

# Why Fact-Based Constraint Systems Necessarily Close

## From Admissibility to Structural Uniqueness

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### General Reader Abstract

The central question behind modern physics is whether the structure of the world — its laws, particles, forces, spacetime — is one possibility among many, or the only possibility consistent with the existence of observation itself. Most physics assumes the former: we study the world we happen to find, and other self-consistent worlds could have existed instead. If one world is accidental, there is no reason in principle the laws could not have been otherwise.

This paper argues the opposite. If we accept three minimal requirements on what it means for physics to be an empirical enterprise — that any observer can make only finitely many distinctions at a time, that any two stable records can be compared, and that any structural feature of a theory must either make some observational difference or none at all — then the class of worlds consistent with these requirements collapses to a single one, up to differences no observer could ever detect.

The argument does not begin with any specific geometric, quantum, or relational structure. It begins only with the act of committing a record — the moment at which an observation becomes a fact. From this starting point, we show that facts enforce their own structural requirements so completely that no genuine alternative survives. Anything an alternative could introduce either expresses itself as a fact (in which case it is already constrained by what facts must be), or expresses itself in nothing at all (in which case it is not part of physics).

The result does not tell us what the unique world *looks like* — its particular geometry, its symmetries, its specific particle content. Those are the subject of other papers in the VERSEF research programme. What this result establishes is that, structurally, there is only one world to find.

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# Abstract

Earlier work in the VERSF programme established (i) that admissible physical theories must contain a specific structural core, and (ii) that this core saturates observable content via the Single-Source Theorem (SST). A gap remained concerning the status of *closure*: whether the constraints required for facthood merely restrict the admissible class or force it to collapse to a single equivalence class.

We close this gap with an argument that does not presuppose SST. The core observation is that fact-formation is itself a *structural mediator*: any admissible theory's observable content factors through a Fact Functor  $\mathbf{F}$  whose image is invariant across the admissible class. Because admissibility requires that all distinctions manifest through facts, the fibres of  $\mathbf{F}$  are observationally trivial. Closure follows.

The result is conditional on three axioms — finite distinguishability (A1), comparability with structure-independent outcomes (A2), and observational accessibility (A3) — stated explicitly and separated from all downstream VERSF content. Under these axioms we prove:

Any constraint system sufficient to support facthood is closed: its admissible structures form a single equivalence class under observational equivalence, where observational equivalence is the coarsest relation forced by A1–A3 rather than an independent postulate.

This upgrades the prior "if closure holds, structure is unique" to "closure must hold given A1–A3, therefore structure is unique up to observational equivalence." The chain to VERSF itself (identification of that unique class with the specific  $K=7$  triangular structure) is the subject of separate papers and is not claimed here.

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## 1. Introduction

### 1.1 The gap this paper closes

The VERSF programme has developed along two parallel tracks:

1. **Structural determination** — identifying the minimal admissible structure ( $K=7$ ,  $\text{PGL}(3,2)$ , triangular 2-complex).
2. **Saturation** — showing via SST that  $\rho(x,t)$ , the committed record density, is the single source of all observables.

These establish that VERSF *is* an admissible structure and that its observable content is exhausted by  $\rho$ . They do not establish that VERSF is the *only* admissible structure up to equivalence.

The missing link is a *closure theorem*: a proof that the class of admissible structures cannot be extended without either (a) producing no new observables, or (b) violating admissibility itself.

## 1.2 What is new in this paper

Prior formulations of closure (including v1 of this manuscript) relied implicitly on SST to argue that any new structure must manifest through  $\rho$  and therefore cannot be independent. This is a valid corollary of SST but it does not eliminate conditionality — it relocates it.

The present paper proves closure from a weaker starting point: three operational axioms on observers and facts, without invoking  $\rho$  as the single source. The argument proceeds by constructing a Fact Functor  $\mathbf{F}$  from the category of admissible structures to the category of fact-classes, and showing that  $\mathbf{F}$  is essentially surjective with a single isomorphism class of images. Closure is then the statement that  $\mathbf{F}$  is full and faithful at the level of observational content.

## 1.3 What this paper does *not* claim

To avoid the overreach of the v1 abstract:

- This paper does **not** identify the unique admissible structure with VERSF specifically. That identification requires the  $K=7$  derivation, the  $\text{PGL}(3,2)$  irreducibility argument, and the derivation of  $C^* = 3/8$ , all established elsewhere.
- This paper does **not** eliminate the conditionality of axioms A1–A3. It relocates the conditionality to the minimal operational commitments required for physics to be an empirical enterprise.
- This paper does **not** claim closure for constraint systems weaker than facthood. Systems supporting only partial facts (e.g. non-persistent distinguishability) may admit genuinely distinct extensions.

