

# Closedness of the Matching Cochain

## Local Orientation-Coherence as the Checkable Form of Closedness for the Gate-2 Matching Cochain

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### General Reader Summary

The Gate-2 verdict reduced its pass condition to the absence of an orientation twist of the vacuum — written  $[s]_{\text{vac}} = 0$  — where  $s$  is the matching cochain on the shared faces of the substrate. A companion line of work asked whether a second rule in the framework (loop-consistency, through the interface dynamics) could save labour in checking that condition by disposing of all the shrinkable loops for free, and left open whether that route was needed.

This paper isolates the one fact that decides the question, and states precisely what kind of fact it is. The matching cochain  $s$  is *closed* — it has no local twist,  $\delta s = 0$  — exactly when the substrate is **locally orientation-coherent**: when the orientation signs the gluing assigns to faces multiply to +1 around the boundary of every elementary 2-cell. The honest description matters here. That local condition is not a separate geometric input from which closedness is *deduced* — on a basis of 2-cells it *is* closedness, written one cell at a time. The theorem's real content is a *localization*: checking each 2-cell separately suffices for the global statement, because the 2-cells span everything. That makes the condition checkable cell by cell, and it is a genuine and useful fact. It is "local closedness is closedness," not "geometry forces closedness" — and certainly not the circular "s represents a characteristic class, therefore it is closed," since representing a class already presupposes closedness and the class is only a name for an already-closed cochain.

Two cautions the paper keeps in view. First, *local* coherence is strictly weaker than *global* orientability: a substrate can be coherent on every 2-cell yet globally twisted, like a Möbius band, and that global twist lives precisely in the loops that wrap all the way around — the very loops the shortcut does *not* dispose of. So closedness frees the shrinkable loops and leaves exactly the global obstruction behind, which is where the real content goes. Second, an operationally-defined sign assignment has no guarantee of closing around each 2-cell at all; if it fails to do so on a cell of the *vacuum* sector, there is no orientation twist to speak of there — the class is simply undefined — and that possibility has to be named, not assumed away.

Once  $s$  is closed, every shrinkable (bounding) loop carries trivial holonomy automatically, by an elementary identity. The labour-saving shortcut therefore needs **no input from the interface dynamics** — it follows from the substrate's own local coherence. The interface programme is not thereby made useless; it simply moves to a different job: not establishing closedness (the substrate already does that), but computing the surviving non-trivial part of the class on the

wrap-around loops and detecting any transport content beyond orientation. Those are Gate-3 questions.

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## Abstract

The Gate-2 verdict identified the matching cochain  $s$  with the orientation cochain  $t$  of the substrate 2-complex (verdict Theorem 6.2). This paper derives the principal structural consequence of that identification, under an explicitly named substrate hypothesis, and corrects a circular argument that would otherwise appear to deliver it for free.

**The hypothesis (H-OC, local orientation-coherence).** The substrate's gluing assigns to each shared face a sign  $t_f \in \{\pm 1\}$  (+1 orientation-preserving,  $-1$  reversing). H-OC is the statement that these signs multiply to  $+1$  around the boundary of every elementary 2-cell  $P$ :  $\langle t, \partial P \rangle = 0$  for all  $P$ . Equivalently, the gluing signs satisfy the  $\mathbb{Z}_2$  cocycle condition on each 2-cell, so that they *assemble into* a genuine  $\mathbb{Z}_2$  line bundle on the substrate. This is the *local* form of orientability — a per-2-cell condition, strictly weaker than global orientability — and it is decidable by inspection of the substrate's gluing data, without the interface dynamics. (That a bundle *assembles* is H-OC; whether it is *trivial* is the separate, global statement  $[s]_{\text{vac}} = 0$  — the Gate-2 pass. The two must not be conflated: H-OC is the existence of the class, the pass is its vanishing.)

