Fock-Space Construction and the Full CAR Algebra in VERSF

Vacuum Structure, the One-Particle Hilbert Space, and the Cross-Relation on the Persistent Spinorial Sector

Keith Taylor *VERSF Theoretical Physics Programme Part V of the Matter-Sector Programme Companion paper to* From Spinorial Source-Carriers to Fermionic Quantization in VERSF

General Reader Summary

Ordinary matter — everything you can see and touch — is made of particles called fermions: electrons, the quarks inside protons and neutrons, and a few others. These particles follow some odd rules. Two electrons can never be in exactly the same state at the same place. Spin one of them around once, and it doesn't return to where it started; you have to spin it around twice. Standard physics writes down rules like these and confirms they match experiment, but it doesn't explain why they have to be true.

This series of papers is trying to do something more ambitious: derive those rules from a deeper layer, the way one might derive the shape of a snowflake from the chemistry of water rather than just observing snowflakes. The "deeper layer" here is called the substrate, and it has its own internal structure that the series builds up step by step.

Earlier papers in the series showed that the two odd rules above can be derived from this deeper layer when certain structural conditions hold. But to complete the picture, three more mathematical pieces had to be filled in. They were the kind of pieces that look unglamorous but are essential — the mathematical equivalent of confirming that a building's foundations are square before adding the roof. The earlier paper flagged them as missing and said: a future paper has to supply these. This is that paper.

It supplies all three. Along the way it adds two pieces of mathematical machinery that go beyond what was strictly missing: a check that the answer doesn't depend on bookkeeping conventions (different reasonable descriptions of the same underlying structure give the same theory), and a more careful mathematical treatment of the case where the substrate's loops are parametrised continuously rather than discretely. Neither was a deferred prerequisite from the earlier paper, but both make the construction less arbitrary-looking and easier to build on.

It also shows that the substrate construction doesn't run into a particular technical trap that snares some other physics theories — a trap that, when it strikes, forces physicists to add elaborate mathematical scaffolding to keep the theory making sense. The substrate construction sidesteps this trap because the structures it's built from are well-behaved in the relevant way.

The paper is also careful about what its main theorem is *not* claiming. It's not claiming that compatibility with special relativity selects the right mathematical setup. It doesn't, and couldn't. That's a separate question, deferred to a future paper.

What this paper *doesn't* do is also worth saying clearly. The version of fermion physics built here is the simplest version — fermions that don't interact with each other or with anything else, sitting in a kind of mathematical idealisation that isn't yet connected to ordinary space and time. To get from here to actual electrons in actual atoms, several more papers are needed: making the construction compatible with Einstein's relativity, adding interactions (and dealing with the infinities those produce), introducing antimatter, identifying which species of fermion is which (electron vs muon vs quark, and so on), and connecting to the rest of the laws of nature. This paper completes one important layer of the foundation; building the upper floors comes later.

The bottom line: this paper shows that the standard mathematical structure used to describe non-interacting fermions can be built up from the deeper substrate rather than postulated. It's not a result anyone outside the field would notice — there's no new prediction, no new experiment. It's a piece of internal scaffolding that, if it had failed, would have made the larger project impossible to complete. It didn't fail.

Abstract

The preceding fermionic-quantisation paper (Part IV) established the substrate-level spin-statistics correspondence and the anticommuting half of the canonical anticommutation relations on the persistent spinorial sector, conditional on the existence of a Fock-space framework and on the Fock-density property. Three structural prerequisites were explicitly deferred: (i) the existence of a positive-definite one-particle Hilbert space $\mathscr{H}_1$ on the persistent spinorial sector; (ii) the antisymmetric Fock-space construction $\mathscr{F}\_A = \bigoplus_n \Lambda^n \mathscr{H}_1$ with vacuum-iterated density; (iii) the cross-relation $\{a_i, a^\dagger_j\} = \delta_{ij}$ that completes the CAR algebra.

This paper supplies the construction, organised in the two-layer architecture inherited from Part IV.

Topological-core† results:

- **Theorem 1† (Admissibility of the positive-definite Hilbert completion).** Theorem 1† does *not* derive Hilbert-space positivity from Lorentz covariance: the Lorentz-invariant Dirac bilinear $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is indefinite and cannot serve as a Hilbert-space norm. What the theorem establishes is more precisely scoped: the persistent spinorial sector admits the standard positive-definite Hilbert completion using the Clifford-internal $\psi^\dagger\psi$ pairing, and within the source-admissibility framework SA1–SA5, no gauge-redundant Krein-space enlargement of the sector is forced, hence no BRST-type quotient is required before Fock completion. Lorentz-covariant realisation of the resulting Fock theory is a separate deliverable (the positivity–covariance bridge, §18 item 10, deferred to Part VI).

- **Theorem 2† (Antisymmetric Fock-space construction with density).** The standard antisymmetric Fock space $\mathscr{F}$_A = $\bigoplus_n \wedge^n \mathscr{H}_1$ exists as a separable Hilbert space; vacuum-iterated states $a^\dagger(\mathcal{C}_1)\cdots a^\dagger(\mathcal{C}_n)|0\rangle$ span a dense subspace.
- **Theorem 3† (Full CAR algebra).** $\{a(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = \langle \mathcal{C}_i \mid \mathcal{C}_j \rangle$, $\{a^\dagger(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = 0$, $\{a(\mathcal{C}_i), a(\mathcal{C}_j)\} = 0$ on $\mathscr{F}$_A.
- **Corollary 4† (Strong Pauli exclusion).** $N(\mathcal{C}) = a^\dagger(\mathcal{C})a(\mathcal{C})$ is a projection with spectrum $\{0, 1\}$ on $\mathscr{F}$_A.
- **Theorem 5† (Vacuum uniqueness and cyclicity).** $|0\rangle$ is the unique unit vector annihilated by every $a(\mathcal{C})$; creation operators generate a dense subspace of $\mathscr{F}$_A from $|0\rangle$.
- **Theorem 6† (Representation-independence under second quantisation).** The substrate-level CAR algebra depends only on the abstract positive-definite Hilbert structure of $\mathscr{H}_1$ for the second-quantisation class of transformations U: $\mathscr{H}_1 \to \mathscr{H}_1'$. Discrete (§5.2) and distributional/rigged-Hilbert (§5.2A) readings yield *-isomorphic CAR C*-algebras under Γ(U). General Bogoliubov-type transformations mixing creation and annihilation operators require the Shale–Stinespring implementability criterion and are deferred to Parts VI–VIII (§13.4).

Physical-realisation result:

- **Corollary 5'† (Physical realisation of CAR observables).** Under the coherence condition of Part IV §4A.3 (set up in §14.1 with emergent-time framing per §3.6), the substrate-level CAR algebra of Theorem 3† is physically manifest as observable creation and annihilation of coherent spinorial transport modes on the coherent entanglement substrate.

Scope. The paper supplies the algebraic-Fock-space level construction completing Part IV's deferred §14 item 1. It does *not* derive: Lorentz-covariant smearing of CAR operators into ψ(x) (Part VI, linked to §18 item 10); microcausality $\{\psi(x), \psi(y)\} = 0$ for spacelike separation (downstream of Part VI smearing); renormalisation (Part VII, where free-vacuum uniqueness no longer applies via Haag's theorem); antiparticle structure (Part VIII); substrate-to-spacetime bridge (Part IX, which supplies the emergent-spacetime semantics implicit in inherited 4-vector notation); species decomposition (Part X); electroweak coupling (Part XI, which would supply the substrate gauge structure that SA5 quotients against); or the full axiomatic-QFT spin-statistics theorem.

Contents

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1. Setting: The §14 Item 1 Deliverable Inherited from Part IV

Part IV established two results that close the spin-statistics question on the persistent spinorial sector at the topological level: Theorem 2† (antisymmetric multi-loop sector exhaustion) and Proposition 4 (anticommuting half of CAR, conditional on a Fock framework and the density property). Both were marked as conditional on the deferred §14 item 1 construction.

This paper supplies the deferred construction: $\mathscr{H}_1$ admitting positive-definite completion (Theorem 1†), antisymmetric $\mathscr{F}$_A with density (Theorem 2†), full CAR including cross-relation (Theorem 3†). With these in hand, Part IV's Proposition 4 becomes unconditional on density, and Part IV's Corollary 6 (Pauli exclusion: $a^\dagger(\mathcal{C})^2 = 0$) strengthens to $N(\mathcal{C}) \in \{0, 1\}$ (Corollary 4† here). Theorem 6† establishes parametrisation-independence under second quantisation.

1.1 The two-layer architecture inherited, and the † convention

The construction inherits Part IV §1.1's two-layer architecture. The topological-core layer consists of the algebraic Fock construction: the abstract $\mathscr{H}_1$ (in either discrete-ONB or rigged-

Hilbert-space-distributional reading), $\mathscr{F}\_A = \bigoplus_n \Lambda^n \mathscr{H}_1$, the CAR operators, the full CAR algebra, and representation-independence. These are standard mathematical constructions on an abstract separable Hilbert space; they admit no substrate-physics dependence at the algebraic level (in the §3.6 sense of "substrate physics" — entanglement-lattice scales ξ, τ_s, m_s, superfluid-transport stability, effective-medium framing). They do, however, carry residual conditionalities flagged below.

The physical-realisation layer consists of the substrate-physical reading: the vacuum as substrate in its no-persistent-loop configuration; $\mathscr{H}_1$ as inner-product completion of coherent single-loop spinorial transport modes; CAR operators as physical creation and annihilation under the coherence condition. The physical-realisation reading inherits the coherence condition of Part IV §4A.3.

The † convention. Throughout this paper, "topological-core" refers to the *layer* in the two-layer architecture, not to a strength claim about being conditionality-free. Topological-core results carry two structural conditionalities flagged in §6:

- **Implicit Part IX inheritance** (§6.5): the SA1–SA5 axioms used by Theorem 1†'s Step 2 currently employ emergent-spacetime 4-vector notation (four-velocity u^μ, four-current J^μ; per §3.5 discipline note); pre-spacetime reformulation is a cross-paper deliverable enabled by Part IX.
- **Substrate-gauge-group conditionality** (§6.4 note, §6.5): SA5's "no gauge-redundant duplicates" performs the gauge-quotient at the loop-label level, but the substrate-level gauge transformations being quotiented are not specified in this paper; the substantive bite of SA1–SA5 depends on the substrate gauge structure being non-trivially restrictive. Specification is a cross-paper deliverable most naturally tied to Part XI.

These conditionalities are inherited by every result that depends on Theorem 1† — which includes Theorems 2–6 and Corollary 4† (transitively). Rather than repeat the qualifier in each section, we adopt:

Convention. A dagger † on a result name (e.g., "Theorem 1†") indicates that the result is topological-core *modulo* the two conditionalities above. The dagger propagates by inheritance: any result whose derivation depends on Theorem 1† also carries †. The two conditionalities are *jointly* discharging — neither alone removes †; both must be discharged (by the cross-paper deliverables of §6.5) before the qualifier can be dropped. Corollary 5'† carries † additionally because it inherits Theorem 3† (and via that Theorem 1†), although its physical-realisation reading carries its own further conditionalities (§14.5) that are noted separately.

This makes the qualification visually scannable without burying the prose. The dagger does not undermine the layer designation; it qualifies the *strength* of "topological-core" by flagging the cross-paper deliverables that would tighten the qualification to unconditional.

1.2 Why admissibility of the positive-definite completion is the central technical question

The standard CAR construction on positive-definite $\mathscr{H}_1$ proceeds without complication. When the inner product is Krein (indefinite), constraint reduction is required (Gupta–Bleuler longitudinal photons; Faddeev–Popov ghosts; covariantly-quantised constrained Lagrangians). The substrate-level question: does P_spin inherit Krein structure? Theorem 1† establishes that it does not. Persistent loops are source-carriers (per Microscopic Origin SA1–SA5), not gauge excitations; standard positive-definite Hilbert completion applies.

1.3 Source-carrier discipline

"Spinorial source-carrier" denotes substrate-level multi-loop spinorial transport with the full algebraic structure derived here, not "physical electron" or "Standard Model fermion." The Fock construction completes the *algebraic* quantisation programme at substrate level; promotion to physical fermions requires species decomposition (Part X), Lorentz-covariant smearing into ψ(x) (Part VI), renormalisation (Part VII).

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2. Notation and Conventions

Conventions of Parts III and IV apply throughout.

Persistent transport manifold. P, as in Parts III §6.3 and IV §2.

Spinorial sector. P_spin, the $U(2\pi) = -\mathbb{1}$ component (Part III Theorem 4).

Single oriented commitment loop. $\mathcal{C} \in$ P_spin, satisfying (L1)–(L4) of Microscopic Origin Definition 2, with closure-orientation frame F(x) and Clifford-internal spinor state $\psi \in \mathbb{C}^4$.

Clifford-internal inner product.

$\langle\psi \mid \varphi\rangle_{\mathbb{C}^4} = \psi^\dagger\varphi$ (positive-definite Hermitian)

distinguished from the Lorentz-invariant indefinite pairing

$(\psi, \varphi)_{Lor} = \bar{\psi}\varphi = \psi^\dagger\gamma^0\varphi$ (signature (2,2)).

The Fock construction uses $\langle \cdot \mid \cdot \rangle_{\mathbb{C}^4}$. The standard textbook precedent is free-Dirac quantisation in the positive-frequency sector (negative-frequency/antiparticle structure deferred to Part VIII). Reconciliation with Lorentz covariance at substrate level is the *positivity–covariance bridge* (§18 item 10).

Substrate-derived measure on P_spin. $d\mu_{spin}(\mathcal{C})$, inherited from Part IV §3.9; used at physical-realisation level only. Topological-core construction is parametrisation-independent (§5.2, Theorem 6†).

One-particle Hilbert space. $\mathscr{H}_1$ — abstract separable Hilbert space with positive-definite inner product, parametrisation-independent (§5).

Antisymmetric Fock space. $\mathscr{F}_A = \bigoplus_{n=0}^\infty \Lambda^n \mathscr{H}_1$, with $\Lambda^0 \mathscr{H}_1 = \mathbb{C}$ spanned by $|0\rangle$.

Creation/annihilation operators. $a^\dagger(\mathcal{C}) : \Lambda^n \to \Lambda^{n+1}$, $a(\mathcal{C}) := (a^\dagger(\mathcal{C}))^*$ (§9).

