Constitutive Sequential Transport in VERSF

Master-Action Variation and the Derivation of the σ-Family

Keith Taylor *VERSF Theoretical Physics Programme*

General Reader Abstract

The preceding paper in this sequence — *Substrate-Generated Sequential Transport in VERSF* — showed that the rule by which the substrate's state changes from one moment to the next is not arbitrary. A specific alternating pattern emerged as the unique non-trivial response compatible with a small set of substrate principles, and four independent mathematical approaches converged on the same answer.

But the previous paper stopped at a particular point. It identified the form of the substrate's response and showed it was natural, but it did not derive that form by direct variation of the underlying VERSF master action. Two postulates carried the construction: the specific four-term structure of the response functional A_cl, and the closure-current conservation law that fixes its coefficients. Both were structurally motivated; neither was derived.

This paper closes both gaps.

We restrict the VERSF master action to the sequential transport sector — perturbations of the substrate that act only on the spoke sector of the K = 7 closure architecture — and vary directly. The four-term decomposition of A_cl is then shown to be the unique leading-order admissible local quadratic functional under stated methodological principles. Closure-current conservation is shown to be the cycle-level statement that is equivalent, under standard homological projection, to commitment conservation at the cell level — not an extra postulate but a translation between formulations of the same substrate principle.

We then study the resulting non-linear flow and confirm what the previous paper conjectured: the alternating mode is the unique persistent direction of the gradient flow on the admissible spoke sector, with all other modes decaying. Integrality of the response (a quantisation condition) selects the specific minimal pattern $\lambda_i = \pm(-1)^i$.

Finally we revisit the residual D_3(vtx) symmetry. The previous paper proposed — conditional on two unstated structural hypotheses about the K = 7 ontology — that this symmetry belongs to the transient transport between committed states, not to the committed states themselves. The present paper retains one of those hypotheses (that committed states are D_6-symmetric, which is a structural fact about the K = 7 architecture stated separately in the broader programme) and *derives* the other from it. The transport-gap reading is therefore upgraded from "proposed conditional on two hypotheses" to "derived from one structural input."

The conceptual picture that emerges is increasingly unified. The σ-family is no longer an imposed rule. It is the constitutive transport response of the substrate, derived from the same master-action methodology that produces the constitutive-current sector of the broader VERSF programme. The two sectors share a structural architecture, and the comparison between them in §10 makes this convergence explicit.

What remains open is the full continuum-limit derivation, the connection to non-Abelian transport sectors, the coupling to matter sectors, and the unification of the transport, gauge, and cohomology sectors under one master-action variation. These are now sharply posed questions, each pointing toward a specific further paper.

Abstract

The preceding paper, *Substrate-Generated Sequential Transport in VERSF*, established that the family of transition morphisms

$W_7^{(0)} \xrightarrow{\sigma_0} W_7^{(1)} \xrightarrow{\sigma_1} W_7^{(2)} \longrightarrow \cdots$

admits a canonical non-trivial alternating spoke correction

$\sigma(s_i) = s_{i+1} + (-1)^i C$

under closure-current conservation, integrality, and any one of four converging sharpness criteria. Two load-bearing elements of that construction remained postulated rather than derived: the four-term decomposition of the closure-response functional

A_cl = A_inc + A_hub + A_circ + A_comp,

introduced at leading-order EFT level but not directly varied from the master action; and the closure-current conservation law

$\Sigma_i \lambda_i = 0$,

postulated as the cellular-shadow expression of commitment conservation but without a worked formal bridge.

This paper closes both gaps.

We restrict the VERSF master action to the sequential transport sector — admissible spoke-supported perturbations of a committed K = 7 closure surface — and identify the reduced action with the Hessian of S_VERSF at a committed state, restricted to that sector. We then prove four results.

Theorem 1. *Under (P1a) bilinear locality + (P1b) topological-conservation sectors, (P2) D_6-covariance, (P3) leading-order quadratic structure, (P4) non-negativity with admissibility-zero, and (P5') sum-squared factorisation over the K = 7 constraint catalogue (closure-incidence, hub anchoring, closure-current conservation, closure-competition), the leading-order admissible local quadratic transport functional on the admissible spoke sector is uniquely*

$A\_cl(\lambda) = \alpha \cdot A\_circ(\lambda) + \gamma \cdot A\_comp(\lambda),\ \alpha, \gamma > 0,$

with $A\_circ = (\Sigma_i \lambda_i)^2$ and $A\_comp = \Sigma_i (\lambda_i + \lambda_{i+1})^2$. The constraint functionals A_inc and A_hub vanish identically on this sector and contribute only off-sector. The closure-smoothness candidate $A\_grad = \Sigma_i (\lambda_i - \lambda_{i+1})^2$ is excluded by (P5') since closure-smoothness is not a K = 7 substrate constraint.

Theorem 2. *On the admissible-cycle sector $\mathbb{Z}\langle C\rangle \subset Z_1(W_7)$ of the K = 7 architecture, closure-transport-sector commitment conservation (CCC) — the H_1-class conservation of spoke-correction currents — is equivalent to closure-current conservation (CC), $\Sigma_i \lambda_i = 0$, under the cycle projection $\Pi\_cyc : \mathbb{Z}\langle C\rangle \to H_1(W_7)$. (CC) is therefore the explicit cycle-class form of (CCC) on the admissible-cycle sector. The load-bearing structural input is the admissible-cycle restriction $Z_1 \to \mathbb{Z}\langle C\rangle$, taken from the K = 7 architecture papers.*

Theorem 3. *Under the gradient flow $\partial\_\tau \lambda = -\nabla A\_cl$ on $\Lambda_0 \otimes \mathbb{R}$, all spoke-correction modes orthogonal to the alternating direction decay exponentially; the alternating subspace $\mathbb{R} \cdot \lambda\_alt$ is the unique persistent (zero-gradient) direction. Imposing integrality $\lambda_i \in \mathbb{Z}$ and minimal normalisation selects the canonical attractor $\lambda_i = \pm(-1)^i$.*

Theorem 4. *Given the structural input (H1) that committed states of the K = 7 architecture have stabiliser D_6 (a property of the closure architecture stated separately in the broader programme), and that the K = 7 substrate constraints are themselves D_6-symmetric: the predecessor's hypothesis (H2) disambiguates into (H2a) image-D_6-symmetry of Π_adm and (H2b) D_6-equivariance of Π_adm. (H2a) follows from (H1) plus the definition of Π_adm; (H2b) follows from the D_6-invariance of all four constraint functionals A_inc, A_hub, A_circ, A_comp — itself a consequence of the substrate constraints being D_6-symmetric — via standard equivariance-of-gradient-flow reasoning. The residual D_3(vtx) symmetry of the alternating mode attaches naturally to the transport-excitation sector — the kernel of $M = \nabla^2 A\_cl$ on Λ_0 — rather than to either committed surface. The transport-gap reading is thereby upgraded from "proposed conditional on (H1) and (unspecified) (H2)" to "derived from (H1) plus the D_6-symmetry of the K = 7 constraint catalogue, with both (H2a) and (H2b) derived within this paper."*

The σ-family is thereby upgraded from a constrained admissibility construction to a derived constitutive transport response of the VERSF substrate. The architecture parallels the constitutive-current sector of the broader programme; the comparison is made explicit in §10.

Epistemic status. *Proven (within this paper):* Theorems 1–4 above; the equivalence of closure-transport-sector commitment conservation and (CC) under the K = 7 cell structure and admissible-cycle restriction; the Fourier analysis of the gradient-flow spectrum on Λ_0; the disambiguation and separate derivation of (H2a) and (H2b). *Structural inputs (not derived here):*

(H1) D_6-symmetric committed states, taken from the K = 7 architecture papers; the K = 7 cell structure of W_7 as a 2-complex giving $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$ at the architecture level, together with the admissible-cycle restriction further selecting $\mathbb{Z}\langle C\rangle \subset Z_1(W_7)$ as the substrate-admissible 1-cycles (both from the K = 7 architecture papers); the D_6-symmetry of the K = 7 substrate constraints (closure-incidence, hub anchoring, closure-current conservation, closure-competition), from the architecture's symmetry structure; the K = 7 constraint catalogue (P5′), closing at these four constraints; integrality $\lambda_i \in \mathbb{Z}$, taken as a substrate quantisation condition. *Suggestive but open:* the formal commonalities between the σ-sector and the constitutive-current sector listed in §10, which motivate but do not establish a master-action unification. *Open:* full continuum-limit derivation; vertex × tick-window σ-duality; non-Abelian transport sectors; coupling to matter sectors; full sequential-transport / gauge / cohomology master-action unification.

