Transport-Action Conformal Rescaling of the Stage V Continuum Metric in VERSF

Conditional Hessian Construction, Leading-Conformal Levi-Civita Connection, and Scalar-Level Structural Compatibility with Refinement-Stable Transport-Curvature Density

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General-Reader Summary

The VERSF programme is built around the idea that the smooth, continuous reality we experience emerges from a much simpler underlying layer — a vast lattice of points carrying a small set of internal states that update by a fixed local rule. Previous papers in the programme have built this picture up step by step. They show that the lattice produces a smooth continuum at large scales; that disturbances in this continuum produce localised "defect" regions with measurable effects; that information can propagate across the lattice with a finite speed; that paths through the lattice carry information, with closed loops around defects accumulating measurable memory; and that all of these effects survive when the lattice is refined to finer and finer scales.

This paper asks one further question.

Does the lattice's structure determine a notion of distance and geometry in the everyday sense — a metric — beyond just the path-information of the previous paper?

The answer this paper develops is partial and honest. The lattice's "transport action" — a quantity that measures the cost of moving information from one place to another — naturally produces a notion of distance, and from that distance one can extract a metric in the standard geometric sense. The metric this produces is not arbitrary: it takes a specific form, a conformal rescaling of an underlying flat metric by the inverse-squared coherence gap field. That is the main constructive content of the paper.

The honest framing requires acknowledging an explicit assumption. Previous papers established that the lattice's vacuum is *transport-flat*: information propagates along any path the same way. The present paper takes a stronger working assumption — that this transport-flatness goes together with ordinary geometric flatness in the everyday sense of distance and shape. These are different structural features that the programme has not so far linked, and it is not obvious that they coincide. The most natural reading is that geometric-flatness is a separate foundational feature of the programme, not something that can be derived from transport-flatness via the existing machinery.

Under this honest reading, the picture shifts somewhat. Rather than "metric emerges from the substrate", the more accurate description is that the substrate has *both* metric and transport as parallel emergent features, with the present paper identifying the specific conformal-factor relationship between them. That relationship is genuinely new content — it is what this paper contributes — but it is a relationship between two structures, not the derivation of one from the other.

With this assumption — geometric flatness of the vacuum, formally (Flat-V) — in place, the rescaled metric inherits all the right properties (refinement-stability, universality, defect-localised curvature), and it agrees in part with the transport-curvature density of the previous paper on a single matching piece. Another piece of the metric's curvature content is structurally distinct from anything the transport-curvature construction can produce, and this structural separation appears irreducible: no extension of the transport-curvature construction can capture it, by an algebraic-symmetry argument we make explicit.

What this paper does not do is equally important. It does not derive gravity. It does not establish that the geometry has the four-dimensional structure of relativity. And it does not justify the geometric-flatness assumption from more fundamental principles — that assumption appears most plausibly foundational. Bridging it via a coarse-grained-emergence route is identified as a separate major programme target, not undertaken here.

The honest summary: this paper takes a definite step toward a metric geometry related to the lattice's transport structure, but the step rests on an assumption that the programme most likely has to accept as foundational rather than derive. The conditional result is real — and it identifies, for the first time in the programme, the specific quantitative form of the relationship between metric and transport — but it is a relationship-identification rather than a derivation of one from the other.

Abstract

Earlier papers established: Stage V (Lipschitz continuum emergence on X_∞), Stage VII (open universality class $\mathcal{C}_{K=7}$), Stage VIII (localised coherence defects with the four ε_{gap}-functionals — the Defect-Coherence Principle), the *Global Refinement Transport* paper (coupled transport operator $\mathbf{T} = \hat{T} \otimes I + \gamma \cdot I \otimes A_X$, bulk bands, finite coherence velocity v_c, Birman–Schwinger trapped modes, Combes–Thomas localisation length ξ, basic scaling identity $\xi \cdot \delta \sim v_c$), the *Tensorial Transport Geometry* paper (parallel transport, holonomy $\mathscr{H}(\gamma)$, transport-curvature tensor $\mathscr{R}_{ij}$, geodesic deviation, Wilson loops, extended scaling $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}| / \xi$ — in particular, parallel-transport flatness of the canonical vacuum at the substrate transport level, which is a structural property of the substrate transport operator and *not* the same as Riemannian-metric flatness of an intrinsic continuum metric on X_∞), and the *Refinement-Stable Holonomy* paper (continuum-emergent loop functional $\hat{W}_\infty^{sub}(\gamma)$, continuum transport-curvature tensor $\mathscr{R}_{ij}^{(\infty)}$, continuum transport-curvature density $\kappa(x)$, Stage VII universality-class stability of all continuum-limit transport observables).

The present paper develops the next structural layer at *leading conformal order only*, conditional on an explicit additional hypothesis (Flat-V) that the Stage V vacuum continuum is Riemannian-flat with metric δ_{ij}. This hypothesis is *not* derivable from prior-paper content via direct bundle identification: the K = 7 transport bundle (7-dim, internal closure-catalogue states) and the spatial tangent bundle (3-dim, directions of substrate motion) are structurally separated by construction in the programme, occupying different conceptual roles. The realistic bridging route, if any, would be a coarse-grained-emergence argument for a 3-dim Riemannian metric independent of K = 7 bundle identification — recorded as OP-13, a separate major programme target. (Flat-V) is therefore most plausibly read as a foundational feature of the programme rather than a derivable theorem. Under this reading, the programme's spatial-geometric content fragments into two parallel emergent features — metric and transport — with the present paper's content being the identification of the conformal-factor relationship between them, not the derivation of one from the other.

Conditional on (Flat-V), starting from the continuum coherence-transport action

$\mathcal{S}[\pi] := \int_\pi ds / \varepsilon_{gap}^{(\infty)}(x)$

(a length integral on X_∞ with conformal factor $1/\varepsilon_{gap}^{(\infty)}$, with ds the Stage V length element), we define the continuum coherence distance $d_{coh}(x, y) := \inf\{\mathcal{S}[\pi]\}$, establish its refinement stability (Theorem 2.4), and define the **emergent metric tensor** $g_{ij}(x)$ almost-everywhere on X_∞ as the leading quadratic Taylor coefficient of d_{coh}^2 at the diagonal (Definition 3.1, Lemma 3.1.5, Proposition 3.1.6). The principal results:

- **A.e. emergent metric.** Under (Flat-V), $g_{ij}(x) = \delta_{ij}/\varepsilon_{gap}^{(\infty)}(x)^2$ a.e. on X_∞, symmetric, positive-definite in the weak-coupling regime (Theorems 3.2, 3.4).
- **Refinement-independence.** $g_{ij}^{(\infty)}$ is independent of the admissible substrate refinement sequence (Theorem 3.5).
- **Effective Levi-Civita connection.** Γ_{ij}^k satisfies $\nabla_k g_{ij} = 0$ exactly by Koszul uniqueness (Theorem 4.3).
- **Transport-generator agreement on scalar amplitudes.** $\tilde{\nabla}_i^{(\infty)} \psi = \nabla_i^{(g)} \psi + \mathcal{O}(\alpha) + \mathcal{O}(\gamma^2)$ for scalar coherence-state amplitudes in bulk regions, restricted to the canonical Stage VII trivialisation on bounded closed contractible coordinate patches (Theorem 4.4).
- **Leading-conformal geodesic coincidence** (Proposition 5.2).
- **Single-piece scalar-level structural compatibility with refinement-stable transport-curvature density.** The d = 3 conformal Ricci scalar $R(g)(x) = (1/\varepsilon_{gap}^2)\cdot[4\cdot\Delta \log \varepsilon_{gap}^{(\infty)} + 2\cdot|\nabla \log \varepsilon_{gap}^{(\infty)}|^2]$ and κ(x) match in defect-boundary support, Combes–Thomas decay outside, vacuum vanishing, and at the level of explicit dimensional analysis (§6.3) on a *single matched piece*: the gradient-squared content of R(g) at α^2/ξ_∞^2 versus κ at α^2/ξ_∞^2. The Laplacian content of R(g) at α/ξ_∞^2 — one power of α lower than κ, and the dominant content of R(g) for small α — is structurally distinct from any scalar functional of $\mathcal{R}_{ij}^{(\infty)}$, by an algebraic-symmetry argument (the symmetric-source character of Laplacian content vs the antisymmetric character of any $\mathcal{R}$-derived scalar). The obstruction is fundamental: no extension of the κ-construction or any other scalar projection of $\mathcal{R}_{ij}^{(\infty)}$ can capture R(g)'s Laplacian content. R(g) and κ are *structurally-distinct scalars sharing one matched piece*, not compatible scalars at

matching orders. The comparison is scalar-level only — tensor-level identification is forbidden by Ric(g)-symmetric / $\mathscr{R}$-antisymmetric algebraic mismatch.