Fock states. $|\mathcal{C}_1, ..., \mathcal{C}_n\rangle_A := a^\dagger(\mathcal{C}_1)\cdots a^\dagger(\mathcal{C}_n)\, |0\rangle \in \Lambda^n \mathscr{H}_1$.

Number operator. $N(\mathcal{C}) = a^\dagger(\mathcal{C})\, a(\mathcal{C})$; $N = \Sigma_{\mathcal{C}} N(\mathcal{C})$.

Vacuum-iterated dense subspace. $\mathscr{F}_A^0 = \text{span}\{|\mathcal{C}_1, ..., \mathcal{C}_n\rangle_A : n \geq 0, \mathcal{C}_i \in P_{spin}\}$.

Gram kernel. $K(\mathcal{C}, \mathcal{C}') := \langle \mathcal{C} \mid \mathcal{C}'\rangle_{\mathscr{H}_1}$. *The notation $\delta(\mathcal{C}, \mathcal{C}')$ in older inheritance equations is reinterpreted here as the Gram-kernel notation: Kronecker* $\delta_{\mathcal{C},\mathcal{C}'}$ in the discrete reading; in the continuum reading, K is the reproducing kernel of the identity operator on $L^2(P_{spin}, d\mu_{spin})$ — i.e., the Dirac kernel associated with $d\mu_{spin}$ (defining property and positivity stated explicitly in Definition 4).

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3. Structural Dependencies

3.1 Spinorial-sector decomposition (Part III Theorem 4)

Conditional on C1–C3. P_{spin} is the persistence-stable $U(2\pi) = -\mathbb{1}$ sector providing the carriers for the Fock construction.

3.2 Clifford-compatible spinorial structure (Part III Theorem 1)

Each loop $\mathcal{C} \in P_{spin}$ has associated Clifford-internal spinor $\psi \in \mathbb{C}^4$ on which the standard Dirac operator $i\gamma^\mu \partial_\mu - m$ acts. The pairing $\langle \psi \mid \varphi\rangle_{\mathbb{C}^4} = \psi^\dagger\varphi$ is positive-definite (textbook).

3.3 Antisymmetric multi-loop wavefunctions (Part IV Theorem 2)

Persistent multi-loop wavefunctions on P_{spin} are antisymmetric: $\Psi(\mathcal{C}_2, \mathcal{C}_1) = -\Psi(\mathcal{C}_1, \mathcal{C}_2)$. Symmetric sector empty. *Conditional on Part III Theorem 4 + C1–C3 + K=7 → d=3.*

3.4 Anticommuting half of CAR (Part IV Proposition 4)

Given a CAR framework + Fock-density: $\{a^\dagger(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = 0$, $\{a(\mathcal{C}_i), a(\mathcal{C}_j)\} = 0$. Cross-relation deferred. The present paper supplies both prerequisites (Theorems 1, 2) and the cross-relation (Theorem 3†).

3.5 Source-admissibility framework SA1–SA5 (Microscopic Origin paper, Lemma 1)

Persistent loops are source-carriers of the persistent current $J^\mu = \rho_{pers} u^\mu = \Pi_{pers} C^\mu$, satisfying SA1–SA5. The framework distinguishes source-carriers from gauge-redundant excitations — persistent loops are not subject to gauge-constraint reduction analogous to Faddeev–Popov ghosts or Gupta–Bleuler longitudinal photons. This is the structural input used in §6 to establish admissibility of the standard positive-definite Hilbert completion.

Reminder of SA1–SA5 (recalled from Microscopic Origin without re-derivation).

- **SA1 (Current-carrying.)** Each persistent loop $\mathcal{C} \in P_{spin}$ carries a well-defined commitment weight $q_{\mathcal{C}}$ and four-velocity $u_{\mathcal{C}}^\mu$, contributing a four-current $J^\mu_{\mathcal{C}} = q_{\mathcal{C}} u_{\mathcal{C}}^\mu$. Persistent loops are *carriers* of measurable substrate current.
- **SA2 (Orientation-admissible frames.)** $F_{\mathcal{C}}$ satisfies OA1–OA4 (Part III §4.2). The frame is part of the loop's source content, not gauge-redundant labelling.
- **SA3 (Refinement-stability.)** Under the refinement coarsening map δ^*, source-admissibility status is preserved.
- **SA4 (Locality.)** The loop's source-current contribution depends only on its local spatial path and frame structure, not on coupling to gauge fields or auxiliary degrees of freedom.
- **SA5 (No gauge-redundant duplicates.)** Distinct source-admissible loops carry distinct source content. Loops differing only by formal gauge transformations are identified as the same loop in P_{spin}. The gauge-quotient is performed at the loop label space level, not at the Hilbert-space level.

SA1–SA5 together distinguish source-carriers from gauge-redundant excitations: any excitation analogous to a longitudinal photon or Faddeev–Popov ghost fails at least one of SA1–SA5 and does not lie in P_{spin}. This is the structural fact Theorem 1†'s Step 2 exploits.

Discipline note on the 4-vector indices. The indices μ on $u_{\mathcal{C}}^\mu$ and J^μ are *emergent-spacetime descriptions* of substrate-level objects: these quantities acquire their Lorentzian-tensor interpretation via the substrate-to-spacetime bridge of Part IX (§18 item 4). At the substrate level the underlying objects are pre-spacetime; the 4-vector notation here is the emergent-physics interpretation under which the present construction interfaces with standard QFT terminology. Convention inherited from the Microscopic Origin paper.

Note on the substrate-level reformulation desideratum. An ideal version of SA1–SA5 would be stated in genuinely pre-spacetime terms, with the 4-vector form (current/four-velocity) recovered under the Part IX bridge as an emergent consequence. As currently inherited from the Microscopic Origin paper, the axioms use the emergent notation directly. This means Theorem 1†'s conditionality on SA1–SA5 carries an implicit Part IX inheritance, which is acknowledged in §6.5; future revision of the Microscopic Origin paper to a pre-spacetime statement of SA1–SA5 is a desirable cross-paper deliverable that would tighten this conditionality. The present paper proceeds with the inherited notation and flags the dependency.

3.6 Substrate-physics inheritances (Part IV §3.9–§3.11)

For the physical-realisation layer only:

- **Coherent entanglement substrate (entanglement-lattice papers).** Coherence scale ξ, characteristic substrate scale τ_s ("spinorial response time" in emergent description), effective inertial scale m_s.
- **Superfluid persistent transport stability.** Persistent loops are dynamically stable on substrate scales $\gg \tau_s$.
- **Effective-medium framing.** Substrate is an effective coherence medium, not a preferred-frame mechanical ether.

These do *not* enter Theorems 1, 2, 3, Corollary 4†, Theorem 5†, or Theorem 6†.

Discipline note on characteristic scales and emergent time. The scales ξ, τ_s are substrate-derived characteristics. In VERSF's emergent-time discipline, the substrate-level primitives underlying τ_s are sequential-interface-transport count structures (per the σ-duality construction); τ_s is the *emergent-time interpretation* of these substrate-level counts under the substrate-to-spacetime bridge of Part IX. The terminology "time" / "timescale" / "response time" / "exchange timescale" applied to τ_s throughout this paper is the emergent description, inherited from the entanglement-lattice paper's convention. The same discipline applies to ξ: substrate-derived characteristic whose emergent-space interpretation is a spatial coherence length under the same Part IX bridge.

3.7 Standard antisymmetric Fock-space construction (mathematical background)

Standard textbook (Reed–Simon Vol. II §X.7; Bratteli–Robinson Vol. II §5.2). Supplies $\Lambda^n \mathscr{H}_1$, bounded CAR operators with $\|a(f)\| = \|a^\dagger(f)\| = \|f\|$, the full CAR algebra including cross-relation, number-operator spectrum, vacuum uniqueness/cyclicity, and the CAR uniqueness theorem (Bratteli–Robinson Vol. II Theorem 5.2.5) covering second-quantisation $\Gamma(U)$ for unitary $U: \mathscr{H}_1 \to \mathscr{H}_1'$. General Bogoliubov implementability uses the Shale–Stinespring criterion (Shale 1962, Stinespring 1959) — invoked in §13.4. *Inherited once Theorem 1† supplies positive-definite $\mathscr{H}_1$.*

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4. The Vacuum State |0⟩

4.1 Topological-core definition

Definition 1 (Vacuum, topological core). $|0\rangle$ is the unique unit vector spanning $\Lambda^0 \mathscr{H}_1 \cong \mathbb{C}$ in $\mathscr{F}$_A. Equivalently, the unique unit vector with $a(\mathcal{C})\,|0\rangle = 0$ for all $\mathcal{C} \in$ P_spin, characterised up to phase by this annihilation condition + normalisation $\langle 0 \mid 0\rangle = 1$. (Uniqueness is Theorem 5†; existence is standard.)

4.2 Physical-realisation reading

The vacuum corresponds to the coherent entanglement substrate in its *no-persistent-loop configuration*: the substrate with its characteristic structure (coherence at scales $\lesssim \xi$, characteristic scale τ_s in the emergent-time framing, effective inertial scale m_s) but no persistent spinorial transport mode excited. Substrate-level zero-point structure persists; what is absent is *persistent spinorial winding patterns*.

This reading is *interpretive*; the topological-core construction proceeds independently.

4.3 Why vacuum existence is non-trivial in some constructions, and what survives in the present free construction

In some quantisation programmes, vacuum existence is a substantive question rather than a definitional one. The clearest example is Haag's theorem in interacting quantum field theory: the assumption that the interacting and free theories share a common Fock space (and therefore a common vacuum) leads to contradiction; the interacting theory's vacuum is unitarily inequivalent to the free vacuum. Concretely, the interacting vacuum is related to the free vacuum by a Bogoliubov transformation whose off-diagonal "β" part (mixing creation and annihilation operators) fails to satisfy the Hilbert–Schmidt implementability condition identified by Shale–Stinespring (Shale 1962, Stinespring 1959); the framework is developed in §13.4. Because no unitary on the free Fock space can implement the transformation, the interacting vacuum lives in a different representation of the CAR algebra entirely.

For the present construction these obstructions do not arise. The free-Fock construction supplies a unique non-trivial $\Lambda^0 \mathscr{H}_1 \cong \mathbb{C}$ (Theorem 2†), and the annihilation-characterised vacuum is unique on this Fock space (Theorem 5†). Two distinct phenomena should be kept apart:

- *Γ(U)-related representations.* Fock representations $\mathscr{F}$_A and $\mathscr{F}$_A' related by the second-quantisation lift Γ(U) for some unitary $U: \mathscr{H}_1 \to \mathscr{H}_1'$ have unitarily equivalent vacua — $\Gamma(U)|0\rangle = |0'\rangle$. There is "one vacuum up to representation choice." This is precisely Theorem 6†.
- *General Bogoliubov-related representations (β ≠ 0).* Representations related by general Bogoliubov transformations mixing creation and annihilation operators have *genuinely inequivalent* vacua when β fails Hilbert–Schmidt — each representation carries its own vacuum sector with no unitary lift connecting them.

Theorem 5†'s uniqueness of $|0\rangle$ on $\mathscr{F}$_A is consistent with the genuine multiplicity in the $\beta \neq 0$ case but does not address it: each Bogoliubov-inequivalent representation has its own annihilation-characterised vacuum on its own Fock space, and Theorem 5† speaks to $\mathscr{F}$_A only. The §12.1 scope note makes this explicit.

Haag-type obstructions of the kind invoked in interacting QFT arise only when interacting field theory is brought in. At the present free-Fock-construction level they do not appear; the interacting-theory question is deferred to Part VII. The unification statement: Lorentz-boost-induced representation changes (Part VI), antiparticle structure (Part VIII charge conjugation), and vacuum-existence obstructions in interacting QFT (Haag, Part VII) are all instances of a single underlying issue — which transformations of the CAR algebra are Fock-implementable under the Shale–Stinespring criterion. The Γ(U) class to which Theorem 6† applies is precisely the Fock-implementable subclass without recourse to inequivalent representations; everything else lives in §13.4 territory.

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5. The One-Particle Hilbert Space $\mathscr{H}_1$

5.1 Construction

Definition 2 (Single-loop spinorial states). For each $\mathcal{C} \in$ P_spin with spinor $\psi\_\mathcal{C} \in \mathbb{C}^4$ and frame F_$\mathcal{C}$(x), the *single-loop spinorial state* is the formal object

$| \mathcal{C} \rangle \equiv (\mathcal{C}, \psi\_\mathcal{C}, F\_\mathcal{C})$.

It encodes the loop's spatial path in P, its Clifford-internal spinor data, and its orientation frame structure.

Definition 3 (Pre-Hilbert space H_1^pre). Complex vector space of formal finite linear combinations $\Sigma_i c_i | \mathcal{C}_i \rangle$.

5.2 The inner product on H_1^pre — Gram-kernel formulation

Definition 4 (Single-loop inner product as Gram kernel). For $|\mathcal{C}\rangle$, $|\mathcal{C}'\rangle$:

$\langle \mathcal{C} | \mathcal{C}' \rangle := K(\mathcal{C}, \mathcal{C}') \cdot \langle\psi\_\mathcal{C} | \psi\_\{\mathcal{C}'\}\rangle\_\mathbb{C}^4$,

with $K(\mathcal{C}, \mathcal{C}')$ the *Gram kernel* of single-loop states in $\mathscr{H}_1$ and $\langle\psi\_\mathcal{C} | \psi\_\{\mathcal{C}'\}\rangle\mathbb{C}^4 = \psi\_\mathcal{C}\dagger\psi\{\mathcal{C}'\}$ the standard positive-definite Hermitian pairing on $\mathbb{C}^4$.

Explicit form of K in the two readings.

- *Discrete reading.* When P_spin is treated as a discrete label space or the $\{|\mathcal{C}\rangle\}$ are chosen as an ONB of $\mathscr{H}_1$, $K(\mathcal{C}, \mathcal{C}') = \delta\_\{\mathcal{C}, \mathcal{C}'\}$ (Kronecker delta). The $|\mathcal{C}\rangle$ are genuine vectors in $\mathscr{H}_1$; bounded operators construct directly.
- *Continuum / distributional reading.* When P_spin is treated as a continuum with substrate-derived measure dμ_spin, K is the *reproducing kernel of the identity operator on L^2(P_spin, dμ_spin)* — i.e., the Dirac kernel associated with dμ_spin, characterised by

  $\int\_\{P\_spin\}\ K(\mathcal{C}, \mathcal{C}')\ f(\mathcal{C}')\ d\mu\_spin(\mathcal{C}') = f(\mathcal{C})$ for all $f \in \Phi$,

  where Φ is the dense test-function space of the rigged-Hilbert-space construction in §5.2A. This is the substrate-level analogue of the textbook free-Dirac convention $K(k, k') = (2\pi)^3\ \delta^3(k - k')$ for momentum-eigenstate generators.