Contents

1. Introduction

The preceding paper *Substrate-Generated Sequential Transport in VERSF* established four converging characterisations of the alternating transport mode on the K = 7 wheel: maximal residual symmetry (Lemma 6.2, structural), spectral selection (§9.2, T-eigenmode), variational minimisation (Proposition 3, A_comp-minimisation), and Laplacian extremality (Proposition 4, maximum-eigenvalue eigenmode of the standard graph Laplacian L). All four picked out the same lattice direction in Λ_0.

That construction was a substantial constraint on the sequential transport sector, but it stopped short of direct derivation from the VERSF master action. Two load-bearing elements remained postulated:

- The four-term decomposition A_cl = A_inc + A_hub + A_circ + A_comp was identified in §9.1a of the preceding paper as the leading-order admissibility functional under the constrained-EFT methodology established for other constitutive sectors of the master action. This was a *postulation with methodological standing* — the decomposition was not arbitrary, but it was not derived from action variation either.
- The closure-current conservation law $\Sigma_i \lambda_i = 0$ was postulated in §5.2 of the preceding paper as the cellular-shadow expression of commitment conservation in the closure-response sector. The bridge from cell-level commitment to cycle-level (CC) was acknowledged as needing a worked formal computation.

In §10 of the preceding paper a transport-gap interpretation of the residual D_3(vtx) symmetry was proposed, conditional on two structural hypotheses about the K = 7 ontology — (H1) that committed closure surfaces have D_6 stabiliser, and (H2) that the admissibility projection Π_adm preserves D_6 symmetry. The interpretation was structurally consistent but not derived.

The purpose of the present paper is to perform the master-action variation in the sequential transport sector, and thereby:

1. Derive the four-term decomposition of A_cl from explicit constrained-EFT analysis (Theorem 1).
2. Identify (CC) as the H_1-shadow of closure-transport-sector commitment conservation under the K = 7 cell structure and admissible-cycle restriction (Theorem 2).
3. Analyse the resulting non-linear gradient flow and confirm the alternating mode as the unique persistent direction (Theorem 3).
4. Upgrade the transport-gap interpretation from conditional on (H1) + (H2) to conditional on (H1) alone, with (H2) becoming a derived consequence (Theorem 4).

The resulting picture is substantially sharper. The σ-family becomes constitutive rather than merely admissible; closure-current conservation becomes a derived continuity relation; the transport-gap reading becomes a structural consequence rather than a proposed interpretation.

The σ-sector of VERSF takes on the same structural shape as the constitutive-current sector of the broader programme — a parallel made explicit in §10.

---

2. Open Problems Inherited from the σ-Family Paper

The preceding paper isolated load-bearing unresolved questions:

Open problem	*Status in the σ-family paper*
P1a — uniqueness of A_cl decomposition under master-action variation	postulated with methodological standing
P1b — bridge from commitment conservation to (CC)	postulated; formal bridge open
P1c — non-linear gradient flow recovers alternation as dynamical attractor	conjectured by analogy with linearised spectrum
P1e — physical content of the D_3(vtx) residual symmetry	proposed reading conditional on (H1) + (H2)

The present paper addresses all four directly: P1a → Theorem 1; P1b → Theorem 2; P1c → Theorem 3; P1e → Theorem 4. The remaining open problems listed in §12 of the preceding paper (continuum limit, vertex × tick-window duality, non-Abelian sectors, matter coupling, full master-action unification) remain open after this paper and are addressed in §12 below.

---

3. The Sequential Transport Sector of the Master Action

3.1 Setup

Let S_VERSF[ρ] denote the admissibility master action of the VERSF substrate, regarded as a functional on the space of committed substrate states ρ. A committed state at update step t is a map

$\rho_t : V(W_7^{(t)}) \longrightarrow \mathcal{C}_8$

assigning a closure-direction element to each vertex of the K = 7 wheel (cf. §3 of the preceding paper).

Cell structure of W_7. Throughout this paper W_7 is treated as a *2-complex* with the cell structure specified by the K = 7 closure architecture papers: seven 0-cells (six outer vertices v_i and the hub h), twelve 1-cells (six outer edges e_i and six spokes s_i), and a collection of 2-cells whose attaching maps are fixed by the architecture. The 2-cell structure is what gives W_7 its specific homological signature: under the architecture-specified 2-cells, the first homology is

$H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle,$

with $C = e_0 + e_1 + \cdots + e_5$ the primitive 1-cycle around the outer hexagon. (As a bare 1-complex with no 2-cells attached, the wheel graph would have $Z_1 \cong \mathbb{Z}^6$ and $B_1 = 0$, giving H_1 of rank 6; the K = 7 architecture's 2-cell structure is what reduces H_1 to rank 1, with [C] as the surviving generator.) We do not derive the cell structure here; it is taken as a structural input from the foundational K = 7 architecture papers of the broader programme. The specific homological signature $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$ is doing significant load-bearing work in §6 below.

A *sequential transport perturbation* is a σ-action of the form

$$\sigma : s_i \longmapsto s_{i+1} + \Delta_i,\ v_i \longmapsto v_{i+1},\ h \longmapsto h,$$

with spoke corrections Δ_i in the admissible cycle sector:

$$\Delta_i \in \mathbb{Z}\langle C\rangle,\ \Delta_i = \lambda_i\, C,\ \lambda_i \in \mathbb{Z}.$$

The restriction $\Delta_i \in \mathbb{Z}\langle C\rangle$ is a substrate admissibility condition: the K = 7 architecture restricts admissible spoke corrections to the outer-cycle subspace generated by C, ruling out corrections that would correspond to interior cycle excitations not picked up by the architecture's 2-cell structure. This restriction is taken as a structural input from the K = 7 admissibility papers of the broader programme. The restriction is doing significant load-bearing work throughout this paper — in §6 it determines the cycle sector on which CCC reduces to (CC), and in §9 it determines (via (H1)) the stabiliser of committed surfaces; see §6.2 and §9.1 for explicit acknowledgement.

In this paper we work throughout on the admissible sector

$$\Lambda\_full := \mathbb{R}^6 \ni \lambda.$$

The zero-sum sublattice

$$\Lambda_0 := \{ \lambda \in \mathbb{Z}^6 : \Sigma_i\, \lambda_i = 0 \} \subset \Lambda\_full$$

was imposed by hand in the preceding paper as the (CC) constraint. In this paper Λ_0 will emerge as the constraint surface induced by commitment conservation (§6).

3.2 The reduced action

A committed state ρ_committed at step t is, by definition of "committed," a stable extremum of S_VERSF restricted to its admissible neighbourhood. The first-order variation of S_VERSF at ρ_committed therefore vanishes:

$$\delta S\_VERSF / \delta\rho\ |\_\{\rho\_committed\} = 0.$$

A sequential transport perturbation σ_λ moves the substrate from ρ_committed to a perturbed state σ_λ(ρ_committed). The change in master action is

$$\Delta S[\lambda] := S\_VERSF[\sigma\_\lambda(\rho\_committed)] - S\_VERSF[\rho\_committed].$$

By the extremum condition, the linear-in-λ term vanishes. The leading non-trivial contribution is quadratic:

$$\Delta S[\lambda] = \tfrac{1}{2} \langle \lambda, H \lambda \rangle + O(\lambda^3),$$

where H is the Hessian of S_VERSF at ρ_committed, restricted to the spoke-perturbation sector. The reduced sequential transport action is therefore

$$\mathbf{S\_seq[\lambda] := \tfrac{1}{2} \langle \lambda, H \lambda \rangle}$$

at leading order, with H a symmetric 6×6 operator on Λ_full.

Note on the constrained Hessian. The Hessian H is the *constrained* Hessian: spoke perturbations σ_λ of the form considered here lie in the admissible spoke sector by construction (they preserve closure incidence A1 by $\partial C = 0$, and they preserve hub anchoring A2 since $h \mapsto h$). The constraints A1 and A2 are therefore satisfied automatically on the sector we are varying over; the constraint-violation penalties A_inc and A_hub contribute to S_VERSF off this sector but vanish on it. Concretely: when we restrict S_VERSF to the admissible spoke sector and identify the leading quadratic content, the on-sector Hessian is determined by the constraints that are non-trivially varied within the sector — closure-current conservation (CC) and closure-competition. These are exactly the two on-sector survivors in Theorem 1.

The closure-response functional A_cl(λ) of the preceding paper is identified with twice this:

$$A\_cl(\lambda) = \langle \lambda, H \lambda \rangle = 2\, S\_seq[\lambda].$$

The substrate dynamics in the τ-direction (the admissibility-restoring update within a single substrate step) is the gradient flow of S_seq under admissibility projection:

$$\mathcal{R}[\rho] = \Pi\_adm \circ \exp(-\eta\, \nabla A\_cl)[\rho],$$

equivalently in the spoke sector

$$\partial\_\tau\, \lambda = -\nabla A\_cl(\lambda) = -2\, H\, \lambda,$$

projected onto Λ_full at each step. This is exactly the form proposed in §9.1 of the preceding paper.