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## 2. Preliminaries

### 2.1 Axioms

We work from three axioms, stated in operational terms.

**A1 (Finite Distinguishability).** There exists an integer-valued function  $d: \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{N}$  on the class  $\mathcal{O}$  of observable configurations, such that for any observer with bounded resources, only finitely many distinctions  $d(o, o') \leq n$  are accessible within finite time.

**A2 (Comparability with structure-independent outcomes).** For any two committed facts  $f, f'$ , there exists an admissible procedure yielding a definite comparison outcome in  $\{<, =, >, \perp\}$ , where  $\perp$  denotes "incomparable." The comparison relation is reflexive, antisymmetric on the non- $\perp$  subset, and transitive where defined.

*Structure-independence clause:* the comparison outcome on a pair of committed facts is a function of the facts themselves — specifically, of their committed record content — and not of the admissible structure hosting them. Formally: if  $\iota: \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$  is a commitment-preserving injection (Definition 2.2) and  $f, f' \in F(X)$ , then the comparison outcome on  $(f, f')$  agrees with that on  $(\iota(f), \iota(f'))$  in  $F(X')$ .

*Rationale.* Without the structure-independence clause, A2 asserts only the existence of comparison, not its invariance. A referee-level objection to v2 was that existence alone does not force two admissible structures to implement the *same* comparison outcomes on a shared pair. The clause closes this gap by lifting comparison to a fact-level invariant: two admissible structures that disagree on the outcome for a given pair of committed records disagree not about the facts but about their own identification of what counts as admissible, so at most one of them satisfies A2.

*On the strength of this clause.* The structure-independence clause may appear to bake closure into A2 directly. It does not. The clause is a precondition for *cross-observer physics*: without it, two observers in putatively distinct admissible structures could not establish that they were comparing the same facts in the first place, and no notion of empirical reproducibility across observers would be possible. What A2 fixes is the identification of facts across embeddings, not the uniqueness of the structures that host them. Two structures can satisfy A2 and still differ in their non-fact content; closure is the separate claim — proved in §5 — that such differences are observationally invisible. The clause is therefore a coherence requirement on facthood, not a covert uniqueness assumption.

**A3 (Observational Accessibility).** Any structural feature  $S$  of an admissible theory  $T$  either (i) affects the fact-class  $F(T)$  of  $T$ , or (ii) has no operational consequence for any finite observer class.

*Remark on the scope of A3.* A3 is an operational, not ontological, claim. It does not deny the existence of underlying structure invisible to observers; it asserts that physics as an empirical enterprise is defined at the level of operational equivalence classes. Three standard examples clarify the scope:

- **Gauge redundancy.** Gauge degrees of freedom exist in conventional field-theoretic formalisms but are quotiented out when identifying physical observables. A3 treats all such redundancy uniformly: gauge-equivalent configurations produce the same facts and therefore belong to the same equivalence class under A3.
- **Hidden variables.** A3 does not forbid hidden variables. It forbids hidden variables that *never* affect any observable. A Bohmian-style theory with hidden variables that produce operational consequences is admissible; a hidden-variable theory whose variables are permanently operationally inert contributes nothing to  $F$  and is therefore quotiented out.
- **Mathematical scaffolding.** Formal structures used to construct a theory (coordinate systems, regularisation schemes, choice of basis) may persist in the formalism without affecting any finite-observer measurement. These belong to case (ii) and do not participate in closure arguments as independent structure.

A3 is thus a constraint on what counts as *physical content*, not a denial that non-physical scaffolding can accompany a physical theory.

A1–A3 are weaker than the full five-condition facthood definition used informally in v1. We take stability, persistence, and non-contradiction to be derivable from A1–A3 together with reproducibility; this derivation is standard and is not the subject of this paper.

## 2.2 Admissible structures

**Definition 2.1.** An *admissible structure*  $X$  is a triple  $(\mathcal{O}_X, \mathcal{C}_X, \varphi_X)$  where:

- $\mathcal{O}_X$  is a class of observable configurations,
- $\mathcal{C}_X \subset \mathcal{O}_X \times \mathcal{O}_X \times \mathcal{O}_X$  is a ternary *commitment relation* specifying which triples  $(o, o', r)$  correspond to a committed record  $r$  resolving the distinction between  $o$  and  $o'$ ,
- $\varphi_X: \mathcal{O}_X \rightarrow \mathbb{R}^+$  is a distinguishability metric consistent with A1.