**The Geometric Cocycle Theorem, correctly named: a localization (Theorem 3.1, [Proven under H-OC]).** Under H-OC, the matching cochain is closed:

$$\delta s = 0.$$

On a 2-chain basis  $\delta t = 0$  is *equivalent* to  $\langle t, \partial P \rangle = 0$  for every plaquette  $P$ , so H-OC is not a premise from which closedness is *derived* — **H-OC is closedness, written one 2-cell at a time** — and the theorem is a *localization*: the per-cell condition suffices for global  $\delta s = 0$  because plaquettes span  $C_2$ , making closedness checkable cell by cell. This is "local closedness is closedness," not "geometry implies closedness," and in particular not the circular "s represents  $w_1$ , hence  $\delta s = 0$ " (representing a class presupposes closedness;  $w_1$  is a class, not a cochain). The honest order is  $\text{H-OC} \Rightarrow \delta s = 0 \Rightarrow [s] \text{ defined} \Rightarrow [s] = w_1$ . Remark 3.2 makes both points once, at full strength.

**Consequence (Corollary 3.2, [Proven under H-OC( $\Gamma_{\text{vac}}$ )]).** Every bounding loop carries trivial orientation holonomy:  $\langle s, \partial F \rangle = \langle \delta s, F \rangle = 0$ . The bounding-loop content of  $[s]_{\text{vac}}$  is exhausted automatically, with no input from the interface dynamics. The residual content of  $[s]_{\text{vac}}$  lies entirely in the non-bounding loop structure of the vacuum sector.

**What this settles, and what it does not.** Under H-OC the verification shortcut is *geometric*: it follows from the substrate alone, and the interface-orientation identification is **idle** for it. This places the substrate squarely in the "geometric route" horn of the transport-holonomy programme

and renders that programme's residual-case apparatus (the operational-cochain analysis) inapplicable to the shortcut. The interface programme retains a role, but a strictly downstream one: computing  $[s]_{\text{vac}}$  on non-bounding loops, determining whether transport reproduces that class, and detecting any transport surplus  $\kappa$ . These are Gate-3 questions, not Gate-2 questions.

**The open dependency, as a trichotomy.** Whether the result is *unconditional* turns on what the verdict supplies; §4 states three branches. **(i)** the verdict establishes H-OC — Theorem 3.1 is unconditional; **(ii)** it fixes only the operational identification  $s = t$  with H-OC open — conditional, the interface route may settle closedness; **(ii')** it fixes  $s = t$  operationally and H-OC *fails* — and here scope is decisive: failure on a plaquette of  $\Gamma_{\text{vac}}$  leaves  $s$  non-closed there, so  **$[s]_{\text{vac}}$  is undefined** and the pass criterion needs restating, whereas failure only outside  $\Gamma_{\text{vac}}$  loses Theorem 3.1's global statement but leaves  $[s]_{\text{vac}}$  and the vacuum corollaries intact. Branch (ii') is the substrate pathology H-OC exists to exclude; an operational  $s$  carries no a priori guarantee of closing on  $\Gamma_{\text{vac}}$ . We localize the entire dependency to H-OC and prove everything else.

Labels: [Proven], [Proven under H-OC], [Methodological], [Open: substrate reading].

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## 1. Position and the Two Cochains

The Gate-2 verdict returned, unconditionally under C1–C3,

$B5 \Leftrightarrow [s]_{\text{vac}} = 0 \Leftrightarrow \text{Gate 2 PASS on the vacuum,}$

with  $[s]_{\text{vac}} \in H^1(\Gamma_{\text{vac}}; \mathbb{Z}_2)$  — *when  $s$  is closed* — the restriction to the vacuum sector of the class of the matching cochain  $s$ . We do not modify this. Everything here sits strictly downstream.