Positivity condition on the Gram kernel. K($\mathcal{C}$, $\mathcal{C}'$) is Hermitian and positive-semidefinite in the standard reproducing-kernel sense:

$\Sigma_{i,j}\, \bar{c}_i\, c_j\, K(\mathcal{C}_i, \mathcal{C}_j) \geq 0$ for every finite $\{\mathcal{C}_i\} \subset$ P_spin and coefficients $c_i \in \mathbb{C}$.

In the discrete reading this is automatic ($K = \delta_{ij}$ is the kernel of the identity on ℓ^2(P_spin)). In the continuum reading it is also automatic: K is the reproducing kernel of the identity operator on L^2(P_spin, dμ_spin), which is a positive operator, so its kernel is positive-semidefinite by the spectral theorem applied to identity-kernel constructions. The positivity of K together with the positive-definiteness of ⟨ · | · ⟩_$\mathbb{C}^4$ jointly produce positive-semidefiniteness of ⟨ · | · ⟩ on H_1^pre via the Schur/Hadamard product of positive kernels; equality-iff-zero follows as in §6.3 Step 1.

The form of K($\mathcal{C}$, $\mathcal{C}'$) depends on the parametrisation of P_spin as specified above; the topological-core construction (Theorems 1, 2, 3, Corollary 4†, Theorems 5, 6) is stated parametrisation-independently. The substrate-derived measure dμ_spin enters only at the physical-realisation level (§14). Theorem 6† (§13) makes parametrisation-independence rigorous for the second-quantisation class of transformations.

Sesquilinear extension:

$\langle \Sigma_i\, c_i \,|\, \mathcal{C}_i \rangle \,|\, \Sigma_j\, d_j \,|\, \mathcal{C}'_j \rangle\rangle = \Sigma_{i,j}\, \bar{c}_i\, d_j \,\langle\, \mathcal{C}_i \,|\, \mathcal{C}'_j \,\rangle.$

Remark on the choice of Clifford-internal pairing. Definition 4 uses $\psi^\dagger\psi$ rather than the indefinite $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ pairing. This is the standard textbook choice (positive-frequency sector; antiparticle structure to Part VIII) and is necessary for the Fock space to be a true positive-definite Hilbert space. Lorentz invariance is implemented at the level of unitary representations on $\mathscr{F}$_A — the positivity–covariance bridge (§18 item 10), deferred to Part VI.

5.2A Rigged Hilbert-space formulation of the distributional reading

In the continuum parametrisation, single-loop "states" $|\mathcal{C}\rangle$ should not be interpreted as ordinary Hilbert vectors. They are distributional generators analogous to momentum eigenstates $|k\rangle$ in standard free-field theory, satisfying a Dirac-distributional Gram kernel rather than belonging directly to $\mathscr{H}_1$.

The mathematically appropriate framework is a *rigged Hilbert space* (Gelfand triple):

$\Phi \subset \mathscr{H}_1 \subset \Phi^\times$,

where:

- Φ is a dense nuclear test-function space on P_spin with a distinguished positive-definite quadratic form compatible with dμ_spin (the abstract characterisation is preferred to a specific topology on P_spin at this stage, since P_spin's analytical structure is itself a downstream deliverable; see *note on P_spin topology* below),
- $\mathscr{H}_1$ is the Hilbert completion (Definition 5),

- $\Phi^\times$ is the topological dual distribution space.

Note on P_spin topology. The textbook Schwartz space construction in standard free-Dirac quantisation works because momentum space carries a smooth manifold structure with Lebesgue measure. For P_spin the underlying topology and analytical structure are not specified at the level needed to make "Schwartz space" directly meaningful; specifying these is partly a Part IX issue (substrate-to-spacetime bridge). The present §5.2A construction therefore uses the abstract nuclear-space-with-distinguished-positive-definite-quadratic-form characterisation, which preserves the formal rigour of the rigged-Hilbert-space machinery without committing to a particular topology on P_spin. Once Part IX supplies P_spin's analytical structure (via the emergent-spacetime mapping), Φ will acquire a concrete Schwartz-type realisation; the present construction is consistent with that downstream specification.

The distributional generators $|\mathcal{C}\rangle$ lie in $\Phi^\times$ — they are *distributions* on the test-function space, not generically vectors of $\mathscr{H}_1$. Smearing produces genuine Hilbert vectors:

$$|f\rangle := \int_{P\_spin} f(\mathcal{C})\, |\mathcal{C}\rangle\, d\mu\_spin(\mathcal{C}) \in \mathscr{H}_1,\ f \in \Phi.$$

The smeared inner product:

$$\langle f \mid g\rangle_{\mathscr{H}_1} = \int\int \bar{f}(\mathcal{C})\, g(\mathcal{C}')\, K(\mathcal{C}, \mathcal{C}')\, \langle\psi\_\mathcal{C} \mid \psi_{\mathcal{C}'}\rangle\_\mathbb{C}^4\, d\mu\_spin(\mathcal{C})\, d\mu\_spin(\mathcal{C}').$$

Under Definition 4's explicit form of K in the continuum reading (K is the Dirac kernel for dμ_spin), the K-integration evaluates trivially via the reproducing property, reducing the double integral to:

$$\langle f \mid g\rangle_{\mathscr{H}_1} = \int_{P\_spin} \bar{f}(\mathcal{C})\, g(\mathcal{C})\, \langle\psi\_\mathcal{C} \mid \psi\_\mathcal{C}\rangle\_\mathbb{C}^4\, d\mu\_spin(\mathcal{C})$$

— manifestly positive (the integrand is non-negative; positive on supp f ∩ supp g where f, g are non-zero and spinors are non-zero). This is the clean form one expects from the textbook treatment.

The CAR algebra in the continuum reading is therefore understood distributionally. Define smeared CAR operators:

$$a(f) := \int \bar{f}(\mathcal{C})\, a(\mathcal{C})\, d\mu\_spin(\mathcal{C}),\quad a^\dagger(f) := \int f(\mathcal{C})\, a^\dagger(\mathcal{C})\, d\mu\_spin(\mathcal{C}),\ f \in \Phi.$$

These smeared operators are bounded on $\mathscr{F}\_A$ with norm $\|a(f)\| = \|a^\dagger(f)\| = \||f\rangle\|\_{\mathscr{H}_1}$. The CAR relations take the smeared form:

$$\{a(f), a^\dagger(g)\} = \langle f \mid g\rangle_{\mathscr{H}_1} \cdot \mathbb{1}_{\mathscr{F}\_A},\ \{a(f), a(g)\} = 0,\ \{a^\dagger(f), a^\dagger(g)\} = 0,\ f, g \in \Phi.$$

The unsmeared symbols $a(\mathcal{C})$, $a^\dagger(\mathcal{C})$ are operator-valued distributions; rigorous meaning is given by the smeared operators acting on Φ. This is precisely analogous to standard distributional treatment of free Dirac fields in momentum space (Reed–Simon Vol. II §X.7; Streater–Wightman §3.1).

The discrete reading of §5.2 corresponds to the degenerate case $\Phi = \mathscr{H}_1 = \Phi^x$; the distributional reading is the genuine application. By Theorem 6† the two readings produce *-isomorphic CAR algebras under second quantisation.

5.3 The one-particle Hilbert space $\mathscr{H}_1$

Definition 5 (One-particle Hilbert space). $\mathscr{H}_1$ is the completion of H_1^pre under the norm induced by Definition 4. Well-defined provided positive-definiteness holds (Theorem 1†). Separable Hilbert space; H_1^pre dense by construction (discrete reading), or the smeared image of Φ dense (distributional reading).

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6. Theorem 1† — Admissibility of the Positive-Definite Hilbert Completion

6.1 Clarification on what Theorem 1† proves

Theorem 1† should not be read as claiming that Lorentz covariance alone selects a positive-definite spinorial inner product. It does not. The Lorentz-invariant Dirac bilinear $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is indefinite. The claim is more precise: once the persistent spinorial sector is identified as a source-carrier sector rather than a gauge-redundant constrained sector, the appropriate one-particle Hilbert norm is the positive-definite $\psi^\dagger\psi$, not the indefinite Lorentz bilinear.

The theorem has two components:

- *Hilbert positivity by construction:* completion using $\psi^\dagger\psi$ + positive-semidefinite Gram kernel K. Positivity of $\psi^\dagger\psi$ is standard textbook; positivity of K is automatic in both readings (Definition 4).
- *No ghost/Krein obstruction:* SA1–SA5 rules out the need for indefinite constrained-space enlargement.

Lorentz covariance is deferred to the positivity–covariance bridge of §18 item 10 (Part VI). This §6.1 paragraph is the primary statement of the Lorentz-vs-positivity disclaimer in the body of the paper; subsequent sections refer back here rather than repeating.

6.2 The theorem

Theorem 1† (Admissibility of the positive-definite Hilbert completion). The persistent spinorial sector admits the standard positive-definite Hilbert completion using the Clifford-internal $\psi^\dagger\psi$ pairing. Within the source-admissibility framework SA1–SA5, no gauge-redundant Krein-space enlargement of the sector is forced, and therefore no BRST-type quotient is required before Fock completion. Lorentz-covariant realisation of the resulting Fock theory is a separate deliverable (the positivity–covariance bridge, §18 item 10, deferred to Part VI).

Equivalently, with two components established by §6.3:

- *(Algebraic positivity.)* Definition 4's inner product is positive-semidefinite on H_1^pre with equality-iff-zero. H_1^pre completes to a positive-definite separable Hilbert space $\mathscr{H}_1$.
- *(No Krein-space obstruction.)* No covariantly-quantised Krein-space enlargement $\tilde{P}$_spin ⊃ P_spin followed by BRST-type physical-subspace selection is required. SA5 performs the gauge-quotient at the loop-label level.

Conditional on: Part III Theorem 4 (under C1–C3) + source-admissibility SA1–SA5 (Microscopic Origin Lemma 1, with implicit Part IX inheritance per §6.5).

6.3 Proof

Step 1 — Algebraic positivity of Definition 4.

Take $\Psi = \Sigma_i c_i \mid \mathcal{C}_i \rangle \in H_1$^pre with the $\mathcal{C}_i$ distinct:

$\langle \Psi \mid \Psi \rangle = \Sigma_{i,j} \bar{c}_i c_j K(\mathcal{C}_i, \mathcal{C}_j) \langle \psi\_\{\mathcal{C}_i\} \mid \psi\_\{\mathcal{C}_j\} \rangle\_\mathbb{C}^4$.

In the discrete reading $K = \delta_{ij}$ for distinct $\mathcal{C}_i$, so $\langle \Psi \mid \Psi \rangle = \Sigma_i |c_i|^2 \cdot \|\psi\_\{\mathcal{C}_i\}\|^2\_\mathbb{C}^4 \geq 0$, with equality iff every $c_i = 0$.

For non-distinct collections, the Schur-product structure (positive-semidefinite K Hadamard-multiplied with positive-definite $\langle \cdot \mid \cdot \rangle\_\mathbb{C}^4$) supplies positivity directly.

In the distributional reading (§5.2A) with K as the Dirac kernel for dμ_spin, the double integral collapses via the reproducing property to $\langle f \mid f \rangle = \int |f(\mathcal{C})|^2 \|\psi\_\mathcal{C}\|^2\_\mathbb{C}^4 \, d\mu\_spin(\mathcal{C}) \geq 0$, with equality iff f = 0 almost everywhere.

Step 2 — Why the positive-definite Hilbert pairing is admissible rather than arbitrary.

The algebraic positivity follows from the ψ†ψ choice + Gram-kernel positivity. The theorem is not deriving positivity from Lorentz invariance (§6.1); it identifies which of two standard pairings is physically appropriate.

There are two pairings available: positive-definite ψ†ψ (suitable as Hilbert norm) and Lorentz-covariant but indefinite $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$. Standard relativistic fermion quantisation uses the first as Hilbert norm, with Lorentz covariance implemented via field transformation; the present construction follows.

The only reason this separation would fail is if P_spin were a constrained gauge-redundant sector requiring indefinite auxiliary state space. SA1–SA5 excludes this: persistent loops are source-carriers (SA1) with admissible frames (SA2), refinement-stable (SA3), local (SA4), no gauge-redundant duplicates (SA5). The gauge-quotient is at the loop label space level (SA5), not the Hilbert-space level. No Gupta–Bleuler/BRST/ghost-type construction required.

Concretely, the analogue of "gauge redundancy" would be a state $|\mathcal{C}\rangle$ failing one or more of SA1–SA5; such states do not lie in P_spin by construction. H_1^pre inherits no Krein-space structure.

The role of SA1–SA5 is not to make ψ†ψ positive — it already is. Its role is to justify ψ†ψ as the correct physical Hilbert pairing because the sector is gauge-non-redundant at the loop-label level. ■

6.4 Structural reading

Theorem 1† closes the §14 item 1 prerequisite. The substrate-level construction takes the standard form rather than indefinite-metric variant.

Note on what Step 2 is actually doing. Step 2 is a structural consequence of how P_spin is defined, not an independent derivation. Source-admissibility selects the gauge-non-redundant sector at the loop-label level; the Hilbert-space construction inherits gauge-non-redundancy by inheritance rather than by quotient. The substrate analogue of the Gupta–Bleuler covariantly-quantised state space — a hypothetical P̃_spin ⊃ P_spin with indefinite Lorentz-covariant pairing — is not constructed here, and whether such an enlargement exists as a natural substrate-physics object is open. Within the SA-selected sector, the standard positive-definite Fock construction applies directly; whether the substrate admits a parallel covariantly-quantised enlargement is a separate question linked to §18 item 10.

The §6.5 admissibility claim should not be read as "the construction has avoided gauge-fixing work." More accurately: the gauge-fixing has been performed at the loop-label level by SA5, and the Hilbert-space construction inherits gauge-non-redundancy as a downstream consequence. Structural work moved, not skipped.

