Continuum Limit of Sequential Transport in VERSF

From the σ-Family Gradient Flow to its Field-Theoretic Image via the Carrier–Envelope Decomposition

Keith Taylor *VERSF Theoretical Physics Programme*

General Reader Abstract

The previous papers in the σ-family sequence established that the rule by which the VERSF substrate updates from one committed state to the next — the σ-family of sequential transport morphisms on the K = 7 closure wheel — is not arbitrary. A specific alternating spoke pattern emerged as the unique constitutively-derived transport response under the master-action variation, with the alternating structure interpretable as a transient feature of the transport excitation between committed substrate states. Those constructions, however, were entirely discrete.

The natural next question is what happens in the continuum. The σ-family lives on the six spokes of the K = 7 wheel; substrate updates occur one at a time. What is the field-theoretic image of this discrete dynamics?

This paper answers the question — but in doing so it makes precise a question that the previous papers left implicit: *what does "continuum limit" mean for a discrete substrate architecture?*

The paper distinguishes three different continuum-limit framings (refinement of the spoke lattice; refinement only in the internal flow parameter; refinement with the alternating mode treated as a carrier wave), explains why the first two give either ill-defined or trivial answers, and commits to the third. The K = 7 architecture is embedded in a family K_N of N-spoke architectures with K = 7 the N = 6 instance; the continuum limit takes N → ∞ with appropriate spatial and temporal rescalings.

The central technical move is a *carrier–envelope decomposition*: the spoke variable λ_i is written as a rapidly-alternating carrier $(-1)^i$ multiplying a slowly-varying envelope ψ_i. This decomposition has two beautiful properties. First, the discrete operator governing the σ-flow — the signless Laplacian on the spoke cycle, which had the alternating mode in its kernel — transforms under the carrier–envelope decomposition into the *standard* graph Laplacian acting on the envelope, which has *constants* in its kernel. The carrier transformation interchanges "alternating" and "constant" as the persistent directions. Second, the closure-competition functional that drove the discrete σ-flow becomes, in envelope variables, the standard discrete Dirichlet energy — a familiar object whose continuum limit is well-understood.

The continuum limit then becomes calculable. The envelope continuum field $\phi(x, \tilde{\tau})$ satisfies the standard heat equation; its persistent direction (kernel of the continuum Laplacian) is constant ϕ, which corresponds in original variables to the pure alternating mode. The persistence of the alternating mode in the discrete σ-sector becomes the persistence of the constant envelope under continuum diffusive flow.

The paper proves four results in this framework: the continuum effective action of the σ-sector is a gradient-energy functional; the continuum dynamics is the heat equation on the envelope field; closure-current conservation translates to an automatic suppression of Brillouin-edge components in the continuum envelope (i.e., it is automatic for slowly-varying fields); and the alternating mode of the discrete σ-sector corresponds, under the K_N → ∞ refinement, to the Brillouin-zone-edge plane wave at $k = \pi/a$.

One conceptual result is worth flagging. The continuum theory is a *dissipative parabolic* equation (heat-equation-like), not a wave equation. The σ-sector does not, in its continuum limit, give a Lorentz-invariant propagating field. This is structurally consistent: the σ-sector is the substrate's *admissibility-restoring response*, which is dynamically a dissipative process, not a conservative one. Lorentz invariance, if it emerges anywhere in VERSF, must come from a different sector — most likely the persistent cohomological/gauge transport sector, where conservative dynamics is naturally encoded.

The paper does not claim to derive the cohomological/gauge sector from the σ-sector continuum limit. It establishes that the σ-sector has a well-defined continuum field-theoretic image with explicit dynamics and conservation law, and that this image has a sensible relation to standard field-theoretic objects. The bridge to the persistent transport sector — i.e., whether the constant envelope of the σ-sector continuum limit is the same object as the persistent cohomology class identified elsewhere in VERSF — remains an open question, and the natural subject of subsequent work.

What the paper does establish is the bridge from discrete substrate dynamics to continuum field theory in the σ-sector, with each derivation honestly earned and the structural inputs explicit.

Abstract

The preceding σ-family papers established that admissibility-restoring sequential transport on the $K = 7$ closure wheel W_7 admits a canonical alternating spoke sector

$\lambda_i = (-1)^i$

selected by four converging characterisations (maximal residual symmetry, spectral selection, variational minimisation, Laplacian-extremality) and derived from master-action variation under the $K = 7$ constraint catalogue. The σ-family was thereby established as the unique leading-order constitutive transport response of the substrate, derived via the gradient flow

$\partial_\tau \lambda = -\nabla A_{cl}(\lambda)$, $A_{cl} = \alpha \cdot A_{circ} + \gamma \cdot A_{comp}$,

with $A_{circ} = (\Sigma_i \lambda_i)^2$ and $A_{comp} = \Sigma_i (\lambda_i + \lambda_{i+1})^2$.

The present paper studies the continuum limit of this construction. We make three scoping decisions explicit:

(L1) Refinement framework. We embed the $K = 7$ architecture in a family K_N of N-spoke closure architectures, with $K = 7$ being the $N = 6$ instance. The continuum limit takes $N \to \infty$ with the wheel rim radius held fixed (lattice spacing $a = 2\pi R/N \to 0$).

(L2) Temporal rescaling. The internal flow parameter τ is rescaled as $\tilde{\tau} = \tau/a^2$ (diffusive rescaling) to extract non-trivial continuum dynamics from the discrete heat-equation-like flow.

(L3) Carrier–envelope decomposition. The spoke variable is written $\lambda_i = (-1)^i \psi_i$, where $(-1)^i$ is the rapidly-alternating *carrier* and ψ_i is the slowly-varying *envelope*. The continuum field $\phi(x, \tilde{\tau})$ is the envelope, not the spoke amplitude directly.

Under (L1)–(L3) we prove four results.

Theorem 1 (Continuum effective action). *Under the carrier–envelope decomposition and the* $K_N \to \infty$ *refinement with* $\gamma \to \gamma/a$ *rescaling of the coupling, the closure-competition functional* A_{comp} *transforms as*

$A_{comp}(\lambda) = \Sigma_i (\psi_i - \psi_{i+1})^2$ *in envelope variables*

(discrete Dirichlet energy of the envelope) and its continuum limit is the gradient-energy functional

$A_{eff}[\phi] = \kappa \int (\partial_x \phi)^2 \, dx$, $\kappa > 0$.

Theorem 2 (Continuum constitutive flow). *Under the diffusive temporal rescaling* $\tilde{\tau} = \tau/a^2$, *the discrete admissibility-restoring flow on the envelope variables converges to the continuum heat equation*

$\partial_{\tilde{\tau}} \phi = D \cdot \partial^2_x \phi$, $D > 0$.

The kernel of the continuum flow is the space of constant envelopes $\phi(x) = const$; *this is the continuum-limit image of the discrete alternating mode (whose constant envelope corresponds to pure alternating* $\lambda_i = c \cdot (-1)^i$*).*

Theorem 3 (Closure-current conservation in the continuum). *The discrete closure-current conservation law* $\Sigma_i \lambda_i = 0$ *translates under the carrier–envelope decomposition to suppression of the Brillouin-edge Fourier component of the envelope. For slowly-varying envelope fields (the natural domain of the continuum limit), this constraint is automatically satisfied to exponential accuracy in a. The continuum conserved quantity is*

$Q[\phi] := \int \phi(x, \tilde{\tau})\, dx,\ \partial_{\tilde{\tau}} Q = 0,$

following from the divergence form of the heat equation.

Theorem 4 (Brillouin-edge identification of the alternating mode). *Under the K_N → ∞ refinement, the discrete alternating spoke mode* $\lambda_i = (-1)^i$ *of K_N corresponds to the Brillouin-zone-edge plane wave* $\lambda(x) \sim \exp(i\pi x/a)$ *of the refined lattice — the highest-frequency mode admissible on the discrete substrate. The carrier–envelope decomposition extracts this plane wave as the structural carrier and renders the envelope as the continuum dynamical field. For finite* $K = 7$ $(N = 6)$*, the alternating mode is the* $k = 3$ *Fourier mode of* $\mathbb{Z}/6$*, the unique highest-frequency mode on the discrete cyclic group; the "Brillouin-edge" identification is the structural analogue at finite N.*

The σ-family therefore possesses a non-trivial continuum field-theoretic image. The continuum field theory is:

- a *parabolic* (heat-equation-like) field theory of the envelope of an alternating carrier, **not** a wave equation;
- a *dissipative* admissibility-restoring dynamics, **not** a conservative propagating dynamics;
- a *constitutive* image of substrate transport, **not** a fundamental matter field.

The continuum theory is not Lorentz-invariant. This is consistent with the σ-sector being the substrate's admissibility-restoring response, which is dynamically dissipative. Lorentz invariance, if it emerges anywhere in VERSF, must come from a different sector — most likely the persistent cohomological/gauge transport sector. The relation between the σ-sector continuum limit and the persistent transport sector is left as an open problem (P2 of the preceding paper).

Epistemic status. *Proven (within this paper):* Theorems 1–4 above, under the scoping decisions (L1)–(L3); the carrier–envelope decomposition's spectral transformation properties; the Fourier-analytic structure of the discrete and continuum spectra. *Structural inputs (not derived here):* the K_N refinement framework (a methodological choice for the continuum limit; the K = 7 architecture itself is structural input from the foundational architecture papers); all structural inputs of the preceding paper (the K = 7 constraint catalogue, the D_6-symmetry of the constraints, the admissible-cycle restriction, integrality); the diffusive temporal rescaling $\tilde{\tau} = \tau/a^2$ (a continuum-limit choice required for non-trivial dynamics). *Open:* identification of the σ-sector continuum field with the persistent cohomological/gauge transport sector; emergence of Lorentz invariance (likely from a different sector, not the σ-sector); non-Abelian generalisations; matter coupling; higher-order EFT corrections; the relation between the K_N family and the K = 7 architecture's structural rigidity.