The remainder of §4–§5 determines H (equivalently A_cl) by direct constrained-EFT analysis at leading order.

---

4. The Most General Admissible Local Quadratic Functional

4.1 The constrained-EFT principles

We constrain A_cl by the following principles, which encode the same constrained-EFT methodology established in the *Leading-Order Unique Record Current* paper and applied schematically in §9.1a of the preceding paper. Here we state them precisely, with the locality principle split into local and topological clauses and the sum-squared principle amended to invoke the K = 7 constraint catalogue explicitly.

(P1a) Bilinear local sectors. A_cl includes sectors of the form Σ_v f_v, where each f_v is a local bilinear in spoke variables within nearest-neighbour radius r = 1 of a cell v of the underlying complex.

(P1b) Topological / conservation sectors. A_cl includes sectors corresponding to H_1-class conservation: one global quadratic penalty per generator of the admissible $H_1(W_7)$. For the K = 7 architecture with $H_1 \cong \mathbb{Z}\langle[C]\rangle$, this is one such sector.

(P1a) is standard nearest-neighbour locality. (P1b) is the methodological principle that conservation laws — which are global, not local — enter the constrained-EFT framework as topological sectors with one penalty per H_1 generator. The two clauses together encode "locality up to topologically-protected zero-mode constraints." Note that $(\Sigma_i \lambda_i)^2$ is not nearest-neighbour local in the sense of (P1a) — it contains pair-products at all separations — but is the unique non-trivial H_1-conservation quadratic for $\mathbb{Z}\langle[C]\rangle$, falling under (P1b).

(P2) D_6-covariance. A_cl is invariant under the standard D_6 permutation representation on $\mathbb{R}^6$ — cyclic rotations T and reflections σ_v (vertex axes), σ_e (edge-midpoint axes).

(P3) Leading order. A_cl is at most quadratic in λ. Nontrivial contributions are at least linear; the constant term vanishes.

(P4) Non-negativity with admissibility-zero. A_cl(λ) ≥ 0 with equality if and only if λ corresponds to a fully admissible substrate configuration (no constraint violation). In particular A_cl(0) = 0.

(P5′) Sum-squared factorisation over the K = 7 constraint catalogue. A_cl decomposes as a positive linear combination of independent sum-squared sectors,

A_cl(λ) = Σ_a c_a · Q_a(λ), c_a > 0, Q_a(λ) = Σ_x (L_a(λ; x))²,

where each linear functional L_a measures the violation of a specific *named* substrate constraint drawn from the K = 7 constraint catalogue:

Constraint	***Substrate origin***	***Violation functional***	***Locality class***	***On-sector status***
closure-incidence	*admissibility condition (A1)*	*A_inc = Σ ‖∂σ(s_i) − σ∂(s_i)‖²*	*local, (P1a)*	*vanishes on admissible spoke sector*

Constraint	***Substrate origin***	***Violation functional***	***Locality class***	***On-sector status***
hub anchoring	*admissibility condition (A2)*	*$A\_hub = \|\sigma(h) - h\|^2$*	*local, (P1a)*	*vanishes on admissible spoke sector*
closure-current conservation	*cycle-projected commitment conservation (CCC; §6)*	*$A\_circ = (\Sigma_i \lambda_i)^2$*	*topological, (P1b)*	*vanishes on Λ_0*
closure-competition	*vertex-local closure imbalance*	*$A\_comp = \Sigma_i (\lambda_i + \lambda_{i+1})^2$*	*local, (P1a)*	*vanishes on alternating subspace*

The sectors are independent (no cross-term mixes two constraint violations).

Why (P5′) is the right load-bearing principle. Without restricting the L_a to the K = 7 constraint catalogue, the sum-squared sectors satisfying (P1)–(P4) are not exhausted by {A_inc, A_hub, A_circ, A_comp}: a fourth candidate at r = 1, namely

$$A\_grad(\lambda) := \Sigma_i (\lambda_i - \lambda_{i+1})^2 = 2 \Sigma_i \lambda_i^2 - 2 \Sigma_i \lambda_i \lambda_{i+1} = \langle \lambda, L \lambda \rangle,$$

with L the standard graph Laplacian of C_6, satisfies (P1a) and is D_6-invariant, non-negative, and sum-squared. It corresponds to a *closure-smoothness* constraint: the requirement that adjacent spokes carry similar transport $\lambda_i \approx \lambda_{i+1}$. The K = 7 architecture does *not* impose closure-smoothness as a substrate constraint — indeed the alternating mode itself violates smoothness maximally, and the substrate does not penalise this. Closure transport between adjacent spokes can be (and in the canonical σ-family *is*) opposite-signed. The constraint catalogue therefore excludes A_grad as a structural input.

(P5′) is the substantive load-bearing principle: it ties the admissible sum-squared sectors to the *named* substrate constraints of the K = 7 architecture rather than to arbitrary D_6-invariant locally-supported sum-squared functionals. The catalogue is itself a structural input from the K = 7 foundational papers, not a derivation in this paper. The epistemic status of Theorem 1 is correspondingly: *unique under the K = 7 constraint catalogue (P5')*, with the catalogue itself as the upstream structural input.

4.2 Enumeration under (P1a)+(P1b), (P2), (P3)

We enumerate D_6-invariant quadratic functionals on $\mathbb{R}^6$ at the level admitted by (P1a) and (P1b).

Local sector (P1a, r = 1). The most general bilinear functional supported within nearest-neighbour radius is

$$f\_bilin(\lambda) = c_1 \Sigma_i \lambda_i + c_2 \Sigma_i \lambda_i^2 + c_3 \Sigma_i \lambda_i \lambda_{i+1},$$

with constants $c_1, c_2, c_3 \in \mathbb{R}$. Each term is D_6-invariant by inspection. By (P4), $c_1 = 0$. The remaining two-parameter family is

$$f\_bilin(\lambda) = c_2 \Sigma_i \lambda_i^2 + c_3 \Sigma_i \lambda_i \lambda_{i+1}.\ \textbf{(*)}$$

Topological / conservation sector (P1b). The unique non-trivial H_1-conservation quadratic on $\mathbb{R}^6$ is

$$f\_topo(\lambda) = c_4 (\Sigma_i \lambda_i)^2 = c_4 \cdot A\_circ(\lambda),\ c_4 \in \mathbb{R},$$

corresponding to the single generator [C] of admissible $H_1(W_7)$. By (P4), $c_4 \geq 0$.

The most general candidate A_cl satisfying (P1a)+(P1b), (P2), (P3), (P4) is therefore the three-parameter family

$$A\_cl(\lambda) = c_2 \Sigma_i \lambda_i^2 + c_3 \Sigma_i \lambda_i \lambda_{i+1} + c_4 (\Sigma_i \lambda_i)^2,$$

with $c_2, c_4 \geq 0$ and c_3 free (subject to making the form positive semidefinite).

(P5′) — sum-squared factorisation over the K = 7 constraint catalogue — now picks out the specific decomposition.

---

5. Derivation of the Closure-Response Functional

5.1 The unique decomposition under the K = 7 constraint catalogue

Under (P5′), the local part f_bilin(λ) must decompose as a positive linear combination of independent sum-squared sectors L_a, where each L_a measures the violation of a *named* substrate constraint from the K = 7 catalogue.

On the admissible spoke sector (A1 and A2 satisfied by construction), the constraints with non-trivial local content are reduced to:

- **closure-competition at vertices**: at the vertex between spokes i and i+1, the substrate principle is that adjacent spoke corrections should not reinforce the same closure direction. The violation linear functional at vertex (i, i+1) is $L\_comp(\lambda; i) = \lambda_i + \lambda_{i+1}$, with violation penalty $(\lambda_i + \lambda_{i+1})^2$. Summing over vertices gives $A\_comp(\lambda) = \Sigma_i (\lambda_i + \lambda_{i+1})^2$.

No other local linear functional in the catalogue corresponds to a named substrate constraint. In particular, the alternative candidate

$$L\_grad(\lambda; i) := \lambda_i - \lambda_{i+1} \rightarrow A\_grad(\lambda) = \Sigma_i (\lambda_i - \lambda_{i+1})^2$$

would correspond to a *closure-smoothness* constraint — that adjacent spoke corrections should be similar in magnitude and sign — and the K = 7 architecture does **not** impose this. The substrate explicitly permits opposite-signed transport between adjacent spokes (and in fact the canonical alternating σ-family realises this maximally). A_grad is therefore excluded from the catalogue.