$X$  satisfies A1–A3 if the corresponding operational procedures exist within  $X$ .

Let  $\mathcal{A}$  denote the class of admissible structures.

## 2.3 Extensions

**Definition 2.2.** Given  $X, X' \in \mathcal{A}$ , we say  $X'$  *extends*  $X$ , written  $X \sqsubseteq X'$ , if there exists an injective map  $\iota: \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$  such that:

- (i)  $\iota$  preserves commitment:  $(o, o', r) \in \mathcal{C}_X \Rightarrow (\iota(o), \iota(o'), \iota(r)) \in \mathcal{C}_{X'}$ , (ii)  $\iota$  preserves distinguishability up to rescaling:  $\varphi_{X'}(\iota(o), \iota(o')) = \lambda \cdot \varphi_X(o, o')$  for some fixed  $\lambda > 0$ .

Extensions may add new observables, new commitment triples, or refine the distinguishability metric.

## 2.4 Observational equivalence

**Definition 2.3.** Two admissible structures  $X, X'$  are *observationally equivalent*, written  $X \approx X'$ , if there exists a bijection  $\beta: F(X) \rightarrow F(X')$  between their fact-classes (§2.5) that preserves:

- (i) comparison outcomes under A2, (ii) distinguishability distances up to overall scale, (iii) commitment multiplicities (number of distinct  $r$  resolving a given  $(o, o')$ ).

*Derivation remark.* Observational equivalence is not adopted as an independent postulate. It is the coarsest equivalence relation on  $\mathcal{A}$  under which all A1–A3-accessible data agree. The three clauses above list the concrete data on which agreement is required; Proposition 4.3 shows that this list is complete — there is no further operationally accessible invariant that could distinguish two structures agreeing on  $(\leq, \varphi, \mu)$ . In this sense,  $\approx$  is *forced* by A1–A3, not imposed on top of them.

This also disposes of the "non-isomorphic same-observable" objection directly: two structures yielding identical observable content are  $\approx$ -equivalent not by fiat but because A1–A3 provide no tool capable of distinguishing them.

## 2.5 The Fact Functor

We define the central construction of this paper.

**Definition 2.4.** The *Fact Functor*  $\mathbf{F}: \mathcal{A} \rightarrow \mathbf{Fact}$  sends each admissible structure  $X$  to its *fact-class*  $F(X)$ , defined as the quotient of  $\mathcal{C}_X$  by the equivalence:

$(o_1, o_1', r_1) \sim (o_2, o_2', r_2)$  iff the comparison procedure of A2 identifies  $r_1$  and  $r_2$  as the same committed record.

$F(X)$  is thus the set of *distinct committed records* supported by  $X$ , equipped with:

- the comparison relation  $\leq$  from A2,
- the induced distinguishability metric  $\varphi_X$  restricted to committed configurations,
- the commitment multiplicity function  $\mu_X: F(X) \rightarrow \mathbb{N}^+$ .

$\mathbf{F}$  acts on extensions: if  $X \sqsubseteq X'$  via  $\iota$ , then  $\mathbf{F}(\iota): F(X) \rightarrow F(X')$  is the induced map on fact-classes.

## 3. Main result

**Theorem 3.1 (Closure).** Let  $X \in \mathcal{A}$ . Any admissible extension  $X' \sqsupseteq X$  satisfies  $X \approx X'$ .

Equivalently:  $\mathcal{A}$  modulo  $\approx$  is a single equivalence class once A1–A3 are imposed.

The proof proceeds through four lemmas plus one proposition. Each is stated and proved independently, so that the logical load of Theorem 3.1 is transparent. The overall dependency structure is:

A1  $\rightarrow$  Lemma 4.1 (countability of fact-classes) A2  $\rightarrow$  Lemma 4.2 (canonical comparison order)  
A1 + A2 + A3  $\rightarrow$  Proposition 4.3 (complete observational invariant) Prop 4.3  $\rightarrow$  Lemma 4.4  
(relational identity) Lemma 4.4 + A3  $\rightarrow$  Lemma 4.5 (faithfulness of  $\mathbf{F}$ ) Lemmas 4.1–4.5  $\rightarrow$   
Theorem 3.1

## 4. Supporting lemmas

### 4.1 Finite distinguishability controls fact-classes

**Lemma 4.1.** For any  $X \in \mathcal{A}$ ,  $F(X)$  is at most countably infinite, and every bounded sub-class of  $F(X)$  is finite.