Contents

---

1. Introduction

The preceding papers in the σ-family sequence — *Substrate-Generated Sequential Transport in VERSF* and *Constitutive Sequential Transport in VERSF: Master-Action Variation and the Derivation of the σ-Family* — established the discrete sequential-transport sector of the K = 7 closure architecture as a uniquely derived constitutive response of the VERSF substrate. Four converging characterisations of the alternating spoke mode were established; the master-action derivation of the four-term closure-response functional was performed; the gradient-flow dynamics was analysed and its kernel structure identified; and the transport-gap reading of the residual D_3(vtx) symmetry was supplied conditional on stated structural inputs.

All of that work was performed in the discrete setting: the σ-family lives on the six spokes of the K = 7 wheel W_7, the substrate updates one step at a time, and the closure-response functional A_cl is a quadratic form on $\mathbb{R}^6$.

The natural next question is the continuum limit. What field-theoretic structure, if any, does the σ-sector approach when its discrete substrate is refined and its internal flow parameter is continuised? Does the alternating mode survive in any form? Does the closure-current conservation become a continuum continuity equation? What kind of field theory — wave, diffusion, neither — emerges?

These are not idle questions. The discrete σ-sector has no direct observational signature; the continuum field theory, if any, is the natural place where the σ-sector's physical content would become testable and where its relation to other VERSF sectors (the persistent cohomological/gauge sector in particular) could be made precise.

The purpose of this paper is to establish the continuum field-theoretic image of the σ-sector. It is structured as a careful continuum-limit derivation, with three scoping decisions made explicit (§3), the key technical decomposition introduced (§5), the limit performed (§6), and four theorems derived (§7–10) along with explicit discussion of what the continuum theory is and is not (§11–12).

A guiding observation: the discrete σ-sector dynamics is a *gradient flow*, which is intrinsically *dissipative*. Continuum limits of dissipative flows give *parabolic* (heat-equation-like) PDEs, not *hyperbolic* (wave-equation-like) PDEs. The continuum σ-sector should therefore be expected to be a dissipative field theory, not a relativistic one — and indeed Theorem 2 will establish this directly. We flag the observation up front because it has implications for what the σ-sector continuum limit can and cannot deliver: in particular, it cannot deliver Lorentz invariance by itself, and any such symmetry must enter via a different sector. We return to this point in §11–12.

---

2. Open Problems Inherited from the Preceding Paper

The preceding paper closed (P1a, P1b, P1c, P1e of the σ-family paper) under stated structural inputs. The remaining open problems, restated:

Open problem	***Status going into this paper***
P2 — master-action unification of σ-sector and constitutive-current sector	open
P3 — continuum-limit derivation of the σ-family	this paper
P4 — non-Abelian transport sectors	open
P5 — coupling to matter sectors	open
P6 — numerical coefficient ratio α/γ	open
P7 — vertex × tick-window σ-duality	open (carried from original σ-family paper)

The present paper addresses P3 directly. The result has implications for P2 (the continuum limit is the natural place where the σ-sector and the persistent cohomological/gauge sector can be compared), but does not close P2. P4–P7 remain open after this paper.

3. Scoping the Continuum Limit: Three Framings

A "continuum limit" of the σ-sector is not a single well-defined operation. The discrete sector has structure at multiple scales, and different limit framings refine different scales while holding others fixed. We distinguish three:

3.1 Limit-A: spatial refinement of the spoke lattice (K_N → ∞)

The K = 7 architecture has six spokes around a hexagonal wheel of fixed radius R. Limit-A embeds this in a family of architectures K_N with N spokes at angular spacing 2π/N around a wheel of radius R, with K = 7 the N = 6 instance. The continuum limit is N → ∞, equivalently lattice spacing a = 2πR/N → 0.

Limit-A is the natural framing for treating the spoke index as a spatial coordinate becoming continuous. It requires:

- A definition of the K_N architecture for arbitrary N, with inheritance of the K = 7 constraint catalogue (closure-incidence, hub anchoring, closure-current conservation, closure-competition);
- A justification of K = 7 as a special instance of K_N rather than a structurally distinct architecture.

This requires care: the K = 7 architecture has structural features (the specific seven-fold cell count, the D_6 symmetry of the hexagon) that are not generically preserved by K_N. We treat K_N as a *methodological scaffolding* — a family of architectures used for the limit-taking, not a claim about the physical primacy of K_N over K = 7.

3.2 Limit-B: temporal refinement of the internal τ-flow only

Holding the K = 7 architecture fixed and treating only the internal flow parameter τ as continuous already gives a continuum dynamics in a trivial sense: the gradient flow $\partial\_\tau \lambda = -2 M\lambda$ is already a continuous dynamics on the six-component lattice variable $\lambda \in \mathbb{R}^6$. No spatial continuum.

Limit-B is technically well-defined but does not produce a field theory in the usual sense (it gives a six-component ODE system, not a PDE). It does not address the spatial structure that the question "what is the continuum field theory of the σ-sector?" is asking about.

3.3 Limit-C: refinement with the alternating mode as carrier

The alternating mode is the highest-frequency Fourier component of the discrete spoke lattice — at $k = \pi/a$ in the K_N refinement framework, the literal Brillouin-zone edge. Treating the alternating structure as a *carrier wave* and the slow modulation of its amplitude as the continuum field is the framing used in condensed-matter physics for similar problems (Néel-ordered systems with slow magnetisation modulation, etc.).

Limit-C is the combination of Limit-A with the additional structural move of factoring the alternating carrier out before taking the spatial continuum limit. The continuum field is then the envelope of the alternating carrier, not the carrier amplitude directly.

3.4 The chosen framing

We commit to **Limit-A + Limit-C** combined: the K_N → ∞ refinement (spatial) with the carrier–envelope decomposition (alternating mode as carrier). The reason is structural: the persistent direction of the discrete σ-sector lies in the alternating mode, which is the highest-frequency lattice mode on the discrete spoke cycle. The carrier–envelope decomposition exposes this persistent direction as a low-frequency envelope mode amenable to continuum description; without it, the persistent direction sits at the lattice cutoff and is not naturally rendered as a continuum field.

The alternative framings:

- **Limit-A alone** (without the carrier decomposition) is a well-defined continuum construction that gives a continuum operator with the alternating mode as its maximum-eigenvalue eigenmode. Under the gradient flow, this mode decays at the fastest rate; the persistent direction (kernel) corresponds in original variables to the constant-λ mode, which is excluded by closure-current conservation $\Sigma\lambda_i = 0$. So Limit-A alone gives a continuum theory whose admissible sector has *no kernel direction* — the flow asymptotically takes everything to zero, with no persistent mode in the admissible sector. This is not a contradiction with the discrete result; it is a different continuum question (the decay-spectrum framing) than the one we are asking (the persistent-dynamics framing).
- **Limit-B alone** gives no spatial field theory (only a six-component ODE system in continuous τ).
- **Limit-C alone** requires Limit-A's spatial scaffolding to produce a continuum field.

The choice of Limit-A + Limit-C is determined by what we wish to model: the long-time persistent dynamics of the σ-sector (where the alternating mode does most of the structural work). For other purposes — modelling the decay spectrum of admissibility-violating perturbations, or studying short-time relaxation — Limit-A alone may be the appropriate framing. We pursue Limit-C because it captures the persistent dynamics the discrete σ-sector papers identified as the σ-family's physical content.

We refer henceforth to the combined Limit-A + Limit-C as **the K_N → ∞ refinement with carrier–envelope decomposition**, or simply "the continuum limit" when context is clear.

---

4. The K_N Architecture Family

4.1 Definition of K_N

For each integer $N \geq 3$ we define the closure architecture K_N as follows:

- **0-cells:** N outer vertices $v_0, \ldots, v_{N-1}$ arranged at angles $2\pi i/N$ around a circle of radius R, plus one hub vertex h at the centre.
- **1-cells:** N outer edges e_i connecting v_i to v_{i+1} (indices mod N), plus N spokes s_i connecting h to v_i.
- **2-cells:** N triangular 2-cells σ_i filling the triangles spanned by s_i, e_i, s_{i+1}; together with structure such that the resulting $H_1(K_N) \cong \mathbb{Z}\langle[C_N]\rangle$, with $C_N = e_0 + e_1 + \cdots + e_{N-1}$ the primitive 1-cycle around the outer rim. (For each N, the specific 2-cell structure is the natural N-fold dihedral generalisation of the K = 7 case; details are inherited from the K = 7 architecture papers.)
- **Constraint catalogue:** the four substrate constraints (closure-incidence, hub anchoring, closure-current conservation, closure-competition) inherited from K = 7 by N-fold dihedral generalisation. The D_N symmetry replaces D_6 as the architecture's dihedral symmetry group.

The lattice spacing along the outer rim is $a = 2\pi R/N$. For N = 6 (the K = 7 instance), $a = 2\pi R/6 = \pi R/3$.

4.2 K = 7 as the N = 6 instance

The K = 7 architecture studied in the preceding papers is exactly the N = 6 case of the K_N family: six outer vertices, six spokes, one hub (the "7" of K = 7 being the count of vertices), six outer edges, six triangular 2-cells, $H_1 \cong \mathbb{Z}\langle[C]\rangle$, D_6 symmetry. We take the K = 7 architecture as the physical architecture and K_N as the *methodological* scaffolding for taking the continuum limit. The N → ∞ refinement is not a claim that the physical substrate has more than seven cells; it is a calculational tool for identifying the continuum behaviour that the K = 7 instance approximates.

This is analogous to how, in condensed-matter physics, one studies a physical lattice (with a definite atomic structure) by embedding it in a family with arbitrary lattice spacing and taking limits — without claiming that the physical lattice has more atoms than it actually does.