The unique local sector under (P5′) is thus

$\gamma \cdot A\_comp(\lambda) = \gamma \cdot \Sigma_i (\lambda_i + \lambda_{i+1})^2 = 2\gamma \cdot \Sigma_i \lambda_i^2 + 2\gamma \cdot \Sigma_i \lambda_i \lambda_{i+1}, \gamma \geq 0.$

Matching to the local part of the three-parameter family of §4.2: $c_2 = 2\gamma$, $c_3 = 2\gamma$, so $c_2 = c_3$ *under the catalogue restriction*. The constraint $c_2 = c_3$ — which is what reduces the three-parameter family to the canonical two-coefficient form α · A_circ + γ · A_comp — follows from (P5′), the restriction of admissible sum-squared sectors to the K = 7 constraint catalogue. Without this restriction, the candidate A_grad satisfying (P5) but excluded by (P5′) would contribute an independent δ-parameter; the catalogue restriction is what excludes it.

The topological sector contributes

$\alpha \cdot A\_circ(\lambda) = \alpha \cdot (\Sigma_i \lambda_i)^2, \alpha \geq 0,$

corresponding to the closure-current conservation constraint with violation linear functional $L\_circ(\lambda) = \Sigma_i \lambda_i$.

The unique admissible local quadratic functional under (P1a)+(P1b), (P2)–(P4), (P5′) is therefore

$A\_cl(\lambda) = \alpha \cdot A\_circ(\lambda) + \gamma \cdot A\_comp(\lambda), \alpha, \gamma > 0,$

with positivity of α and γ following from non-trivial substrate-constraint contribution.

5.2 Interpretation of the four terms

In the formulation of §9.1 of the preceding paper, A_cl was decomposed as

A_cl = A_inc + A_hub + A_circ + A_comp,

with A_inc and A_hub penalising violations of closure incidence (A1) and hub anchoring (A2) respectively. These constraint functionals do not appear in *() because ()* was already restricted to *admissible* spoke perturbations — perturbations of the form $\sigma(s_i) = s_{i+1} + \lambda_i C$, which preserve A1 (since $\partial C = 0$) and A2 (since $h \mapsto h$ by construction).

In other words:

A_inc(λ) = 0 and A_hub(λ) = 0 on the admissible spoke sector.

The constraint functionals A_inc and A_hub contribute non-trivially only when the substrate is perturbed *off* the admissible sector. They serve as constraint penalties that drive the system back to admissibility under the gradient flow (the role of Π_adm in §9.1 of the preceding paper), but they vanish identically on Λ_full.

The four-term decomposition

A_cl = A_inc + A_hub + A_circ + A_comp

is therefore unified across on-sector and off-sector behaviour: the four terms together generate the admissibility-restoring dynamics, with A_inc and A_hub vanishing on-sector while A_circ and A_comp determine the dynamics on-sector.

5.3 Theorem 1: uniqueness of the leading-order admissible functional

Theorem 1 (Uniqueness of the leading-order admissible transport functional under the K = 7 constraint catalogue). *Let A : A_full → ℝ satisfy (P1a)+(P1b), (P2), (P3), (P4), and (P5′). Then there exist α, γ > 0 such that*

A(λ) = α · A_circ(λ) + γ · A_comp(λ) on Λ_full,

where A_circ(λ) = $(\Sigma_i \lambda_i)^2$ and A_comp(λ) = $\Sigma_i (\lambda_i + \lambda_{i+1})^2$. The decomposition is unique up to overall positive rescaling and the relative coefficient α/γ. The full closure-response functional including off-sector constraint terms is

A_cl = A_inc + A_hub + A_circ + A_comp,

with A_inc, A_hub ≡ 0 on the admissible spoke sector and contributing only as off-sector constraint penalties.

Proof. By the enumeration of §4.2, any candidate A satisfying (P1a)+(P1b), (P2), (P3), (P4) has the three-parameter form

$$A(\lambda) = c_2\, \Sigma_i\, \lambda_i^2 + c_3\, \Sigma_i\, \lambda_i\, \lambda_{i+1} + c_4\, (\Sigma_i\, \lambda_i)^2,\quad c_2, c_4 \geq 0.$$

By (P5′), the local part $c_2\, \Sigma_i\, \lambda_i^2 + c_3\, \Sigma_i\, \lambda_i\, \lambda_{i+1}$ must decompose as a positive linear combination of sum-squared sectors corresponding to *named* substrate constraints in the K = 7 catalogue. On the admissible spoke sector the only catalogue-resident local sector is closure-competition, with violation linear functional L_comp(λ; i) = $\lambda_i + \lambda_{i+1}$ and quadratic A_comp = $\Sigma_i (\lambda_i + \lambda_{i+1})^2 = 2\, \Sigma_i\, \lambda_i^2 + 2\, \Sigma_i\, \lambda_i\, \lambda_{i+1}$. The alternative candidate L_grad(λ; i) = $\lambda_i - \lambda_{i+1}$ (closure-smoothness) is excluded by (P5′) since closure-smoothness is not a K = 7 substrate constraint. Hence $c_2 = c_3 = 2\gamma$ for some $\gamma \geq 0$. The topological sector contributes $c_4 = \alpha \geq 0$ corresponding to closure-current conservation. Positivity of α and γ follows from non-trivial substrate-constraint contribution. Off-sector constraint terms A_inc, A_hub are required for the full admissibility-restoring functional but vanish identically on Λ_full. □

Remark on epistemic status. Theorem 1 closes P1a *under the $K = 7$ constraint catalogue* (P5′), with the catalogue itself as the upstream structural input. The four-term decomposition is no longer the unique decomposition by D_6 + locality + sum-squared structure alone — that would also admit the closure-smoothness sector A_grad. It is the unique decomposition under the additional restriction that admissible sum-squared sectors correspond to *named* $K = 7$ substrate constraints, of which there are exactly four (closure-incidence, hub anchoring, closure-current conservation, closure-competition). The constrained-EFT methodology of §9.1a of the preceding paper is thereby placed on a verifiable footing: (P1a)+(P1b), (P2)–(P4), (P5′) are precise principles with explicit structural inputs, and the theorem follows by direct enumeration plus catalogue restriction.

---

6. Closure-Transport-Sector Commitment Conservation and Closure-Current Conservation

6.1 Closure-transport-sector commitment conservation

The substrate principle of commitment conservation states that the substrate response rule $\mathcal{R}$ cannot manufacture committed structure ex nihilo: it can rearrange or redistribute existing commitments, but the *total committed content* across the substrate is preserved under one substrate update step. In the closure-transport sector, this principle takes a specific form involving the H_1-class of spoke-correction currents.

Closure-transport-sector commitment conservation (CCC). *Let σ be a sequential transport perturbation with spoke-correction current*

$J(\sigma) := \Sigma_i \Delta_i = \Sigma_i \lambda_i \cdot C \in Z_1(W_7),$

where $\Delta_i = \lambda_i C \in \mathbb{Z}\langle C\rangle$ is the admissible spoke correction at position i (§3.1). Then σ preserves the H_1-class of the spoke sector under one substrate update:

$\Pi\_cyc(J(\sigma)) = 0$ *in* $H_1(W_7)$,

where $\Pi\_cyc : Z_1(W_7) \to H_1(W_7)$ is the cycle-class projection in the $K = 7$ cell complex.

In words: σ cannot generate any net 1-cycle homology class from a configuration that originally had none. The H_1-class of closure-transport history is conserved.

CCC is a constraint formulated at the H_1 level of the $K = 7$ cell complex. The $K = 7$ architecture is a 2-complex (§3.1) with $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$; the H_1-class statement is therefore intrinsically tied to the cell structure of W_7 rather than being a generic graph-theoretic property.

6.2 The cell structure and the load-bearing role of $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$

By §3.1, W_7 in the $K = 7$ architecture is a 2-complex whose 2-cell structure produces

$H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$,

with $[C] = [e_0 + e_1 + \cdots + e_5]$ the primitive generator. The cycle-class projection

$\Pi\_cyc : Z_1(W_7) \longrightarrow H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle, \gamma \longmapsto [\gamma]$,

is the standard cellular cycle-to-homology map under the architecture's cell structure.

Two distinct structural restrictions are operative here:

1. **Architecture-level cell structure** (taken from $K = 7$ architecture papers): W_7 has 2-cells specified such that $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$. This makes H_1 one-dimensional with [C] as primitive generator.
2. **Admissible-cycle restriction** (also from $K = 7$ architecture): admissible substrate currents in the spoke sector lie in the $\mathbb{Z}\langle C\rangle \subset Z_1(W_7)$ sublattice (§3.1).

Together these restrictions collapse the H_1-shadow of any admissible spoke-correction current to a single integer.

Applying Π_cyc to J(σ):

$\Pi\_cyc(J(\sigma)) = \Pi\_cyc((\Sigma_i \lambda_i) \cdot C) = (\Sigma_i \lambda_i) \cdot [C] \in \mathbb{Z}\langle[C]\rangle$.

CCC requires this to vanish, equivalent to

$\Sigma_i \lambda_i = 0$,

which is the closure-current conservation law (CC).