Note on the substrate-level gauge group. SA5 ("no gauge-redundant duplicates") performs the gauge-quotient at the loop-label level, but the present paper does not specify *what gauge transformations are being quotiented out* at the substrate level. In standard gauge theory, this would be the local gauge group action on field configurations; at the substrate level, this presumably involves frame-bundle automorphisms (acting on the OA1–OA4 frame structure of persistent loops) and/or symmetries internal to the K=7 architecture. The substantive specification of the substrate gauge group is a deliverable for a future paper — most naturally tied to Part XI (the substrate-to-electroweak bridge), since electroweak gauge structure would be the most direct emergent counterpart, though the substrate-level question is logically prior. Until that specification is in place, the bite of SA1–SA5 as a "substantive characterisation of source-carriers" is itself partially conditional on the substrate gauge structure being non-trivially restrictive; Step 2 should be read with this understanding. This is the structural counterpart to the §6.5 note on SA1–SA5's emergent-spacetime content.

6.5 Conditionality

Theorem 1† inherits:

- **Part III Theorem 4** (spinorial-sector existence; conditional on C1–C3).
- **Microscopic Origin Lemma 1** (SA1–SA5; gauge-non-redundancy for Step 2).
- **Standard positivity of $\psi^\dagger\psi$ on $\mathbb{C}^4$** (textbook). Standard input across all fermionic quantisation programmes; not derived from deeper structure in any of them; *not a circularity concern*.
- **Positive-semidefiniteness of K** (Definition 4). Automatic in both discrete and continuum readings — discrete via Kronecker δ being the kernel of identity on ℓ^2; continuum via K being the reproducing kernel of the identity operator on $L^2(d\mu\_spin)$ (Definition 4 specification).

Cross-paper deliverables jointly discharging †. Theorem 1† carries two structural conditionalities that together constitute †. Per the §1.1 convention, they are *jointly* discharging — neither alone removes †; both must be addressed (by their respective cross-paper deliverables) before the qualifier can be dropped. The two conditionalities are:

- **Implicit Part IX inheritance.** The §3.5 discipline note observes that SA1's 4-vector content (four-velocity $u\_\mathcal{C}^\mu$, four-current $J^\mu\_\mathcal{C}$) is an emergent-spacetime description acquired via the Part IX substrate-to-spacetime bridge. The inheritance of SA1–SA5 by Theorem 1†'s Step 2 therefore carries an implicit Part IX dependency: the strength of Theorem 1† as a topological-core result is conditional on the structural content of SA1–SA5 being preservable in a pre-spacetime form (with the 4-vector notation recovered as emergent under the Part IX bridge). The §3.5 note flags this as a desirable but currently unmet cross-paper deliverable. *Discharge:* pre-spacetime reformulation of SA1–SA5 in the Microscopic Origin paper (or a successor) such that the 4-vector form follows downstream under Part IX.
- **Substrate-gauge-group conditionality (per §6.4 note).** Theorem 1†'s Step 2 derives gauge-non-redundancy from SA1–SA5, but the substantive content of SA1–SA5 itself depends on the substrate gauge structure being non-trivially restrictive — i.e., the substrate-level gauge transformations being quotiented by SA5 must be a genuine, non-trivial group action, since otherwise SA5 has no bite. The present paper does not specify the substrate gauge group. *Discharge:* substrate-gauge-group specification, most naturally as part of Part XI's electroweak bridge, though the substrate-level question is logically prior.

Discharging either alone leaves † in place; discharging both removes it. The two are not independent in any operationally useful sense — both are structural-content questions about SA1–SA5, and until both are addressed SA1–SA5 is not fully grounded as the substrate-level axiomatic input it is treated as here.

Modulo these joint conditionalities, Theorem 1† is topological-core. It does *not* depend on the substrate-physics inheritances of Part IV §3.9–§3.11. Per the §1.1 propagation rule, † inherits to every downstream result depending on Theorem 1†.

6.6 What Theorem 1† does not establish

Scope. Genuine out-of-scope items. Cross-paper deliverables that would discharge † are listed separately in §6.5, since they are upstream prerequisites rather than things Theorem 1† was attempting to establish.

- *Lorentz-invariance of the inner product.* See §6.1 for the precise scoping. Deferred to Part VI positivity–covariance bridge (§18 item 10).
- *Inner products for non-persistent-sector states.* Theorem 1† applies to P_spin only.
- *Inner products under interaction.* Theorem 1† is a free-construction result; deferred to Part VII.
- *Existence of the covariantly-quantised enlargement $\tilde{P}$_spin.* Open; connects to §18 item 10.

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7. The Antisymmetric Fock Space $\mathscr{F}$_A

7.1 Antisymmetric n-particle spaces

Definition 6 (Antisymmetric n-fold tensor product). For $n \geq 1$:

$\wedge^n \mathscr{H}_1 = \{T \in \mathscr{H}_1^{\otimes n} : (E\pi)\, T = \mathrm{sign}(\pi) \cdot T \text{ for all } \pi \in S_n\}$,

the image of the antisymmetrisation projector $P\_A = (1/n!)\, \Sigma\_\pi\, \mathrm{sign}(\pi) \cdot E\pi$. For $n = 0$, $\wedge^0 \mathscr{H}_1 = \mathbb{C}$.

Inner product on $\wedge^n \mathscr{H}_1$: $\langle f_1 \wedge \cdots \wedge f_n \mid g_1 \wedge \cdots \wedge g_n\rangle = \det[\langle f_i \mid g_j\rangle\_\mathscr{H}_1]$. Standard Gram-determinant formula (Reed–Simon Vol. II §X.7); real, non-negative, positive-definite on $\wedge^n \mathscr{H}_1$.

7.2 The antisymmetric Fock space $\mathscr{F}$_A

Definition 7 (Antisymmetric Fock space). $\mathscr{F}\_A = \bigoplus\_\{n=0\}^\infty \wedge^n \mathscr{H}_1$ with orthogonal direct-sum inner product. Vector $\Psi = (\Psi^{(0)}, \Psi^{(1)}, ...)$ with $\Psi^{(n)} \in \wedge^n \mathscr{H}_1$, $\Sigma\_n \|\Psi^{(n)}\|^2 < \infty$. Separable (inherits from $\mathscr{H}_1$).

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8. Theorem 2† — Fock-Space Construction with Density

8.1 The theorem

Theorem 2† (Antisymmetric Fock-space construction with vacuum-iterated density). $\mathscr{F}$_A is a separable Hilbert space. The vacuum-iterated subspace $\mathscr{F}\_A^0 = \mathrm{span}\{|\mathcal{C}_1, ..., \mathcal{C}_n\rangle\_A : n \geq 0, \mathcal{C}_i \in P\_spin\}$ is dense in $\mathscr{F}$_A. Fock-density (required by Part IV Proposition 4) holds.

8.2 Proof

Separability. $\mathcal{H}_1$ separable (Definition 5); $\wedge^n \mathcal{H}_1$ inherits from $\mathcal{H}_1^{\otimes n}$ (Reed–Simon II Theorem II.10); direct sum of countably many separable spaces is separable. Hence $\mathcal{F}_A$ separable.

Density of $\mathcal{F}_A^0$. Suffices to show $\mathcal{F}_A^0$ dense in each $\wedge^n \mathcal{H}_1$. For $n = 0$: $|0\rangle$ spans $\wedge^0 \mathcal{H}_1$. For $n \geq 1$: $|\mathcal{C}_1, ..., \mathcal{C}_n\rangle_A = \sqrt{n!} \cdot P_A(|\mathcal{C}_1\rangle \otimes \cdots \otimes |\mathcal{C}_n\rangle)$; single-loop states (or their smeared versions) span dense subspace of $\mathcal{H}_1$; antisymmetric tensor products span dense subspace of $\wedge^n \mathcal{H}_1$. By orthogonal direct sum, $\mathcal{F}_A^0$ dense in $\mathcal{F}_A$. ■

8.3 Structural reading

Theorem 2† closes the Fock-density prerequisite. Part IV Proposition 4 becomes unconditional on density:

$\{a^\dagger(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = 0$, $\{a(\mathcal{C}_i), a(\mathcal{C}_j)\} = 0$ on $\mathcal{F}_A$.

8.4 Conditionality

Inherits Theorem 1† + standard Fock construction. Purely topological-core (modulo §6.5 implicit Part IX inheritance propagated through Theorem 1†).

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9. Creation and Annihilation Operators on $\mathcal{F}_A$

9.1 Definition

Definition 8 (Creation operators). $a^\dagger(\mathcal{C}) : \mathcal{F}_A \to \mathcal{F}_A$ with $a^\dagger(\mathcal{C})\,|0\rangle = |\mathcal{C}\rangle$, $a^\dagger(\mathcal{C})\,|\mathcal{C}_1, ..., \mathcal{C}_n\rangle_A = |\mathcal{C}, \mathcal{C}_1, ..., \mathcal{C}_n\rangle_A$. Sesquilinear + continuous extension to $\mathcal{F}_A$ in discrete reading; smeared $a^\dagger(f)$ for $f \in \Phi$ in distributional reading.

Definition 9 (Annihilation operators). $a(\mathcal{C}) := (a^\dagger(\mathcal{C}))^*$. On vacuum-iterated states (index $k = 1, ..., n$):

$$a(\mathcal{C})\,|\mathcal{C}_1, ..., \mathcal{C}_n\rangle_A = \sum_{k=1}^{n} (-1)^{k-1} \langle \mathcal{C} \mid \mathcal{C}_k\rangle \cdot |\mathcal{C}_1, ..., \mathcal{C}_k, ..., \mathcal{C}_n\rangle_A,$$

$a(\mathcal{C})\,|0\rangle = 0$. Smeared $a(f)$ similarly.

9.2 Boundedness and domain — discrete vs distributional readings

Discrete reading. When $\{|\mathcal{C}\rangle\}$ forms an ONB (or lies in $\mathcal{H}_1$ directly), CAR operators are *bounded on all of $\mathcal{F}_A$* with norm $\|a^\dagger(\mathcal{C})\| = \|a(\mathcal{C})\| = \||\mathcal{C}\rangle\|_{\mathcal{H}_1}$ (Reed–Simon Vol. II §X.7). Antisymmetry forces single-mode states to vanish after one application — algebraic origin of Pauli exclusion at the operator level.

Distributional reading. When P_spin is treated as a continuum and $|\mathcal{C}\rangle$ are distributional generators in $\Phi^\times$ (§5.2A), the unsmeared symbols $a(\mathcal{C})$, $a^\dagger(\mathcal{C})$ are *operator-valued distributions*, not bounded operators on $\mathscr{F}$_A. Bounded operators arise only after smearing against test functions: for $f \in \Phi$,

$a(f) := \int \bar{f}(\mathcal{C})\, a(\mathcal{C})\, d\mu\_spin(\mathcal{C})$, $a^\dagger(f) := \int f(\mathcal{C})\, a^\dagger(\mathcal{C})\, d\mu\_spin(\mathcal{C})$,

are bounded on $\mathscr{F}$_A with norm $\|a(f)\| = \|a^\dagger(f)\| = \||f\rangle\|\_\mathscr{H}_1$. The present paper follows standard QFT convention treating unsmeared symbols distributionally.

In either reading, substrate-level CAR operators (discrete or smeared) are bounded on a separable Hilbert space. By Theorem 6† the two readings yield *-isomorphic CAR algebras under second quantisation.

9.3 The CAR algebra as a C*-algebra

The bounded operators $\{a(\mathcal{C}), a^\dagger(\mathcal{C})\}$ (discrete) or $\{a(f), a^\dagger(f) : f \in \Phi\}$ (distributional) generate, via the *norm-closure of the -algebra they generate inside B($\mathscr{F}$_A), a unital C*-algebra. Explicitly: A_0 = *-algebra under linear combinations, products, Hermitian conjugation; $\mathcal{A}$_CAR = $A_0 \subseteq$ B($\mathscr{F}$_A) in operator norm.

This C*-algebraic structure allows embedding in the operator-algebraic QFT formulation (Haag, Bratteli–Robinson) for Part VI's Lorentz-invariant extension.

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10. Theorem 3† — The Full CAR Algebra

10.1 The theorem

Theorem 3† (Full canonical anticommutation relations). On $\mathscr{F}$_A:

$\{a(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = \langle \mathcal{C}_i \mid \mathcal{C}_j\rangle \cdot \mathbb{1}\_\{\mathscr{F}\_A\}$, $\{a^\dagger(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = 0$, $\{a(\mathcal{C}_i), a(\mathcal{C}_j)\} = 0$,

for all $\mathcal{C}_i, \mathcal{C}_j \in$ P_spin. With substrate-derived Gram kernel: $\{a(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = K(\mathcal{C}_i, \mathcal{C}_j) \cdot \langle\psi\_\{\mathcal{C}_i\} \mid \psi\_\{\mathcal{C}_j\}\rangle\mathbb{C}^4 \cdot \mathbb{1}\{\mathscr{F}\_A\}$, reducing to $\{a_i, a^\dagger_j\} = \delta_{ij}$ in ONB (discrete) reading or $\{a(f), a^\dagger(g)\} = \langle f \mid g\rangle\_\mathscr{H}_1$ in smeared (distributional) reading.

10.2 Proof

Anticommuting half: Part IV Proposition 4, promoted by Theorem 2† (density). Inherited.

Cross-relation: standard antisymmetric tensor-product calculus. The argument is case-independent — $\alpha = \beta$ and $\alpha \neq \beta$ are handled by the same calculation, with only the leading inner product differing:

For $\mathcal{C}_\alpha, \mathcal{C}_\beta \in P_{spin}$ (general, allowing $\alpha = \beta$ or $\alpha \neq \beta$), compute $a(\mathcal{C}_\alpha)\, a^\dagger(\mathcal{C}_\beta)\, |\mathcal{C}_1, \ldots, \mathcal{C}_n\rangle_A$. After $a^\dagger(\mathcal{C}_\beta)$ prepends $\mathcal{C}_\beta$ to give the (n+1)-loop state $|\mathcal{C}_\beta, \mathcal{C}_1, \ldots, \mathcal{C}_n\rangle_A$, applying $a(\mathcal{C}_\alpha)$ by Definition 9 with ℓ-indexing (ℓ=1 on the prepended $\mathcal{C}_\beta$; ℓ=k+1 ↔ $\mathcal{C}_k$ for k = 1, ..., n) yields:

- *ℓ=1:* contributes $\langle \mathcal{C}_\alpha | \mathcal{C}_\beta\rangle \cdot |\mathcal{C}_1, \ldots, \mathcal{C}_n\rangle_A$.
- *ℓ=k+1:* contributes $(-1)^k \cdot \langle \mathcal{C}_\alpha | \mathcal{C}_k\rangle \cdot |\mathcal{C}_\beta, \mathcal{C}_1, \ldots, \hat{\mathcal{C}}_k, \ldots, \mathcal{C}_n\rangle_A$.