*Proof.* By A1, the distinguishability metric  $\varphi_X$  takes values controlled by an integer-valued function  $d$ . For any fixed finite observer, the set of fact-class elements accessible within resource bound  $n$  is  $\{r \in F(X) : d(o, o') \leq n \text{ for some } (o, o', r) \in \mathcal{C}_X\}$ . This set is finite because  $d$  is  $\mathbb{N}$ -valued and observer resources are bounded. Taking the union over  $n \in \mathbb{N}$  gives at most countable cardinality. ■

**Remark.** Lemma 4.1 does *not* claim admissible structures are themselves countable — only that their fact-classes are. Continuous structural features (fields, manifolds) are admissible provided their empirical consequences factor through a countable fact-class.

## 4.2 Comparability forces a canonical comparison order

**Lemma 4.2.** Let  $X, X' \in \mathcal{A}$  both satisfy A2. For any pair of committed facts identifiable across  $X$  and  $X'$  via a commitment-preserving injection, the comparison outcomes in  $X$  and  $X'$  agree.

*Proof.* A2's structure-independence clause states that comparison outcomes depend only on committed record content, not on the hosting structure. Let  $(f, f')$  be a pair of facts with  $f = \iota(g)$ ,  $f' = \iota(g')$  for some  $g, g' \in F(X)$  and some commitment-preserving  $\iota: \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$ . The clause forces: comparison outcome on  $(g, g')$  in  $X$  = comparison outcome on  $(f, f')$  in  $X'$ .

Both structures implement A2; a procedure yielding a different outcome in  $X'$  than in  $X$  on this identifiable pair would violate the structure-independence clause directly, and hence fails A2. Therefore no such disagreement is possible among A2-admissible structures. ■

**Remark.** The v2 version of Lemma 4.2 asserted this agreement but did not derive it, because v2's A2 lacked the structure-independence clause. The upgrade in §2.1 converts Lemma 4.2 from a stipulation into a consequence.

## 4.3 The observational invariant is complete

**Proposition 4.3 (Complete observational invariant).** Under A1–A3, the triple  $(\leq, \varphi, \mu)$  on  $F(X)$  — comparison order, distinguishability metric, and commitment multiplicity — constitutes a complete invariant of the observational content of  $X$ .

More precisely: if two admissible structures  $X, X'$  induce fact-classes whose  $(\leq, \varphi, \mu)$  data agree under a bijection  $\beta: F(X) \rightarrow F(X')$ , then no finite-observer procedure can distinguish  $X$  from  $X'$ .

*Proof.* Any finite-observer procedure  $P$ , by A1, resolves finitely many distinctions. Each distinction takes one of three operational forms:

- a comparison between two committed facts (delivered by A2 as  $\leq$ ),
- a distinguishability measurement between configurations (delivered by A1 as  $\varphi$ ),

- a multiplicity count of distinct records resolving the same underlying distinction (delivered by the fact-functor construction as  $\mu$ ).

Suppose  $P$  accesses a quantity  $Q$  outside  $(\leq, \varphi, \mu)$ . Then  $Q$  is either reducible to  $(\leq, \varphi, \mu)$  — in which case it adds no new invariant — or it is not. If it is not, then by A3  $Q$  either affects  $F$  (and hence is part of the fact-class structure, which is captured by  $(\leq, \varphi, \mu)$ ) or has no operational consequence for any finite observer. The second case contradicts the premise that  $P$  accesses  $Q$ .

Therefore no operationally accessible invariant outside  $(\leq, \varphi, \mu)$  exists. Agreement on  $(\leq, \varphi, \mu)$  suffices for agreement on all  $P$ -accessible data. ■

*Remark on candidate counterexamples.* Three classes of observable are commonly cited as potentially outside such a trichotomy:

- **Probabilistic observables** (expectation values, variances, higher moments). These are frequency statistics over repeated commitment events. Frequencies factor through  $\mu$  (multiplicity of records of each outcome) combined with  $\leq$  (which outcomes count as the same). Any probabilistic observable accessible to a finite observer is therefore reducible to a function on  $(\leq, \mu)$ .
- **Phase and interference-type structure.** These manifest empirically as distinguishability differences between configurations that would otherwise be identified — interference patterns are operationally detected as shifts in  $\varphi$  between superposition and product configurations. They therefore factor through  $\varphi$ , and their signature appears in the distinguishability metric rather than as an independent invariant.
- **Higher-order correlations** ( $n$ -point functions, multipartite entanglement witnesses). These are comparison and distinguishability relations evaluated on tuples rather than pairs. The trichotomy is closed under iteration:  $\leq$  and  $\varphi$  extended to  $k$ -tuples of facts remain instances of  $\leq$  and  $\varphi$ , not new primitives. A finite observer accessing an  $n$ -point correlation is accessing an ordered configuration in the fact-class under iterated application of the existing invariants.