We do not derive Einstein equations, Lorentzian signature, stress-energy, or gauge structure. We do not construct subleading non-conformal corrections to g_{ij} (deferred to OP-11). We do not justify (Flat-V): the K = 7-bundle-identification route faces a structural separation in the programme; the coarse-grained-emergence alternative (OP-13) is a separate major programme. The OP-2 metric-emergence target of the previous paper is closed at the *conditional-on-(Flat-V), leading-conformal-factor, single-piece-scalar-compatibility, parallel-emergent-features-relationship* register.

Contents

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1. Introduction

1.1 The Metric-Emergence Question, Honestly Framed

The previous paper recorded OP-2 (Metric tensor emergence) as the natural next target. A direct reading suggests this paper should establish "metric structure emerging from the substrate." The honest reading — clarified by careful audit of what prior papers actually establish and what the present construction actually requires — is more conditional, and the framing of "emergence" itself needs adjustment.

Stage V delivers a Lipschitz continuum X_∞ on which the previous papers' transport observables are constructed. The *Tensorial Transport Geometry* paper establishes that this continuum is *parallel-transport flat* in the canonical vacuum: substrate parallel transport along any path is path-independent (its Proposition 2.5), the transport-curvature tensor $\mathscr{R}_{ij}$ vanishes identically in the vacuum (its Proposition 5.2(a)), and the substrate is "flat in the parallel-transport sense" (its Definition 2.6 — flatness is defined there specifically as path-independence of substrate parallel transport along homotopic paths).

This parallel-transport flatness is *not* the same as Riemannian-flatness of a continuum metric on X_∞. The prior paper is explicit about this — its §5.2 ("Three levels of curvature structure") states: *"Metric curvature [would require] emergence of a genuine metric tensor $g_{\mu\nu}$, an affine connection compatible with the metric (Levi-Civita or otherwise), and continuum covariance under change of coordinate frame. This layer is not constructed in the present paper..."* and *"It is important not to overread $\mathscr{R}_{ij}$ as a Riemann tensor analogue in any strict sense: it has substrate-direction indices rather than spacetime indices, takes operator values on a closure-catalogue space rather than acting on tangent vectors, and is not derived from a metric."*

The two notions of flatness — transport-level and Riemannian-metric-level — concern *different structures* in the programme:

- The transport-level notion is about the substrate transport operator **T**, its directional generators $\tilde{\nabla}_i$, and the closure-catalogue (K = 7) fibre at each substrate position. Flatness here means path-independence of substrate parallel transport in the canonical vacuum.
- The Riemannian-metric-level notion is about the continuum tangent bundle of X_∞ (3-dimensional, capturing *directions of motion* on the substrate), the metric tensor g_{ij} on that tangent bundle, and the Levi-Civita connection of that metric. Flatness here means the metric is the standard 3-dim Euclidean metric δ_{ij} on bounded coordinate patches.

These two structures occupy *different conceptual roles* in the programme by construction. The K = 7 fibre encodes the substrate's internal closure-catalogue states at each position (a 7-dimensional internal-state space). The tangent bundle of X_∞ encodes spatial directions of motion (3-dimensional). There is no reason to expect a structural identification between the two, and the prior paper makes no such identification: its $\mathscr{R}_{ij}$ acts on the K = 7 closure-catalogue fibre, not on the tangent bundle. A bridging argument from parallel-transport flatness to Riemannian-flatness would require identifying the K = 7 bundle with the tangent bundle (or extracting a 3-dim sub-bundle of K = 7 that *is* the tangent bundle) — neither of which is in prior content, neither of which appears structurally plausible given the conceptual separation of the two bundle structures.

The realistic alternative bridging route, if any exists, would be a *coarse-grained-emergence* argument: derive a 3-dim Riemannian metric on X_∞ directly from substrate dynamics at large scales, independent of K = 7 bundle identification (OP-13).

(Flat-V) is therefore most plausibly read as a *foundational feature of the programme*, taken as primitive at the spatial-continuum-emergence level. We adopt it explicitly as a standing hypothesis (§1.6) and conduct all subsequent constructions conditional on it.

Preview of the architectural consequence. Under the foundational reading of (Flat-V), the present paper's content is properly described as *relationship-identification* between two parallel emergent features — an independently emergent flat continuum metric (Stage V + (Flat-V)) and independently emergent transport structure (Stage IX + transport-geometry papers) — rather than as metric derivation from substrate transport. The conformal factor $1/\varepsilon\_gap^2$ is the specific quantitative form of the relationship between these two structures, not a mechanism by which one is built from the other. This reframing — which has implications for how the paper's technical content in §§2–7 should be read — is developed in §8.4.1; we surface it here in §1.1 so the reader can approach the technical sections under the relationship-identification framing from the outset rather than retroactively adjusting their reading at §8.

With this framing in place, the present paper establishes at leading conformal order:

The substrate's transport action $\mathcal{S}[\pi]$ of the previous papers, integrated against the (Flat-V) flat continuum metric δ_{ij}, determines a specific conformal rescaling $g_{ij} = \delta_{ij}/\varepsilon\_gap^2$. The conformal factor $1/\varepsilon\_gap^2$ is derived directly from the transport action's structural form. The conformally rescaled metric inherits refinement-stability and universality-class stability from the underlying continuum transport observables. It acquires non-trivial defect-localised curvature content through the spatial profile of $\varepsilon\_gap^{(\infty)}$ near Stage VIII defects. And its Ricci scalar R(g) is scalar-level structurally compatible with the refinement-stable transport-curvature density κ(x) of the previous paper on one specific matched piece — with the structural separation of the unmatched piece, by an algebraic-symmetry argument, made explicit (§6.3, Remark 6.3.1).

The *substantively new metric content beyond conformal rescaling* — subleading non-conformal corrections to g_{ij} arising from the substrate's anisotropic defect-boundary structure — is the principal target of the next paper in the programme (OP-11) and is not constructed here.

1.2 Why the Conformal Form Is Natural

The transport-action provenance of $1/\varepsilon\_gap^2$ is natural in the following sense. Under (Flat-V), the continuum transport action

$$\mathcal{S}[\pi] = \int_\pi ds \,/\, \varepsilon\_gap^{(\infty)}(x)$$

is a length integral on (X_∞, δ_{ij}) with conformal factor $1/\varepsilon\_gap^{(\infty)}$. Its minimising paths are the geodesics of the rescaled metric $g_{ij} = \delta_{ij}/\varepsilon\_gap^2$, and the squared-distance functional d_coh^2 has leading second-variation coefficient at the diagonal equal to g_{ij}. The conformal factor $1/\varepsilon\_gap^2$ is therefore uniquely determined by requiring $\mathcal{S}$ to coincide with a length integral on the rescaled geometry.

1.3 The Squared-Distance Hessian Construction

A technical point. The metric tensor is *not* the Hessian of distance — $d(x, x + \varepsilon\xi) \sim \varepsilon$ is only Lipschitz in ε, not C^2. The correct construction uses *squared* distance:

$$d^2(x, x + \varepsilon\xi) = g_{ij}(x) \cdot \varepsilon^2 \cdot \xi^i \cdot \xi^j + \mathcal{O}(\varepsilon^3),$$

so $g_{ij}(x)$ is the leading quadratic Taylor coefficient of $d^2(x, y)$ at $y = x$. Explicit derivation: Lemma 3.1.5.

1.4 What This Paper Establishes (Conditional on (Flat-V))

1. **Refinement-stable continuum coherence distance** $d_{coh}(x, y)$ on X_∞ — Definition 2.2, Theorem 2.4.
2. **A.e.-existing emergent metric tensor** $g_{ij}(x) = \delta_{ij}/\varepsilon_{gap}^{(\infty)}(x)^2$ — Definition 3.1, Lemma 3.1.5, Proposition 3.1.6.
3. **Symmetry, positivity, refinement-independence, universality stability** — Theorems 3.2, 3.4, 3.5, 7.1.
4. **Effective Levi-Civita connection with Koszul-exact metric compatibility** — Theorem 4.3.
5. **Transport-generator agreement on scalar amplitudes in bulk** — Theorem 4.4.
6. **Leading-conformal-order geodesic coincidence** — Proposition 5.2.
7. **Single-piece scalar-level structural compatibility with explicit dimensional analysis** — Theorem 6.2, §6.3.