Meanwhile:

$a^\dagger(\mathcal{C}_\beta)\, a(\mathcal{C}_\alpha)\, |\mathcal{C}_1, \ldots, \mathcal{C}_n\rangle_A = \Sigma_{k=1}^n (-1)^{k-1} \cdot \langle \mathcal{C}_\alpha | \mathcal{C}_k\rangle \cdot |\mathcal{C}_\beta, \mathcal{C}_1, \ldots, \hat{\mathcal{C}}_k, \ldots, \mathcal{C}_n\rangle_A$.

Summing the two: only the ℓ=1 term survives; the ℓ=k+1 contributions cancel against the $a^\dagger(\mathcal{C}_\beta)\, a(\mathcal{C}_\alpha)$ terms identically via the $(-1)^k + (-1)^{k-1} = 0$ pairing. Hence

$\{a(\mathcal{C}_\alpha), a^\dagger(\mathcal{C}_\beta)\} = \langle \mathcal{C}_\alpha | \mathcal{C}_\beta\rangle \cdot \mathbb{1}$

on vacuum-iterated states, in both the $\alpha = \beta$ and $\alpha \neq \beta$ cases. (Case-independence is intrinsic: the only difference between cases is whether the surviving leading inner product reads $\langle \mathcal{C}|\mathcal{C}\rangle$ or $\langle \mathcal{C}_i|\mathcal{C}_j\rangle$, which is incidental to the cancellation argument.) Extends to operator identity on $\mathscr{F}_A$ by Theorem 2† (density) + bounded continuation. ■

10.3 Structural reading

Theorem 3† closes the §14 item 1 algebraic deliverable from Part IV. The chain from Part III through this paper is complete:

K=7 (d ≥ 3) + κ-field uniqueness + Schrödinger→Dirac → Clifford structure (Part III Theorem 1) → SU(2) lift (Part III SI) → spinorial U(2π) = −𝟙 (Part III Theorem 4) → F–R holonomy (Part IV Theorem 1) → antisymmetric multi-loop (Part IV Theorem 2) → spin-statistics (Part IV Theorem 3) → admissibility (Theorem 1† here) → Fock space + density (Theorem 2†) → full CAR (Theorem 3†).

Every step inherited from named result or derived structural consequence.

10.4 Conditionality

Inherits Theorem 2† + Part IV Proposition 4 + standard cross-relation derivation. Topological-core modulo §6.5 implicit Part IX inheritance.

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11. Corollary 4† — Strong Pauli Exclusion

11.1 The corollary

Corollary 4† (Strong Pauli exclusion). For $\mathcal{C} \in$ P_spin with $\langle\mathcal{C}|\mathcal{C}\rangle = 1$, $N(\mathcal{C}) := a^\dagger(\mathcal{C})\, a(\mathcal{C})$ is a projection ($N^2 = N$) with spectrum {0, 1} on $\mathscr{F}$_A.

11.2 Argument

From Theorem 3†: $a(\mathcal{C})\, a^\dagger(\mathcal{C}) = \mathbb{1} - N(\mathcal{C})$. Then $N(\mathcal{C})^2 = a^\dagger(\mathcal{C})\, a(\mathcal{C})\, a^\dagger(\mathcal{C})\, a(\mathcal{C}) = a^\dagger(\mathcal{C})[\mathbb{1} - N(\mathcal{C})]\, a(\mathcal{C}) = a^\dagger(\mathcal{C})\, a(\mathcal{C}) - a^\dagger(\mathcal{C})\, N(\mathcal{C})\, a(\mathcal{C})$. Since $N(\mathcal{C})\, a(\mathcal{C}) = a^\dagger(\mathcal{C})\, a(\mathcal{C})^2 = 0$ (anticommuting half), $N(\mathcal{C})^2 = N(\mathcal{C})$. Self-adjoint projection has spectrum ⊆ {0, 1}. Both eigenvalues realised: $N(\mathcal{C})\, |0\rangle = 0$; $N(\mathcal{C})\, |\mathcal{C}\rangle = |\mathcal{C}\rangle$. ■

11.3 Structural reading

Corollary 4† strengthens Part IV Corollary 6 to the spectral form. Each single-loop mode is occupied (eigenvalue 1) or unoccupied (eigenvalue 0). Spectral form of Pauli exclusion underlying atomic structure, chemical bonding, Fermi–Dirac statistics at substrate level.

11.4 Total number operator

$N := \Sigma_{\{\mathcal{C} \in P\_spin\}} N(\mathcal{C})$ is well-defined on finite-particle-number sector with integer spectrum {0, 1, 2, ...} on $\mathscr{F}$_A. Each $\Lambda^n \mathscr{H}_1$ is the eigenspace with eigenvalue n.

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12. Theorem 5† — Vacuum Uniqueness and Cyclicity

12.1 The theorem

Theorem 5†. *(Uniqueness on $\mathscr{F}$_A.)* $|0\rangle$ is the unique (up to phase) unit vector in $\mathscr{F}$_A with $a(\mathcal{C})\, |\Psi\rangle = 0$ for all $\mathcal{C} \in$ P_spin. *(Cyclicity.)* $\mathscr{F}\_A^0$ is dense in $\mathscr{F}$_A; $|0\rangle$ is cyclic for $\mathcal{A}$_CAR.

Scope note. Uniqueness here is *on the specific Fock space $\mathscr{F}$_A* constructed in §7 over the specific $\mathscr{H}_1$ of §5. It is not uniqueness across all Fock representations of the CAR algebra over $\mathscr{H}_1$; distinct representations arising from non-Γ(U) Bogoliubov transformations (β ≠ 0; §13.4) carry their own inequivalent vacuum sectors. Γ(U)-related representations have unitarily equivalent vacua (per Theorem 6†) — $\Gamma(U)|0\rangle = |0'\rangle$ — and are not a source of genuine multiplicity; only the β ≠ 0 case produces inequivalent vacua. See §4.3 for the conceptual distinction.

12.2 Argument

Uniqueness via ONB argument. Suppose $a(\mathcal{C})\, |\Psi\rangle = 0$ for all $\mathcal{C}$, $\langle\Psi|\Psi\rangle = 1$. Decompose $|\Psi\rangle = \Sigma_n \Psi^{(n)}$, $\Psi^{(n)} \in \Lambda^n \mathscr{H}_1$. Condition forces $a(\mathcal{C})\, \Psi^{(n)} = 0$ separately for each $n \geq 1$.

Pick ONB $\{e_i : i \in \mathbb{N}\}$ of $\mathscr{H}_1$. Expand $\Psi^{(n)} = \Sigma_{\{i_1 < \cdots < i_n\}} c_{\{i_1\cdots i_n\}} \cdot e_{\{i_1\}} \wedge \cdots \wedge e_{\{i_n\}}$. For each j:

$a(e_j)\,\Psi^{(n)} = \Sigma_{i_1 < \cdots < i_n : j \in \{i_k\}}\, \varepsilon_{k(j)} \cdot c_{i_1 \cdots i_n} \cdot e_{i_1} \wedge \cdots \wedge \hat{e}_{i_{k(j)}} \wedge \cdots \wedge e_{i_n}$,

with k(j) the position of j and $\varepsilon_{k(j)} = (-1)^{k(j)-1}$. Image vectors are basis elements of $\wedge^{n-1} \mathscr{H}_1$, orthogonal across distinct (n−1)-tuples. $a(e_j)\,\Psi^{(n)} = 0$ forces every $c_{i_1 \cdots i_n}$ with $j \in \{i_1, ..., i_n\}$ to vanish. Running over all j: every coefficient vanishes, so $\Psi^{(n)} = 0$ for all $n \geq 1$. Combined with normalisation: $|\Psi\rangle = e^{i\alpha}\,|0\rangle$.

Cyclicity. Established by Theorem 2†. ■

12.3 Structural reading

$|0\rangle$ uniquely characterised by annihilation on $\mathscr{F}_A$; creation operators generate $\mathscr{F}_A$ from $|0\rangle$. Fock construction complete and minimal on this representation. Together with Theorems 3, 6 + Corollary 4† + §11.4: substrate-level free fermionic theory has all standard structure of textbook free Dirac fermion theory at the algebraic level.

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13. Theorem 6† — Representation-Independence Under Second Quantisation

13.1 The theorem

Theorem 6† (Representation-independence under second quantisation). The substrate-level CAR algebra of Theorem 3† depends only on the abstract positive-definite Hilbert structure of $\mathscr{H}_1$ for the class of transformations arising from second quantisation of unitary maps on $\mathscr{H}_1$.

Specifically: let $\mathscr{H}_1$ and $\mathscr{H}_1'$ be two separable Hilbert spaces of the same Hilbert dimension, with positive-definite inner products derived from different parametrisations of the persistent spinorial sector (e.g., discrete reading §5.2 vs continuum/distributional reading §5.2A, or two different $d\mu_{spin}$ calibrations). Let $U : \mathscr{H}_1 \to \mathscr{H}_1'$ be any unitary isomorphism. The *second-quantisation lift* $\Gamma(U) : \mathscr{F}_A \to \mathscr{F}_A'$ defined by

$\Gamma(U)\,|0\rangle = |0'\rangle$, $\Gamma(U)(f_1 \wedge \cdots \wedge f_n) = U(f_1) \wedge \cdots \wedge U(f_n)$

extended by linearity and continuity is a unitary isomorphism, inducing a *-isomorphism

$\alpha_U : \mathcal{A}_{CAR} \cong \mathcal{A}_{CAR}'$

between the CAR C*-algebras of §9.3, with $\alpha_U(a(f)) = a'(Uf)$, $\alpha_U(a^\dagger(f)) = a'^\dagger(Uf)$.

13.2 Argument

Standard CAR uniqueness theorem (Bratteli–Robinson Vol. II Theorem 5.2.5): the CAR C*-algebra over a separable Hilbert space is determined up to *-isomorphism by the Hilbert dimension, with the second-quantisation functor Γ supplying the lifting. Any unitary $U : \mathscr{H}_1 \to \mathscr{H}_1'$ lifts canonically to:

- Unitary $\Gamma(U) : \mathscr{F}\_A \to \mathscr{F}\_A'$ on antisymmetric Fock spaces by the above formula.
- *-isomorphism $\alpha\_U : \mathcal{A}\_CAR \to \mathcal{A}\_CAR'$ with $\alpha\_U(A) = \Gamma(U)\ A\ \Gamma(U)^{-1}$.

CAR relations preserved because U preserves the inner product: ⟨Uf | Ug⟩{𝓗₁'} = ⟨f | g⟩{𝓗₁}, hence

{α_U(a(f)), α_U(a†(g))} = {a'(Uf), a'†(Ug)} = ⟨Uf | Ug⟩{𝓗₁'} = ⟨f | g⟩{𝓗₁} = α_U({a(f), a†(g)});

similarly for the anticommuting halves.

Inherited from standard CAR uniqueness; applies directly to substrate-level $\mathscr{H}_1$ in either discrete or distributional reading. ■

13.3 Structural reading

Theorem 6† elevates the substrate-level Fock construction from depending on a specific substrate parametrisation to depending only on the abstract Hilbert structure of $\mathscr{H}_1$ — *for the second-quantisation class of transformations*. The structural consequences:

- Discrete reading (§5.2, ONB) and distributional/rigged-Hilbert reading (§5.2A, Gelfand triple $\Phi \subset \mathscr{H}_1 \subset \Phi^\times$) yield *-isomorphic CAR algebras. Choice of reading is parametrisation convenience.
- Different reasonable substrate-physics calibrations of dμ_spin yield CAR algebras isomorphic up to standard CAR uniqueness, provided the calibrations are related by a unitary on $\mathscr{H}_1$ (which is the generic case for reasonable calibrations).
- The substrate-level fermionic-CAR structure is uniquely determined (up to standard CAR isomorphism on this class) by the Hilbert dimension of P_spin's Hilbert space.

This is the substrate-level analogue of the standard textbook fact that free-Dirac CAR algebras over momentum-space and over position-space (related by Fourier transform — a unitary on $\mathscr{H}_1$) are *-isomorphic.

On parametrisation-dependence of physical observables. Theorem 6† does *not* claim physical observables are independent of parametrisation — only that the *abstract algebraic structure* is. Particular observables (position-localised vs momentum-localised mode counts) live in different concrete representations of the same abstract CAR algebra; the correspondence is α_U.

13.4 Scope of Theorem 6† vs general Bogoliubov implementability

Theorem 6† covers the second-quantisation functor Γ(U) for unitaries $U : \mathscr{H}_1 \to \mathscr{H}_1'$. The broader CAR-uniqueness landscape involves *Bogoliubov transformations* that mix creation and annihilation operators — i.e., transformations of the form

$a(f) \mapsto a(\alpha f) + a^\dagger(\beta\bar{f}),\ a^\dagger(f) \mapsto a^\dagger(\alpha f) + a(\beta\bar{f}),$

with α, β bounded operators on $\mathscr{H}_1$ satisfying $\alpha\alpha + \beta\bar{\beta} = \mathbb{1}$ (and a parallel relation ensuring CAR preservation). Γ(U) is the special case $\beta = 0$.

For general Bogoliubov transformations ($\beta \neq 0$), the question of Fock-space implementability is governed by the *Shale–Stinespring criterion* (Shale, 1962; Stinespring, 1959): the transformation lifts to a unitary on $\mathscr{F}_A$ if and only if β is Hilbert–Schmidt. When β fails Hilbert–Schmidt — which occurs generically in infinite dimensions — the transformation produces a *unitarily inequivalent* Fock representation of the CAR algebra over $\mathscr{H}_1$, with its own vacuum sector.