4.3 Inheritance of the constraint catalogue

The four-element constraint catalogue of $K = 7$ has natural N-fold dihedral generalisations:

- **Closure-incidence:** $A\_inc(\sigma) = \Sigma_i \|\partial\sigma(s_i) - \sigma\partial(s_i)\|^2$, summed over the N spokes.
- **Hub anchoring:** $A\_hub(\sigma) = \|\sigma(h) - h\|^2$, independent of N.
- **Closure-current conservation:** $A\_circ(\lambda) = (\Sigma_i \lambda_i)^2$, summed over N spokes.
- **Closure-competition:** $A\_comp(\lambda) = \Sigma_i (\lambda_i + \lambda_{i+1})^2$, summed over N adjacent pairs (indices mod N).

The constrained-EFT principles (P1a)+(P1b), (P2)–(P4), (P5′) from the preceding paper generalise with D_6 replaced by D_N.

The catalogue extension is methodological, not derived. The preceding paper established (P5′) for $K = 7$ *specifically* — closure at exactly four constraints is a structural input from the $K = 7$ architecture papers, not a derivation. We posit here that the catalogue extends to K_N by N-fold dihedral generalisation: closure-incidence, hub-anchoring, and closure-current conservation generalise straightforwardly (they are local/topological structures with manifest N-fold versions); closure-competition extends by the same vertex-local construction with N nearest-neighbour pairs replacing 6. This extension is a *methodological choice* for the refinement framework, not a derivation. The K_N catalogue is itself postulated, not independently established by an architecture paper analogous to the $K = 7$ foundational papers.

The consequence: results obtained in the $K\_N \to \infty$ refinement are conditional on the postulated K_N catalogue. The continuum-limit results (Theorems 1–4) describe the leading dynamics of the $K = 7$ instance under this extension. Whether the extension is the correct way to refine $K = 7$ — or whether some other refinement (with a different catalogue or non-dihedral symmetry) would give a structurally different continuum limit — is itself an open question (P10 below).

Even-N restriction. The gradient flow $\partial\_\tau \lambda = -\nabla A\_cl$ on the admissible spoke sector $\Lambda\_N,0 := \{\lambda \in \mathbb{R}^N : \Sigma_i \lambda_i = 0\}$ is governed by the signless Laplacian M_N of the cycle graph C_N:

$(M\_N \lambda)_i = \lambda_{i-1} + 2\lambda_i + \lambda_{i+1}, \ i \in \mathbb{Z}/N.$

For N even, M_N has the alternating mode $\lambda_i = (-1)^i$ in its kernel (eigenvalue 0). This mode is well-defined on $\mathbb{Z}/N$ only for N even — for N odd, no consistent assignment of alternating signs around the cycle exists, and M_N has no alternating-mode kernel direction.

The K_N refinement framework therefore proceeds **along the even integers only**, with $N = 6, 8, 10, 12, \ldots$ and $a = 2\pi R/N \to 0$. The "K_N family" is not a smooth family parametrised by all integers $N \geq 3$ but a subsequence indexed by even N. This is not a smoothness defect of the limit — the alternating-mode structure is the load-bearing feature being refined, and it exists only at even N — but it is worth flagging that the refinement framework selects even N by the existence of the persistent direction it intends to track.

---

5. The Carrier–Envelope Decomposition

5.1 The decomposition

For each spoke variable λ_i in the admissible sector of K_N (N even), introduce the decomposition

$\boldsymbol{\lambda_i = (-1)^i \psi_i}$**,**

where:

- **$(-1)^i$ is the alternating carrier** — a fixed pattern on the spoke cycle, identified with the Fourier mode at $k = \pi/a$ (the Brillouin-zone-edge mode);
- **ψ_i is the slowly-varying envelope** — a new variable that is to become the continuum field $\phi(x, \tilde{\tau})$ in the limit $N \to \infty$.

The decomposition is a bijection $\mathbb{R}^N \to \mathbb{R}^N$ for even N (the map $\lambda_i \mapsto \psi_i = (-1)^i \lambda_i$ is its own inverse). It re-parametrises the admissible spoke sector in a way that explicitly factors out the alternating structure.

5.2 Transformation of A_comp

We compute the closure-competition functional in envelope variables:

$\text{A\_comp}(\lambda) = \Sigma_i (\lambda_i + \lambda_{i+1})^2 = \Sigma_i ((-1)^i \psi_i + (-1)^{i+1} \psi_{i+1})^2 = \Sigma_i ((-1)^i (\psi_i - \psi_{i+1}))^2 = \Sigma_i (\psi_i - \psi_{i+1})^2$.

In envelope variables, A_comp is the **discrete Dirichlet energy** of ψ on the cycle C_N — the sum of squared nearest-neighbour differences. This is exactly the quadratic form of the standard graph Laplacian L_N of C_N:

$\text{A\_comp}(\lambda) = \langle \psi, \text{L\_N}\, \psi \rangle$, where $(\text{L\_N}\, \psi)_i = 2\psi_i - \psi_{i-1} - \psi_{i+1}$.

This is a remarkable simplification. The closure-competition functional, which in original variables was the quadratic form of the *signless* Laplacian M_N (with alternating in its kernel), becomes in envelope variables the quadratic form of the *standard* graph Laplacian L_N (with constants in its kernel). The carrier transformation has interchanged the two operators.

5.3 The carrier transformation interchanges M and L

Concretely: under $\lambda_i = (-1)^i \psi_i$, the operator M_N acts as

$(M\_N\,\lambda)_i = \lambda_{i-1} + 2\lambda_i + \lambda_{i+1} = (-1)^{i-1}\,\psi_{i-1} + 2(-1)^i\,\psi_i + (-1)^{i+1}\,\psi_{i+1} = (-1)^i\,(-\psi_{i-1} + 2\psi_i - \psi_{i+1}) = (-1)^i\,(L\_N\,\psi)_i$.

Let *J* denote the diagonal operator on $\mathbb{R}^N$ (N even) with $J_{ii} = (-1)^i$. Then J is *involutory*: $J^2 = I$ (since $(-1)^{2i} = 1$). The carrier transformation $\lambda = J\psi$ is its own inverse, $\psi = J\lambda$.

The intertwining computation above says J M_N J = L_N. Combined with $J^2 = I$, this is equivalent to:

$J\,M\_N\,J = L\_N,\ J\,L\_N\,J = M\_N.$

Carrier conjugation is an involutory similarity transformation that interchanges M_N and L_N as operators on $\mathbb{R}^N$. It is therefore a symmetry of the discrete spectral theory: M_N and L_N share the same set of eigenvalues (which is true — both have spectrum $\{2 + 2\cos(2\pi k/N) : k = 0, \ldots, N-1\} = \{2 - 2\cos(2\pi k/N) : k = 0, \ldots, N-1\}$ for N even, since these sets are equal under $k \leftrightarrow N/2 - k$), but with their eigenmode-labellings interchanged via J.

The kernel correspondence — alternating in $\lambda \leftrightarrow$ constant in ψ — follows from the involution structure of J.

5.4 Kernel correspondence

The kernel of M_N restricted to the admissible sector $\Lambda\_N{,}0$ is the alternating subspace $\mathbb{R} \cdot (1, -1, 1, -1, \ldots, 1, -1)$. Under the carrier transformation this maps to the kernel of L_N: the constant subspace $\mathbb{R} \cdot (1, 1, 1, \ldots, 1)$ restricted by some compatibility condition with the admissibility sector.

More carefully: the original admissibility $\Sigma_i\,\lambda_i = 0$ translates under $\lambda_i = (-1)^i\,\psi_i$ to $\Sigma_i\,(-1)^i\,\psi_i = 0$ — i.e., the envelope has zero Brillouin-edge Fourier component. For N even, this excludes the alternating-ψ mode (which is at $k = \pi/a$, the Brillouin edge in envelope variables), but *not* the constant-ψ mode (which is at $k = 0$). The constant-ψ subspace is therefore in the admissibility sector and is the kernel of L_N restricted to it.

Kernel correspondence summary:

Original variables (λ)	***Envelope variables (ψ)***
Alternating mode $(-1)^i$ — kernel of M_N	*Constant mode 1 — kernel of L_N*
Constant mode 1 — excluded by $\Sigma\lambda_i = 0$	*Alternating mode $(-1)^i$ — excluded by $\Sigma(-1)^i\psi_i = 0$*
Highest-frequency Fourier mode = alternating	*Highest-frequency Fourier mode = alternating ψ (lives at $k = \pi/a$)*

The persistent direction of the discrete σ-sector is the alternating λ mode, which corresponds in envelope variables to the constant ψ mode. This will become the constant continuum field $\phi(x) =$ const in the continuum limit — the zero mode of the continuum Laplacian.

6. Continuum Scaling

We now perform the continuum limit explicitly, with all scaling choices stated.

6.1 Spatial rescaling

In K_N with lattice spacing $a = 2\pi R/N$, the envelope ψ_i at spoke position i is identified with a continuous function ϕ at position $x_i = ia$:

$\psi_i \longmapsto \phi(x_i, \tilde{\tau})$, $x_i = ia$.

For slowly-varying envelope (the assumption underlying the carrier–envelope framing), neighbouring envelope values relate via Taylor expansion:

$\psi_{i+1} = \phi(x_i + a) = \phi(x_i) + a \cdot \partial_x\phi(x_i) + (a^2/2) \cdot \partial^2_x\phi(x_i) + O(a^3)$.

The cycle structure imposes periodic boundary conditions:

$\phi(x + 2\pi R, \tilde{\tau}) = \phi(x, \tilde{\tau})$.

The spatial domain is therefore the circle S^1 of circumference $2\pi R$. We work on this circle throughout.

Lattice sums convert to integrals via:

$\Sigma_i \longmapsto a^{-1} \int_0^{2\pi R} dx$,

the standard Riemann-sum-to-integral correspondence as $a \to 0$.

6.2 Coupling rescaling

The discrete A_comp(λ) = $\Sigma_i (\psi_i - \psi_{i+1})^2$ becomes, using $\psi_i - \psi_{i+1} = -a \cdot \partial_x\phi + O(a^2)$:

A_comp = $\Sigma_i\, a^2(\partial_x\phi)^2 + O(a^3) = a^{-1} \int dx \cdot a^2(\partial_x\phi)^2 + O(a^2) = a \cdot \int (\partial_x\phi)^2\, dx + O(a^2)$.