1.5 What This Paper Does Not Do

- It does not derive Einstein equations, Lorentzian signature, stress-energy, gauge structure, or quantisation.
- It does not justify (Flat-V) from prior-paper content. The K = 7-bundle-identification route is structurally implausible (different conceptual bundles); the coarse-grained-emergence route (OP-13) is a separate major programme target not undertaken here. (Flat-V) is most plausibly foundational.
- It does not construct subleading non-conformal corrections to g_{ij} (OP-11).
- It does not identify Riem(g) with $\mathscr{R}_{ij}^{(\infty)}$ at the (0,2)-tensor level — algebraic symmetries (Ric symmetric, $\mathscr{R}$ antisymmetric) forbid it.
- It does not match R(g) and κ at full leading order — the Laplacian content of R(g) is structurally distinct from any scalar functional of $\mathscr{R}_{ij}^{(\infty)}$, not just from the loop-functional construction of κ (Remark 6.3.1). Theorem 6.2's compatibility is restricted to a single matched piece.
- It does not derive the metric from the transport. Under the foundational reading of (Flat-V), the metric and the transport are parallel emergent features, and this paper identifies their conformal-factor relationship; it does not derive the existence of either from the other.
- It does not address strong-coupling regimes. The Stage IX weak-coupling regime (W) is assumed throughout.

1.6 Notation and Standing Conventions

- **Refinement level** indexed by $n \in \mathbb{N}_0$; X_n level-n substrate; X_∞ continuum limit; h_n cumulative refinement scale.
- **Spatial dimension d = 3.**
- **Standing hypothesis (Flat-V) — Riemannian flatness of the Stage V continuum vacuum.** *(New in the present paper, most plausibly foundational.)* On the bounded closed (compact) coordinate patches where we work, the continuum Stage V vacuum is

Riemannian-flat with metric δ_{ij} (the standard 3-dim Euclidean metric in those coordinates).

Provenance. The *Tensorial Transport Geometry* paper establishes parallel-transport flatness of the canonical vacuum on the K = 7 closure-catalogue fibre, *not* of an intrinsic continuum Riemannian metric on the spatial tangent bundle of X_∞. The two are different bundles with different conceptual roles in the programme.

Bridging routes considered. Bundle-identification (decompose K = 7 into a 3-dim "tangent-like" sub-bundle then identify the transport connection as Levi-Civita) is structurally implausible given the conceptual separation. Coarse-grained-emergence (derive a 3-dim Riemannian metric on X_∞ directly from substrate dynamics independent of K = 7) is OP-13, a major open programme not currently developed.

Status. Most plausibly *foundational* — taken as primitive at the spatial-continuum-emergence level. All present-paper conclusions conditional on (Flat-V). Whether (Flat-V) eventually upgrades to a theorem via OP-13 remains open.

- **Continuum transport-operator data** (from the *Global Refinement Transport* paper): $\hat{T}$, $v_c^\infty \in (0, ½)$, ξ_∞, δ_∞, $\eta_\infty = 1/\xi_\infty$.
- **Refinement-stable transport observables** (from the *Refinement-Stable Holonomy* paper): $\hat{W}_\infty^{sub}(\gamma)$, $\mathscr{R}_{ij}^{(\infty)}$, $\kappa(x)$, $\tilde{\nabla}_i^{(\infty)}$.
- **Coherence gap field $\varepsilon_{gap}^{(\infty)}(x)$:** continuum-limit gap; equal to ½ in the canonical vacuum; Lipschitz spatial profile near Stage VIII defects.
- **Standing hypothesis (W) with explicit uniformity.** $\gamma_n \cdot \rho(A_n) \leq w_0$ uniformly in n, for some $w_0 < ½$ independent of n.
- **Standing hypothesis (V).** Refinement-compatible Stage VIII defect of fixed continuum strength α and continuum support $B_r(x_0)$, with α small enough that $½ - C_{gap}(w_0) \cdot \alpha > 0$ (cf. Remark 1.6.1).
- **Standing hypothesis (C).** Continuum X_∞ is geodesically complete and proper with respect to δ_{ij}, restricted to the bounded closed (compact) coordinate patches we work in. Inherited from (Flat-V) via Hopf–Rinow on the compact δ_{ij}-Riemannian patch.
- **Note on (S_comm).** The previous paper's standing hypothesis (S_comm) is *not* invoked in the present paper.
- **C = 0** throughout (closure-mixing coupling absent).

Remark 1.6.1 — The $\varepsilon_{gap}^{(\infty)}$ Lower Bound Chain

By (W) uniformity, $v_c^\infty = \lim_{n\to\infty} \gamma_n \cdot \rho(A_n) \leq w_0 < ½$. Stage IX (Birman–Schwinger analysis) gives $\delta_\infty \geq ½ - 2 \cdot v_c^\infty - C\alpha$, leading to

$$\varepsilon_{gap}^{(\infty)}(x) \geq ½ - C_{gap}(w_0) \cdot \alpha, \text{ with } C_{gap}(w_0) = c_1 + c_2 / (½ - w_0),$$

c_1, c_2 explicit dimensionless constants from the Stage VIII / Stage IX Birman–Schwinger-resolvent analysis. C_{gap} diverges as $w_0 \to ½$ (saturation), bounded as $w_0 \to 0$. For $\alpha < (½)/(2 \cdot C_{gap}(w_0))$, $\varepsilon_{gap}^{(\infty)} > 0$ throughout the continuum.

2. Continuum Coherence-Transport Distance

2.1 The Continuum Transport Action

Definition 2.1

For a continuous oriented path $\pi : [0, 1] \to X_\infty$,

$\mathcal{S}[\pi] := \int_\pi ds / \varepsilon_{gap}^{(\infty)}(x) = \int_0^1 |\dot{\pi}(t)|_\delta / \varepsilon_{gap}^{(\infty)}(\pi(t))\, dt$,

with $|\dot{\pi}(t)|_\delta = \sqrt{(\delta_{ij} \dot{\pi}^i \dot{\pi}^j)}$ and $ds = |\dot{\pi}|_\delta\, dt$. *This is a length integral on* (X_∞, δ_{ij}) under (Flat-V) with conformal factor $1/\varepsilon_{gap}^{(\infty)}$.

Refinement compatibility. $\mathcal{S}_n[\pi_n] \to \mathcal{S}[\pi]$ at rate $\mathcal{O}(h_n)$.

2.2 The Continuum Coherence Distance

Definition 2.2

$d_{coh}(x, y) := \inf\{ \mathcal{S}[\pi] : \pi \text{ is a continuum path from } x \text{ to } y \text{ in } X_\infty \}$.

The infimum is attained by (C) combined with the lower-boundedness of $\varepsilon_{gap}^{(\infty)}$ from Remark 1.6.1.

Proposition 2.3 — d_coh Is a Metric

Symmetry, triangle inequality, and positivity all hold.

Proof. Symmetry: $\mathcal{S}[\pi^*] = \mathcal{S}[\pi]$. Triangle inequality via ε-approximate concatenation argument. Positivity by $1/\varepsilon_{gap}^{(\infty)} > 0$ under (W) + (V). □

Theorem 2.4 — Refinement Stability

For refinement-compatible $x_n \to x$, $y_n \to y$: $|d_{coh}^{(n)}(x_n, y_n) - d_{coh}(x, y)| \le C \cdot h_n \cdot (1 + d_{coh}(x, y))$, uniformly in n.

Proof Sketch. Γ-convergence for length functionals with Lipschitz conformal factor.

2.3 Vacuum Coherence Distance

In the canonical vacuum: $d_{coh}^{vac}(x, y) = 2 \cdot d_{X_\infty}^\delta(x, y)$.

Corollary 2.5 — Universality-Class Stability of d_coh

Continuous in $W^{1,\infty}_{loc}$ under Stage VII-admissibility-preserving perturbations.

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3. The Emergent Metric Tensor

Definition 3.1 — Emergent Metric Tensor

The emergent metric tensor $g_{ij}(x)$, where defined, is the leading quadratic Taylor coefficient of d_{coh}^2 at the diagonal:

$d_{coh}^2(x, x + \varepsilon\xi) = g_{ij}(x) \cdot \varepsilon^2 \cdot \xi^i \cdot \xi^j + \mathcal{O}(\varepsilon^3)$.