This issue becomes relevant downstream of the present paper at three points:

- **Part VI (Lorentz boosts):** in the relativistic theory, Lorentz boosts mix positive- and negative-frequency components of the Dirac field, producing Bogoliubov transformations on the Fock space. Their Fock-implementability under Shale–Stinespring is a non-trivial question even before antiparticles are explicitly introduced.
- **Part VIII (antiparticles + charge conjugation):** charge conjugation is genuinely a Bogoliubov transformation mixing a and a†. Its implementability is part of the construction of the doubled particle/antiparticle Fock structure.
- **Part VII (interactions; Haag's theorem):** interacting vacua are related to free vacua by Bogoliubov transformations whose off-diagonal parts generically fail Hilbert–Schmidt. No unitary on the free Fock space implements the transformation; the interacting vacuum lives in a different representation. This is Haag's theorem in concrete form, and it is the reason Theorem 5†'s vacuum-uniqueness result is scoped to $\mathscr{F}_A$ specifically (see §4.3 and §12.1's scope note).

The present paper does not address Bogoliubov transformations with $\beta \neq 0$, and Theorem 6† should not be read as making any claim about them. The Γ(U) subclass is exactly what is needed for the parametrisation-independence question this paper is concerned with — different substrate parametrisations of the same persistent spinorial sector — and the present construction is honest about its scope.

13.5 Conditionality

Inherits Theorem 1† + Theorem 3† + standard CAR uniqueness theorem (Γ(U) subclass). Topological-core modulo §6.5 implicit Part IX inheritance.

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14. Corollary 5'† — Physical Realisation on the Coherent Substrate

14.1 The coherence condition setup

Setup. The coherence condition is inherited from Part IV §4A.3. In substrate-level terms, it requires that coherent multi-mode frame transport be supported by the substrate over the relevant extent: spatial extent $\lesssim \xi$ (the entanglement-lattice coherence length), and exchange ordering $\lesssim \tau_s$ characteristic counts (where the τ_s "exchange timescale" framing is the emergent-time description of substrate-level sequential-interface-transport ordering, per the §3.6 discipline note). When this condition holds, the substrate physically supports the coherent multi-loop frame transport that makes the CAR algebra observable as creation and annihilation events; outside this regime, the algebra remains a valid algebraic identity on $\mathscr{F}$_A but is not physically manifest as observable transport.

The condition is inherited; it is not derived in the present paper. Its substrate-level grounding lies in the entanglement-lattice and superfluid-transport papers (Part IV §3.9–§3.10), and is conditional on the current scale-estimate calibrations established there.

14.2 The corollary

Corollary 5'† (Physical realisation of the CAR algebra). Under the coherence condition set up in §14.1, the substrate-level CAR algebra of Theorem 3† is physically manifest as observable creation and annihilation of coherent spinorial transport modes on the coherent entanglement substrate.

- Vacuum $|0\rangle$ ↔ coherent substrate in its no-persistent-loop configuration.
- Single-loop states $|\mathcal{C}\rangle = a^\dagger(\mathcal{C})|0\rangle$ ↔ coherent spinorial transport excitations.
- CAR cross-relation ↔ substrate-level commutation structure between creation and annihilation, observable in any process where the coherence condition is maintained.

14.3 Argument

By Theorem 3†, CAR algebra holds at topological level on abstract $\mathscr{F}$_A. By Part IV Corollary 1', under the coherence condition the topological frame-bundle holonomy is physically manifest as observable substrate-level wavefunction phase structure.

CAR operators are constructed from the antisymmetric multi-loop wavefunction structure on $\mathscr{F}$_A. The same coherence-condition argument that promotes Part IV Theorem 1 to its physical-realisation form (Corollary 1' there) promotes substrate-physical analogues of CAR operators to physically observable creation and annihilation: within the coherence regime, substrate supports coherent multi-mode frame transport over the necessary extent; CAR algebra is physically manifest.

Outside coherence regime (substrate-level counterparts of spatial extent $\gg \xi$ or exchange ordering $\gg \tau_s$), topological CAR algebra still holds as algebraic identity, but substrate does not coherently support multi-mode frame transport; CAR observables become inaccessible at the substrate level. ■

14.4 Structural reading

Corollary 5'† is the physical-realisation companion to Theorem 3†, in the same way Part IV Corollary 1' is to Part IV Theorem 1. CAR algebra is topological-core; physical observability requires the coherence condition from substrate-physics layer.

For physical fermion-pair systems at observable laboratory scales (electrons, quarks in standard quantum chemistry, condensed-matter, high-energy phenomenology), the coherence condition is comfortably satisfied (Part IV §4A.3 footnote 1 inheritance) — *under entanglement-lattice scale estimates as currently calibrated*. If calibrations shift, Corollary 5'†-level observability is affected, not the topological-core results: Theorems 1, 2, 3, 5, 6 and Corollary 4† hold as algebraic identities regardless.

14.5 Conditionality

Inherits Theorem 3† + Part IV Corollary 1' + Part IV §3.9 + §3.10–§3.11 + §3.6 emergent-time framing. Specifically conditional on current scale-estimate calibrations. Only physical-realisation result; all numbered theorems are topological-core.

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15. Structural Consequences, Hierarchy Chain, and Comparison to Standard Free Dirac Quantisation

15.1 Catalogue of structural consequences

15.1.1 Persistent spinorial sector admits standard positive-definite Hilbert completion; no Krein-space modifications required. *Derived (Theorem 1†).*

15.1.2 Antisymmetric Fock space $\mathscr{F}$_A with vacuum-iterated density. *Derived (Theorem 2†).*

15.1.3 Full CAR algebra at substrate level. *Derived (Theorem 3†).*

15.1.4 Pauli exclusion strengthens to spectral form: $N(\mathcal{C}) \in \{0, 1\}$. *Derived (Corollary 4†).*

15.1.5 Vacuum unique on $\mathscr{F}$_A and cyclic. *Derived (Theorem 5†; with §12.1 scope note on Fock-representation-specific uniqueness.)*

15.1.6 Substrate CAR physically manifest under coherence condition. *Derived (Corollary 5'†).*

15.1.7 CAR C*-algebra inside B($\mathscr{F}$_A). *Derived (§9.3).*

15.1.8 No interaction-picture obstructions at substrate free-Fock level. *Derived (§4.3).* Haag-type obstructions = Bogoliubov-implementability obstructions per §13.4; deferred to Part VII.

15.1.9 CAR algebra is parametrisation-independent under second quantisation. *Derived (Theorem 6†, with §13.4 scope note on general Bogoliubov implementability.)*

15.1.10 Distributional reading is mathematically legitimate. *Derived (§5.2A).* Rigged-Hilbert-space (Gelfand triple $\Phi \subset \mathscr{H}_1 \subset \Phi^\times$) formulation with abstract nuclear-space characterisation; K explicitly identified as reproducing kernel of identity on $L^2(d\mu\_spin)$ (Definition 4).

15.2 The strengthened hierarchy chain

Logical chain (topological-core throughout; † qualifier carried by all boxes from "Admissibility of positive-definite $\mathscr{H}_1$" onwards per §1.1, §6.5 propagation rule):

```
K=7 architecture (d ≥ 3) + κ-field uniqueness + Schrödinger→Dirac
        ↓
Clifford structure on spinorial loops (Part III Theorem 1)
        ↓
Closure-frame bundle SU(2) lift (Part III SI, conditional on C1–C3)
        ↓
Spinorial sector U(2π) = −𝟙 (Part III Theorem 4)
        ↓
F–R exchange holonomy at topological level (Part IV Theorem 1)
        ↓
Antisymmetric multi-loop sector (Part IV Theorem 2)
        ↓
Spin-statistics correspondence (Part IV Theorem 3)
        ↓
Anticommuting half of CAR (Part IV Proposition 4 + Theorem 2†)
        ↓
Admissibility of positive-definite ℋ₁ (Theorem 1†)  ← † enters (see §6.5)
        ↓
Antisymmetric Fock space ℱ_A with vacuum-iterated density (Theorem 2†)
        ↓
Full CAR algebra (Theorem 3†)
        ↓                ↓                        ↓
Strong Pauli         Vacuum uniqueness     Representation-
exclusion            on ℱ_A and cyclicity   independence under
N(𝒞) ∈ {0, 1}        (Theorem 5†)            second quantisation
(Corollary 4†)                                (Theorem 6†, §13.4 scope)
```

Physical-grounding chain:

```
Void
  ↓
Entanglement lattice: scales ξ, τₛ, mₛ (Part IV §3.9; §3.6 emergent-time
framing)
  ↓
```

```
Coherent persistent transport (Part IV §3.10, Proposition 5)
  ↓
Persistent loops as coherent transport modes (Part IV §4A)
  ↓
Coherence condition (Part IV §4A.3; emergent-time framing, set up in §14.1)
  ↓
Topological holonomy physically manifest (Part IV Corollary 1')
  ↓
CAR observables physically manifest (Corollary 5'†)
```

Logical chain closed: from K=7 architecture, κ-field uniqueness, source-admissibility (plus C1–C3 + the two cross-paper conditionalities of † on SA1–SA5: Part IX pre-spacetime reformulation and substrate-gauge-group specification), the full parametrisation-independent CAR algebra (in the second-quantisation sense) follows.

The dagger enters at the "Admissibility" box and is inherited by every downstream box. Discharging *both* the implicit Part IX inheritance (via pre-spacetime reformulation of SA1–SA5) and the substrate-gauge-group conditionality (via Part XI specification) strips † from the entire downstream chain; discharging either alone leaves † in place (per §6.5 joint-discharge rule).

15.3 What remains: from Fock to QFT

- Lorentz-invariant fermionic QFT (Part VI), with positivity–covariance bridge (§18 item 10) substructural prerequisite; microcausality downstream; Lorentz-boost Bogoliubov implementability (§13.4).
- Interacting fermionic theory (Part VII), with Haag-type Bogoliubov obstructions (§13.4).
- Antiparticle structure (Part VIII), with charge conjugation as Bogoliubov transformation (§13.4).
- Substrate-to-spacetime bridge (Part IX), which supplies the emergent-spacetime semantics implicit in SA1–SA5's 4-vector notation (§3.5, §6.5).
- Species decomposition (Part X).
- Substrate-to-electroweak bridge (Part XI), which may supply the substrate gauge structure that SA5 quotients against (§6.4 note).

15.4 Comparison to standard free Dirac quantisation

The present substrate-level construction recovers the standard free-Dirac CAR/Fock structure at the algebraic level:

Standard free Dirac theory	Present substrate-level construction
One-particle Hilbert space $\mathscr{H}_1$	$\mathscr{H}_1$ from persistent spinorial sector P_spin (Definition 5, Theorem 1†)
Positive-definite Hilbert norm $\psi^\dagger\psi$	Same; admissibility established by Theorem 1†
Antisymmetric Fock space $\mathscr{F}$_A = $\bigoplus_n \Lambda^n \mathscr{H}_1$	Same; existence and density by Theorem 2†
CAR algebra	Theorem 3†

Standard free Dirac theory	Present substrate-level construction
Strong Pauli exclusion (spectrum ⊆ {0, 1})	Corollary 4†
Vacuum uniqueness on free Fock space; cyclicity	Theorem 5†
CAR uniqueness / second-quantisation representation-independence	Theorem 6†
CAR C*-algebra (unital) inside B(ℱ_A)	§9.3 (norm-closure of *-algebra)
Rigged Hilbert space / distributional momentum generators	§5.2A (Gelfand triple over P_spin, abstract nuclear formulation)
Bounded smeared operators a(f), a†(f)	§5.2A / §9.2 (smearing on P_spin)
Lorentz-covariant smeared fields ψ(x), ψ̄(x)	**Deferred to Part VI** (linked §18 1 ↔ 10)
Microcausality {ψ(x), ψ(y)} = 0 spacelike	**Deferred to Part VI** (downstream of smearing)
Antiparticle / negative-frequency reinterpretation	**Deferred to Part VIII**
Bogoliubov-implementability for general (β ≠ 0) transformations	**Deferred to Parts VI–VIII via §13.4**
Interacting renormalised QFT (with Haag-type Bogoliubov obstructions)	**Deferred to Part VII**
Physical fermion species	**Deferred to Part X**
Electroweak coupling	**Deferred to Part XI**
Embedding in continuous spacetime	**Deferred to Part IX**

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16. Falsifiability Channels

The present paper's claims fall into three categories, with correspondingly different things at stake under failure. Distinguishing them honestly is necessary to avoid inflating the falsifiability surface beyond what is substantive.

16.1 Structural-consistency channels — tests of whether load-bearing assumptions actually hold

These test whether the assumptions on which the construction rests are themselves correct. Failure of any of these is a substantive failure: the underlying assumption is empirically or structurally wrong.

Channel A: Admissibility of positive-definite completion fails. Tests Theorem 1†'s substrate-level content. Would occur if SA1–SA5 admits a hidden Krein-space extension propagating indefinite-metric structure to P_spin despite source-carrier identification. Signature: negative-norm components on persistent spinorial loop states satisfying source-admissibility.

Channel B: Fock-density property fails. Tests Theorem 2†'s assumption that vacuum-iterated states span a dense subspace. Would occur if there are non-trivial "vacuum-isolated" sectors in the antisymmetric multi-loop Hilbert space not reachable from |0⟩ by creation operators. Signature: persistent spinorial multi-loop sectors not generated by the standard Fock construction.

Channel E: Substrate admits sectors beyond 𝓕_A (superselection structure not captured by the free-construction). Tests whether the 𝓕_A constructed here is the whole story for the substrate. (Distinct from inequivalent vacua across different Fock representations — the §13.4 Bogoliubov issue — which is *expected*. Also distinct from non-uniqueness within 𝓕_A, which is ruled out mathematically by Theorem 5†'s ONB-based proof.) Channel E concerns the possibility that the substrate's true antisymmetric multi-loop Hilbert space exceeds 𝓕_A — for example, with additional superselection sectors arising from substrate-level topological invariants, conserved charges, or symmetry-broken phases not visible at the present free-construction level. If such sectors exist, Theorem 2†'s identification of 𝓕_A as the complete antisymmetric Fock space is incomplete (not wrong: 𝓕_A would still be one sector, just not the only one), and the construction needs an extension to a superselection-graded sum ⊕_s 𝓕_A^(s). Signature: substrate-level identification of conserved sector labels not currently accounted for, or empirical signatures of selection rules between such sectors. Distinct from Channel B (Fock-density within 𝓕_A) in that Channel E asks whether 𝓕_A is the right *target* for density, not whether density holds in it.

Channel G: Inheritance from Part IV fails. Tests the inheritance chain. If Part IV's Theorems 1, 2, 3 or Proposition 4 fail (F–R argument breakdown, antisymmetric sector exhaustion failure, etc.), the present construction's load-bearing inheritances fail. Previous paper's calibrated significance of C1–C3 carries over.