This vanishes as $a \to 0$ if the coupling γ is held fixed. To obtain a finite continuum effective action, we rescale the coupling:

$\gamma \longmapsto \gamma/a$, equivalently A_eff[$\phi$] := A_comp($\lambda$) / a.

This rescaling has a clean structural interpretation: the discrete sum Σ_i counts $N = 2\pi R/a$ terms (one per spoke), so dividing by a (equivalently, multiplying by $N/(2\pi R)$) rescales the sum to an integral *per unit length* of the rim. The resulting A_eff has finite limit:

$A\_eff[\phi] \to \kappa \cdot \int_0^{2\pi R} (\partial_x\phi(x, \tilde{\tau}))^2 \, dx$ as $a \to 0$,

with $\kappa = \gamma\_\infty$, the limiting value of γ after the rescaling. We take $\kappa > 0$ throughout.

The closure-current functional A_circ rescales similarly, but on slowly-varying envelopes (no Brillouin-edge component) it vanishes exponentially in a (see §9 below), and contributes nothing to the leading continuum action.

6.3 Temporal rescaling

The discrete envelope-variable flow is

$\partial\tau\, \psi_i = - \nabla\{\psi_i\}\ A\_comp / \gamma = -2\, (L\_N\ \psi)_i$.

(The factor $1/\gamma$ from the rescaling cancels the overall coupling, giving a normalised flow rate.)

In the continuum, using $(L\_N\ \psi)_i \to -a^2 \cdot \partial^2_x\phi(x_i) + O(a^4)$:

$\partial\_\tau\ \phi = 2a^2 \cdot \partial^2_x\phi + O(a^4)$.

This vanishes as $a \to 0$ if τ is held fixed. To extract non-trivial continuum dynamics, we rescale time diffusively:

$\tilde{\tau} := \tau \cdot a^{-2}$, equivalently $\tilde{\tau} = \tau\, N^2/(4\pi^2R^2)$.

This is the standard diffusive rescaling: short discrete times correspond to short continuum times of order a^2, so long continuum times $\tilde{\tau}$ correspond to *many* substrate-update lattice times. With $\tilde{\tau}$ as the continuum time variable, $\partial\_\tau\ \phi = a^2\partial\_\tilde{\tau}\phi$, so:

$a^2 \cdot \partial\_\tilde{\tau}\ \phi = 2a^2 \cdot \partial^2_x\phi + O(a^4)$,

equivalent to

$\partial\_\tilde{\tau}\ \phi = 2 \cdot \partial^2_x\phi + O(a^2)$.

This is the standard heat equation on the circle, with diffusion coefficient $D = 2$ (after the rescalings) and corrections vanishing as $a \to 0$. The continuum limit is non-trivial, and the resulting PDE is well-defined.

On the numerical value $D = 2$. The specific value $D = 2$ reflects the chosen scaling conventions: the factor of 2 from $\nabla A\_comp = 2\ L\_N\ \psi$ (the gradient of a quadratic form $\langle\psi, L\_N\ \psi\rangle$), the diffusive rescaling $\tilde{\tau} = \tau/a^2$, and the coupling rescaling $\gamma \longmapsto \gamma/a$. It is not a derived structural constant. The physically meaningful continuum quantity is the combination $\kappa = \gamma\_\infty$ (the limiting rescaled coupling); the diffusion coefficient D is fixed once κ and the rescalings are chosen, and the two are not independent. The "$D > 0$" in the theorem statements below is a structural claim about the sign and non-degeneracy of the diffusion; the specific numerical value $D = 2$ should be read as scaling-convention-dependent.

---

7. Theorem 1: Leading-Order Continuum Limit of A_comp

Theorem 1 (Leading-order continuum effective action of the σ-sector). *Take as given the discrete closure-response functional* $A\_cl = \alpha \cdot A\_circ + \gamma \cdot A\_comp$ *of* K_N*, established at leading order under the* $K = 7$ *constraint catalogue (preceding paper, Theorem 1) and extended to* K_N *by N-fold dihedral generalisation (§4.3 caveat). Under the* $K\_N \to \infty$ *refinement (§3, §4) with carrier–envelope decomposition* $\lambda_i = (-1)^i \psi_i$ *(§5) and coupling rescaling* $\gamma \longmapsto \gamma/a$ *(§6.2), the closure-competition functional converges, at leading order in the lattice spacing a, to the continuum gradient-energy functional*

$$A\_eff[\phi] = \kappa \int\{S^1\} (\partial_x\phi(x))^2\, dx + O(a),\ \kappa := \lim\{a \to 0\}\ \gamma \cdot a > 0,$$

on the space of slowly-varying envelope fields $\phi : S^1 \to \mathbb{R}$*. The closure-incidence and hub-anchoring functionals A_inc and A_hub vanish on the admissible spoke sector (cf. preceding paper, §5.2); the closure-current functional A_circ vanishes to exponential accuracy on real-analytic envelopes (cf. §9 below). The continuum effective action on the admissible envelope sector is therefore* $A\_eff[\phi]$ *alone at leading order.*

Proof. §5.2 established $A\_comp(\lambda) = \Sigma_i (\psi_i - \psi_{i+1})^2$ in envelope variables. §6.1–6.2 performed the continuum limit:

$$A\_comp = a \cdot \int\_\{S^1\} (\partial_x\phi)^2\, dx + O(a^2).$$

Under the rescaling $\gamma \longmapsto \gamma/a$, the effective continuum action is

$$A\_eff[\phi] = A\_comp / a = \int\_\{S^1\} (\partial_x\phi)^2\, dx + O(a),$$

which has the stated finite limit. The vanishing of A_circ on real-analytic envelopes is established in §9. The decomposition into on-sector and off-sector pieces is inherited directly from the preceding paper. □

Remark on what this theorem does and does not establish. Theorem 1 is a *convergence result*: given the discrete A_cl of the preceding paper, its continuum limit is $A\_eff = \kappa \int(\partial_x\phi)^2\, dx + O(a)$. It is not a uniqueness result in the sense of the preceding paper's Theorem 1 (which established A_cl as the unique leading-order admissible quadratic functional under the $K = 7$ constraint catalogue (P5′)). The two are different in character:

- The preceding paper's Theorem 1 was about *uniqueness of the discrete A_cl* given the constraint catalogue.
- The present Theorem 1 is about *leading-order continuum convergence of A_comp* given the discrete A_cl.

In particular, Theorem 1 here does *not* exclude higher-derivative corrections in the continuum action — e.g., terms of the form $\int(\partial^2_x\phi)^2\,dx$ that would arise at sub-leading order in the lattice expansion. These corrections are O(a) or smaller in the chosen scaling and are not picked up by the leading-order calculation. They may correspond to:

- Higher-derivative terms in the discrete A_cl that vanish at finite N but survive in the continuum limit (none such are derived from the K = 7 catalogue at the order treated by the preceding paper, but their absence at sub-leading order is not established);
- Lattice-discretisation corrections to the continuum operator (universal features of any lattice → continuum limit, present regardless of the architecture's substrate content).

Whether either of these contributes structurally at sub-leading order is open (P9 below). The leading-order convergence of Theorem 1 is independent of this open question; the higher-derivative status is part of what "leading-order" excludes.

Remark on the absence of a continuum mass term. $A\_eff[\phi] = \kappa \int(\partial_x\phi)^2\,dx$ contains no continuum mass term $\int m^2\phi^2\,dx$. This is a structural consequence of the carrier–envelope decomposition combined with the absence of any λ^2 penalty in the discrete A_cl: the alternating-carrier structure of the discrete σ-sector is factored into the carrier, leaving the envelope dynamics governed entirely by gradient energy. The continuum σ-sector is therefore *massless* in the natural envelope-field sense — under leading-order continuum analysis.

---

8. Theorem 2: The Continuum Constitutive Flow

Theorem 2 (Continuum constitutive flow of the σ-sector). *Under the K_N → ∞ refinement with carrier–envelope decomposition and the diffusive temporal rescaling* $\tilde{\tau} = \tau/a^2$, *the discrete admissibility-restoring flow on the envelope variables*

$$\partial\_\tau\, \psi_i = -2(L\_N\, \psi)_i$$

converges to the continuum heat equation on the envelope field:

$$\partial\tilde{\tau}\, \phi(x, \tilde{\tau}) = D \cdot \partial^2_x\, \phi(x, \tilde{\tau}),\ D = 2,\ x \in S^1,\ \tilde{\tau} \in \mathbb{R}\{\geq 0\}.$$

The kernel of the continuum flow is the space of constant envelope fields $\{\phi(x) = c : c \in \mathbb{R}\}$. *Under the inverse carrier transformation, this corresponds to the alternating mode* $\lambda_i = c \cdot (-1)^i$ *in original spoke variables — the persistent direction of the discrete σ-sector. The continuum image of the discrete alternating-mode kernel is therefore the constant-envelope kernel of the heat equation.*

Proof. §6.3 performed the temporal rescaling. The discrete envelope-variable flow $\partial\_\tau\, \psi = -2\, L\_N\, \psi$ becomes $\partial\_\tilde{\tau}\phi = 2\,\partial^2_x\phi + O(a^2)$ in the K_N → ∞ limit with $\tilde{\tau} = \tau/a^2$. The kernel of ∂^2_x on S^1 is the space of constant functions, which is one-dimensional (consistent with the discrete kernel of L_N on the admissible sector being one-dimensional). The inverse carrier transformation maps constant ϕ to alternating λ; this is the kernel correspondence of §5.4. □

Remark on parabolic structure. The continuum equation is a *heat equation* — a parabolic PDE. It is *not* a wave equation (Klein–Gordon, d'Alembertian) and is *not* Lorentz-invariant. This is the correct continuum image of the σ-sector dynamics, which in the discrete setting is a *gradient flow* (an intrinsically dissipative process). The σ-sector is the substrate's admissibility-restoring response, not a conservative propagating field; its continuum image inherits this character.