The verdict additionally identified the matching cochain with a geometric object (verdict Theorem 6.2):

$s = t$ ,

where  $t$  is the **orientation cochain** of the substrate 2-complex, assigning to each shared face  $f$  the sign

$t_f = +1$  if the gluing across  $f$  preserves induced orientation,  $-1$  if it reverses it,

written additively as a  $\mathbb{Z}_2$  1-cochain ( $+1 \leftrightarrow 0, -1 \leftrightarrow 1$ ). We write  $\delta$  for the coboundary,  $\partial$  for the boundary, and  $\langle \cdot, \cdot \rangle$  for the  $\mathbb{Z}_2$  pairing of cochains with chains, related by the adjunction  $\langle c, \partial F \rangle = \langle \delta c, F \rangle$ .

Two distinct questions must be kept apart, since they are easily run together:

- **Is  $t$  a cochain that represents a class?** — i.e. is  $\delta t = 0$ ? This is the question of *closedness*, and it can genuinely answer *no* for an operationally-defined  $t$  (§4, branch ii'). It is the content of this paper.
- **If  $t$  is closed, which class does it represent?** — the orientation obstruction  $w_1 \in H^1(\Gamma; \mathbb{Z}_2)$ . This is a *labelling* of an already-closed cochain, downstream of closedness, and is not in question here.

The error to avoid is treating the second as if it answered the first. " $t$  represents  $w_1$ " cannot establish " $\delta t = 0$ ", because being a representative *means* being a cocycle: the implication runs  $\delta t = 0 \Rightarrow [t]$  defined  $\Rightarrow [t] = w_1$ , never backwards.  $w_1$  is an element of  $H^1$  — a class — and has no coboundary; only a cochain representative does. Closedness must be earned from the geometry of  $t$  directly, and — crucially — earned *locally*: the condition  $\delta t = 0$  is equivalent, on a basis of 2-cells, to  $t$  closing around each 2-cell separately. The whole content of §3 is that this local, per-cell condition (H-OC) is the global closedness, made checkable. That is a localization, not a derivation of closedness from some independent geometric premise.

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## 2. The Hypothesis: Orientation-Coherence of the Substrate

**Definition 2.1 (elementary plaquette).** An elementary plaquette  $P$  is a single 2-cell of the substrate;  $\partial P$  is its boundary 1-cycle, the cyclic sequence of shared faces bounding  $P$ . The plaquettes  $\{P\}$  form a basis of the 2-chain group  $C_2(\Gamma; \mathbb{Z}_2)$ .

**Definition 2.2 (local orientation-coherence, H-OC) [substrate hypothesis].** The substrate is *locally orientation-coherent* on a subcomplex  $\Sigma$  ( $\Sigma = \Gamma$  for the global statement,  $\Sigma = \Gamma_{\text{vac}}$  for the vacuum statements) if the orientation signs close around every plaquette of  $\Sigma$ :

$\langle t, \partial P \rangle = 0$  for every elementary plaquette  $P$  of  $\Sigma$ ,

equivalently  $\prod_{f \in \partial P} t_f = +1$  for each such  $P$  — the product of gluing signs around the boundary of each 2-cell is  $+1$ . On a 2-chain basis of  $\Sigma$  this is *equivalent* to  $\delta t = 0$  on  $\Sigma$  — hence  $\delta s = 0$ , since  $s = t$ ; H-OC( $\Sigma$ ) is thus the **local form of closedness on  $\Sigma$** , one per-2-cell equation, and Theorem 3.1 is the observation that it suffices for the global statement. **Scope matters and is tracked throughout:** Theorem 3.1's  $\delta s = 0$  on all of  $\Gamma$  needs H-OC( $\Gamma$ ); the vacuum statements (Corollaries 3.2–3.3, every claim about  $[s]_{\text{vac}}$ ) need only H-OC( $\Gamma_{\text{vac}}$ ), the restriction H-OC( $\Gamma$ ) implies. We state H-OC on  $\Gamma$  as the cleaner hypothesis and note where only the  $\Gamma_{\text{vac}}$  restriction is used.