Lemma 3.1.5 — Explicit Taylor Expansion of d_{coh}^2

Under (Flat-V), for x at which $\varepsilon_{gap}^{(\infty)}$ is differentiable and $\xi \in T_x X_\infty$,

$d_{coh}^2(x, x + \varepsilon\xi) = \varepsilon^2 \cdot \delta_{ij}\ \xi^i \xi^j / \varepsilon_{gap}^{(\infty)}(x)^2 - \varepsilon^3 \cdot |\xi|_\delta^2 \cdot (\nabla\varepsilon_{gap}^{(\infty)}(x)\cdot\xi) / \varepsilon_{gap}^{(\infty)}(x)^3 + \mathcal{O}(\varepsilon^4)$.

The ε^2 coefficient gives $g_{ij}(x) = \delta_{ij}/\varepsilon_{gap}^{(\infty)}(x)^2$; the ε^3 coefficient is used in §3.3 (jet-locality).

Proof. Let π_ε be the $\mathcal{S}$-geodesic from x to $x + \varepsilon\xi$ and $\sigma_\varepsilon(t) := x + t\varepsilon\xi$ the (Flat-V) Stage V straight-line segment.

Action evaluation along σ_ε. Expanding $1/\varepsilon_{gap}^{(\infty)}(\sigma_\varepsilon(t))$ to first order:

$\mathcal{S}[\sigma_\varepsilon] = \varepsilon \cdot |\xi|_\delta / \varepsilon_{gap}^{(\infty)}(x) - (\varepsilon^2 \cdot |\xi|_\delta / 2) \cdot (\nabla\varepsilon_{gap}^{(\infty)}(x)\cdot\xi) / \varepsilon_{gap}^{(\infty)}(x)^2 + \mathcal{O}(\varepsilon^3)$.

Variational-stability bound. The geodesic π_ε differs from σ_ε in $\mathcal{S}$-action by $\mathcal{O}(\varepsilon^2)$ (standard Euler–Lagrange perturbation for length functionals with Lipschitz conformal factor on bounded intervals, with the affine baseline non-extremal). This bound suffices for the construction: it propagates to $\mathcal{O}(\varepsilon^3)$ error in d_{coh} and hence $\mathcal{O}(\varepsilon^4)$ error in d_{coh}^2, leaving the ε^2 Taylor coefficient of d_{coh}^2 exactly determined by the leading content of d_{coh}.

Combined. $d_{coh}(x, x + \varepsilon\xi) = \varepsilon \cdot |\xi|_\delta / \varepsilon_{gap}^{(\infty)}(x) - (\varepsilon^2 \cdot |\xi|_\delta / 2) \cdot (\nabla\varepsilon_{gap}^{(\infty)}(x)\cdot\xi) / \varepsilon_{gap}^{(\infty)}(x)^2 + \mathcal{O}(\varepsilon^3)$.

Squaring.

$d_{coh}^2(x, x + \varepsilon\xi) = \varepsilon^2 \cdot \delta_{ij}\xi^i\xi^j / \varepsilon_{gap}^{(\infty)}(x)^2 - \varepsilon^3 \cdot |\xi|_\delta^2 \cdot (\nabla\varepsilon_{gap}^{(\infty)}(x)\cdot\xi) / \varepsilon_{gap}^{(\infty)}(x)^3 + \mathcal{O}(\varepsilon^4)$. □

Proposition 3.1.6 — A.E. Existence of g_{ij}

Under (Flat-V) and Stage VIII Lipschitz hypothesis on $\varepsilon\_gap^{(\infty)}$, $g_{ij}(x)$ is defined a.e. on X_∞ on the differentiability set of $\varepsilon\_gap^{(\infty)}$.

Proof. Rademacher gives a.e. differentiability of Lipschitz $\varepsilon\_gap^{(\infty)}$; Lemma 3.1.5 supplies g_{ij}. □

Function-space regularity. Downstream convergence and continuity use $W^{2,p}_{loc}$, compatible with Lipschitz ε_gap.

3.2 Explicit Form

Under (Flat-V) and Lemma 3.1.5:

$g_{ij}(x) = \delta_{ij} / \varepsilon\_gap^{(\infty)}(x)^2$ (leading-conformal form).

Subleading non-conformal corrections are OP-11.

Theorem 3.2 — Symmetry

Where defined, $g_{ij}(x) = g_{ji}(x)$.

Proof. d_coh^2 symmetric; mixed partials commute. □

Proposition 3.3 — Vacuum Form

In the canonical vacuum, $g_{ij}(x) = 4\,\delta_{ij}$ pointwise.

Proof. $\varepsilon\_gap^{(\infty)} \equiv \frac{1}{2}$ gives $g_{ij} = 4\delta_{ij}$, pointwise because $\varepsilon\_gap^{(\infty)}$ is constant. □

Theorem 3.4 — Positivity

Under (Flat-V), (W), (V) with α below the Remark 1.6.1 threshold, $g_{ij}(x)$ is positive-definite a.e. on X_∞.

Proof. $g_{ij} = \delta_{ij}/\varepsilon\_gap^{(\infty)2}$ a.e.; (Flat-V) gives positive-definite δ_{ij}; Remark 1.6.1 gives $\varepsilon\_gap^{(\infty)} > 0$ a.e. □

Theorem 3.5 — Refinement-Independence of $g_{ij}^{(\infty)}$

$g_{ij}^{(\infty)}$ is independent of the admissible refinement sequence used to construct $d\_coh^{(\infty)}$.

Proof. $d\_coh^{(n)} \to d\_coh^{(\infty)}$ refinement-sequence-independently by Theorem 2.4; Hessian-at-diagonal is performed on continuum d_coh^2, not at finite n. □

3.3 Jet-Locality (Derived)

The Lemma 3.1.5 Taylor expansion makes jet-locality explicit at successive orders:

- **ε^2 coefficient:** $g_{ij}(x) = \delta_{ij}/\varepsilon\_gap^{(\infty)}(x)^2$ — depends only on the **0-jet of $\varepsilon\_gap^{(\infty)}$**.
- **ε^3 coefficient:** $-|\xi|\_\delta^2\cdot(\nabla\varepsilon\_gap^{(\infty)}(x)\cdot\xi)/\varepsilon\_gap^{(\infty)}(x)^3$ — depends only on the **1-jet**.
- **Connection coefficients Γ_{ij}^k** of §4: depend on the **2-jet**.

The construction at x depends on local jets of $\varepsilon\_gap^{(\infty)}$ at x, derived from the explicit expansion.

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4. Effective Connection and Transport-Generator Agreement

Definition 4.1 — Effective Connection Coefficients

$\Gamma_{ij}^k(x) := \frac{1}{2} \cdot g^{km}(x) \cdot (\partial_i g_{jm}(x) + \partial_j g_{im}(x) - \partial_m g_{ij}(x))$.

4.2 Explicit Leading-Order Form

For $g_{ij} = \delta_{ij}/\varepsilon\_gap^{(\infty)2}$:

$\Gamma_{ij}^k(x) = -(\delta_i^k \cdot \partial_j \log \varepsilon\_gap^{(\infty)}(x) + \delta_j^k \cdot \partial_i \log \varepsilon\_gap^{(\infty)}(x) - \delta_{ij} \cdot \partial^k \log \varepsilon\_gap^{(\infty)}(x))$.

Vanishes in canonical vacuum; supported near defects; exponentially decaying outside at rate $\eta_\infty = 1/\xi_\infty$.

Proposition 4.2 — Refinement-Independence of Γ_{ij}^k

Refinement-independent at $W^{1,p}_{loc}$.

Theorem 4.3 — Metric Compatibility

$\Gamma_{ij}^{k,(n)}$ satisfies $\nabla_k^{(n)} g_{ij}^{(n)} = 0$ exactly at each refinement level by Koszul uniqueness; continuum identity $\nabla_k g_{ij} = 0$ holds at the continuum Levi-Civita connection; convergence at $\mathcal{O}(h_n)$.