A frequently-confused point: the discrete alternating mode is the *persistent* direction of the σ-flow (it remains under gradient descent because it is in the kernel of M). One might therefore expect the continuum alternating image to be "persistent" in some directly-mapped sense. The carrier–envelope decomposition shows that this persistence is correctly inherited, but the persistent direction takes different forms at each level of description:

Level	***Persistent direction***	***Operator/equation***
Discrete original variables	*alternating* $\lambda_i = (-1)^i$	*kernel of M_N*
Discrete envelope variables	*constant* ψ_i	*kernel of L_N*
Continuum	*constant* $\phi(x)$	*kernel of* ∂^2_x *on* S^1

The persistence survives the continuum limit; what changes is the form the persistent direction takes once the alternating-carrier structure is factored out.

---

9. Theorem 3: Closure-Current Conservation in the Continuum

Theorem 3 (Continuum image of closure-current conservation). *The discrete closure-current conservation law*

$\Sigma_i \lambda_i = 0$ *in K_N*

translates under the carrier–envelope decomposition $\lambda_i = (-1)^i \psi_i$ *to suppression of the Brillouin-zone-edge Fourier component of the envelope:*

$\Sigma_i (-1)^i \psi_i = 0.$

For real-analytic envelope fields (those whose Fourier coefficients decay exponentially at high momentum), this constraint is automatically satisfied to exponential accuracy in a. The heat equation flow of Theorem 2 instantaneously smooths arbitrary L^2 *initial data to real-analytic for* $\tilde{\tau} > 0$*; the real-analytic class is therefore the natural domain of the continuum envelope theory at positive flow time, and the Brillouin-edge constraint is automatic on this domain. In the continuum limit* $a \to 0$*, the constraint becomes vacuous on the natural domain of the envelope field theory.*

The continuum conserved quantity associated with the heat-equation dynamics is the total envelope integral

$$Q[\phi] := \int_{S^1} \phi(x, \tilde{\tau})\, dx, \quad \partial_{\tilde{\tau}} Q[\phi] = 0,$$

following from the divergence form of the continuum equation and periodic boundary conditions on S^1. $Q[\phi]$ is the conserved continuum charge associated with the constant-mode direction (the kernel of ∂^2_x on S^1).

Proof. Under $\lambda_i = (-1)^i \psi_i$:

$$\Sigma_i \lambda_i = \Sigma_i (-1)^i \psi_i.$$

If $\psi_i = \phi(x_i)$ for a real-analytic function ϕ on S^1, the right-hand side is a Riemann sum approximation to

$$a^{-1} \int_{S^1} (-1)^{x/a} \phi(x)\, dx = a^{-1} \int_{S^1} \exp(i\pi x/a)\, \phi(x)\, dx \text{ (real part)},$$

which is the Brillouin-edge Fourier component of ϕ. For ϕ real-analytic on S^1 — i.e., extending holomorphically to a complex strip of width $\delta > 0$ — the standard Paley–Wiener bound gives Fourier coefficients $|\hat{\phi}(k)| \leq \text{const} \cdot \exp(-\delta|k|)$. At $k = \pi/a$, this gives $|\Sigma_i (-1)^i \psi_i| \leq \text{const} \cdot \exp(-\delta\pi/a)$, exponentially small in a.

For C^∞ envelopes (smooth but not real-analytic), the suppression is super-polynomial in $1/a$ (faster than any power); for C^k envelopes, the suppression is polynomial of order k. The heat equation flow $\partial_{\tilde{\tau}}\phi = D\, \partial^2_x\phi$ smooths arbitrary L^2 initial data to real-analytic instantaneously for $\tilde{\tau} > 0$ (a standard property of analytic heat semigroups on compact domains), so the real-analytic class is the natural domain at any positive flow time. We work on this class throughout.

For the continuum conserved charge: from $\partial_{\tilde{\tau}} \phi = D\, \partial^2_x\phi$, we have

$$\partial_{\tilde{\tau}} \int_{S^1} \phi\, dx = D \int_{S^1} \partial^2_x \phi\, dx = D\, [\partial_x\phi]_{S^1}^{\text{periodic}} = 0,$$

by periodicity of ϕ and $\partial_x\phi$ on S^1. Hence $Q[\phi]$ is conserved. □

Remark. The continuum image of closure-current conservation is *not* a constraint imposed on the field theory (in the sense of restricting the configuration space); it is a *conservation law* satisfied automatically by the heat-equation dynamics. This is structurally cleaner than the discrete picture, where $\Sigma\lambda_i = 0$ was an explicit admissibility constraint. The constraint becomes the conservation law because:

- On slowly-varying envelope fields, the discrete constraint is automatic (no Brillouin-edge component);
- The conserved quantity $Q[\phi] = \int\phi\, dx$ is naturally non-zero in general (the envelope has a constant component);
- The heat equation preserves $Q[\phi]$ by its divergence structure.

The persistent direction (constant ϕ) of the continuum dynamics is exactly the direction along which $Q[\phi]$ is non-trivial. The conserved charge labels the kernel direction of the heat flow.

---

10. Theorem 4: The Brillouin-Edge Identification

Theorem 4 (Brillouin-edge identification of the alternating mode).

(i) For finite K_N with N even, the alternating spoke mode $\lambda_i = (-1)^i$ is the unique Fourier eigenmode of the cyclic shift T_N acting on $\mathbb{R}^N$ at wavenumber $k = \pi$ — i.e., the unique mode at the highest available frequency on the discrete group $\mathbb{Z}/N$. This is a structural property of the discrete Fourier transform on $\mathbb{Z}/N$.

(ii) Under the K_N → ∞ refinement with lattice spacing $a = 2\pi R/N \to 0$, this discrete highest-frequency mode corresponds to the continuum plane wave at $k = \pi/a$:

$\lambda(x) \sim \exp(i\pi x/a),\ x \in S^1,$

i.e., the Brillouin-zone-edge mode of the refined lattice.

(iii) The carrier–envelope decomposition $\lambda_i = (-1)^i \psi_i$ explicitly factors this Brillouin-edge plane wave out of the spoke variable, identifying:

$(-1)^i \leftrightarrow$ carrier wave at $k = \pi/a$ (Brillouin edge); $\psi_i \leftrightarrow$ slow envelope at $k \ll \pi/a$.

The continuum field $\phi(x, \tilde{\tau})$ of Theorems 1–3 is the envelope of this Brillouin-edge carrier, not the spoke amplitude directly.

Proof. (i) The cyclic shift operator T_N (the discrete lattice translation by one spoke position) acting on $\mathbb{R}^N$ has Fourier eigenmodes $v_k = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k})$ for $k = 0, 1, \ldots, N-1$, with $\omega = \exp(2\pi i/N)$. For N even, $k = N/2$ gives $\omega^{N/2} = \exp(i\pi) = -1$, so $v_{N/2} = (1, -1, 1, -1, \ldots, 1, -1)$ — exactly the alternating mode. This is the unique highest-frequency mode (largest $|1 - \omega^k|$) on the discrete group $\mathbb{Z}/N$.

(ii) Under $x = ia$ with $a = 2\pi R/N$, the mode $v_{N/2} = (-1)^i$ at spoke position $x_i = ia$ equals $(-1)^{x/a} = \exp(i\pi \cdot x/a)$ (real part), the plane wave at $k = \pi/a$. As $a \to 0$, this is the Brillouin-zone-edge mode of the refined 1D lattice. (*The Brillouin zone of a 1D lattice with spacing a is the momentum interval $k \in [-\pi/a, \pi/a]$; modes at $k = \pm\pi/a$ sit at the edge of this zone and are the highest-frequency modes the lattice can support. The standard reference is any solid-state physics text; the term originates in the band-structure theory of periodic crystals.*)

(iii) The decomposition $\lambda_i = (-1)^i \psi_i$ explicitly multiplies the envelope ψ_i by the Brillouin-edge plane wave $(-1)^i$; this is, by construction, the carrier–envelope decomposition with the alternating mode as the carrier. The envelope ψ_i is then the slowly-varying modulation, and its continuum image $\phi(x, \tilde{\tau})$ is the field of Theorems 1–3. □

Remark on the discrete-vs-continuum distinction. Theorem 4(i) is a structural statement about ℤ/N for finite N — including N = 6 (the K = 7 case). For finite N = 6, the alternating mode is the k = 3 Fourier mode of ℤ/6, the unique highest-frequency mode on this discrete group. Calling this mode "the Brillouin-zone edge" is a *physical analogy* to the continuum (lattice with N → ∞) case, where the highest-frequency mode literally is the BZ edge at $k = \pi/a$. For finite N the BZ-edge language is meaningful only in the limiting sense — under the K_N → ∞ refinement framework.

The K = 7 paper's identification of the alternating mode by four converging characterisations (Lemma 6.2 stabiliser-maximality, §9.2 spectral, Proposition 3 variational, Proposition 4 Laplacian-extremal) is consistent with the Brillouin-edge interpretation in (ii), but does not require the refinement framework — those characterisations are structural properties of the finite ℤ/6 spectrum. The K_N refinement framing of this paper makes the BZ-edge identification literal, but it is a methodological scaffolding for the continuum limit, not a claim about the K = 7 architecture's physical primacy.

---

11. What the Continuum Theory Is — and Is Not

The continuum field theory of the σ-sector, established by Theorems 1–4, has several specific features that deserve explicit acknowledgement.

11.1 Parabolic, not hyperbolic

The continuum equation $\partial_{\tilde{\tau}} \phi = D\, \partial^2_x \phi$ is a parabolic PDE. It is *not* a wave equation; it is *not* second-order in $\tilde{\tau}$. The continuum σ-sector is a dissipative field theory, not a propagating one.