**Remark 2.3 (H-OC is local, strictly weaker than global orientability — its single statement) [Methodological].** H-OC is the condition that the per-face gluing signs satisfy the  $\mathbb{Z}_2$  cocycle condition around each 2-cell. This is exactly bundle assembly: a  $\mathbb{Z}_2$  line bundle is transition data  $\{t_f\}$  on faces subject to the cocycle identity around each 2-cell, so " $\langle t, \partial P \rangle = 0$  for every  $P$ " is that condition, and a closed  $t$  is by construction a Čech representative of  $w_1$  — this standard cellular-to-Čech correspondence is the hinge from H-OC to the  $w_1$  labelling, and we name it rather than let "assemble into" carry it silently.

H-OC is the *local* part of orientability and **strictly weaker than global orientability**. A complex can satisfy H-OC on every 2-cell yet be globally non-orientable: the Möbius obstruction lives in the *non-bounding* loop structure, not in plaquette closure, so it is invisible to H-OC and survives as a non-trivial class. This is not a defect — it is precisely why closedness frees only the bounding loops (Corollary 3.2) and leaves a residue (Corollary 3.3): the residue Corollary 3.3 isolates is exactly the global obstruction that local H-OC does not see. Equating H-OC with global orientability would overclaim and contradict 3.3. This is the sole place we make the local/global distinction at length; later sections invoke it by reference.

H-OC is **not** automatic: a gluing whose signs fail to close around some 2-cell defines no line bundle and no orientation class — branch (ii') of §4. Its content is a genuine *local* fact about the substrate, checkable cell by cell from the gluing data alone, without the interface dynamics; where the relevant 2-skeleton is finite or locally finite this is a finite per-cell check, and otherwise it remains a per-cell (locally checkable) condition.

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## 3. The Geometric Cocycle Theorem

**Theorem 3.1 (Geometric Cocycle Theorem — a localization) [Proven under H-OC].** If the substrate is locally orientation-coherent on  $\Gamma$  (Definition 2.2, H-OC( $\Gamma$ )), then the matching cochain is closed:

$$\delta s = 0.$$

*Proof.* By verdict Theorem 6.2,  $s = t$ . By H-OC( $\Gamma$ ),  $\langle t, \partial P \rangle = 0$  for every elementary plaquette  $P$  of  $\Gamma$ . By the boundary–coboundary adjunction,  $\langle \delta t, P \rangle = \langle t, \partial P \rangle = 0$  for every  $P$ . Since the

plaquettes  $\{P\}$  are a basis of  $C_2(\Gamma; \mathbb{Z}_2)$ , the cochain  $\delta t$  vanishes on a basis of 2-chains, hence  $\delta t = 0$ ; and  $s = t$  gives  $\delta s = 0$ .

**Remark 3.2 (what the theorem is, and what it is not — the single statement)**

**[Methodological].** Two points, stated once here at full strength and referenced rather than re-argued elsewhere.

*It is a localization, not a derivation.* On a 2-chain basis,  $\delta t = 0 \Leftrightarrow \langle t, \partial P \rangle = 0$  for all  $P$ ; the proof is the adjunction plus "a basis spans." H-OC is not an external geometric premise *implying*  $\delta s = 0$  — H-OC *is*  $\delta s = 0$ , written one 2-cell at a time. The content is that the per-cell form suffices for the global one, which is what makes closedness checkable cell by cell. This is "local closedness is closedness," not "geometry implies closedness"; dressing it as the latter would, in mild form, re-commit the naming move the next point catches.

*Closedness is earned from the gluing, not from the characteristic class.* The tempting " $s = t$  represents  $w_1$ , hence  $\delta s = 0$ " is circular twice over: " $t$  represents  $w_1$ " already asserts  $\delta t = 0$  (a representative is a cocycle by definition), so it cannot be a premise for it; and  $w_1$  is a class, not a cochain, so " $\delta w_1 = 0$ " is a category error — only representatives have coboundaries.