Channel H: Source-admissibility framework fails to exclude gauge-redundancy. Tests Theorem 1† Step 2's substantive content. Would occur if SA1–SA5 admits gauge-redundant excitations on P_spin that the framework does not in fact exclude. Theorem 1†'s Step 2 argument would fail; Krein-space structure could arise. Signature: identification of gauge-redundant excitations satisfying SA1–SA5. Related: the substrate gauge structure that SA5 is implicitly quotienting against (§6.4 note) being trivial or absent, in which case SA5 has no bite.

16.2 Derivational-consistency channels — referee-quality cross-checks

These follow as logical consequences of the structural assumptions (Theorems 1, 2 + standard CAR machinery + standard CAR uniqueness for Γ(U)). If any of these fail under the assumptions, the derivation contains an error rather than the physics being wrong. Useful as cross-checks; not substantive falsifiability channels in the standard sense.

Channel C: Cross-relation of CAR fails (assuming Theorems 1, 2). Would mean Theorem 3†'s derivation contains an error (e.g., the §10.2 ℓ-indexing computation), since the cross-relation follows from standard antisymmetric tensor-product calculus once positive-definite $\mathscr{H}_1$ and Fock density are in hand. Cross-check, not substantive channel.

Channel D: Number-operator spectrum exceeds {0, 1} (assuming Theorem 3†). Would mean Corollary 4†'s derivation contains an error, since {0, 1} spectrum follows from $a(\mathcal{C})^2 = 0$ + the projector identity. Cross-check.

Channel F: Representation-independence under Γ(U) fails (assuming Theorems 1, 3). Would mean standard CAR uniqueness theorem doesn't apply at substrate level — which would itself require some substrate-specific obstruction not captured by abstract Hilbert-space theory. Cross-check. (Note: Bogoliubov-implementability failures with $\beta \neq 0$ are *expected* per §13.4 and are not a Channel-F-style failure.)

16.3 Empirical channels — tests of substrate-physics inheritances

Channel I: Coherence condition violated at scales where CAR is empirically tested. Tests Corollary 5'†. If the substrate-level coherence scale ξ or characteristic count τ_s turn out incompatible with the regime of standard fermionic-matter observables (under emergent-spacetime mapping), Corollary 5'† would not apply at observable scales; substrate-level CAR observability lost in physical-fermion regime. Topological-core results (Theorems 1, 2, 3, 5, 6 and Corollary 4†) unaffected at the algebraic level.

General empirical posture

Structural channels (16.1) test load-bearing claims and are the substantive falsifiability surface for this paper. Derivational channels (16.2) are referee-quality cross-checks that the math is correct rather than tests of the physics. Empirical channel I (16.3) tests substrate-physics inheritance underlying Corollary 5'† and is shared with Part IV. The present paper does not establish new directly-empirical predictions beyond the CAR algebra itself, whose physical observability is inherited from Part IV's substrate-physics framing.

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17. What This Paper Achieves, and What It Does Not

17.1 What is achieved

Topological-core† results:

1. **Admissibility of standard positive-definite Hilbert completion on the persistent spinorial sector** (Theorem 1†).
2. **Antisymmetric Fock space $\mathscr{F}$_A with vacuum-iterated density** (Theorem 2†).
3. **Full CAR algebra at substrate level** (Theorem 3†, with i = j and i ≠ j cases both explicit in §10.2).
4. **Strong Pauli exclusion** (Corollary 4†).
5. **Vacuum uniqueness on $\mathscr{F}$_A and cyclicity** (Theorem 5†; ONB-based argument; scope on Fock-representation specificity per §12.1).

6. **Representation-independence of the CAR algebra under second quantisation** (Theorem 6†; Γ(U) subclass; general Bogoliubov implementability deferred per §13.4).
7. *CAR C-algebra structure on B(𝓕_A)** (§9.3; norm-closure of *-algebra).
8. **Rigged-Hilbert-space formulation of the distributional reading** (§5.2A; Gelfand triple $\Phi \subset \mathscr{H}_1 \subset \Phi^{x}$, abstract nuclear-space characterisation; K explicitly identified as Dirac kernel for dμ_spin per Definition 4).

Physical-realisation result:

9. **Substrate-level CAR algebra physically manifest under coherence condition** (Corollary 5'†; coherence condition set up cleanly in §14.1; current scale-estimate calibrations; §3.6 emergent-time framing).

17.2 What is not achieved

The present paper supplies the Fock-construction layer completing Part IV's §14 item 1. It does not derive:

- **Part VI deliverables (Lorentz covariance, smearing, microcausality, boost Bogoliubov implementability).** The Lorentz-vs-positivity disclaimer is scoped in §6.1; reconciliation is the positivity–covariance bridge (§18 item 10). Lorentz-covariant smearing of CAR operators into quantum fields ψ(x), ψ̄(x) is §18 item 1, paired with item 10. Microcausality {ψ_α(x), ψ_β(y)} = 0 for spacelike separation is downstream of smearing; the present CAR algebra is an *equal-"time" substrate algebra* (equal-"time" being the emergent-spacetime description of substrate-level simultaneous-fold configuration per §3.6). Lorentz-boost implementability requires the Shale–Stinespring framework of §13.4.
- **Parts VII–XI deliverables (renormalisation, antiparticles, substrate-to-spacetime bridge, species, electroweak).** Renormalised fermionic QFT (Part VII; Haag-type Bogoliubov obstructions per §13.4 and §4.3 — interacting vacua live in unitarily inequivalent CAR representations). Antiparticle structure and charge conjugation (Part VIII; charge conjugation is genuinely a Bogoliubov transformation mixing a and a†, per §13.4). Substrate-to-spacetime bridge (Part IX) supplies the emergent-spacetime semantics implicit in SA1–SA5's 4-vector notation (§3.5, §6.5) and in characteristic scales ξ, τ_s (§3.6). Species decomposition into Standard Model fermions (Part X). Substrate-to-electroweak bridge (Part XI) may supply the substrate gauge structure that SA5 quotients against (§6.4 note).
- **Cross-paper deliverables (pre-spacetime SA1–SA5, substrate gauge group, P_spin topology).** Pre-spacetime reformulation of SA1–SA5 and substrate-gauge-group specification together discharge † (§6.5). P_spin topology specification is logically prior to Parts VI / IX (§18 item 9a) and is currently treated in abstract nuclear form (§5.2A).
- **Existence of the covariantly-quantised enlargement P̃_spin** (open; connects to §18 item 10) and **the full axiomatic-QFT spin-statistics theorem** (substrate-restricted analogue in Part IV Theorem 3; full statement requires Part VI).
- **Direct empirical predictions from this paper alone.** Empirical embedding via Corollary 5'† + Part IV §4A.3.

Deliverable table:

Deliverable	Status
Admissibility of positive-definite Hilbert completion	**Theorem 1†**
Antisymmetric Fock space 𝒻_A with vacuum-iterated density	**Theorem 2†**
Full CAR algebra (both i = j and i ≠ j cases explicit)	**Theorem 3†**
Strong Pauli exclusion ($N(\mathcal{C}) \in \{0, 1\}$)	**Corollary 4†**
Vacuum uniqueness on 𝒻_A and cyclicity	**Theorem 5†** (with §12.1 scope)
Representation-independence under second quantisation	**Theorem 6†** (with §13.4 Bogoliubov scope)
Rigged-Hilbert-space formulation (Gelfand triple, abstract nuclear)	**Derived (§5.2A)**
Gram-kernel positivity condition with explicit form of K	**Stated (Definition 4, §5.2)**
CAR C*-algebra structure (norm-closure inside B(𝒻_A))	**Derived (§9.3)**
Physical-realisation of CAR observables	**Corollary 5'†** (coherence condition + scale calibrations + §3.6)
Part IV §14 item 1 deliverable	**Closed**
Part IV Proposition 4 (unconditional on density)	**Strengthened**
Part IV Corollary 6 (Pauli) strengthened to spectral form	**Strengthened (Corollary 4†)**
Part VI linked deliverables (items 1 ↔ 10, microcausality downstream):	
↳ Lorentz-covariant smearing into $\psi(x)$, $\bar{\psi}(x)$	**Open** (Part VI, linked)
↳ Positivity–covariance bridge	**Open** (Part VI, linked)
↳ Microcausality $\{\psi(x), \psi(y)\} = 0$ spacelike	**Open** (Part VI, downstream)
↳ Lorentz-boost Bogoliubov implementability	**Open** (Part VI, §13.4)
Renormalised fermionic QFT (Haag-type Bogoliubov obstructions)	**Open** (Part VII, §13.4)
Antiparticle structure and charge conjugation (Bogoliubov mixing)	**Open** (Part VIII, §13.4)
Substrate-to-spacetime bridge	**Open** (Part IX)
Pre-spacetime reformulation of SA1–SA5	**Open** (cross-paper, §6.5)
Substrate gauge group specification	**Open** (cross-paper, §6.4 note)
Species decomposition (leptons, quarks)	**Open** (Part X)
Substrate-to-electroweak bridge	**Open** (Part XI)
Full axiomatic-QFT spin-statistics theorem	**Open** (requires Part VI)

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18. Open Problems

1. Lorentz-covariant smearing of CAR operators (Part VI deliverable, linked to item 10; microcausality downstream). The dominant next-paper target.

Target deliverable. Construct Lorentz-covariant fermion field operators $\psi(x)$, $\bar{\psi}(x)$ on $\mathscr{F}_A$ by smearing substrate-level CAR operators against substrate-level mode functions, such that the resulting fields satisfy textbook Dirac field commutation relations $\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta} \cdot \delta^3(x - y)$, satisfy microcausality $\{\psi_\alpha(x), \psi_\beta(y)\} = 0$ for spacelike separation, and transform covariantly under unitary representations of the Lorentz group on $\mathscr{F}_A$.

Requires: (i) substrate-level Lorentz-invariance derivation; (ii) substrate-to-spacetime bridge linking discrete loops to emergent-spacetime points; (iii) positivity–covariance bridge of item 10; (iv) microcausality downstream of (i)–(iii); (v) Lorentz-boost Bogoliubov implementability (§13.4). Items 1 and 10 should be treated together in Part VI as a single linked deliverable.

2. Renormalised fermionic QFT (Part VII deliverable). Substrate-level fermionic interactions, renormalisation theory, chiral anomaly structure. Includes Haag-type Bogoliubov obstructions per §13.4 and §4.3: interacting vacua live in unitarily inequivalent representations of the CAR algebra; Theorem 5†'s vacuum uniqueness is on $\mathscr{F}_A$ specifically and does not survive the interacting context.

3. Antiparticle structure and charge conjugation (Part VIII deliverable). Doubling into particle/antiparticle Fock sectors via charge conjugation (genuine Bogoliubov transformation per §13.4); substrate-level charge-conjugation symmetry.

4. Substrate-to-spacetime bridge (Part IX deliverable). Explicit connection between substrate-level P and emergent physical spacetime coordinates x^μ. Under this bridge: substrate-derived characteristic scales (ξ, τ_s) acquire their emergent-spacetime interpretations as spatial coherence length and response timescale respectively (§3.6); SA1–SA5's 4-vector notation (u^μ, J^μ) acquires its emergent-spacetime interpretation (§3.5, §6.5). The pre-spacetime reformulation of SA1–SA5 is a cross-paper deliverable that the Part IX bridge would enable.

5. Species decomposition into Standard Model fermions (Part X deliverable). Spinorial sector P_{spin} is uniform here; physical fermion species (electron, muon, tau, quarks) decompose into distinct species. Requires substantive additional structural input.

6. Substrate-to-electroweak bridge (Part XI deliverable). Connection between substrate-level fermionic theory and electroweak gauge sector. May supply the substrate gauge structure that SA5 quotients against (§6.4 note), tightening the bite of SA1–SA5.

7. Non-abelian fermionic sectors (incorporating QCD). SU(3) colour on quarks introduces internal S_n-non-trivial structure bearing on Part IV §11.1.5. Requires species decomposition + substrate-level treatment of confinement and colour gauge.

8. Confinement-fermion bridge. Inherited from Part IV §14 item 9.

9a. P_spin topology specification (cross-paper deliverable, logically prior to Parts VI / IX). Specification of P_spin's topological/measurable structure, needed for §5.2A's nuclear-space Φ to be concretely realised. Currently in abstract form. Logically prior to the Part VI / IX deliverables that build on it.

9b. Lorentz-invariance of substrate-derived measure (Part VI prerequisite). Lorentz-invariance and translation-invariance of dμ_spin under the emergent-Lorentz group, downstream of P_spin topology (9a). Part of Part VI's positivity–covariance bridge / Part IX's substrate-to-spacetime construction.

10. Positivity–covariance bridge (Part VI prerequisite, linked to item 1). The present paper separates positive-definite Hilbert norm from Lorentz-covariant bilinear (standard in fermionic quantisation; substrate-level version requires dedicated bridge).

Target deliverable. Part VI must show CAR/Fock construction based on $\psi^\dagger\psi$ supports a unitary representation of the emergent Lorentz group on $\mathscr{F}$_A and admits Lorentz-covariant smeared fields ψ(x), $\bar{\psi}$(x).

Items 1 and 10 should be treated together; microcausality is downstream consequence.

11. General Bogoliubov implementability (cross-paper, §13.4). Theorem 6† covers the Γ(U) subclass; general Bogoliubov transformations mixing a and a† require Shale–Stinespring criterion. Relevant in Part VI (Lorentz boosts), Part VII (Haag's theorem on interacting vacua), Part VIII (charge conjugation). Cross-paper deliverable threading through these papers.

12. Substrate gauge group specification (cross-paper, §6.4 note). SA5 performs gauge-quotient at loop-label level, but the substrate-level gauge transformations being quotiented are not specified in this paper. Most natural locus is Part XI's electroweak bridge, though the substrate-level question is logically prior. Cross-paper deliverable.

13. Pre-spacetime reformulation of SA1–SA5 (cross-paper, §6.5). Microscopic Origin paper's SA1–SA5 currently uses emergent-spacetime 4-vector notation; ideal version would be in pre-spacetime terms with 4-vector form recovered downstream under Part IX. Cross-paper deliverable; together with item 12 (substrate gauge group specification) jointly discharges †, leaving the downstream chain conditional only on the upstream Part III / Part IV inheritances (C1–C3, K=7 → d=3, etc.) rather than on the cross-paper deliverables of §6.5.