This is the correct continuum image of the discrete gradient flow $\partial_\tau \lambda = -\nabla A_{cl}$, which is intrinsically dissipative. The substrate's admissibility-restoring response — embodied by the σ-flow — moves the substrate toward the kernel of M_N along the gradient of A_cl; it does not oscillate, propagate as a wave, or conserve any "kinetic" quantity. The continuum image inherits all of these features.

A reader expecting the continuum σ-sector to be a wave equation should reset expectations. The σ-sector is not the field-theoretic analogue of a propagating particle; it is the field-theoretic analogue of an order-parameter relaxation in condensed matter — specifically, of *Model A* in the Hohenberg–Halperin classification of dynamic critical phenomena, in which a non-conserved scalar order parameter relaxes by gradient flow of a Ginzburg–Landau-like functional. The σ-sector is structurally analogous: an envelope field ϕ relaxes toward the kernel of ∂^2_x under gradient flow of $A_{eff}[\phi] = \kappa \int (\partial_x \phi)^2\, dx$.

11.2 Dissipative, not conservative

The heat equation is dissipative: A_eff[ϕ] decreases monotonically along the flow except on the kernel:

$$dA_{\text{eff}}/d\tilde{\tau} = \int_{\{S^1\}} (\partial_x\phi)\,(\partial_{\tilde{\tau}}\,\partial_x\phi)\cdot 2\,dx = -D\int_{\{S^1\}} (\partial^2_x\phi)^2\cdot 2\,dx \leq 0,$$

with equality only when $\partial^2_x\phi = 0$, i.e., ϕ is constant. The flow drives ϕ toward the constant kernel direction; the dissipation is governed by A_{eff} itself.

This means the σ-sector continuum theory has no conserved energy in the conventional Hamiltonian sense. The only conserved quantity is the constant-mode amplitude $Q[\phi] = \int\phi\,dx$, as established by Theorem 3. The "action" A_{eff} is a Lyapunov functional, not an energy — it monotonically decreases under the flow.

11.3 Constitutive, not fundamental; and 1-dimensional, not higher-dimensional

The field $\phi(x, \tilde{\tau})$ is the continuum envelope of the discrete alternating spoke-correction pattern; it is the field-theoretic image of *how the substrate redistributes closure transport*, not a propagating excitation of an underlying field. ϕ is to the σ-sector what a continuum order parameter is to a microscopic spin model: an emergent field that captures the slowly-varying degrees of freedom of an underlying discrete dynamics.

The σ-sector continuum theory is therefore *constitutive*: it describes the substrate's response to admissibility-violation, not the propagation of substrate states themselves. Physical observables in the σ-sector are response coefficients (transport, relaxation rates), not particle masses or scattering amplitudes.

Dimensional limitation. The σ-sector continuum is a *one-dimensional* field theory: ϕ lives on the rim circle S^1 of the wheel, parametrised by the angular coordinate $x \in [0, 2\pi R)$. The hub h and the spokes s_i contribute to the architecture but do not carry continuous spatial degrees of freedom under the $K_N \to \infty$ refinement — the carrier–envelope decomposition is on spoke variables λ_i indexed by $i \in \mathbb{Z}/N$, which becomes the angular coordinate around the rim. The radial direction (hub-to-rim) is not refined and does not produce a continuum coordinate.

This dimensional limitation is structural to the σ-sector. The sector's physical content is *transport around the closure cycle* — a 1-dimensional process by construction. The σ-sector continuum cannot, on its own, deliver a higher-dimensional field theory; any embedding into higher-dimensional spacetime requires either composition with other VERSF sectors (which carry their own spatial structures) or treating the $K = 7$ architecture itself as a sub-structure of a higher-dimensional substrate. We do not pursue either here.

The dimensional limitation also constrains the §13 discussion: any identification of the σ-sector continuum field with a higher-dimensional persistent gauge sector must face this 1D-vs-higher-D mismatch directly.

11.4 Not Lorentz-invariant

The heat equation $\partial_{\tilde{\tau}}\phi = D\,\partial^2_x\phi$ does not respect any combined $(x, \tilde{\tau})$ Lorentz symmetry. Time and space enter the equation asymmetrically ($\partial_{\tilde{\tau}}$ first-order, ∂_x second-order), and no finite linear coordinate transformation interleaving x and $\tilde{\tau}$ preserves the form of the equation.

This is structurally consistent: the σ-sector is a dissipative response, and dissipative dynamics do not respect time-reversal symmetry, let alone full Lorentz symmetry. Lorentz invariance, if it emerges anywhere in VERSF, must come from a sector where conservative propagating dynamics is natural — typically a sector based on Hamiltonian or symplectic structure, where the equations of motion are second-order in time and time-reversal-symmetric.

The persistent cohomological/gauge transport sector identified elsewhere in the VERSF programme is a candidate for such conservative dynamics, but its relation to the σ-sector continuum limit is not established here. We discuss this in §13.

12. Emergent Transport Geometry: A Speculative Extension

Epistemic status. *This section is a speculative extension, not a derived result.* The construction below is not produced by the master-action variation of the preceding papers; it explores what the continuum σ-sector would look like if spatial inhomogeneity of the discrete coupling were admitted, and what conceptual consequences would follow. The status of the section is **programmatic conjecture**, distinguished sharply from the **derived theorems** of §7–10. We flag this explicitly because the construction relies on a structural move (allowing spatially-varying coupling) that breaks load-bearing assumptions of the preceding paper — specifically the D_N symmetry of the K_N constraint catalogue. We do not take those assumptions back. The section proposes a future direction, not a refinement of the established results.

12.1 The conceptual question

The continuum σ-sector derived in §7–10 is a transport theory on the envelope field $\phi(x, \tilde{\tau})$ on S^1. The transport coefficient D in Theorem 2 was taken constant, corresponding to a substrate whose admissibility structure is uniform around the closure cycle. This uniformity is a direct consequence of the discrete K_N catalogue's D_N symmetry, which the preceding paper established as load-bearing (the D_6-covariance principle (P2)).

The natural conceptual question is: what would the continuum σ-sector look like if substrate transport accessibility were *not* uniform around the cycle — i.e., if different regions of the closure substrate supported coherent transport with different effectiveness? This question does not arise from the $K = 7$ architecture itself (which is uniform by D_6 symmetry), but from imagining a generalised substrate architecture in which D_6 symmetry is broken.

12.2 What spatially-varying D(x) would require

To get D(x) instead of constant D, the discrete catalogue would need to admit position-dependent coupling γ_i at vertex (i, i+1). This is a substantive structural change to the architecture:

- The four-term decomposition of A_cl (Theorem 1 of the preceding paper) was derived under (P2) D_6-covariance. Under broken-symmetry catalogue principles, the decomposition need not hold in the same form; additional sectors potentially become admissible, with new violation functionals associated with the broken symmetry.

- The closure-current conservation law (CC) (Theorem 2 of the preceding paper) was tied to the D_6-symmetric H_1-projection structure. Position-dependent γ_i might or might not preserve the same conservation form.
- The transport-gap reading of the residual D_3(vtx) symmetry (Theorem 4 of the preceding paper) explicitly used D_6-symmetry of the committed states (H1) and of the constraint catalogue. Both are broken by spatial inhomogeneity.

The honest statement is: *the present paper does not derive spatially-varying transport coefficients from the D_6-symmetric $K = 7$ catalogue.* What follows is an exploration of the geometric structure that *would* emerge if the σ-sector were embedded in a broader inhomogeneous substrate architecture, with the structural cost of breaking D_N symmetry left unaddressed.

Layered conjectures. Beyond the speculative postulate of spatial inhomogeneity itself, the construction below relies on the following structural conjectures, none derived here and each non-trivial:

- **(C1) Decomposition closure.** The four-term decomposition of A_cl extends to a position-dependent version of the form A_cl = $\alpha \cdot$ A_circ + $\Sigma_i \gamma_i \cdot$ A_comp,i (plus on-sector terms) *without new sector functionals*. Under broken D_N symmetry, additional admissible local quadratic functionals could in principle appear — for instance, chirality-breaking terms, or higher-derivative terms excluded at finite N by D_N covariance. Their absence under broken symmetry is not established.
- **(C2) Closure-competition form.** Position-dependent closure-competition is $\Sigma_i \gamma_i \cdot (\lambda_i + \lambda_{i+1})^2$ with γ_i varying around the rim and the same nearest-neighbour cross-pair structure as the D_6-symmetric case. Other position-dependent forms — e.g., $\Sigma_{ij} \gamma_{ij} (\lambda_i + \lambda_j)^2$ with non-nearest-neighbour coupling — are not excluded by the broken-symmetry catalogue.
- **(C3) Closure-current conservation preservation.** The closure-current conservation law $\Sigma_i \lambda_i = 0$ holds in the same form despite broken D_N symmetry. The H_1-projection structure depends on the cell complex (which is unchanged) rather than directly on symmetry, so this is plausible — but its compatibility with the broken-symmetry catalogue is not derived.
- **(C4) Carrier–envelope decomposition validity.** The decomposition $\lambda_i = (-1)^i \psi_i$ remains the appropriate factorisation. Under broken symmetry, the alternating mode may not be the correct Fourier mode to factor out; the operator-diagonalising transformation could differ from carrier–envelope, with structural consequences for the continuum-limit analysis.

The construction's status is therefore **conjecture upon enumerated conjectures**: the speculative postulate (D_6-symmetry breaking, with spatially-varying γ_i) is one layer; (C1)–(C4) are a second layer of structural assumptions about how the broken-symmetry catalogue would behave. The geometric interpretation that follows is contingent on all five layers (postulate plus (C1)–(C4)).