6.3 Theorem 2: CCC ⇒ (CC) under the K = 7 cell structure and admissible-cycle restriction

Theorem 2 (Closure-current conservation as the H_1-shadow of closure-transport-sector commitment conservation). *In the $K = 7$ architecture (W_7 as a 2-complex with $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$, admissible spoke currents in $\mathbb{Z}\langle C\rangle \subset Z_1(W_7)$), closure-transport-sector commitment conservation (CCC) is equivalent to closure-current conservation (CC):*

$\Pi\_cyc(J(\sigma)) = 0$ in $H_1(W_7) \Leftrightarrow \Sigma_i \lambda_i = 0$.

Proof. Direct computation. By the $K = 7$ cell structure, $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$, so the cycle-class projection on the admissible spoke sector reduces to $\Pi\_cyc(\lambda C) = \lambda \cdot [C]$. The spoke-correction current is $J(\sigma) = (\Sigma_i \lambda_i) \cdot C$, with image $(\Sigma_i \lambda_i) \cdot [C] \in \mathbb{Z}\langle[C]\rangle$. This vanishes if and only if $\Sigma_i \lambda_i = 0$. The forward and reverse implications are immediate. □

Remark on scope. Theorem 2 closes P1b of the preceding paper *in the following precise sense*. The closure-current conservation law (CC), which the preceding paper postulated as the "natural cellular-shadow" of commitment conservation, is identified as the H_1-shadow of closure-transport-sector commitment conservation (CCC) under (i) the $K = 7$ cell structure giving $H_1 \cong \mathbb{Z}\langle[C]\rangle$, and (ii) the admissible-cycle restriction $Z_1 \to \mathbb{Z}\langle C\rangle$. The two statements are not at genuinely different levels of formal description — CCC is itself an H_1-class constraint, and (CC) is its explicit form on the one-dimensional H_1 shadow under these structural restrictions. The novelty is identifying (CC) as the cycle-class shadow of a more general principle: the substrate principle that closure-transport history is H_1-conservatively redistributed under σ. The load-bearing structural inputs are the $K = 7$ cell structure and the admissible-cycle restriction, both established in the $K = 7$ architecture papers separately.

Corollary 5. *(CC) is no longer an independent postulate beyond closure-transport-sector commitment conservation in the $K = 7$ architecture. The constraint surface $\Lambda_0 \subset \mathbb{R}^6$ on which the sequential transport dynamics lives is the kernel of the cycle-class projection of the spoke-correction current under the $K = 7$ cell structure and admissible-cycle restriction.*

In the rest of the paper we work on Λ_0 rather than Λ_full = $\mathbb{R}^6$, with the understanding that Λ_0 is the *derived* admissible spoke-perturbation sector under CCC plus the $K = 7$ structural restrictions, not an externally imposed constraint.

---

7. The Nonlinear Sequential Transport Flow

7.1 Gradient flow of A_cl on Λ_0

Under Theorem 1, the leading-order admissible closure-response functional restricted to Λ_0 is

$A\_cl(\lambda) |\_\{\Lambda_0\} = \gamma \cdot A\_comp(\lambda), \lambda \in \Lambda_0,$

since $A\_circ(\lambda) = (\Sigma_i \lambda_i)^2 = 0$ on Λ_0 by definition. (The factor α drops out.) Rescaling τ to absorb γ, the gradient flow becomes

$\boldsymbol{\partial\_\tau \lambda = -\nabla A\_comp(\lambda).}$

We compute the gradient explicitly. With $A\_comp(\lambda) = \Sigma_i (\lambda_i + \lambda_{i+1})^2$,

$\partial A\_comp / \partial\lambda_j = \partial/\partial\lambda_j \Sigma_i (\lambda_i + \lambda_{i+1})^2 = \Sigma_i 2(\lambda_i + \lambda_{i+1}) \cdot (\delta_{ij} + \delta_{i,j-1}) = 2 (\lambda_j + \lambda_{j+1}) + 2 (\lambda_{j-1} + \lambda_j) = 2 (\lambda_{j-1} + 2\lambda_j + \lambda_{j+1}).$

So

$\nabla A\_comp = 2 M \lambda,$

where M is the discrete operator

$(M \lambda)_j := \lambda_{j-1} + 2\lambda_j + \lambda_{j+1}$ (indices mod 6).

The gradient flow is therefore

$\partial_\tau \lambda = -2 M \lambda$ on Λ_0.

7.2 The signless Laplacian and its spectrum

The operator M is the **signless Laplacian** of the six-cycle C_6 (also called the "co-Laplacian" or "Q-matrix" in spectral graph theory). It is closely related to but *distinct from* the standard graph Laplacian L of the predecessor paper (Proposition 4 of *Substrate-Generated Sequential Transport in VERSF*).

The two operators are linked by

$M + L = 4 I$, equivalently $M = 4 I - L$,

where I is the 6 × 6 identity. Both are positive semi-definite and circulant, hence diagonalisable in the Fourier basis on $\mathbb{Z}/6\mathbb{Z}$. Letting $\omega = \exp(2\pi i/6)$, the Fourier eigenmodes are

$v_k = (1, \omega^k, \omega^{2k}, \omega^{3k}, \omega^{4k}, \omega^{5k})$, $k = 0, 1, \ldots, 5$,

with eigenvalues

k	***L eigenvalue*** $\mu_k^L = 2 - 2\cos(2\pi k/6)$	***M eigenvalue*** $\mu_k^M = 2 + 2\cos(2\pi k/6)$	***Notes***
0	*0*	*4*	*constant; excluded by* Λ_0
1, 5	*1*	*3*	*complex Fourier pair*
2, 4	*3*	*1*	*complex Fourier pair*
3	*4*	*0*	*alternating mode* $\lambda_{alt} = (1, -1, 1, -1, 1, -1)$

The two spectra are *reversed*: where L has eigenvalue 4 (maximum), M has eigenvalue 0 (minimum), and vice versa. The alternating mode is the *maximum-eigenvalue* eigenmode of L and the *minimum-eigenvalue* (in fact zero) eigenmode of M.

The relation $A_{comp} + \langle \lambda, L \lambda \rangle = 4 |\lambda|^2$ noted in the predecessor paper (§9.2b) is exactly the identity $M + L = 4 I$ evaluated as a quadratic form on λ.

Note on generality. The relation $M + L = 4 I = (\text{graph-degree}) \cdot I$ is specific to the *k-regular* graph structure: for the 6-cycle each vertex has degree 2, both L and M differ from the adjacency matrix by ±A, and the sum is $2 \cdot 2 I = 4 I$. For arbitrary graphs $M + L = D + D$ where D is the degree-diagonal, and the clean "spectra-add-to-degree" reversal only holds for k-regular graphs. The identity is therefore a feature of the cyclic K = 7 architecture, not a general spectral-graph-theoretic fact.

7.3 Relation to the standard Laplacian of the predecessor paper

It is important to be explicit about which Laplacian is doing the work at which point in the argument.

- **Predecessor paper, Proposition 4 (Laplacian-extremal characterisation).** The alternating mode is the *unique nontrivial real-integer eigenmode of L with maximal eigenvalue* $\mu = 4$. This is the standard graph Laplacian, in which alternation corresponds to the highest-frequency Fourier mode.
- **Present paper, Theorem 3 (gradient-flow attractor).** The alternating mode is the *unique nontrivial real-integer eigenmode of M with minimal eigenvalue* $\mu = 0$. This is the signless Laplacian arising directly as the Hessian of A_comp.

Both statements are correct and complementary. They use different operators because they extract different structural information: the standard Laplacian L gives the *curvature spectrum* on the spoke cycle (alternation is the most-curved mode, with maximal "graph Dirichlet energy" $\langle\lambda, L\lambda\rangle = \Sigma_i (\lambda_i - \lambda_{i+1})^2$); the signless Laplacian M gives the *Hessian spectrum of the action* (alternation is the unique zero-mode, the persistent direction under gradient flow). The two characterisations are dual via $M + L = 4I$.

---

8. Stability of the Alternating Mode

8.1 Fourier decomposition of the flow

The gradient flow $\partial_\tau \lambda = -2M\lambda$ on $\Lambda_0 \otimes \mathbb{R}$ decomposes in the Fourier basis as

$$\partial_\tau \hat{\lambda}(k) = -2\mu_k^M \cdot \hat{\lambda}(k),$$

where $\hat{\lambda}(k)$ is the k-th Fourier component. The solution is

$$\hat{\lambda}(k, \tau) = \hat{\lambda}(k, 0) \cdot \exp(-2\mu_k^M \tau).$$

Restricted to Λ_0 ($k = 0$ excluded):

- $k = 1, 5$ modes ($\mu^M = 3$): decay with rate 6;
- $k = 2, 4$ modes ($\mu^M = 1$): decay with rate 2;
- $k = 3$ (alternating; $\mu^M = 0$): **invariant**.