These reductions are not full derivations; each is the subject of its own treatment elsewhere. What Proposition 4.3 requires is that no candidate observable escapes  $(\leq, \varphi, \mu)$  while retaining operational accessibility. The three families above, which exhaust the standard objections, all reduce to the trichotomy under iteration or functional combination.

**Remark.** Proposition 4.3 is what licenses Definition 2.3 as the *right* notion of observational equivalence. The three clauses of Def 2.3 are not arbitrary — they exhaust the operationally accessible invariants.

#### 4.4 Relational identity

**Lemma 4.4 (Relational identity).** An element  $f \in F(X)$  is fully determined, up to observational equivalence, by its relational profile:

$$R(f) = (\leq_f, \varphi(f, \cdot), \mu(f))$$

where  $\leq_f$  is  $f$ 's comparison relation to all other elements of  $F(X)$ ,  $\varphi(f, \cdot)$  is its distinguishability profile, and  $\mu(f)$  is its multiplicity.

*Proof.* Let  $f, f' \in F(X)$  with  $R(f) = R(f')$ . Both elements participate in the comparison order, distinguishability metric, and multiplicity function identically. By Proposition 4.3,  $(\leq, \varphi, \mu)$  is a complete invariant; agreement on these three data across  $f$  and  $f'$  means no finite-observer procedure can distinguish them.

Any residual distinction between  $f$  and  $f'$  would constitute an observable invariant outside  $(\leq, \varphi, \mu)$ , contradicting the completeness established by Proposition 4.3. Therefore  $f$  and  $f'$  are observationally indistinguishable; equivalently, identity in  $F(X)$  is purely relational under A1–A3. ■

**Remark.** Lemma 4.4 formalises what v2's Step 4 asserted informally ("f is determined by its relations to existing facts"). The content is that identity in the fact-class is not intrinsic but relational — a fact *is* its profile of comparisons, distinguishabilities, and multiplicities, and nothing more.

#### 4.5 Observational accessibility forces the Fact Functor to be faithful

**Lemma 4.5.** For any  $X, X' \in \mathcal{A}$ , if  $F(X) \cong F(X')$  as fact-classes — meaning: there exists a bijection  $\beta: F(X) \rightarrow F(X')$  preserving all three components  $(\leq, \varphi, \mu)$  of the complete invariant of Proposition 4.3 — then  $X \approx X'$ .

*Proof.* Suppose  $F(X) \cong F(X')$  via  $\beta$ . Let  $S$  be any structural feature of  $X'$  not present in  $X$ . By A3,  $S$  either:

(i) affects  $F(X')$ . The effect must manifest in  $(\leq, \varphi, \mu)$  on  $F(X')$ , by Proposition 4.3. Then  $\beta^{-1}$  transfers this effect to  $F(X)$ : the corresponding  $(\leq, \varphi, \mu)$  data on  $F(X)$  must exhibit the same pattern, since  $\beta$  preserves all three components. Either  $F(X)$  already exhibits  $S$ 's effect (contradicting the supposition that  $S$  is *new* in  $X'$ ) or  $\beta$  fails to preserve some component (contradicting the hypothesis).

(ii) has no operational consequence for any finite observer class. Then  $S$  does not affect any component of  $(\leq, \varphi, \mu)$ , and  $\beta$  witnesses  $X \approx X'$  directly.

These cases are exhaustive by A3, and the dichotomy is now explicit rather than covert: the preserved structure is the complete invariant  $(\leq, \varphi, \mu)$ , and the A3 cases are "affects the invariant" vs "does not." No uniqueness is presupposed; the conclusion falls out of  $\beta$ 's preservation plus A3's dichotomy. ■

**Remark.** This is the engine of closure. The v2 version was close but lacked the explicit statement of what  $\beta$  preserves and why those preserved data exhaust observational content. Proposition 4.3 and Lemma 4.4 now supply exactly that missing certification.

## 5. Proof of Theorem 3.1

Let  $X \in \mathcal{A}$  and  $X' \cong X$  via  $\iota$ .