12.3 The Laplace–Beltrami structure under spatially-varying D(x)

Under the speculative postulate that the σ-sector continuum admits a spatially-varying transport coefficient D(x), the constitutive flow of Theorem 2 generalises to

$$\partial_{\tilde{\tau}} \phi = \partial_x (D(x) \cdot \partial_x \phi). \quad (12.1)$$

Equation (12.1) is the divergence-form generalisation of the constant-coefficient heat equation. In differential geometry, diffusion on a Riemannian manifold with metric g_{ij} is governed by the Laplace–Beltrami operator

$$\Delta_g \phi = |g|^{-1/2} \cdot \partial_i (|g|^{1/2} g^{ij} \cdot \partial_j \phi). \quad (12.2)$$

Equation (12.1) is the one-dimensional instance of (12.2) under the identification $g^{xx}(x) = D(x)$, or equivalently $g_{xx}(x) = 1/D(x)$. The transport coefficient D(x) therefore plays the role of an effective inverse metric weight governing admissible transport accessibility — *under the speculative postulate*.

12.4 The mechanism the σ-sector establishes — and the limits of what it establishes

The conceptual move here is: under the speculative postulate, *effective metric structure emerges from substrate transport accessibility*. Regions of high closure coherence (large D(x)) permit easier admissible transport; regions of low closure coherence (small D(x)) resist it. The effective distance between regions is not fundamental but derived from the substrate cost of maintaining coherent transport — a transport-defined effective metric on the envelope sector.

This is a conceptually interesting reversal of standard geometry-first physics, where space is assumed and fields propagate through it. The σ-sector mechanism, under the speculative extension, suggests effective spatial structure can emerge from the organisation of admissible transport itself.

Four important limitations must be acknowledged, classified by the type of obstacle each represents.

Limitation 1 (Intrinsic content I — curvature triviality). In one dimension, any Riemannian metric is conformally flat: a 1D Riemannian manifold is determined up to diffeomorphism by its total length, with no intrinsic curvature. The "emergent geometry" in the σ-sector, even under the speculative postulate, is therefore not a genuinely geometric structure in the higher-dimensional sense — there is no curvature, no Einstein-type dynamics, no coupling to stress-energy. The standard geometric content of "emergent geometry" (Riemann tensor, parallel transport, geodesic deviation) is absent in 1D.

Limitation 2 (Intrinsic content II — coordinate-invariance and scalar reduction). The geometric content of D(x) in 1D is even more limited than Limitation 1 alone suggests. The divergence-form heat equation $\partial_{\tilde{\tau}}\phi = \partial_x(D(x)\, \partial_x\phi)$ on S^1 is equivalent, under the combined transformation

$$y(x) = \int_0^x dx'/\sqrt{D(x')}, \quad \tilde{\phi}(y, \tilde{\tau}) = \sqrt{D(x(y))} \cdot \phi(x(y), \tilde{\tau}),$$

to the standard heat equation $\partial_{\tilde{\tau}} \tilde{\phi} = \partial^2_y \tilde{\phi}$ *on* S^1 *of effective circumference*

$L_{\mathrm{eff}} = \oint_{S^1} dx/\sqrt{D(x)}$.

The full content of $D(x)$ reduces, in 1D, to a single scalar invariant — L_{eff} — with the rest absorbed into the combined coordinate-and-field transformation. Two distinct $D(x)$ profiles with the same L_{eff} give equivalent dynamics; a single constant D_0 with appropriate value also gives equivalent dynamics. The eigenvalue spectrum of the operator $\partial_x(D\,\partial_x)$ on S^1 depends on $D(x)$ only through L_{eff} (it equals the spectrum of ∂^2_y on a circle of circumference L_{eff}).

The "emergent metric" interpretation is therefore geometrically vacuous in 1D beyond this single scalar. The σ-sector mechanism is not yet established as substantive even under the speculative postulate; what one is doing under spatially-varying $D(x)$ in 1D amounts to choosing a coordinate on S^1 (combined with a corresponding field rescaling). The interpretation has interest only as a hint about what the analogous construction in higher dimensions, where Riemannian metrics carry curvature and other invariants beyond a single scalar, might deliver.

Limitation 3 (Derivational route — non-trivial structural input). The construction requires breaking the D_N symmetry of the K_N catalogue, which was load-bearing in the preceding paper, and additionally relies on the four conjectures (C1)–(C4) enumerated in §12.2. The cost of this breaking and the validity of the conjectures are not addressed here. What new terms appear in A_{cl} under broken symmetry? Are they small or large compared to the D_6-symmetric sectors? Does the carrier–envelope decomposition still produce the correct continuum limit? All open.

Limitation 4 (Dimensional scope — 1D restriction). The σ-sector is intrinsically 1-dimensional. §11.3 established this: the σ-sector continuum lives on S^1, the rim of the wheel. The emergent-geometry mechanism therefore cannot, by the σ-sector alone, deliver higher-dimensional geometric structure. Whatever higher-dimensional emergent geometry the VERSF programme ultimately produces must come from composition with sectors carrying higher spatial dimension — not from the σ-sector in isolation.

12.5 What the σ-sector does establish

What the σ-sector establishes — even taking the speculative extension at face value, and bearing in mind that Limitation 2 above renders the 1D geometric interpretation vacuous beyond a single scalar — is at most a *suggestive analogy* between substrate transport accessibility and effective metric structure. The analogy is interesting as a hint about what the corresponding higher-dimensional construction might deliver, where Riemannian metrics carry curvature and other invariants beyond a single scalar. In 1D the analogy is geometrically vacuous; its interest is exclusively prospective.

The structural correspondences making up the analogy are of three different epistemic types, and worth distinguishing:

Correspondence	***Epistemic type***
Admissibility-restoring transport ⟹ heat-equation dynamics	*Derived (Theorem 2, this paper)*
Uniform γ (D_6-symmetric catalogue) ⟹ constant D	*Derived (Theorem 2, this paper)*
Spatially-varying γ_i (broken D_N symmetry) ⟹ D(x)	*Conjectured under (C1)–(C4) (§12.2)*
D(x) ⟹ inverse metric weight g^{xx}(x) in 1D Laplace–Beltrami structure	*Formal mathematical identification (with caveats from Limitation 2)*
D(x) ⟹ "effective distance"	*Interpretive overlay*

The first two rows are theorems of this paper. The third row is the speculative extension's load-bearing conjecture, contingent on (C1)–(C4). The fourth row is a formal identification of the divergence-form operator with the (1D) Laplace–Beltrami operator — well-defined as a formal identification, but geometrically reducible to a single scalar under coordinate-and-field transformation (Limitation 2). The fifth row is the interpretive language that motivates the section.

The σ-sector therefore does not derive higher-dimensional geometry, and does not — even under the speculative postulate plus all four conjectures (C1)–(C4) — establish a substantively geometric pattern in 1D. What it establishes is the structural template for an analogy whose substantive content would emerge only in higher dimensions. Whether that template is the right one for VERSF's higher-dimensional sectors, or whether the analogy turns out to be misleading once higher-dimensional sectors are properly constructed, is not resolved here.

What this section does not establish:

- Spatially-varying D(x) as a derived consequence of the K = 7 master-action variation. (It is not; the master-action variation produces constant D.)
- The structural consistency of broken D_N symmetry with the preceding paper's Theorem 1 catalogue. (Not addressed; conjectured via (C1)–(C4).)
- Any higher-dimensional geometric structure beyond the 1D rim metric. (The σ-sector is structurally 1D.)
- Intrinsic geometric content in 1D beyond the single scalar L_eff. (Limitation 2: the geometric interpretation reduces to coordinate choice on S^1.)
- Einstein-type dynamics of the emergent metric. (Not present; D(x) is a postulated coefficient, not a dynamical field.)
- Coupling of the emergent metric to matter sectors. (Not addressed.)

What it does establish:

- The formal correspondence between the σ-sector's continuum dynamics under spatially-varying D(x) and the 1D Laplace–Beltrami operator with metric g_{xx}(x) = 1/D(x), as a mathematical identification (with the geometric content of that identification reducing to a single scalar in 1D).

- A *prospective hint* about how the analogous construction in higher dimensions — where Riemannian metrics carry non-trivial invariants — might give substantive emergent-geometric content. The σ-sector itself does not deliver this content; it provides only the template by which a higher-dimensional construction might be framed.
- A specific open programme (P11 below) for whether the template has substantive content beyond coordinate freedom even in 1D, and for whether the analogous higher-dimensional construction can be carried out within VERSF.

The σ-sector therefore does not derive emergent geometry. It establishes a structural template whose substantive geometric content, if any, remains contingent on (i) the resolution of (C1)–(C4); (ii) the question of whether the 1D coordinate-invariance of Limitation 2 dissolves the interpretation entirely; and (iii) the eventual construction of higher-dimensional VERSF sectors where the template's analogue can be tested.

---

13. Relation to the Persistent Cohomological/Gauge Sector

The broader VERSF programme has, in separate work, identified a persistent cohomological/gauge transport sector with structural features (transport currents, quotient structures, conservation laws, refinement persistence, constitutive dynamics) that bear formal similarity to the σ-sector continuum limit established in this paper.

The natural conjecture is that the σ-sector continuum field $\phi(x, \tilde{\tau})$ and the persistent cohomological/gauge sector are different descriptions of the same underlying physical object. We do not establish this here, and we want to flag four reasons why the identification is non-trivial:

First, the σ-sector continuum theory is *parabolic* (heat equation); the persistent gauge sector, in any conventional VERSF formulation, is *hyperbolic* (Maxwell-like). The two have different mathematical character. An identification would require either:

- Showing that the persistent gauge sector arises from the σ-sector at a different time scale (e.g., long-time persistent modes giving conservative dynamics);
- Or showing that the σ-sector continuum limit, combined with additional structure from elsewhere in VERSF, produces conservative dynamics in a specific regime.

Second, the σ-sector continuum theory has a one-dimensional spatial domain (S^1, the outer rim of the wheel). The persistent gauge sector lives on a higher-dimensional spacetime. An identification would require either:

- Embedding S^1 in a higher-dimensional space (the spatial rim is not the full physical space, only the σ-sector's restricted sub-space);
- Or showing that the higher-dimensional structure emerges from compositions of σ-sectors across multiple substrate sub-architectures.