Characteristic-class theory says *given a bundle,  $w_1$  is well-defined*; it does not make an arbitrary gluing-sign cochain close. What makes  $t$  a cocycle is H-OC (the bundle assembles), a per-cell substrate fact. We establish closedness from H-OC and label  $w_1$  only afterward, as the class of the now-closed  $t$ .

**Corollary 3.2 (bounding-loop holonomy is trivial) [Proven under H-OC( $\Gamma_{\text{vac}}$ )].** Under H-OC on  $\Gamma_{\text{vac}}$ , every bounding (null-homologous) loop  $\gamma = \partial F$  in  $\Gamma_{\text{vac}}$  carries trivial orientation holonomy:

$$\langle s, \gamma \rangle = \langle s, \partial F \rangle = \langle \delta s, F \rangle = \langle 0, F \rangle = 0.$$

*Proof.* Immediate from the vanishing of  $\delta s$  on  $\Gamma_{\text{vac}}$  and the adjunction.

**Corollary 3.3 (the residue is entirely non-bounding — and is the global obstruction)**

**[Proven under H-OC( $\Gamma_{\text{vac}}$ )].** Under H-OC on  $\Gamma_{\text{vac}}$ , the bounding-loop content of  $[s]_{\text{vac}}$  is exhausted automatically, and  $[s]_{\text{vac}}$  is determined entirely by the non-bounding (non-null-homologous) loops of  $\Gamma_{\text{vac}}$ .

*Proof.* H-OC( $\Gamma_{\text{vac}}$ ) makes  $s$  closed on  $\Gamma_{\text{vac}}$ , so  $[s]_{\text{vac}} \in H^1(\Gamma_{\text{vac}}; \mathbb{Z}_2)$  is defined; it is the pairing of  $s$  with  $H_1(\Gamma_{\text{vac}}; \mathbb{Z}_2)$ , and its bounding part is killed by Corollary 3.2, leaving the non-bounding homology as the sole carrier.

This residue is precisely the *global* orientation obstruction that the *local* H-OC does not detect (Remark 2.3): H-OC guarantees the class exists, the non-bounding loops decide whether it vanishes. The Möbius-type non-orientability that survives local coherence lives exactly here.

## 4. The Status of H-OC Against the Verdict, and What Each Answer Means

The theorem is proven *under H-OC*. Whether the result is **unconditional** turns on a single external fact: what the Gate-2 verdict supplies. This is a reading of verdict Theorem 6.2 and the substrate construction, which we cannot settle inside this paper, so we state all branches exactly and assume none. There are three, not two — a two-way "established vs. open" framing omits a live case, the same non-closure case the transport-holonomy companion had to add when its own trichotomy proved non-exhaustive.

**Branch (i): the verdict establishes H-OC.** If verdict Theorem 6.2 (or the substrate construction it rests on) establishes that the gluing signs close around every 2-cell — local orientation-coherence — then H-OC holds outright, Theorem 3.1 is unconditional, and  $\delta s = 0$  is a genuine consequence of the verdict with no further input. The result is exactly as the title claims, and §5's consequences follow. The correct citation in Theorem 3.1's proof is then to *that* substrate result, not to characteristic-class generality.

**Branch (ii): the verdict fixes only the operational identification, H-OC open.** If verdict Theorem 6.2 establishes  $s = t$  with  $t$  defined operationally by transport-gluing — the orientation the dynamics enacts at each face — but does not establish that those signs close around each 2-cell, then H-OC is **open**. Here  $\delta s = 0$  is not proven; the matching cochain is the "operational cochain of unestablished closedness" of the transport-holonomy programme's residual case, and closedness may be sought by the interface route ( $s - \sigma$  a cocycle) or left open. This is the branch in which that programme's residual-case apparatus is live.