Any initial condition $\lambda(0) \in \Lambda_0 \otimes \mathbb{R}$ decomposes as

$$\lambda(0) = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + c_5 v_5$$

(omitting $k = 0$), and at large τ

$\lambda(\tau) \longrightarrow c\_3 \cdot v\_3 = c\_3 \cdot (1, -1, 1, -1, 1, -1)$

with all other components exponentially suppressed.

The alternating subspace $\mathbb{R} \cdot \lambda\_alt$ is therefore the unique persistent direction of the flow. Strictly, this is the kernel of M restricted to $\Lambda_0 \otimes \mathbb{R}$.

8.2 Theorem 3: alternation as unique persistent direction

Theorem 3 (Gradient-flow attractor on the admissible sector). *Under the gradient flow $\partial\_\tau \lambda = - \nabla A\_cl$ on $\Lambda_0 \otimes \mathbb{R}$:*

(i) All Fourier modes $k \in \{1, 2, 4, 5\}$ decay exponentially with positive rate $2\, \mu\_k$^M, where μ_k^M $= 2 + 2 \cos(2\pi k/6)$.

(ii) The alternating mode (k = 3) is in the kernel of M restricted to $\Lambda_0 \otimes \mathbb{R}$ and is therefore invariant under the flow.

(iii) The alternating subspace $\mathbb{R} \cdot \lambda\_alt$ is the unique persistent (zero-gradient) direction of the flow.

(iv) Imposing integrality $\lambda_i \in \mathbb{Z}$ and minimal-norm non-trivial normalisation selects

$\lambda_i = \pm (-1)^i$,

i.e. the canonical lattice point of the alternating direction.

Proof. Parts (i)–(iii) follow from the Fourier decomposition of §8.1 and the explicit spectrum of M in §7.2. For (iv), the kernel of M restricted to $\Lambda_0 \otimes \mathbb{R}$ is one-dimensional, spanned by $\lambda\_alt = (1, -1, 1, -1, 1, -1)$. The integer lattice points on this line are $k \cdot \lambda\_alt$ for $k \in \mathbb{Z}$. Non-triviality excludes $k = 0$; minimal norm selects $k = \pm 1$. □

Remark. Theorem 3 closes P1c of the preceding paper. The non-linear gradient flow does indeed select the alternating mode as the unique persistent attractor on the admissible sector, with the four characterisations of the predecessor paper now grounded in the explicit spectral structure of the Hessian of S_seq.

---

9. Transport-Gap Symmetry from Action Variation

The preceding paper's §9.3a proposed a transport-gap reading of the residual D_3(vtx) symmetry, conditional on two structural hypotheses about the K = 7 ontology stated under the single label (H2):

- **(H1)** Committed closure surfaces have stabiliser D_6.

- **(H2)** The admissibility projection Π_adm "preserves D_6 symmetry."

The predecessor's phrasing of (H2) is ambiguous between two distinct statements:

- **(H2a)** *Image-symmetry.* The image of Π_adm consists of D_6-symmetric states.
- **(H2b)** *Equivariance.* Π_adm commutes with the D_6 action: $\Pi\_adm(g \cdot \rho) = g \cdot \Pi\_adm(\rho)$ for every $g \in D_6$.

These are different: (H2b) implies (H2a), but (H2a) alone does not imply (H2b). A projection can land in a D_6-symmetric subspace without commuting with the group action. The transport-gap argument of the predecessor's §9.3a in fact requires (H2b) — the D_3(vtx) attachment to the transport sector is a per-step equivariant statement, not just an image-symmetry statement.

In §9.1–§9.3 below we treat (H2a) and (H2b) separately. (H2a) follows from (H1) by definition of Π_adm. (H2b) follows from D_6-invariance of all four constraint functionals — which is itself a consequence of the K = 7 substrate constraints being D_6-symmetric — independently of (H1). Both are derived within this paper from inputs available here. The predecessor's two-hypothesis framing therefore collapses to two load-bearing structural inputs — (H1) and the D_6-symmetry of the K = 7 substrate constraints — together with internal-to-this-paper consequences.

9.1 (H1) as structural input

(H1) is a structural property of the K = 7 closure architecture, established separately in the foundational papers of the broader programme. We do not derive it here. We use it as input.

Formally: the master action S_VERSF is D_6-covariant (S_VERSF respects the dihedral symmetry of the wheel), and committed states — by which we mean stable extrema of S_VERSF on the admissible substrate sector — are D_6-symmetric configurations of the K = 7 architecture.

9.2 (H2a) from (H1): image of Π_adm is D_6-symmetric

By definition, Π_adm is the projection onto the admissible substrate sector. By (H1), the admissible substrate sector consists of D_6-symmetric committed states. Therefore the image of Π_adm consists of D_6-symmetric states:

Π_adm(ρ) is D_6-symmetric for every ρ.

This is (H2a). It follows from (H1) and the definition of Π_adm. The derivation is near-tautological — the load is entirely on (H1) and the definition of "admissible."

9.3 (H2b) from D_6-invariance of all four constraint sectors

We now derive (H2b): D_6-equivariance of Π_adm.

Disambiguating Π_adm. Recall the substrate update rule from the predecessor:

$\mathcal{R}[\rho] = \Pi\_adm \circ \exp(-\eta\ \nabla A\_cl)[\rho]$.

Two operations are involved:

- **On-sector flow:** $\exp(-\eta\ \nabla A\_cl)$ acts on the admissible spoke sector and drives perturbations toward the kernel of $M = \nabla^2 A\_cl$ on Λ_0 (i.e., toward the alternating mode). This is governed by A_circ + A_comp.
- **Admissibility projection:** Π_adm acts on the full substrate state space, mapping off-sector states (those violating closure-incidence or hub-anchoring) back to the admissible spoke sector. This is governed by A_inc + A_hub.

For (H2b) — that Π_adm commutes with the D_6 action on the full substrate state space — we need D_6-equivariance of *both* operations, hence D_6-invariance of all four constraint functionals.

D_6-invariance of all four sectors. The four constraint functionals comprising A_cl = A_inc + A_hub + A_circ + A_comp are all D_6-invariant:

- **A_inc** penalises violations of closure-incidence ($\partial\sigma = \sigma\partial$). The closure-incidence constraint is D_6-symmetric — it relates each spoke to the corresponding outer vertex through the boundary structure, which is preserved by the D_6 action permuting spokes and vertices consistently. The violation functional $\Sigma_i \|\partial\sigma(s_i) - \sigma\partial(s_i)\|^2$ is therefore D_6-invariant.
- **A_hub** penalises violations of hub-anchoring ($\sigma(h) = h$). The hub is a D_6-fixed point of the architecture (the centre of the wheel is invariant under all dihedral transformations). The violation functional $\|\sigma(h) - h\|^2$ is therefore D_6-invariant.
- **A_circ** = $(\Sigma_i \lambda_i)^2$ is manifestly D_6-invariant (the total sum is a D_6-scalar).
- **A_comp** = $\Sigma_i (\lambda_i + \lambda_{i+1})^2$ is manifestly D_6-invariant (the cyclic sum and the reflection-invariant pair structure).

The general principle: each constraint in the K = 7 catalogue penalises violation of a *D_6-symmetric substrate constraint*, and the violation functional inherits D_6-invariance from the symmetry of the constraint it penalises.

Equivariance argument. With all four functionals D_6-invariant, ∇A_cl is D_6-equivariant on the full substrate state space (not just on Λ_full):

$\nabla A\_cl(g \cdot \rho) = g \cdot \nabla A\_cl(\rho)$ for every $g \in D_6$.

The gradient flow $\partial\_\tau\ \rho = -\nabla A\_cl(\rho)$ therefore commutes with the D_6 action: if $\rho(\tau)$ solves the flow with initial condition $\rho(0)$, then $g \cdot \rho(\tau)$ solves the flow with initial condition $g \cdot \rho(0)$. In particular, the time-η flow operator $F\_\eta : \rho \mapsto \exp(-\eta\ \nabla A\_cl)(\rho)$ is D_6-equivariant:

$F\_\eta(g \cdot \rho) = g \cdot F\_\eta(\rho)$.

Two readings of Π_adm then yield (H2b):

- If Π_adm is the *admissibility projection* (off-sector → on-sector, driven by A_inc + A_hub), its D_6-equivariance follows from D_6-invariance of A_inc + A_hub alone.
- If Π_adm is the *full asymptotic projector* of the combined flow (driven by all four functionals), its D_6-equivariance follows from D_6-invariance of all four sectors.

In either reading, Π_adm is a limit (or composition) of D_6-equivariant operators and inherits equivariance:

$\Pi\_adm(g \cdot \rho) = g \cdot \Pi\_adm(\rho)$.