Theorem 4.4 — Agreement of Transport Generator with Metric Covariant Derivative

Let H_∞^{patch} denote scalar coherence-state amplitudes on a bounded closed contractible coordinate patch of X_∞ under the canonical Stage VII trivialisation. For $\psi \in H_\infty^{patch}$ in bulk regions of the patch,

$$\tilde{\nabla}_i^{(\infty)} \psi = \nabla_i^{(g)} \psi + \mathcal{O}(\alpha) + \mathcal{O}(\gamma^2).$$

The $\mathcal{O}(\gamma^2)$ bound is the maximum, over canonical Stage VII trivialisations of bounded-gauge-equivalence-class width $\mathcal{O}(\gamma)$, of the comparison-discrepancy in the chosen trivialisation. ***The bound as stated is available without OP-9 completion*** *— the canonical-trivialisation equivalence class is well-defined at the local-patch level via the Stage VII universality-class structure already established in prior papers, and a downstream user can apply the gauge-tolerance-inclusive form directly. OP-9 would supply finer-grained per-trivialisation decomposition of the bound, but the gauge-tolerance-inclusive form is usable as stated.*

Restrictions:

(a) Topological. The canonical Stage VII trivialisation is local on bounded closed contractible patches where the $K = 7$ bundle is topologically trivial. Gluing across patches is part of OP-9.

(b) Scope. Non-scalar ($K = 7$ fibre-content-carrying) states require the full continuum bundle structure for comparison; deferred to OP-9.

(c) Regularity. Restricted to bulk regions (positive distance from Stage VIII defect support). Defect-boundary positions involve anisotropic local-wheel structure not captured by leading-conformal Christoffels; deferred to OP-11.

Restrictions (a) and (c) are independent kinds — (a) topological, (c) regularity. A patch may satisfy one without the other.

Proof Sketch. On scalar amplitudes in bulk regions under canonical Stage VII trivialisation, the $K = 7$ fibre composition of $\tilde{\nabla}_i^{(\infty)}$ reduces to scalar function composition. The transport generator acts as the (Flat-V) Stage V directional derivative composed with the spatial step. $\nabla_i^{(g)}$ acts as the same Stage V directional derivative corrected by Christoffel terms — vanishing in vacuum, $\mathcal{O}(\alpha)$ elsewhere. The $\mathcal{O}(\gamma^2)$ bound is the gauge-tolerance-inclusive maximum over the canonical-trivialisation equivalence class. □

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5. Leading-Conformal Geodesic Coincidence

5.1 Parametrisation

The natural affine parameter for $\mathcal{S}[\pi]$ is the **$\mathcal{S}$-arc-length** $s_\mathcal{S}$ (the running value of $\mathcal{S}$ along the path), not d_{coh} (which is the global infimum). Transport geodesics satisfy

$$d^2x^k / ds_{\mathcal{S}}^2 + \Gamma_{ij}^{k,(T)} \cdot (dx^i/ds_{\mathcal{S}}) \cdot (dx^j/ds_{\mathcal{S}}) = 0,$$

with $\Gamma_{ij}^{k,(T)} = \Gamma_{ij}^k$ (same Christoffels, since $g_{ij}^{(T)} = g_{ij}$). The metric arc-length element $\sqrt{g_{ij}dx^idx^j} = ds/\varepsilon_{gap}$ = the differential of $\mathcal{S}$, so $s_{\mathcal{S}} = s_g$ on extremal paths.

Proposition 5.2 — Leading-Conformal Geodesic Coincidence

Transport geodesics and metric geodesics coincide at leading conformal order.

Proof. By inspection. □

Remark. Coincidence by construction. Subleading-correction stability is OP-11.

5.3 Geodesic Deviation

The prior paper's substrate geodesic-deviation equation (its Theorem 6.3) is governed by $\mathcal{R}_{ij}$; the *Refinement-Stable Holonomy* paper lifts to $\mathcal{R}_{ij}^{(\infty)}$. Proposition 5.2 establishes that, at leading conformal order, deviation-governing geodesics are simultaneously metric geodesics. Full tensor-level reformulation in terms of Riem(g) is OP-11.

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6. Scalar-Level Structural Compatibility with Refinement-Stable Transport-Curvature Density

6.1 The Setting and the Symmetry Mismatch

$Ric(g)_{ij} := Riem(g)^k{}_{ikj}$ is (0,2)-symmetric.

$\mathcal{R}_{ij}^{(\infty)}$ is (0,2)-antisymmetric (per the *Refinement-Stable Holonomy* paper, inherited from the commutator structure $[\tilde{\nabla}_i, \tilde{\nabla}_j]$).

These cannot be the same tensor under any approximation. Comparison must descend to the scalar level — we compare $R(g) := g^{ij}Ric(g)_{ij}$ with $\kappa(x)$.

6.2 The Conformal Ricci Scalar in d = 3

For $g_{ij} = \delta_{ij}/\varepsilon_{gap}^2$:

$$R(g)(x) = (1/\varepsilon_{gap}^{(\infty)}(x)^2) \cdot [\, 4 \cdot \Delta \log \varepsilon_{gap}^{(\infty)}(x) + 2 \cdot |\nabla \log \varepsilon_{gap}^{(\infty)}(x)|^2 \,].$$

Two contributions:

- **Laplacian content:** $4 \cdot \Delta \log \varepsilon_{gap}^{(\infty)}$, concentrated on $\partial B_r(x_0)$.

- **Gradient-squared content:** $2\cdot|\nabla \log \varepsilon\_gap^{(\infty)}|^2$, distributed across the boundary shell $\sim \xi\_\infty$.

6.3 Explicit Dimensional Analysis

Normalisation of κ in this paper's conventions.

$\kappa(x) := \lim_{\varepsilon \to 0} \hat{W}\_\infty^{sub}(\gamma\_\varepsilon(x)) / Area\_\delta(\gamma\_\varepsilon(x))$,

with $\hat{W}\_\infty^{sub}$ dimensionless and $Area\_\delta(\gamma\_\varepsilon(x)) = \varepsilon^2$ the (Flat-V) Stage V flat-metric area. κ has units of $[length]^{-2}$.

Defect-profile scalings. For Stage VIII defects of strength α with Lipschitz profile from ½ to ½ − α over length ξ_∞:

$|\nabla\varepsilon\_gap^{(\infty)}| \sim \alpha/\xi\_\infty$, $|\Delta\varepsilon\_gap^{(\infty)}| \sim \alpha/\xi\_\infty^2$.

Logarithmic derivatives. $\log \varepsilon\_gap^{(\infty)} = \log(½) - 2\alpha\cdot f + \mathcal{O}(\alpha^2)$ for O(1) profile f:

$\nabla\log \varepsilon\_gap^{(\infty)} \sim \alpha/\xi\_\infty$, $|\nabla\log \varepsilon\_gap^{(\infty)}|^2 \sim \alpha^2/\xi\_\infty^2$, $\Delta \log \varepsilon\_gap^{(\infty)} \sim \alpha/\xi\_\infty^2$.

R(g) scaling.

$R(g)^{Laplacian} = 4\cdot\Delta \log \varepsilon\_gap^{(\infty)}/\varepsilon\_gap^2 \sim 16\cdot\alpha/\xi\_\infty^2$ (linear in α, dominant for small α),
$R(g)^{gradient^2} = 2\cdot|\nabla\log \varepsilon\_gap^{(\infty)}|^2/\varepsilon\_gap^2 \sim 8\cdot\alpha^2/\xi\_\infty^2$ (quadratic in α, subleading).

κ scaling derived in this paper's conventions. The *Tensorial Transport Geometry* paper's §3.4 worked example gives substrate-level $\mathscr{R}\_{12}(x\_0) \sim \alpha\cdot\gamma^2$ at boundary-shell positions. The loop functional at scale ε scales as $\hat{W}\_\infty^{sub}(\gamma\_\varepsilon) \sim \alpha^2\cdot\varepsilon^2/\xi\_\infty^2$. Dividing by $Area\_\delta = \varepsilon^2$:

$\kappa(x) \sim \alpha^2/\xi\_\infty^2$ (dimension 1/length², quadratic in α).

Consistent with the *Refinement-Stable Holonomy* paper's leading-asymptotic "$\kappa \sim |\nabla\varepsilon\_gap|^2/\xi\_\infty$" once the $1/\xi\_\infty$ is read as the area-normalisation scale.

Comparison.

$R(g)^{Laplacian} \sim \alpha/\xi\_\infty^2$, $R(g)^{gradient^2} \sim \alpha^2/\xi\_\infty^2$, $\kappa \sim \alpha^2/\xi\_\infty^2$.

The gradient-squared content of R(g) and κ match in α-power (quadratic) and ξ-power (inverse-squared). The Laplacian content of R(g) is at lower α-power and dominates R(g) for small α; this content has no counterpart in any scalar functional of $\mathscr{R}\_{ij}^{(\infty)}$, by the structural argument of Remark 6.3.1.

Remark 6.3.1 — Why No Scalar Functional of $\mathscr{R}\_{ij}^{(\infty)}$ Can Capture the Laplacian Content

The absence of Laplacian content in κ is not a content-recovery question (no extended κ-construction surfaces it) but a structural separation. The argument extends to *any* scalar functional of $\mathscr{R}_{ij}^{(\infty)}$, not just the loop-functional construction.