**Branch (ii)': the verdict fixes the operational identification, and H-OC *fails*.** A two-way "established vs. open" framing assumes closedness is *achievable*. It need not be. The consequence depends on *where* closure fails, and the scope distinction of Definition 2.2 is essential here:

- **Failure on a plaquette of  $\Gamma_{\text{vac}}$**  —  $\langle t, \partial P \rangle \neq 0$  for some  $P$  in  $\Gamma_{\text{vac}}$ . Then  $s$  is not a cocycle on  $\Gamma_{\text{vac}}$ ,  $s$  represents no class there, and  **$[s]_{\text{vac}}$  is undefined**. This is the genuinely dangerous case: the Geometric Cocycle Theorem's conclusion is false where it matters, and the Gate-2 pass criterion itself needs attention — either  $s$  replaced by a closed representative (if the class is to be recovered) or the criterion restated, since " $[s]_{\text{vac}} = 0$ " presupposes  $[s]_{\text{vac}}$  exists.
- **Failure only on  $\Gamma \setminus \Gamma_{\text{vac}}$**  — closure holds throughout  $\Gamma_{\text{vac}}$  but fails on some plaquette outside it. Then  $\delta s \neq 0$  globally, so Theorem 3.1 (which needs H-OC( $\Gamma$ )) fails, **but  $[s]_{\text{vac}}$  remains perfectly well-defined** because H-OC( $\Gamma_{\text{vac}}$ ) still holds, and Corollaries 3.2–3.3 stand. The global theorem is lost; the vacuum statements survive. Global non-closure does not by itself undermine the vacuum class — only non-closure *inside*  $\Gamma_{\text{vac}}$  does.

An operational  $s$ , not being the geometric orientation cochain, carries **no a priori guarantee** of closing on  $\Gamma_{\text{vac}}$  — this is the substrate pathology H-OC exists to exclude, and the non-closure case the companion programme had to add when its trichotomy proved non-exhaustive.

**The diagnostic.** The word "gluing" in the definition of  $t$  is the tell: whether an operational sign closes around a 2-cell is substantive, not definitional, and can answer *no*. The single action item is to read verdict Theorem 6.2 and locate the substrate among the branches. If (i), promote H-OC from "Definition 2.2 hypothesis" to "established by verdict §X" — a one-line edit that changes no logic and makes Theorem 3.1 unconditional. If (ii), the residual interface analysis stands. If (ii'), determine *where* closure fails: inside  $\Gamma_{\text{vac}}$ , the pass-criterion framing must be revisited before anything downstream proceeds; outside it, only the global statement is affected and the vacuum results hold.

We flag, but do not resolve, this dependency. Everything else in the paper is proven outright under the named hypothesis.

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## 5. Consequence for the Transport-Holonomy Programme

This section is stated for Branch (i) (H-OC holds); under Branch (ii) it is contingent on closedness being established by other means.

**The shortcut is geometric, and the identification is idle for it.** The transport-holonomy programme drew a dichotomy: a *geometric route* ( $s$  the closed orientation cochain of the substrate  $\implies \delta s = 0$  by construction; the interface-orientation identification idle for the shortcut) and a *residual route* ( $s$  operational, closedness open, the identification load-bearing). Theorem 3.1 under H-OC places the substrate in the **geometric route**: closedness follows from orientation-coherence, the bounding loops discharge automatically (Corollary 3.2), and the interface-orientation identification supplies *nothing* to the shortcut that the substrate has not already supplied. The residual-case apparatus — the operational-cochain comparison, the cochain-level hypotheses on  $s - \sigma$ , the four-way split — is therefore inapplicable to the *shortcut*: there is no closedness left for it to import.

**The interface programme is not eliminated — it is relocated.** What remains after Corollary 3.3 is the non-bounding content of  $[s]_{\text{vac}}$ , and that is untouched by closedness. The interface programme retains exactly three roles, all strictly downstream of the shortcut and all Gate-3 in character:

1. **Compute  $[s]_{\text{vac}}$  on the non-bounding loops.** Closedness frees the bounding loops; it says nothing about the value of the class on the non-bounding homology, which is where the entire residue now lives (and where any global non-orientability hides — Corollary 3.3).