This is (H2b). It follows from D_6-invariance of the constraint sectors — itself a consequence of the substrate constraints being D_6-symmetric, which is a structural feature of the K = 7 architecture rather than an independent assumption.

Where the load sits. The load on (H2b) is on the D_6-symmetry of the K = 7 constraint catalogue itself, not on (H1). Theorem 1's derivation that A_cl = α · A_circ + γ · A_comp under (P5′) plus the D_6-symmetry of A_inc and A_hub (each of which penalises a D_6-symmetric constraint) is sufficient. The on-sector gradient flow toward the alternating mode requires the D_6-invariance of A_cl established by Theorem 1; the admissibility projection requires the D_6-invariance of A_inc and A_hub stated above. Both pieces are now in place.

9.4 Theorem 4: dynamical attachment of D_3(vtx) to the transport sector

With (H2a) and (H2b) both derived (the former from (H1), the latter from Theorem 1), we can state Theorem 4.

Theorem 4 (Transport-gap symmetry from action variation). *Assume (H1) — that committed states of the K = 7 architecture have stabiliser D_6 — as structural input, and that A_cl satisfies the hypotheses of Theorem 1. Then:*

(i) ***A_cl is D_6-invariant as a functional on Λ_full.*** *This follows from Theorem 1 plus the manifest D_6-invariance of A_circ and A_comp.*

(ii) ***The kernel of M = ∇^2A_cl restricted to Λ_0 is one-dimensional, spanned by λ_alt, with stabiliser D_3(vtx).*** *This follows from Theorem 3 (kernel is the alternating subspace on Λ_0) and Lemma 6.2 of the predecessor paper (Stab(λ_alt) = D_3(vtx)).*

(iii) ***(H2a) follows from (H1):*** *Π_adm projects onto admissible (= D_6-symmetric by (H1)) states; therefore the image of Π_adm consists of D_6-symmetric states.*

(iv) ***(H2b) follows from D_6-invariance of all four constraint sectors, independently of (H1):*** *the four functionals A_inc, A_hub, A_circ, A_comp are each D_6-invariant because each penalises violation of a D_6-symmetric substrate constraint; the gradient flow generated by the full A_cl is therefore D_6-equivariant, as is the admissibility projection Π_adm and the asymptotic projector.*

(v) ***The residual D_3(vtx) symmetry of the alternating mode attaches dynamically to the transport-excitation sector.*** *Committed states have stabiliser D_6 (by (H1)); the σ-update generates a perturbation $\sigma(s_i) - s_{i+1}$ in the kernel of M on Λ_0 (by Theorem 3); the kernel has stabiliser D_3(vtx) $\subsetneq D_6$ (by (ii)). The reduced symmetry attaches to the perturbation generated by the update — the transport excitation — not to either committed surface at the endpoints of the σ-morphism.*

(vi) ***The next committed surface is D_6-symmetric.*** *By (H2a) (image of Π_adm is admissible = D_6-symmetric), the projected state $W_7^{(t+1)}$ has stabiliser D_6.*

(vii) ***The transport-gap structure is D_6-equivariant.*** *By (H2b), the entire σ-update is D_6-equivariant: any D_6-action on the initial committed state commutes with the σ-step and the Π_adm-projection.*

Proof. (i) by Theorem 1 and D_6-invariance of A_circ and A_comp. (ii) by Theorem 3 (Theorem 1 → Hessian M → kernel on $\Lambda_0 = \mathbb{R} \cdot \lambda$_alt) and Lemma 6.2 of the predecessor. (iii) by definition of Π_adm and (H1). (iv) by the substrate constraints being D_6-symmetric → D_6-invariance of each violation functional → D_6-equivariance of the full gradient flow and of Π_adm in either reading (admissibility projection or full asymptotic projector); see §9.3 for the disambiguation. (v) by combining (i)–(iv) and the stabiliser computation. (vi)–(vii) by (iii) and (iv) respectively. □

Remark on epistemic status. Theorem 4 closes P1e of the preceding paper. The transport-gap reading was framed in the predecessor as conditional on (H1) + (H2), with (H2) ambiguous. Disambiguating (H2) into (H2a) and (H2b): (H2a) follows from (H1) plus the definition of Π_adm; (H2b) follows from D_6-invariance of all four constraint functionals — which is itself a consequence of the K = 7 substrate constraints being D_6-symmetric. The remaining structural inputs are (H1) and the D_6-symmetry of the K = 7 constraint catalogue, both from the K = 7 architecture papers. The transport-gap reading is therefore upgraded from "conditional on (H1) + (H2), with (H2) unspecified" to "conditional on the D_6-symmetry of the K = 7 substrate constraints plus (H1), with both (H2a) and (H2b) derived within this paper."

A note on what (H1) does: it is a *structural*, not *dynamical*, input. It is a property of the K = 7 architecture established by the foundational papers, not derivable from sequential transport dynamics alone. If (H1) is established, the transport-gap reading follows. If (H1) were to fail — i.e., if committed K = 7 states had reduced symmetry — then both (H2a) and the transport-gap attachment of D_3(vtx) would fail with it, and the residual symmetry would instead correspond to one of the alternative readings listed in §9.3a of the predecessor (gauge residual of A_cl, sublattice structure of admissible states, etc.).

The reduction from "(H1) + (H2)" to "the D_6-symmetry of the K = 7 substrate constraints plus (H1)" is both an epistemic upgrade and a precise identification of where the remaining structural dependency lives: it is now a question about the K = 7 architecture (specifically, the D_6-symmetry of its constraint catalogue and of its committed states), not about the σ-sector or the projection structure.

10. Formal Features in Common with the Constitutive-Current Programme

The architecture established in §3–§9 has formal features in common with the constitutive-current sector of the broader VERSF programme. Whether these features reflect a deeper unification — i.e., whether both sectors emerge from a single composite master-action variation — is the subject of P2 (§12) and not established here. We list the shared formal features as motivation.

Aspect	***Constitutive-current sector***	***Sequential transport sector (this paper)***
Underlying conservation principle	*commitment conservation in the record sector*	*commitment conservation in the closure-transport sector (CCC)*
Cycle-projected statement	*record-current continuity equation*	*closure-current conservation (CC),* $\Sigma_i \lambda_i = 0$
Constrained-EFT principles	*constrained-EFT methodology of the broader programme*	*(P1a)+(P1b), (P2)–(P4), (P5′) of §4*
Leading-order functional	*record-current functional A_J*	*closure-response functional A_cl*
Decomposition structure	*constraint penalties + dynamical sectors*	*A_inc + A_hub + A_circ + A_comp*
Dynamical sector on the admissible surface	*constitutive record-current response J*	*σ-family with* $\lambda_i = (-1)^i$
EFT-leading-order selected mode	*unique record-current direction*	*alternating closure-competition wave*
Persistence structure	*gauge-persistent record current*	*homological-persistent closure-current (Corollary 2 of predecessor)*

The features in common are at the level of formal structure: both sectors use a constrained-EFT methodology with leading-order admissible functionals; both reduce a substrate-level conservation principle to a cycle-projected continuity equation; both select a specific dynamical mode through gradient-flow analysis; both connect to persistence structures of the broader programme.

What is *not* established by these formal commonalities:

- That the record-current functional A_J and the closure-response functional A_cl arise from a *single* composite functional under one variational principle. This would require explicit construction of the composite and verification that its variation on different substrate sub-configurations recovers both A_J and A_cl.
- That the constrained-EFT principles in the two sectors are formally identical. The principles applied here — (P1a)+(P1b), (P2)–(P4), (P5′) — refer to the K = 7 constraint

catalogue specifically; the analogous catalogue for the record-current sector would need separate statement and matching.
- That the persistence structures of the two sectors are related by an explicit map. The "gauge-persistent record current" and the "homological-persistent closure-current" are analogous but not obviously the same object; the relation requires construction.

These open questions are the load-bearing content of P2 (§12). The formal commonalities listed above are *suggestive* of unification, but unification itself remains to be established. The σ-family is the **constitutive sequential transport response** of the VERSF substrate within the sequential transport sector; whether it composes with the constitutive-current sector under a single master-action variation is the next paper's question, not this one's.

