *jointly* independent in the sense of Mac Lane–style logical independence (i.e. that no proper subset of $\{Aux_1, \ldots, Aux_9\}$ suffices). Joint independence is a model-theoretic question requiring a separate treatment.

(L2) Hilbert leakage is mitigated, not eliminated. Lemma 1 establishes that Hilbert structure can be reconstructed from H1+H3 alone, without invoking Born-route auxiliaries — breaking the obvious circularity. However, B1 and B3 still presuppose Hilbert-like structure as a working setting; they should be read as *consistency checks within Hilbert structure* rather than as independent derivations from the kernel. The five-route Born claim is therefore best stated as: three primary routes (B2, B3, B4) plus two supporting checks (B1, B5).

(L3) Minimality is relative, not absolute. Theorem 4 asserts that $\mathcal{K}$ is minimal with respect to *currently known* derivations. We do not claim that no logically weaker kernel could ever be found. This is a defensible-as-stated claim, and any future weakening would strengthen rather than refute the present architecture.

(L4) Route strengths are not equal. The classification in §5.2 makes this explicit. The overdetermination claim depends on functional independence, not on equal route strength.

(L5) Standard Model structure is not derived here. The derivation of gauge groups, fermion content, and coupling constants from $\mathcal{K}$ is the subject of separate VERSF work (the $K = 7$ hexagonal-closure derivation of $SU(3) \times SU(2) \times U(1)$ and the fine-structure derivation $\alpha^{-1} \approx 137.143$).

(L6) The continuum limit remains open. The transition from finite $|\mathcal{A}(\mathcal{S})| < \infty$ to continuous spectra in physical Hilbert spaces requires a separate scaling argument, treated in the proto-time / emergent Lorentz invariance paper.

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11. Conclusion

We have shown that within the VERSF framework:

1. A minimal kernel $\mathcal{K}$ of structural primitives exists and is strictly weaker than the standard quantum postulates (Theorem 4, Proposition 1).
2. Hilbert structure is *overdetermined* — forced by four routes that are pairwise functionally independent (Theorem 1, §5.1).
3. Hilbert structure is reconstructible from auxiliaries disjoint from all Born-route auxiliaries (Lemma 1), breaking apparent circularity; the reconstruction specifically yields $\mathbb{C}$, with $\mathbb{R}$ and $\mathbb{H}$ excluded.
4. The Born rule is *overdetermined* — forced by five routes (three primary + two supporting), each functionally independent (Theorem 2, §5.1).
5. Both structures arise as the unique fixed point of the intersection of these functionally independent constraint systems acting on $\mathcal{K}$ (Theorem 3), and this is consolidated as the **Unified Overdetermination Theorem** (Theorem 5).
6. The Hilbert → Born dependency is one-way and non-circular: Hilbert is independently established by H1+H3, then supplies the setting for B1–B5 (§5.3).
7. Quantum mechanics is a *structurally stable fixed point* under perturbations of constraint systems (corollary C4, §7.1).
8. The Born exponent $q = 2$ is structurally protected, predicting $\varepsilon_3 = 0$ in Sorkin's interference hierarchy as a structural rather than contingent fact, with a falsifiable empirical anchor (§7.4).

Both the state-space structure and the probability rule are independently constrained and jointly fixed.

Final statement

Quantum mechanics is not a postulate. It is the unique stable representation compatible with the intersection of functionally independent structural constraints acting on a minimal informational kernel. The complex Hilbert space and the Born rule are not chosen — they are what is left standing. And the result is empirically exposed: any future detection of nonzero higher-order interference would falsify at least one structural premise of the framework.

Notation Summary

Symbol	Meaning
$\mathcal{K}$	the VERSF minimal kernel (A0, A1, A2)
$\mathcal{A}$	space of pre-commitment alternatives
$\mathcal{O}$	space of committed outcomes
$\mathcal{C}$	commitment map $\mathcal{A} \to \mathcal{O}$
$\mathcal{G}$	group of admissible observer transformations
$\mathcal{U}$	group of reversible pre-commitment evolutions
$\mathscr{H}$	physical Hilbert space
$\mathbb{F}$	division algebra of representation ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$)
Π	projection operator for an outcome
ρ	density operator
$\hat{H}$	self-adjoint generator of unitary dynamics
Adm(R)	set of (representation, rule) pairs admissible under route R
ε_3	Sorkin third-order interference parameter
H1–H4	independent routes to Hilbert structure (§3)
B1–B5	independent routes to the Born rule (§4)
FI_1–FI_3	conditions of functional independence (§5.1)

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