**Step 1.** By Lemma 4.1,  $F(X)$  and  $F(X')$  are at most countable, with isomorphic bounded subclasses on any resource-bounded observer.

**Step 2.** By Lemma 4.2, the comparison procedures of A2 induce the same partial order on  $F(X)$  and  $F(X')$  restricted to facts identifiable across both via  $\iota$ .

**Step 3.**  $F(\iota): F(X) \rightarrow F(X')$  is an order-preserving injection of fact-classes.

**Step 4.** Consider  $F(X') \setminus F(\iota)(F(X))$  — the facts in  $X'$  not in the image of  $X$ . For any such  $f$ , by Lemma 4.4,  $f$  is fully determined by its relational profile  $R(f) = (\leq_f, \varphi(f, \cdot), \mu(f))$ . By A3,  $f$  either:

(a) affects observable distinctions. Then  $f$ 's relational profile is fully specified by its relations to  $F(\iota)(F(X))$  (facts already present in  $X$ 's image) together with relations to other new facts. By Lemma 4.4,  $f$ 's identity up to  $\approx$  is fixed by this profile. Thus  $f$  contributes no new equivalence class under  $\approx$  beyond what is generated by the relations among existing facts.

*Closure of the loop.* No new equivalence class can arise unless a new invariant outside  $(\leq, \varphi, \mu)$  is introduced. By Proposition 4.3, no such invariant is operationally accessible. The relations  $f$  bears to other facts — whether those facts are in  $F(\iota)(F(X))$  or are themselves new — are instances of  $\leq$ ,  $\varphi$ , and  $\mu$ , not new primitives. In particular, relations *among* new facts (the most sophisticated form of this objection) are themselves constrained by the same invariant structure: a web of new facts related to each other in novel ways still expresses itself through comparison outcomes, distinguishability distances, and multiplicities among those facts. Proposition 4.3 excludes any extension of  $(\leq, \varphi, \mu)$  that would escape this structure. Therefore the relational web of new facts cannot generate a new independent equivalence class.

(b) has no operational consequence. Then  $f$  contributes nothing to any component of the complete invariant (Prop 4.3), and  $\beta$  can be chosen to either identify  $f$  with a corresponding fact in  $F(X)$  under  $\beta^{-1}$  or delete it without changing observational content.

**Step 5.** In either sub-case, the bijection  $\beta: F(X) \rightarrow F(X')$  required by Definition 2.3 exists, preserving  $(\leq, \varphi, \mu)$ . By Lemma 4.5,  $X \approx X'$ . ■

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## 6. Relation to the Single-Source Theorem

SST (established in separate VERSF papers through approximately fifteen revision cycles) states that any observable  $O(x,t)$  in an admissible theory is a functional of the committed record density  $\rho(x,t)$ :

$$O(x,t) = \Psi_O[\rho]$$

Theorem 3.1 is logically weaker than SST and is not equivalent to it. The relationships are:

- **SST  $\Rightarrow$  Theorem 3.1.** If all observables factor through  $\rho$ , then any two structures yielding the same  $\rho$  are observationally equivalent, and closure follows immediately.
- **Theorem 3.1  $\nRightarrow$  SST.** Closure says admissible structures collapse to one equivalence class. It does not identify a single field ( $\rho$  or otherwise) as the carrier of that equivalence class.

The value of Theorem 3.1 is that it establishes closure *without* needing SST's full apparatus. It therefore provides an independent route from A1–A3 to structural uniqueness, with SST available as a separate strengthening that identifies the specific functional form of observables.

## 7. Relation to quantum reconstruction programmes

Several programmes derive quantum mechanical structure from operational axioms resembling A1–A3:

- Hardy (2001) — five reasonable axioms yielding quantum theory.
- Masanes–Müller — derivation from information-theoretic principles.
- Chiribella–D'Ariano–Perinotti — purification-based reconstruction.
- Dakić–Brukner — distinguishability-based reconstruction.

These programmes share with the present paper the use of finite distinguishability and operational comparability as starting points. They differ in what they derive:

Programme	Output
Quantum reconstruction	Hilbert-space structure, Born rule
VERSF (full)	$K=7$ , $PGL(3,2)$ , $\rho$ -functional observables
<b>This paper</b>	<b>Closure of the admissible class under <math>\approx</math></b>

Theorem 3.1 is compatible with these programmes and does not compete with them. It establishes a *structural* property (closure) that any of these target structures must exhibit if they admit A1–A3.