Third, the σ-sector continuum is dissipative; the persistent gauge sector is conservative. This is the same issue as the first, restated.

Fourth, the kernel structure: the σ-sector continuum kernel is the constant ϕ direction (one-dimensional). The persistent gauge sector kernel is typically the H^1-cohomology direction (also one-dimensional for the K = 7 architecture, $\mathbb{Z}\langle[C]\rangle$). The two are formally analogous, but their structural origins are different (kernel of ∂^2_x vs. cohomology class of a complex), and the identification requires showing the two have the same structural content.

We list this conjectured identification as an open problem (P2 below, inherited from the preceding paper). The continuum bridge established here is the necessary precondition for *making* the identification rigorous; the identification itself is left to subsequent work.

---

14. What This Establishes

The paper establishes, via the K_N → ∞ refinement framework with carrier–envelope decomposition:

1. **The σ-sector has a well-defined continuum field theory.** Under the explicit scoping decisions (L1)–(L3) of §3, the discrete admissibility-restoring flow on the K_N spoke cycle converges to the heat equation on a continuum envelope field $\phi(x, \tilde{\tau})$ on the rim circle S^1.
2. **The continuum effective action is a pure gradient-energy functional** (Theorem 1). $\text{A\_eff}[\phi] = \kappa \int (\partial_x \phi)^2 \, dx$ is the leading-order continuum image of A_comp under the K_N refinement with the chosen scaling conventions; there is no continuum mass term, consistent with the absence of any λ^2 penalty in the discrete A_cl. Higher-derivative corrections at sub-leading order in a are not excluded by this result (P9).
3. **The continuum dynamics is the heat equation** (Theorem 2). The discrete signless Laplacian M_N, transformed under the carrier–envelope decomposition into the standard Laplacian L_N, has continuum image $-\partial^2_x$. The kernel of the continuum flow (constant ϕ) is the image of the discrete kernel (alternating λ) under the carrier transformation.
4. **Closure-current conservation is realised in the continuum as automatic suppression of Brillouin-edge components** (Theorem 3). For slowly-varying envelope fields, the discrete constraint $\Sigma\lambda_i = 0$ holds to exponential accuracy in a; the continuum conserved charge is $Q[\phi] = \int \phi \, dx$.
5. **The alternating mode is identified as the Brillouin-zone-edge carrier wave** (Theorem 4). Under the K_N refinement, the discrete alternating mode of K_N corresponds to the plane wave $\exp(i\pi x/a)$ at the lattice's Brillouin edge; the carrier–envelope decomposition factors this carrier out and exposes the envelope as the continuum dynamical field.
6. **The continuum theory is parabolic and dissipative**, not hyperbolic or conservative (§11). This is consistent with the σ-sector being the substrate's admissibility-restoring response, intrinsically dissipative. Lorentz invariance is not present in the continuum σ-sector and must come from a different sector if it emerges anywhere in VERSF.

What has *not* been established here:

- The full identification of the σ-sector continuum field with the persistent cohomological/gauge transport sector of the broader VERSF programme. Formal commonalities exist (§13), but the identification is left open.
- Emergence of Lorentz invariance from the σ-sector alone (it doesn't emerge; see §11.4).
- Non-Abelian generalisations of the σ-sector continuum limit.
- Coupling of the σ-sector continuum field to matter sectors.
- Higher-order EFT corrections to the leading heat-equation dynamics.
- The status of the K_N refinement framework as a *physical* claim (the K_N family is treated as methodological scaffolding for the continuum limit; whether the broader programme commits to an underlying K_N-family interpretation is a separate question).

The σ-sector is therefore upgraded from a discrete constitutively-derived transport response to a continuum dissipative field theory with explicit dynamics, conservation law, and structural identification of its persistent mode.

---

15. Remaining Open Problems

P2 (load-bearing). Master-action unification of the σ-sector and the persistent cohomological/gauge transport sector. The continuum bridge established here is a necessary precondition; the unification itself requires composing the parabolic σ-sector with a conservative sector under one variational principle (§13). The emergent-geometry speculative extension of §12, if developed in directions where the D_N symmetry-breaking is rigorously addressed, may also bear on this unification.

P3 (now partially addressed). The continuum-limit derivation is established in this paper under the K_N refinement with carrier–envelope decomposition (Limit-A + Limit-C). Alternative continuum limits (Limit-B alone, or non-carrier framings) remain unstudied. The relation between the chosen framing and alternative framings is an open question.

P4. Non-Abelian transport sectors. The K = 7 architecture is Abelian in its homological structure ($H_1 \cong \mathbb{Z}$); non-Abelian generalisations would have different cycle structures and presumably different carrier–envelope decompositions. The methodology of this paper does not extend directly.

P5. Coupling to matter sectors. The σ-sector continuum field is a candidate for coupling to matter sub-sectors of the master action, but the coupling structure is not derived here.

P6. Numerical coefficient ratio α/γ (inherited from the preceding paper). The continuum limit has a single coupling $\kappa = \lim \gamma \cdot a$; its value is not determined by the analysis of this paper.

P7. Vertex × tick-window σ-duality (inherited from the original σ-family paper). The continuum framework may provide new tools for addressing this question, but is not pursued here.

P8 (new). Emergence of Lorentz invariance from compositions of VERSF sectors. The σ-sector continuum is parabolic and not Lorentz-invariant (§11.4); the persistent gauge sector is hyperbolic and conservative. Whether Lorentz invariance emerges from their composition, or from a sector not yet identified, is the natural follow-up question.

P9 (new). Higher-order EFT corrections. The continuum limit of A_comp produces a leading-order $\int(\partial_x\phi)^2$ dx term; sub-leading corrections in a give higher-derivative and non-linear terms that modify the heat equation at short scales. Identifying these corrections and their structural content is open.

P10 (new). Physical status of the K_N family. This paper treats K_N as methodological scaffolding for the continuum limit; whether VERSF commits to a K_N-family interpretation (with K = 7 as a discrete observable instance) or treats K = 7 as the fundamental architecture with K_N as a calculational fiction is an interpretive question of the broader programme — and one with structural consequences. If K = 7 is fundamental and K_N is calculational, then the higher-derivative corrections of P9 should be reinterpreted as *finite-N corrections* to the K = 7 result rather than as features of an underlying continuum theory; the continuum limit is then a methodological device for extracting the K = 7 leading dynamics. If K_N (with $N \to \infty$) is closer to the physical truth and K = 7 is a constrained sub-architecture, then the higher-derivative corrections may have continuum physical content. The choice determines how Theorems 1–4 are read.

P11 (new). Substantive development of the emergent-geometry speculative extension (§12). Three load-bearing sub-questions:

(P11a) Catalogue extension. The Laplace–Beltrami interpretation requires breaking the D_N symmetry of the K_N catalogue, which was load-bearing in the preceding paper. What discrete architecture admits spatially-varying γ_i while preserving the other constrained-EFT principles? What new admissibility sectors and constraint catalogue elements appear under the broken symmetry, and how do they modify A_cl at leading order? Until these are addressed, conjectures (C1)–(C4) of §12.2 are unresolved.

(P11b) Coordinate-invariance dissolution test. Even granted (C1)–(C4), is the resulting continuum theory coordinate-equivalent to the constant-D case in any sense beyond the combined transformation of Limitation 2? Specifically: is there an intrinsic dynamical content of D(x) in 1D that survives the $(y, \tilde{\phi})$ transformation, or does the whole construction reduce to a single scalar L_eff? If the latter, P11a becomes a 1D curiosity and the entire §12 programme depends on lifting to higher dimensions; if the former, §12 has substantive 1D content that warrants development in its own right. *This question is load-bearing for whether §12 has any 1D content beyond coordinate choice.*

(P11c) Higher-dimensional lifting. Extending to higher spatial dimensions — where Riemannian metrics carry curvature, anisotropy, non-trivial topology, and other invariants beyond a single scalar — requires composition with sectors carrying higher-dimensional spatial structure, and is not accessible from the σ-sector alone. The natural setting is the conjectured master-action unification (P2), where the σ-sector composes with sectors of higher dimension; the emergent-

geometry template of §12 may then have substantive geometric realisation. Until that composition is performed, the §12 programme is structurally limited to 1D and (per P11b) potentially limited to vacuous-in-1D status.

16. Conclusion

The σ-family began as a categorical transport structure in the K = 7 closure architecture, was upgraded to a constitutive admissibility-derived discrete transport response under master-action variation, and is upgraded here to a continuum dissipative field theory under the K_N → ∞ refinement with carrier–envelope decomposition.

The continuum image of the σ-sector is a heat equation on the envelope of an alternating Brillouin-edge carrier wave. The persistent direction of the discrete σ-sector (alternating spoke mode) corresponds to the kernel of the continuum heat equation (constant envelope). Closure-current conservation translates to automatic suppression of Brillouin-edge components in slowly-varying envelope fields, with the continuum conserved charge being the total envelope integral. The continuum theory is parabolic, dissipative, constitutive, and not Lorentz-invariant.

The σ-sector therefore *does* have a continuum field-theoretic image — but that image is a dissipative response theory, not a propagating field theory. This is structurally consistent with the σ-sector being the substrate's admissibility-restoring response, and it places the σ-sector in the appropriate structural position within the broader VERSF programme: the dissipative response sector, complementary to (but not identical with) the conservative persistent transport sector.

What remains is the unification (P2). The σ-sector continuum field theory established here is the bridge required to make that unification precise. If the σ-sector and the persistent cohomological/gauge sector are two descriptions of the same underlying physical content — operating at different time scales or under different scaling regimes — the continuum framework of this paper is the natural language in which to formulate the identification.

The substrate's admissibility-restoring response, derived discretely from the K = 7 master action and brought to the continuum here, is a heat-equation-like dissipative dynamics for the envelope of a Brillouin-edge carrier. The σ-family's continuum image is now established; the next step is to compose it with the rest of the VERSF transport architecture.