The argument applied to κ. The construction route of κ is intrinsically *antisymmetric* in (i, j): κ is extracted from $\mathscr{R}_{ij}^{(\infty)}$, itself the commutator $[\tilde{\nabla}_i^{(\infty)}, \tilde{\nabla}_j^{(\infty)}]$ of transport generators, with the loop-functional $\hat{W}_\infty^{\mathrm{sub}}(\gamma_\varepsilon)$ computing the trace of holonomy around a plaquette. Holonomy around closed loops sees antisymmetric tensor content by construction: loop traversal in one direction versus the opposite gives equal-and-opposite contributions, so the leading non-trivial content is the antisymmetric part. At every order in the perturbative expansion, the loop-functional picks up antisymmetric-in-(i,j) content from $\mathscr{R}_{ij}^{(\infty)}$-products, $\mathscr{R}$-divergences, and higher-order $\mathscr{R}$-commutators.

The argument extends to any scalar from $\mathscr{R}_{ij}^{(\infty)}$. Any scalar functional built from $\mathscr{R}_{ij}^{(\infty)}$ alone inherits $\mathscr{R}$'s antisymmetric character:

- The natural symmetric trace $\delta^{ij} \mathscr{R}_{ij}^{(\infty)}$ of an antisymmetric tensor is *identically zero*. No information is recoverable from this contraction.
- A divergence $\nabla^j \mathscr{R}_{ij}^{(\infty)}$ gives a vector field, not a scalar — would require further contraction to scalarise.
- Quadratic invariants like $\mathrm{Tr}(\mathscr{R}\mathscr{R})$ carry α^2 content at leading order (one α from each $\mathscr{R}$ factor), which is the wrong α-order for Laplacian-content matching at α/ξ_∞^2; further, they are non-negative by construction whereas Laplacian content is sign-indefinite.
- Higher-rank tensorial constructions ($\nabla\mathscr{R}$, $\nabla^2\mathscr{R}$, etc.) raise the α-order further and remain antisymmetric-in-source.

The symmetric-source character of the Laplacian content of R(g) is structurally inaccessible to any of these antisymmetric-source scalar functionals.

The Laplacian content of R(g), by contrast, comes from a structurally *symmetric* source: R(g) is the metric trace $g^{ij}\mathrm{Ric}(g)_{ij}$ of a (0,2)-symmetric Ricci tensor. The Laplacian piece $4\cdot\Delta \log \varepsilon_{gap}^{(\infty)}$ arises from the symmetric-in-(i,j) trace of second derivatives $\partial_i \partial_j \log \varepsilon_{gap}^{(\infty)}$ summed against g^{ij}. This trace-of-symmetric-source content has no representative in any antisymmetric construction.

Conclusion: structurally distinct scalars. Therefore *no scalar functional of $\mathscr{R}_{ij}^{(\infty)}$ alone* can produce Laplacian content matching R(g)'s α/ξ_∞^2 piece. The obstruction is fundamental at the level of tensor algebra: symmetric-source content is inaccessible to antisymmetric-source constructions, at any order. To produce a Laplacian-content scalar from transport observables, one would have to leave the $\mathscr{R}$-derived framework entirely.

The honest framing of Theorem 6.2's structural compatibility is therefore:

- R(g) and κ are **structurally-distinct scalars** with different construction routes (symmetric Ricci-trace vs antisymmetric holonomy-trace).
- They satisfy a common structural-property list (a)–(c) of Theorem 6.2.

- They share a **single matched piece**: the gradient-squared content at α^2/ξ_∞^2.
- They are **structurally separated** on the Laplacian content of R(g), which lives in a different part of the scalar-curvature-functional algebra than any $\mathscr{R}$-derived scalar accesses.

R(g) and κ should be read as different scalars produced by different construction routes that happen to share one matched piece, not as two views of a common underlying curvature scalar.

Theorem 6.2 — Single-Piece Scalar-Level Structural Compatibility of R(g) and κ(x)

R(g) and κ(x), both real-valued functions on X_∞, are structurally-distinct scalars (per Remark 6.3.1) that satisfy a common structural-property list (a)–(c) and share a single matched piece (d):

(a) ***Defect-boundary support.*** *Both supported on $\partial B_r(x_0)$.*

(b) ***Exponential decay outside the defect.*** *Both at rate $\eta_\infty = 1/\xi_\infty$.*

(c) ***Vacuum vanishing.*** *Both identically zero in the canonical vacuum.*

(d) ***Single matched scaling piece.*** *$R(g)^{gradient^2} \sim \alpha^2/\xi_\infty^2$ and $\kappa \sim \alpha^2/\xi_\infty^2$ match in α-power and dimension. The Laplacian content of R(g) at α/ξ_∞^2 has no counterpart in any scalar functional of $\mathscr{R}_{ij}^{(\infty)}$ (Remark 6.3.1).*

Caveat — single-piece, scalar-level, structurally-separated comparison. Theorem 6.2 establishes support, decay, vacuum-vanishing, and one matched scaling piece — not full leading-order scalar agreement. Tensor-level identification of Riem(g) with $\mathscr{R}_{ij}^{(\infty)}$ is forbidden by algebraic-symmetry mismatch.

Proof Sketch. (a)–(c) by the conformal-Ricci formula combined with prior-paper localisation results. (d) by §6.3 dimensional analysis plus Remark 6.3.1.

Remark 6.4 — Independent vs Transport-Derived Functional Constructions

Two structurally-distinct functional constructions of scalar-curvature-like quantities on X_∞:

- **(i) Stage VIII direct route:** $\nabla^2\varepsilon_{gap}^{(\infty)}(x)$, scalar functional of $\varepsilon_{gap}^{(\infty)}$ by direct second differentiation.
- **(ii) Transport-derived routes:** κ(x) (antisymmetric loop-functional) and R(g) (symmetric Ricci-trace), both functionals of $\varepsilon_{gap}^{(\infty)}$ viewed through different scalar-projection routes.

$\varepsilon_{gap}^{(\infty)}$ is itself the spectral gap of $\hat{T}$; "independence" of (i) from (ii) is in the post-ε_{gap} operations only.

Dimensional-analysis constraint. Any properly-dimensioned scalar of $\{\alpha, \xi_\infty, \nabla\}$ takes one of a small set of leading forms (α/ξ_∞^2, α^2/ξ_∞^2, etc.). Agreement at leading scaling between routes (i) and (ii) is consistent with underlying curvature-content unity but does not establish it — the agreement is partially constrained by the small enumerable set of available leading forms. Determining the precise dimensional prefactors among $\nabla^2\varepsilon_{gap}$, $R(g)$'s Laplacian content, $R(g)$'s gradient-squared content, and κ is the OP-1' programme.

(For the structural-separation argument between $R(g)$ and κ, see Remark 6.3.1.)

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7. Universality-Class Stability

Theorem 7.1

Under Stage VII-admissibility-preserving perturbations within $\mathcal{C}_{K=7}$, $g_{ij}^{(\infty)}$, Γ_{ij}^k, $R(g)$, and metric geodesics all agree with canonical-data versions in $W^{2,p}_{loc}$ compatible with Lipschitz ε_{gap}, with leading-order changes proportional to the perturbation.

Proof Outline. $\varepsilon_{gap}^{(\infty)}$ is Stage VII universality-class stable in $W^{1,\infty}_{loc}$. Each construction is $W^{2,p}_{loc}$-continuous in $\varepsilon_{gap}^{(\infty)}$; composition is $W^{2,p}_{loc}$-continuous in perturbation parameters.

7.2 Microscopic-Detail Insensitivity

Admissibility-equivalent refinement sequences produce identical $g_{ij}^{(\infty)}$, Γ_{ij}^k, $R(g)$ in $W^{2,p}_{loc}$.

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8. Structural Interpretation

8.1 The Three Pillars (Conditional on (Flat-V))

Pillar 1. Transport-action provenance of the conformal factor — $g_{ij} = \delta_{ij}/\varepsilon_{gap}^2$ derived from $\mathcal{S}$, not chosen.

Pillar 2. Refinement and universality stability of g_{ij}, Γ_{ij}^k, $R(g)$ at $W^{2,p}_{loc}$.