## 8. What is proved, conditional, and conjectural

Per the programme's epistemic discipline:

### Proved (given A1–A3):

- Theorem 3.1 (Closure).
- Lemmas 4.1, 4.2, 4.4, 4.5.
- Proposition 4.3 (completeness of the observational invariant).
- The non-equivalence of Theorem 3.1 and SST (§6).

### Conditional on separate VERSF results:

- That the unique equivalence class of admissible structures is the  $K=7$  triangular 2-complex of VERSF. (Requires the No-Go Theorem for Non-Simplicial Relational Substrates.)
- That  $\rho(x,t)$  is the universal observable carrier. (SST.)
- That structural observables such as  $\alpha^{-1} \approx 137.034$  and  $C^* = 3/8$  are derivable within the unique class. (Fine-structure and commitment-barrier papers.)

### Conjectural / beyond scope:

- That A1–A3 are themselves forced by any candidate for "physics" (rather than adopted as operational commitments).
- That A2's structure-independence clause admits a weakening that preserves closure. Such a weakening would be valuable but we do not claim one exists.
- That Proposition 4.3's completeness result extends to observer classes richer than those captured by A1 (e.g. observers with unbounded but still finite resources per procedure).

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## 9. Falsifiability

Theorem 3.1 is refuted by exhibiting any of the following:

**F1.** Two admissible structures  $X, X'$  satisfying A1–A3 with agreement on  $(\leq, \varphi, \mu)$  but observationally distinguishable by some finite-observer procedure. This would falsify Proposition 4.3's completeness claim and closure as a corollary.

**F2.** An experimentally realised observable that does not participate in the comparison partial order induced by A2, does not reduce to a distinguishability measurement, and is not a commitment multiplicity. Such an observable would escape the complete invariant while remaining operationally real.

**F3.** A demonstrable observer class (not merely a hypothetical) with access to distinctions outside the image of  $\mathbf{F}$  for some  $X \in \mathcal{A}$ . This would falsify A3 directly and closure as a corollary.

**F4.** A pair of admissible structures disagreeing on the comparison outcome for an identifiable pair of committed facts. This would falsify A2's structure-independence clause and undermine Lemma 4.2.

F1 and F4 are structural refutations; F2 and F3 are empirical. The empirical tests attach to the same experimental programme as VERSF's testable predictions — any observation that cannot be cast as a component of  $(\leq, \varphi, \mu)$  on a committed-record fact-class is a candidate for F2.

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## 10. Discussion

### 10.1 What closure rules out

Closure eliminates three families of residual freedom that prior VERSF papers left in principle open:

1. **Hidden-sector extensions** coupling to facts only through constraint-preserving channels. Any such sector either affects  $(\leq, \varphi, \mu)$  and is therefore already in the equivalence class, or has no operational effect and is quotiented out under A3.
2. **Non-isomorphic structural refinements** producing the same observational data. These are collapsed by  $\approx$ , which (per §2.4) is forced by A1–A3 rather than stipulated.
3. **Parallel admissible frameworks** yielding the same phenomenology via distinct structural roots. These are forbidden: all admissible structures are  $\approx$ -equivalent.

### 10.2 What closure does not rule out

Closure is a statement about structural uniqueness, not about uniqueness of *description*. Two mathematically distinct formalisms can both describe the same equivalence class. In particular, a reformulation of VERSF in an alternative category (e.g. via higher-category or topos-theoretic language) is not forbidden; it must, however, yield the same  $F(X)$  and hence the same observational content.

Closure also does not rule out the existence of mathematical scaffolding that accompanies the physical content (gauge freedom, coordinate choices, regularisation schemes). A3's operational framing explicitly leaves such scaffolding intact; only features with no operational consequence at all are identified with features of other structures.

### 10.3 Self-enforcement

The deeper intuition, now backed by the chain Lemma 4.1  $\rightarrow$  4.2  $\rightarrow$  Prop 4.3  $\rightarrow$  Lemma 4.4  $\rightarrow$  Lemma 4.5, is that fact-formation is self-referential. Any candidate extension must itself be mediated by facts (by A3), fact-mediation is fully characterised by the complete invariant (by Proposition 4.3), and identity within the fact-class is purely relational (by Lemma 4.4). The constraint system therefore closes on itself: there is no route by which new structure can enter without being captured by the existing invariant.

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## 11. Conclusion

We have proved that the class of admissible structures, given three operational axioms on observers and facts, collapses to a single equivalence class under observational equivalence. The proof does not presuppose the Single-Source Theorem; it constructs an independent route via the Fact Functor and the completeness of its observational invariant.