---

11. What This Establishes

The paper establishes, by direct master-action variation in the sequential transport sector:

1. **The four-term decomposition of A_cl is unique under the K = 7 constraint catalogue** (Theorem 1, under (P1a)+(P1b), (P2)–(P4), (P5′)). The decomposition A_cl = A_inc + A_hub + A_circ + A_comp is no longer a postulate with methodological standing — it is the unique leading-order admissible local quadratic functional under the K = 7 substrate constraints, with A_inc and A_hub vanishing on the admissible sector and contributing only as off-sector constraint penalties. The structural input is the constraint catalogue itself (closure-incidence, hub anchoring, closure-current conservation, closure-competition); the closure-smoothness candidate A_grad is excluded because it is not a K = 7 constraint. *Whether the K = 7 catalogue is closed at exactly these four constraints — and in particular whether closure-smoothness genuinely fails to be a substrate constraint — is established separately in the K = 7 architecture papers and taken as input here.*
2. **Closure-current conservation is the H_1-shadow of closure-transport-sector commitment conservation** (Theorem 2). On the admissible-cycle sector $\mathbb{Z}\langle C\rangle \subset Z_1(W_7)$, (CCC) and (CC) are equivalent under the cycle projection. (CC) is therefore no longer an independent postulate but the explicit cycle-class form of a substrate principle stated at the H_1-class level. The load-bearing structural input is the admissible-cycle restriction $Z_1 \to \mathbb{Z}\langle C\rangle$, taken from the K = 7 architecture papers.
3. **The alternating mode is the unique persistent direction of the non-linear gradient flow** on the admissible sector (Theorem 3). All other Fourier modes decay exponentially; the alternating subspace is the kernel of $M = \nabla^2$A_cl on Λ_0, and integer-minimal non-trivial normalisation gives the canonical lattice point.
4. **The transport-gap reading of the D_3(vtx) symmetry is derivable from (H1) plus the D_6-symmetry of the K = 7 constraint catalogue** (Theorem 4). Disambiguating the predecessor's (H2) into (H2a) image-symmetry and (H2b) D_6-equivariance: (H2a) follows from (H1) plus the definition of Π_adm; (H2b) follows from the D_6-symmetry of *all four* constraint functionals A_inc, A_hub, A_circ, A_comp — itself a consequence of the K = 7 substrate constraints being D_6-symmetric — via standard equivariance-of-gradient-flow reasoning. Theorem 1 alone delivers D_6-invariance of A_cl = A_circ + A_comp on the

admissible sector, which is sufficient for the on-sector flow but not for the admissibility projection driven by A_inc + A_hub; the additional D_6-symmetry of the off-sector constraint functionals is required for (H2b) on the full substrate state space. The remaining structural inputs are (H1) and the D_6-symmetry of the K = 7 substrate constraints, both from the K = 7 architecture papers.

5. **The σ-family is the unique leading-order constitutive transport response under the K = 7 constraint catalogue.** Whether the σ-sector composes with the constitutive-current sector under a single master-action variation is the subject of P2 — the formal commonalities listed in §10 are suggestive but not conclusive.

What has *not* been established here:

- The *unique* numerical values of the coefficients α and γ in Theorem 1. The decomposition is forced (under (P5′)), but the ratio α/γ remains free up to the relative scaling of A_circ and A_comp. Pinning these numerical values requires going beyond leading order — including higher-order EFT corrections — or matching to observable continuum-limit quantities.
- The *closure of the K = 7 constraint catalogue* itself. (P5′) is conditional on the catalogue being closed at four constraints (closure-incidence, hub anchoring, closure-current conservation, closure-competition); the K = 7 architecture papers establish this separately, but the present paper takes it as input rather than deriving it.
- The full continuum-limit derivation of the σ-family. Theorem 3 analyses the discrete gradient flow on Λ_0; the continuum limit, in which the discrete time-step τ becomes a continuous parameter and the spoke index i becomes a continuous angular coordinate, requires separate treatment.
- Non-Abelian transport sectors. The K = 7 architecture is Abelian in its homological structure ($H_1(W_7) \cong \mathbb{Z}$); non-Abelian generalisations involve different cohomological projections and are not addressed here.
- Coupling to matter sectors. The substrate response is treated in isolation; coupling to matter sub-sectors of the master action is left for future work.
- The full master-action unification of the sequential transport, gauge, and cohomology sectors. §10 lists formal commonalities with the constitutive-current sector; whether these reflect a deeper unification is the load-bearing question of P2.

The epistemic status of σ is therefore upgraded from "substrate-generated minimal candidate pinned by four converging characterisations" (the conclusion of the preceding paper) to "unique leading-order constitutive transport response under the K = 7 constraint catalogue (P5′), derived by master-action variation in the sequential transport sector." The structural inputs that remain are precisely identified: (H1), integrality, and the closure of the K = 7 constraint catalogue at four constraints.

---

12. Remaining Open Problems

P2 (load-bearing). Master-action unification of the sequential transport and constitutive-current sectors. The parallel architecture of §10 suggests both sectors emerge from a single composite functional under one variational principle; this needs to be made explicit.

P3. Continuum-limit derivation of the σ-family. Take the discrete gradient flow $\partial_\tau \lambda = - 2 M \lambda$ on Λ_0 to its continuum limit as the discretisation scale of the $K = 7$ wheel goes to zero. Identify the limiting field-theoretic structure and the role of the alternating mode in the continuum.

P4. Non-Abelian transport sectors. Extend the cycle-projection analysis of §6 to non-Abelian generalisations of $H_1(W_7)$, and identify the analogous closure-current conservation law in those sectors.

P5. Coupling to matter sectors. Determine how the closure-response functional A_cl is modified when the substrate dynamics is coupled to matter sub-sectors of the master action, and whether the four-term decomposition survives or is modified at coupled-sector EFT-leading order.

P6. Numerical values of α and γ in Theorem 1. Beyond the leading-order EFT analysis, determine whether the ratio α/γ is constrained by higher-order matching to observable continuum-limit quantities, or fixed by additional substrate principles.

P7. Connection of the telescope σ-family to the vertex × tick-window object $D(i, t)$ of the broader programme, and explicit σ-duality between them. (Carried over from the preceding paper.)

---

13. Conclusion

The preceding σ-family paper established that the sequential transport rule of the $K = 7$ substrate is constrained by four converging characterisations, all selecting the alternating spoke pattern. Two elements of that construction remained postulated: the four-term decomposition of the closure-response functional A_cl, and the closure-current conservation law (CC). A proposed reading of the residual D_3(vtx) symmetry was offered as conditional on two structural hypotheses (H1) and (an unspecified) (H2) about the $K = 7$ ontology.

This paper closes both postulates and reduces the conditional reading to a precisely identified set of structural inputs.

The four-term decomposition is derived by direct enumeration of leading-order admissible local quadratic functionals under (P1a)+(P1b), (P2)–(P4), and the constraint-catalogue principle (P5′) (Theorem 1). The closure-current conservation law (CC) is identified as the H_1-shadow of closure-transport-sector commitment conservation under the $K = 7$ cell structure and the admissible-cycle restriction $Z_1 \to \mathbb{Z}\langle C\rangle$ (Theorem 2). The alternating mode is confirmed as the unique persistent direction of the resulting gradient flow on the admissible sector (Theorem 3). The transport-gap reading of the residual D_3(vtx) symmetry is upgraded from "conditional on (H1) and (unspecified) (H2)" to "conditional on (H1) plus the D_6-symmetry of the $K = 7$

constraint catalogue, with the disambiguated (H2a) image-symmetry derived from (H1) and (H2b) D_6-equivariance derived from D_6-invariance of all four constraint functionals" (Theorem 4).

The σ-family is therefore the unique leading-order constitutive transport response of the VERSF substrate under the K = 7 constraint catalogue (P5′), derived by direct master-action variation in the sequential transport sector. The four-term decomposition of A_cl is forced by the catalogue; the alternating mode is selected by gradient-flow stability; the transport-gap reading of the residual symmetry is derived from the action structure plus the structural inputs (H1) and the D_6-symmetry of the K = 7 substrate constraints.

§10 identifies formal features in common between the σ-sector derived here and the constitutive-current sector of the broader programme. Whether these commonalities reflect a deeper unification — i.e., whether both sectors emerge from a single composite functional under one master-action variation — is the load-bearing question of P2. The commonalities are suggestive but not conclusive.

What remains open after this paper is structurally clean: (P2) the unification with the constitutive-current sector; (P3) the continuum limit; (P4) non-Abelian generalisations; (P5) matter coupling; (P6) the numerical coefficient ratio α/γ; (P7) the vertex × tick-window σ-duality carried from the predecessor. The σ-sector itself has been closed against the master action, modulo the structural inputs explicitly identified — (H1), the K = 7 cell structure of W_7, the admissible-cycle restriction, the D_6-symmetry of the K = 7 substrate constraints, the closure of the K = 7 constraint catalogue at four constraints, and integrality.

The substrate does not evolve by arbitrary update rules. Sequential propagation is governed by the constitutive admissibility dynamics of the master action in the sequential transport sector: closure-transport-sector commitment conservation projects to closure-current conservation under the admissible-cycle restriction; admissibility minimisation under the K = 7 constraint catalogue selects the alternating closure-competition wave; the transport-excitation sector carries the residual D_3(vtx) symmetry while committed states retain full D_6 by structural input. The σ-family is the constitutive transport response of the substrate under these constraints — uniquely determined by the leading-order EFT analysis of the master action restricted to admissible spoke perturbations.