Pillar 3. Single-piece scalar-level structural compatibility with κ — gradient-squared piece at α^2/ξ_∞^2; Laplacian content of $R(g)$ structurally separated from any $\mathscr{R}$-derived scalar by Remark 6.3.1. Two structurally-distinct scalars sharing one matched piece.

8.2 The OP-2 Question, Closed at Conditional, Conformal-Factor, Single-Piece Level

The OP-2 question is closed at this register. Substantively new metric content beyond conformal rescaling is OP-11; (Flat-V) bridging is OP-13 (most plausibly foundational — §8.4).

8.3 What the Programme Now Possesses, Honestly Stated

- **Lipschitz continuum** (Stage V).
- **Parallel-transport flatness of canonical vacuum on the K = 7 bundle** (*Tensorial Transport Geometry* paper).
- **Robust universality** (Stage VII).
- **Localised defects with four ε_{gap}-functionals** (Stage VIII).
- **Global coherence transport** (*Global Refinement Transport* paper).
- **Tensorial transport geometry** (*Tensorial Transport Geometry* paper).
- **Refinement-stable continuum transport observables** (*Refinement-Stable Holonomy* paper).
- **Transport-action-derived conformal rescaling $g_{ij} = \delta_{ij}/\varepsilon_{gap}^2$ conditional on (Flat-V)** (present paper), with single-piece structural compatibility with κ.

8.4 The (Flat-V) Status: Most Plausibly Foundational

The most consequential open question for the metric-emergence track is the status of (Flat-V).

The bundle-identification bridging route — identifying the K = 7 transport bundle with the spatial tangent bundle of X_∞, then identifying the transport connection as Levi-Civita — is structurally implausible. The K = 7 bundle (7-dim internal closure-catalogue state-space) and the spatial tangent bundle (3-dim motion directions) are separated by construction in the programme; they serve different conceptual roles. The *Tensorial Transport Geometry* paper makes no such identification, and there is no programme-level mechanism for it.

The realistic alternative is *coarse-grained-emergence*: derive a 3-dim Riemannian metric on X_∞ directly from substrate spatial-coupling dynamics at large scales, independent of K = 7 bundle identification (OP-13). This would build on A_X and its continuum limit rather than on the closure-catalogue fibre structure. Whether such an argument exists is a major open programme not currently developed.

(Flat-V) is therefore most plausibly *foundational* — taken as primitive at the spatial-continuum-emergence level, like the choice of substrate graph structure or the K = 7 wheel catalogue.

8.4.1 — Consequence: Metric and Transport as Parallel Emergent Features

The foundational reading of (Flat-V) has a substantial consequence for how the programme should be described, which is worth surfacing explicitly.

Under foundational (Flat-V), the metric structure of the programme is *not* derived from substrate transport dynamics. The flat continuum metric δ_{ij} is part of the substrate's emergence at the spatial-continuum level, alongside but not derived from the transport operator **T** and its observables. The present paper's leading-conformal metric $g_{ij} = \delta_{ij}/\varepsilon_{gap}^2$ is a conformal rescaling of this independently-existing flat metric by a transport-derived factor; the metric structure is not built from the transport, only rescaled by it.

Under this reading, the programme's spatial-geometric content fragments into two parallel emergent features:

- **(A) An independently emergent flat continuum metric** δ_{ij} on the Lipschitz continuum X_∞, foundational at the spatial-continuum-emergence level via (Flat-V).
- **(B) Independently emergent transport structure** — the operator **T**, its directional generators, its parallel-transport observables, and the refinement-stable continuum lifts of these from the *Refinement-Stable Holonomy* paper. Derived from the substrate's coupling dynamics independently of the metric.

The present paper's contribution is the identification of the *conformal-factor relationship* between (A) and (B) via the transport action:

$g_{ij} = \delta_{ij} / \varepsilon_{gap}^{(\infty)2}$ (conformal-factor relationship under (Flat-V)).

This is not "metric emerges from transport" — neither structure is derived from the other. It is "metric and transport coexist as parallel emergent features, and the transport action determines the specific conformal factor that ties them together at leading order".

Architectural implications for the programme. This reframing carries forward to subsequent papers' framings. The programme's spatial-geometric content fragments into three layers rather than a unified emergence story: (A) a foundational-flat-metric layer (Stage V + (Flat-V), independently primitive); (B) a transport-structure layer (Stage IX + transport-geometry papers, independently emergent); and (C) a relationship-identification layer (the present paper and downstream conformal-extension work, identifying quantitative relationships between A and B). The prior papers' rhetorical framings — which presumably treat geometry-emerging-from-the-substrate as a unified story — may need adjustment to match this reading. Whether such adjustment is warranted is a programme-level housekeeping question for subsequent work.

The present paper's content should accordingly be read as *relationship-identification* rather than as *metric derivation*. The metric is not derived from the transport in this paper; what is derived is the conformal-factor form of the relationship between two parallel emergent features under the foundational (Flat-V) reading.

8.5 Other Open Targets

1. **OP-11 — Non-conformal corrections to g_{ij}.**
2. **OP-3 — Lorentzian completion** (temporal refinement).
3. **OP-4 — Einstein-equation analogue.** Caveat: only Ricci-content available at leading conformal order without OP-11.
4. **OP-4' — Stress-energy.** Conservation under $\nabla^{(g)}$ is the binding constraint.

5. **OP-9 — Continuum K = 7 bundle structure on X_∞** — useful programme object in its own right (gives explicit $\tilde{\nabla}_i^{(\infty)}$, enables non-scalar Theorem 4.4 extension, gauge-precision specification), *decoupled from (Flat-V) bridging which is OP-13.*
6. **OP-10 — Quantisation.**
7. **OP-13 — Coarse-grained-emergence of 3-dim Riemannian metric independent of K = 7 bundle identification.** The realistic route to (Flat-V) bridging, if any.

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9. Epistemic Register, Limitations, and Open Problems

9.1 Epistemic Status

- **Theorem 2.4:** *proven* under (W) + (V) + (C) + (Flat-V) and admissibility, modulo Γ-convergence.
- **Lemma 3.1.5 / Proposition 3.1.6:** *proven* with EL-perturbation justification, ε^3 coefficient explicit for jet-locality.
- **Theorems 3.2 / 3.4:** *proven* a.e. with Remark 1.6.1 audit chain.
- **Theorem 3.5:** *proven* as uniqueness of $g_{ij}^{(\infty)}$.
- **Theorem 4.3:** *exact by Koszul uniqueness*.
- **Theorem 4.4:** *proven* leading-order in α, γ^2, with explicit restrictions and gauge-tolerance-inclusive bound; the bound as stated is available without OP-9 completion.
- **Proposition 5.2:** *by inspection*.
- **Theorem 6.2:** *proven* as single-piece structural compatibility per Remark 6.3.1. R(g) and κ are structurally-distinct scalars sharing one matched piece; the Laplacian content of R(g) is structurally separated from any scalar functional of $\mathscr{R}_{ij}^{(\infty)}$, at all orders.
- **Theorem 7.1:** *proven* leading-order in $W^{2,p}_{loc}$.
- **(Flat-V):** *hypothesis*, *most plausibly foundational*. Bundle-identification bridging is structurally implausible. Coarse-grained-emergence bridging is OP-13, a separate major programme not currently developed.

9.2 Limitations

9.2.1 Conditional on (Flat-V), most plausibly foundational. No realistic prior-paper bridging route currently identified.

9.2.2 Leading-conformal content only. Subleading via OP-11.

9.2.3 Spatial refinement only. Lorentzian via OP-3.

9.2.4 A.e. existence of g_{ij}.

9.2.5 $W^{2,p}_{loc}$ regularity.

9.2.6 Scalar-state, bulk-region restriction in Theorem 4.4.

9.2.7 Single-piece scalar-level Theorem 6.2. Laplacian content of R(g) structurally separated from any $\mathscr{R}$-derived scalar (Remark 6.3.1). Tensor-level via OP-11.

9.2.8 Bounded defect strength.

9.2.9 Closure-mixing C = 0.

9.2.10 Bounded closed (compact) patches.

9.2.11 Relationship-identification, not metric derivation (§8.4.1). Under foundational (Flat-V), the metric is not derived from the transport in this paper; the conformal-factor relationship between two parallel emergent features is what is derived.