2. **Determine whether transport reproduces the class.** Whether the orientation projection of the interface transport holonomy agrees with  $[s]$  on the non-bounding loops — the image question — is a comparison of the transport's orientation read-out against the now-closed  $s$ .
3. **Detect any transport surplus  $\kappa$ .** Whether the full interface transport carries content beyond its orientation projection — a residual holonomy in the closure channel that the orientation class never sees — is the splitting question, and it is the genuine first higher transport-holonomy layer of Gate 3.

None of these is a Gate-2 verification step. The Geometric Cocycle Theorem thus does to the shortcut what closedness always does: it removes the bounding loops as a site of work and concentrates the remaining content in the non-bounding structure, where the transport programme's real Gate-3 questions live.

**Reconciliation with the residual case's  $s \neq \sigma$  distinction.** A reader holding the transport-holonomy paper will note a possible tension: that paper's Definition 5.0 made its residual case non-vacuous by insisting the matching read-out  $s$  and the holonomy read-out  $\sigma = \pi_{\text{or}}(U)$  are *distinct cochains*. Saying here that the residual apparatus is "inapplicable to the shortcut" must not be read as discarding that distinction. It does not. Under H-OC, what is settled geometrically is the residual case's *closedness question* —  $\delta s = 0$  comes from the substrate, so the interface route's job of supplying closedness is empty. But the  $s \neq \sigma$  distinction *survives untouched* and is exactly what roles 2 and 3 above turn on: role 2 (does transport reproduce the class) is the comparison of  $\sigma$  against the now-closed  $s$  on non-bounding loops, and role 3 (surplus  $\kappa$ ) is the content of the transport beyond  $\sigma$ . So the two papers are consistent — H-OC retires the residual route's closedness task while preserving the  $s$ -versus- $\sigma$  distinction for the downstream image and splitting questions. Idle-for-the-shortcut is not idle-everywhere.

**Relation to the nesting result.** The transport-holonomy programme reached a compatible conclusion — that the geometric and interface routes to the shortcut are nested, the interface route's only possible contribution being to establish  $\delta s = 0$ . But this is shared hypothesis surfacing in two places, *not* two independent routes converging: that programme's geometric horn was *defined* by "s the closed orientation cochain of the substrate" — i.e. it assumed H-OC. So this paper proving  $\delta s = 0$  under H-OC and that paper assuming it are the same assumption seen twice, not corroboration. The accurate statement of the relationship: this paper supplies the transport-holonomy dichotomy's geometric horn with a *local form of its defining hypothesis* (H-OC) and a *per-plaquette decision procedure* for it, and shows the residue is the global obstruction that horn always pushed into the non-bounding loops. That is what the two papers share and how they fit — not an independent agreement.

## 6. Limitations

**Conditional on H-OC; status against the verdict not performed.** Theorem 3.1 is [Proven under H-OC] and is a localization (Remark 3.2), not a derivation. Unconditionality depends on

which §4 branch holds — (i) H-OC established, (ii) open, (ii') failing — a reading of verdict Theorem 6.2 external to this paper. Branch (ii') is genuine for an operational  $s$ : failure on a *vacuum* cell leaves  $[s]_{\text{vac}}$  undefined, failure only outside  $\Gamma_{\text{vac}}$  costs only the global statement. We localize the conditionality to H-OC and prove everything else; we do not assume the branch. [Open: substrate reading.]

**H-OC is local, not global (Remark 2.3).** The residue Corollary 3.3 isolates is exactly the global obstruction local H-OC does not see; reading H-OC as global orientability would contradict 3.3.

**Closedness is not the class.** Theorem 3.1 establishes  $\delta s = 0$ ; it does not compute  $[s]_{\text{vac}}$ , which is exactly the non-bounding content Corollary 3.3 isolates and §5 routes to the interface programme. A closed cochain can have any class; closedness frees the bounding loops and nothing more.