Combined with the separate results of the VERSF programme, the chain from facthood to physical structure is:

A1–A3  $\Rightarrow$  closure (this paper)  $\Rightarrow$  uniqueness up to  $\approx$

- No-Go for Non-Simplicial Substrates  $\Rightarrow$  K=7 triangular structure
- SST  $\Rightarrow$   $\rho$ -functional observables
- specific derivations  $\Rightarrow$  VERSF phenomenology

Each arrow is the subject of a separate paper, and each is stated conditionally on its axioms. What this paper contributes is the previously missing second arrow: the guarantee that admissibility closes, rather than merely restricts. The conditionality of the VERSF programme is thereby localised in the operational axioms A1–A3 and in the specific structural derivations, rather than pervading the programme through an unproven closure assumption.

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## Appendix A — Evolution of the proof

**v1** argued that any new feature S generating facts must obey C and therefore "does not introduce independent structure." This inference is invalid in general: distinct structures can obey the same constraints. v1 rescued the inference by implicitly citing SST, making the closure theorem a corollary rather than an independent result.

**v2** replaced this with a Fact-Functor argument but left five load-bearing assertions unsupported:

1. Lemma 4.2 asserted cross-structure agreement of comparison outcomes without deriving it from A2.
2. A3 was presented in a form that could be read as ontological rather than operational.
3. Lemma 4.3 did not explicitly state what  $\beta$  preserved or why those preserved data were complete.
4. Step 4 of the main proof was intuitive but formally unsupported.
5. Observational equivalence was presented as a definition rather than as a forced consequence of A1–A3.

**v3 (previous version)** closed each of these:

1. A2 is upgraded with a structure-independence clause; Lemma 4.2 is then a direct consequence.
2. A3 is accompanied by an explicit scope remark distinguishing operational from ontological content, with gauge and hidden-variable examples.
3. Proposition 4.3 establishes that  $(\leq, \varphi, \mu)$  is a complete observational invariant; Lemma 4.5 explicitly invokes this completeness.
4. Lemma 4.4 formalises relational identity and is cited in Step 4.
5. §2.4's derivation remark establishes  $\approx$  as the coarsest equivalence forced by A1–A3, not as an independent postulate.

**v3.1 (this version)** addresses four residual attack vectors that remained after v3's hardening:

6. **A2 strength concern.** The structure-independence clause could be read as baking uniqueness into the axiom. §2.1 now carries an "on the strength of this clause" paragraph framing structure-independence as a precondition for cross-observer physics rather than a covert uniqueness assumption.
7. **Proposition 4.3 scope.** The completeness claim could be challenged by citing probabilistic observables, phase/interference structure, or higher-order correlations as outside  $(\leq, \varphi, \mu)$ . A remark after the proof addresses all three families explicitly, showing how each reduces to the trichotomy under functional combination or iteration.
8. **Step 4 compression.** The no-new-equivalence-class conclusion was intuitive but not tied explicitly to Proposition 4.3. A "closure of the loop" clause in Step 4(a) now makes this link explicit.
9. **Relations among new facts.** The most sophisticated category-theoretic attack — that new facts could form a new independent relational structure among themselves — is now addressed directly in Step 4(a), invoking Proposition 4.3 to show that such relations are themselves instances of  $(\leq, \varphi, \mu)$  and do not introduce new primitives.

## Appendix B — Minimal formal statement

Let  $(\mathcal{A}, \sqsubseteq)$  be the poset of admissible structures under extension, and let  $\approx$  be observational equivalence. Let  $\mathbf{F}: \mathcal{A} \rightarrow \mathbf{Fact}$  be the Fact Functor of Definition 2.4.

Under A1–A3:

$$\forall X, X' \in \mathcal{A} : X \sqsubseteq X' \Rightarrow X \approx X'$$

Equivalently,  $(\mathcal{A} / \approx, \sqsubseteq)$  is the one-element poset, where  $\sqsubseteq$  is the induced order on equivalence classes.

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*Submitted for internal peer-review-style refinement within the VERSF programme. Comments, counterexamples, and sharpenings of A1–A3 are welcomed, particularly any demonstration that A2's structure-independence clause admits a genuinely weaker form that preserves closure.*

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### External

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### Internal (VERSF programme)

All internal references are available at [versf-eos.com](http://versf-eos.com) as part of the ongoing VERSF research programme conducted through the AIDA Institute.

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