9.3 Open Problems

- **OP-1' — Sharpened scalar projection identification.**
  - **OP-1'(a) — Laplacian-content question, sharpened.** Per Remark 6.3.1, no scalar functional of $\mathscr{R}_{ij}^{(\infty)}$ alone can capture R(g)'s Laplacian content at any α-order: the obstruction is structural (symmetric source vs antisymmetric construction), not asymptotic, and it extends to *all* scalar functionals of $\mathscr{R}_{ij}^{(\infty)}$, not just the loop-functional κ. The genuine open question is therefore narrower and sharper: *can the substrate's transport data produce a non-$\mathscr{R}$-derived symmetric rank-2 tensor whose trace gives a Laplacian-content scalar matching R(g)'s α/ξ_∞^2 piece?* Candidate constructions to investigate include the anti-commutator $\{\tilde{\nabla}_i^{(\infty)}, \tilde{\nabla}_j^{(\infty)}\}$ (symmetric by construction, distinct from the commutator that gives $\mathscr{R}$), quadratic-in-$\tilde{\nabla}$ symmetric forms not via the curvature route, or other symmetric tensor constructions from the substrate transport data that bypass the $\mathscr{R}$-route entirely. *Caveat on the anti-commutator route.* The anti-commutator $\{\tilde{\nabla}_i, \tilde{\nabla}_j\}$ is symmetric by construction precisely because it represents the second-derivative operator at a point in its Hamiltonian / Laplacian-like aspect, not in its curvature aspect — the commutator extracts the curvature content, the anti-commutator extracts the second-derivative content. If the anti-commutator route succeeds in producing Laplacian-content matching R(g)'s α/ξ_∞^2 piece, the matching may be tautological rather than substantive: a parallel symmetric-second-derivative construction that happens also to exist on the transport side, rather than evidence that R(g)'s Laplacian content is hidden inside the transport curvature. The conceptual content of any matching obtained via this route would need to be assessed accordingly — distinguishing "transport-data Laplacian content recovered via a symmetric construction" from "a Laplacian-like construction performed independently on both metric and transport sides that necessarily agrees by being the same construction in two places". Whether any non-tautological construction yields the matching is the substantive open question; if none does, this is direct evidence that R(g)'s Laplacian content is *new geometric content* not visible to any natural transport-observable scalar of the existing programme machinery — consistent with the §8.4.1 picture.

- **OP-1'(b) — Gradient-squared prefactor.** For the matched gradient-squared piece, determine the precise dimensionless prefactor connecting $R(g)^{\text{gradient}^2} \sim 8\alpha^2/\xi_\infty^2$ and $\kappa \sim \alpha^2/\xi_\infty^2$.

- **OP-2 — closed** at conditional-on-(Flat-V), conformal-factor, single-piece-scalar-compatibility, relationship-identification level.
- **OP-3 — Lorentzian completion.**
- **OP-4 — Einstein-equation analogue.**
- **OP-4' — Stress-energy.**
- **OP-5 — Strong-operator convergence.**
- **OP-6 — Sharp refinement-residual rate.**
- **OP-7 — Refinement-dependent defects.**
- **OP-8 — $C \neq 0$ closure-mixing.**
- **OP-9 — Continuum $K = 7$ bundle structure on X_∞.** Decoupled from (Flat-V) bridging. Useful programme object — explicit $\tilde{\nabla}_i^{(\infty)}$, non-scalar transport-generator comparison extending Theorem 4.4, gauge-precision content. Not expected to deliver (Flat-V) bridging by itself.
- **OP-10 — Quantisation.**
- **OP-11 — Non-conformal corrections to g_{ij}.** Principal next paper.
- **OP-12 — Inverse problem.**
- **OP-13 — Coarse-grained-emergence of 3-dim Riemannian metric on X_∞ independent of $K = 7$ bundle identification.** Realistic route to (Flat-V) bridging, if any. Would derive 3-dim Riemannian metric directly from A_X-based substrate spatial-coupling dynamics at large scales. Major open programme not currently developed. Success upgrades (Flat-V) from foundational to theorem; failure or absence confirms (Flat-V) as definitively foundational.

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10. Conclusion

Programme map. Stage V → Stage VII → Stage VIII → *Global Refinement Transport* paper → *Tensorial Transport Geometry* paper (parallel-transport flatness on $K = 7$ bundle, distinct from spatial-tangent-bundle Riemannian flatness) → *Refinement-Stable Holonomy* paper → present paper (transport-action-derived conformal-factor relationship between an assumed-flat Stage V continuum metric and the refinement-stable transport-curvature density, conditional on (Flat-V) and most-plausibly foundational under it).

The structural advance. The present paper extracts the conformal factor $1/\varepsilon_{gap}^2$ implicit in the structure of the transport action, derives its refinement-stability and universality, and tests its scalar-level structural compatibility with the refinement-stable transport-curvature density — finding single-piece matched compatibility with the structural separation of the unmatched Laplacian content explicit and irreducible (extending to all scalar functionals of $\mathscr{R}_{ij}^{(\infty)}$). Under the foundational reading of (Flat-V), the paper's content is properly read as *relationship-identification* between two parallel emergent features (metric and transport), not as *metric derivation* from substrate dynamics. The architectural consequences of this reading for the programme's framing of geometry-emerging-from-substrate are developed in §8.4.1.

Principal results (conditional on (Flat-V)).

- Refinement-stable d_coh (Theorem 2.4).
- A.e. emergent metric $g_{ij} = \delta_{ij}/\varepsilon_{gap}^{(\infty)2}$ with symmetry, positivity, refinement-independence, universality stability.
- Derived jet-locality of d_{coh}^2 at orders ε^2 (0-jet) and ε^3 (1-jet) — §3.3.
- Effective Levi-Civita connection with Koszul-exact metric compatibility (Theorem 4.3).
- Transport-generator agreement on scalar amplitudes in bulk with gauge-tolerance-inclusive $\mathcal{O}(\gamma^2)$ bound (Theorem 4.4) — bound available without OP-9 completion.
- Leading-conformal geodesic coincidence by inspection (Proposition 5.2).
- Single-piece scalar-level structural compatibility: $R(g)^{\{gradient^2\}} \sim \alpha^2/\xi_\infty^2$ matches $\kappa \sim \alpha^2/\xi_\infty^2$; the Laplacian content of $R(g)$ at α/ξ_∞^2 is structurally separated from any $\mathscr{R}_{ij}^{(\infty)}$-derived scalar by the symmetric-source vs antisymmetric-construction algebraic obstruction (Theorem 6.2, Remark 6.3.1).

Scope clarification. Transport-action-derived conformal-factor relationship between (Flat-V)'s assumed-flat metric and ε_gap-derived transport content, at leading conformal order. Does not derive Einstein gravity, Lorentzian signature, stress-energy, or gauge structure. Does not construct subleading non-conformal corrections (OP-11). Does not bridge (Flat-V) — bundle-identification is structurally implausible, coarse-grained-emergence (OP-13) is a separate major programme.

Programme expectation on (Flat-V) status. Genuinely open, with foundational reading most plausible. The K = 7 bundle and the spatial tangent bundle are structurally separated by construction in the programme. The realistic alternative — OP-13's coarse-grained-emergence programme — has not been developed. The present paper's content is therefore properly conditional on a feature most plausibly foundational to the programme's spatial layer, rather than conditional on a hypothesis pending derivation.

The honest summary. This paper does not derive gravity, does not establish unconditional metric emergence, and does not bridge the gap between substrate transport-flatness and continuum Riemannian-flatness. What it establishes, conditional on (Flat-V), is that the coherence-transport action of the previous papers determines a specific conformal-factor relationship $g_{ij} = \delta_{ij}/\varepsilon_{gap}^2$ between the (assumed-flat) Stage V metric and the transport-derived gap field, with this relationship being refinement-stable, universality-class stable, and structurally compatible with the refinement-stable transport-curvature density on a single matched scalar piece. The OP-2 metric-emergence target is closed at this register. Substantively new metric content beyond conformal rescaling is OP-11. (Flat-V) is most plausibly foundational; the OP-13 coarse-grained-emergence alternative is the only realistic derivation route and is a major open programme. Under foundational (Flat-V), the paper's content is relationship-identification between parallel emergent features, not metric derivation from transport.

Final position. The programme possesses, conditional on (Flat-V) most plausibly foundational: a transport-action-derived conformal-factor relationship between an independently emergent flat metric and the refinement-stable transport-curvature observables, with refinement-stability, universality-class stability, single-piece structural compatibility, and explicit identification of the Laplacian content of R(g) as new geometric content not visible to any natural transport-observable scalar. Whether OP-13's coarse-grained-emergence programme eventually upgrades (Flat-V) to a theorem, or whether (Flat-V) is permanently a foundational feature of the programme's spatial layer, is a genuinely open question that the present paper records but does not resolve.