**The pass criterion is untouched.** Nothing here modifies  $B5 \Leftrightarrow [s]_{\text{vac}} = 0$  (unconditional, under C1–C3). The result concerns only the *verification shortcut* for that criterion, never the criterion itself.

**The transport surplus is not addressed here.** Whether the interface transport carries content beyond orientation (the  $\kappa$  / splitting question of §5.3) is a Gate-3 matter treated in the transport-holonomy programme, conditional there on the structure of the closure channel; it is named here only to locate the interface programme's relocated role, not resolved.

**The framework is taken as given.** The cohomology used — the boundary–coboundary adjunction,  $\mathbb{Z}_2$  cochains, cocycle versus coboundary, the line-bundle reading of orientation signs — is standard and applied within the VERSF/substrate framework; the Gate-2 verdict (including the identification  $s = t$ ) is taken as established, not re-derived. "Established," where used, means within that framework and under the stated hypothesis.

## 7. Conclusion

The Gate-2 verdict identified the matching cochain  $s$  with the orientation cochain  $t$  of the substrate. The principal structural consequence is that  $s$  is closed — but only under an explicit *local* hypothesis, the consequence is a localization rather than a derivation, and it is not closed for the reason a quick argument would suggest.

The matching cochain is closed,  $\delta s = 0$ , **provided the substrate is locally orientation-coherent** (Theorem 3.1, H-OC): provided the gluing signs close around every elementary 2-cell. On a basis of 2-cells this local condition *is* closedness; the theorem's content is that the per-cell check suffices for the global statement because the cells span  $C_2$  — a localization, checkable cell by cell, not a derivation from independent geometry and not the circular appeal to a characteristic class (Remark 3.2). Closedness is earned from the gluing one cell at a time; the class  $w_1$  is only its name afterward.

Under that hypothesis every bounding loop carries trivial holonomy (Corollary 3.2), the bounding-loop content of the vacuum orientation class is exhausted automatically, and the residue lies entirely in the non-bounding loop structure (Corollary 3.3) — which is exactly the *global* obstruction that local coherence does not see, the Möbius-type twist that survives per-cell consistency. The verification shortcut is then *geometric*, following from the substrate alone, and the interface-orientation identification is idle for it. The interface programme is not retired; it moves to its proper, downstream tasks: computing  $[s]_{\text{vac}}$  on the non-bounding loops, deciding whether transport reproduces that class, and detecting any transport surplus. Those are Gate-3 questions — and the  $s$ -versus- $\sigma$  distinction that made the residual case non-vacuous survives untouched to serve them, even as its closedness question is settled geometrically.

The one fact that decides whether this result is unconditional is what the verdict supplies, and there are three possibilities: (i) H-OC established — the theorem is unconditional and the title earned with a one-line citation; (ii) only the operational identification  $s = t$ , H-OC open — the residual interface analysis stands; (ii') the operational signs *fail* to close — and scope decides the damage: failure on a vacuum cell leaves  $[s]_{\text{vac}}$  undefined and the pass-criterion framing in need of repair, while failure only outside  $\Gamma_{\text{vac}}$  costs only Theorem 3.1's global statement and leaves the vacuum results intact. We have localized the entire dependency to the single per-2-cell hypothesis H-OC and proven everything else. The next concrete step is to read verdict Theorem 6.2 against H-OC and locate the substrate among the three branches.

In one line:

**The matching cochain is closed —  $\delta s = 0$  — exactly when the substrate's gluing signs close around every 2-cell (H-OC), which is local closedness made checkable per cell, not a derivation from independent geometry and not the circular appeal to a characteristic class; under it the bounding loops discharge for free and the shortcut owes nothing to the interface dynamics, while the global obstruction survives in the non-bounding loops where Gate-3's real questions — including the still-live  $s$ -versus- $\sigma$  distinction — live; and whether the result is unconditional, conditional, or *false* turns on whether the verdict supplies H-OC, leaves it open, or admits its failure.**