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19. Relation to Earlier VERSF Papers, and the Dependency Graph

Direct Part IV inheritances (load-bearing throughout):

- Part IV Theorem 2 (antisymmetric multi-loop sector) — used via Part IV Proposition 4.
- Part IV Proposition 4 (anticommuting half of CAR) — inherited in Theorem 3†, now unconditional on density.
- Part IV Corollary 1' (physical realisation under coherence condition) — pattern inherited in Corollary 5'†.
- Part IV §4A (physical-realisation layer) — pattern inherited for §4.2.
- Part IV's two-layer architecture (§1.1) — preserved.

Part III inheritances:

- Part III Theorem 4 (spinorial-sector decomposition) — §3.1, §6.5.
- Part III Theorem 1 (Clifford-compatible spinorial structure) — §3.2, §5.2.
- Part III SI under C1–C3 (SU(2) lift) — via Part IV Theorem 1.

Other VERSF inheritances:

- κ-field uniqueness; Schrödinger→Dirac; K=7 minimal fact architecture; triangular closure. Via Part III.
- Microscopic Origin paper, especially Lemma 1 (SA1–SA5). *Load-bearing for Theorem 1†.* §3.5 reminder makes inheritance explicit; §6.5 acknowledges implicit Part IX inheritance via 4-vector notation.
- Substrate-dynamics / σ-duality programme. Substrate-level primitives underlying τ_s (sequential-interface-transport counts) inherited via §3.6.
- Topological-threshold paper. Via Part III.

Substrate-physics inheritances (physical-realisation layer only):

- Entanglement-lattice papers — scales ξ, τ_s, m_s via Part IV §3.9 (with §3.6 emergent-time interpretation).
- Superfluid-transport paper — via Part IV §3.10.
- *When Space Itself Has Mass* — effective-medium framing via Part IV §3.11.

Standard mathematical background (not VERSF):

- Standard antisymmetric Fock-space construction (Reed–Simon Vol. II §X.7; Bratteli–Robinson Vol. II §5.2). Theorems 2, 3, §9.3.
- Standard CAR uniqueness / second-quantisation functor (Bratteli–Robinson Vol. II Theorem 5.2.5). Theorem 6†.
- Shale–Stinespring criterion for general Bogoliubov implementability (Shale 1962; Stinespring 1959). §13.4.
- Standard CAR-cross-relation derivation. Theorem 3† proof.
- Standard positivity of $\psi^\dagger\psi$ on $\mathbb{C}^4$ (textbook). Theorem 1† Step 1.
- Standard rigged-Hilbert-space (Gelfand triple) formulation (Gelfand–Vilenkin; Streater–Wightman §3.1). §5.2A; abstract nuclear-space variant used pending P_spin topology specification.

- Reproducing-kernel Hilbert-space theory (Aronszajn 1950). Definition 4's explicit form of K in continuum reading.

Dependency graph (key results):

Result	Depends on	Layer
$\mathscr{H}_1$ pre-Hilbert structure (Definition 3)	Part III Theorem 1	Topological
Gram-kernel inner product (Definition 4)	ψ†ψ on $\mathbb{C}^4$ + positive-semidefinite K (Kronecker discrete / identity-reproducing-kernel continuum)	Topological
Rigged Hilbert space (§5.2A)	Standard Gelfand triple (abstract nuclear pending P_spin topology)	Topological
Theorem 1†	Part III Theorem 4 + SA1–SA5 (with §6.5 implicit Part IX inheritance) + ψ†ψ positivity + K positivity	Topological†
$\mathscr{H}_1$ as separable Hilbert space	Theorem 1† + standard completion	Topological†
$\mathscr{F}$_A (Definition 7)	Standard tensor algebra + Theorem 1†	Topological†
Theorem 2†	Theorem 1† + standard Fock construction	Topological†
CAR operators (Definitions 8, 9)	Theorem 2† + standard tensor calculus	Topological†
Boundedness / smeared bounded operators (§9.2)	Standard Fock construction; §5.2A + smearing	Topological†
CAR C*-algebra (§9.3)	Theorem 3† + boundedness	Topological†
Theorem 3† (both i = j and i ≠ j cases)	Theorem 2† + Part IV Proposition 4 + standard cross-relation	Topological†
Corollary 4†	Theorem 3† with i = j	Topological†
Total N spectrum (§11.4)	Corollary 4† + standard	Topological†
Theorem 5† (uniqueness on $\mathscr{F}$_A; cyclicity)	Theorem 2† + Theorem 3† + ONB-based uniqueness	Topological†
Theorem 6† (Γ(U) subclass)	Theorem 1† + Theorem 3† + standard CAR uniqueness	Topological†
Corollary 5'†	Theorem 3† + Part IV Corollary 1' + Part IV §3.9 + §3.6 + scale calibrations	Physical-realisation

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20. Epistemic-Status Labelling and Representation-Theoretic Status Table

20.1 Epistemic-status labelling

Derived at the topological level (under named inheritance).

- Theorem 1†: two components — (a) algebraic positivity (ψ†ψ + K positivity, standard inputs), (b) no Krein-space obstruction (from SA1–SA5, with §6.5 implicit Part IX inheritance). Combined: derived under inheritance.
- Theorem 2† (under Theorem 1† + standard Fock construction).
- Theorem 3† (under Theorem 2† + Part IV Proposition 4 + standard cross-relation; both $i = j$ and $i \neq j$ cases derived explicitly in §10.2).
- Corollary 4† (under Theorem 3† with $i = j$).
- Theorem 5† (under Theorem 2† + Theorem 3† + ONB-based uniqueness; scope on Fock-representation specificity per §12.1).
- Theorem 6† (under Theorem 1† + Theorem 3† + standard CAR uniqueness; Γ(U) subclass per §13.4).

Derived at the physical-realisation level. Corollary 5'† (under Theorem 3† + Part IV Corollary 1' + Part IV §3.9, with conditionality on current scale-estimate calibrations and §3.6 emergent-time framing).

Inherited (VERSF). Substrate scales ξ, τ_s, m_s (Part IV §3.9; §3.6 emergent-time framing). Effective-medium framing (Part IV §3.11). Source-admissibility 4-vector notation (Microscopic Origin paper, as emergent-spacetime descriptions per §3.5). Substrate-level primitives underlying τ_s (sequential-interface-transport counts; σ-duality programme, §3.6).

Inherited (standard mathematics). Standard antisymmetric Fock construction (Reed–Simon, Bratteli–Robinson). Standard CAR uniqueness theorem (Bratteli–Robinson Vol. II Theorem 5.2.5; Γ(U) subclass). Shale–Stinespring criterion (Shale 1962, Stinespring 1959; for general Bogoliubov, §13.4). Standard positivity of ψ†ψ on $\mathbb{C}^4$ (textbook; not circularity per §6.5). Standard rigged-Hilbert-space formulation (Gelfand–Vilenkin; abstract nuclear-space variant). Reproducing-kernel Hilbert-space theory (Aronszajn; Definition 4 continuum reading).

Conditional on framework assumptions. All topological-core derivations conditional on Part IV inheritances (C1–C3, K=7 → d=3, κ-field uniqueness, Schrödinger→Dirac) propagating through Part III Theorem 4 and Part IV Theorem 2 / Proposition 4. Theorem 1† specifically conditional on SA1–SA5 correctly excluding gauge-redundant excitations from P_spin, with implicit Part IX inheritance per §6.5. Standard-mathematical inputs (Reed–Simon, Bratteli–Robinson, Shale–Stinespring, Gelfand–Vilenkin, Aronszajn) do not introduce additional VERSF-internal conditionalities; the framework-assumption conditionalities derive entirely from VERSF inheritance.

Interpretive. §4.2 (physical-realisation reading of vacuum). §15.1.7 (CAR C*-algebra as downstream foundation).

Synthetic. §15.2 (strengthened hierarchy chain). §15.4 (comparison table to standard free Dirac quantisation).

Conjectural / open. Lorentz-covariant smearing (Part VI, linked to item 10; microcausality downstream; Lorentz-boost Bogoliubov implementability). Positivity–covariance bridge (Part VI item 10). Microcausality (Part VI, downstream). Renormalised fermionic QFT (Part VII, Haag-type Bogoliubov obstructions per §13.4). Antiparticle structure (Part VIII, charge conjugation as Bogoliubov per §13.4). Substrate-to-spacetime bridge (Part IX). Species decomposition (Part X). Substrate-to-electroweak bridge (Part XI; may supply substrate gauge group per §6.4 note). Full axiomatic-QFT spin-statistics theorem. Lorentz-invariance of substrate-derived measure / P_spin topology specification (Part VI prerequisite). Existence of covariantly-quantised enlargement P̃_spin (open). Pre-spacetime reformulation of SA1–SA5 (cross-paper, §6.5).

20.2 Representation-theoretic status table

Result	Status	Source	Layer
Pre-Hilbert structure H_1^pre	Derived	§5.1	Topological
Gram-kernel inner product (Definition 4)	Defined (with explicit K form in both readings)	§5.2	Topological
Gram-kernel positivity condition	Automatic (Kronecker discrete; identity-reproducing-kernel continuum)	§5.2	Topological
Rigged Hilbert space $\Phi \subset \mathscr{H}_1 \subset \Phi^\times$	Standard Gelfand triple (abstract nuclear pending P_spin topology)	§5.2A	Topological
Theorem 1†	**Derived under inheritance** (modulo §6.5 implicit Part IX)	**§6.2**	**Topological†**
↳ Component (a): Algebraic positivity (ψ†ψ + K positivity)	Standard textbook + automatic	§6.3 Step 1	Topological
↳ Component (b): Gauge-non-redundancy	Inherited from SA1–SA5	§6.3 Step 2	Topological
$\mathscr{H}_1$ as separable Hilbert space	Standard completion	Definition 5	Topological†
Antisymmetric $\Lambda^n \mathscr{H}_1$	Standard tensor algebra	§7.1	Topological†
Antisymmetric Fock space $\mathscr{F}$_A	Standard direct sum	§7.2	Topological†
Vacuum-iterated density	**Derived under inheritance**	**Theorem 2†**	**Topological†**
Bounded / smeared bounded CAR operators	Standard Fock construction	§9.2	Topological†
CAR C*-algebra (norm-closure inside B($\mathscr{F}$_A))	Derived	§9.3	Topological†
Anticommuting half of CAR	**Derived under inheritance** (Part IV Prop 4 + Theorem 2†)	**§8.3 / Theorem 3†**	**Topological†**
Cross-relation (both i = j and i ≠ j explicit)	**Derived under inheritance**	**Theorem 3†**	**Topological†**
Full CAR algebra	**Derived under inheritance**	**Theorem 3†**	**Topological†**

Result	Status	Source	Layer
Strong Pauli exclusion $N(\mathcal{C}) \in \{0, 1\}$	**Derived**	**Corollary 4†**	**Topological†**
Total number operator integer spectrum	Derived	§11.4	Topological†
Vacuum uniqueness on $\mathscr{F}_A$ (ONB-based)	**Derived under inheritance**	**Theorem 5†**	**Topological†**
Vacuum cyclicity	Derived	Theorem 5†	Topological†
CAR representation-independence (Γ(U) subclass)	**Derived under inheritance** (standard CAR uniqueness)	**Theorem 6†**	**Topological†**
General Bogoliubov implementability ($\beta \neq 0$)	Open; deferred to Parts VI–VIII	§13.4	—
Physical-realisation of CAR observables	**Derived under topological + coherence + scale calibrations + §3.6**	**Corollary 5'†**	**Physical-realisation**
Physical-realisation reading of vacuum	Interpretive	§4.2	Physical-realisation
Strengthened hierarchy chain	Synthetic	§15.2	Synthesis
Comparison table to standard free Dirac	Synthetic	§15.4	Synthesis
Lorentz-covariant smearing $\psi(x)$, $\bar{\psi}(x)$	Open (linked to §18 item 10; microcausality downstream)	§18 item 1 ↔ 10	—
Positivity–covariance bridge	Open (Part VI prerequisite)	§18 item 10	—
Microcausality $\{\psi(x), \psi(y)\} = 0$ spacelike	Open (Part VI, downstream)	§17.2, §18 item 1	—
Lorentz-invariance of $\mathscr{H}_1$ inner product	Open (Part VI)	§6.6 → §6.1	—
Existence of $\tilde{P}_{spin}$	Open	§6.4 note, §18 item 10	—
Renormalised fermionic QFT (Haag/Bogoliubov)	Open	§18 item 2	—
Antiparticle structure (Bogoliubov mixing)	Open	§18 item 3	—
Substrate-to-spacetime bridge	Open	§18 item 4	—
Pre-spacetime reformulation of SA1–SA5	Open (cross-paper)	§6.5, §18 item 13	—
Substrate gauge group	Open (cross-paper)	§6.4 note, §18 item 12	—
Species decomposition	Open	§18 item 5	—
Substrate-to-electroweak bridge	Open	§18 item 6	—

Result	Status	Source	Layer
Full axiomatic-QFT spin-statistics	Open	§18	—
Confinement-fermion bridge	Open	§18 item 8	—
P_spin topology specification	Open (cross-paper)	§5.2A, §18 item 9	—

Status glossary. *Derived under inheritance* — derived given explicit inheritance from named theorems (modulo §6.5 implicit Part IX inheritance where applicable). *Standard tensor algebra / standard Fock construction / standard CAR uniqueness / Shale–Stinespring* — background mathematics. *Standard textbook fact* — well-established input not re-derived; not a circularity concern. *Standard Gelfand triple / reproducing-kernel theory* — standard rigged-Hilbert-space machinery. *Automatic* — follows from standard machinery without additional substrate-level input. *Inherited* — from another VERSF paper or standard reference. *Defined* — supplied as definition. *Interpretive* — physical or conceptual reading. *Synthetic* — combines multiple results. *Open* — identified deliverable.

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21. Conclusion

Part IV deferred three prerequisites for the spin-statistics correspondence on the persistent spinorial sector: existence of a positive-definite one-particle Hilbert space $\mathscr{H}_1$; antisymmetric Fock-space construction with vacuum-iterated density; the cross-relation $\{a_i, a^\dagger_j\} = \delta_{ij}$. These were the §14 item 1 deliverable.

The present paper supplies all three, plus representation-independence under second quantisation.

Theorem 1† establishes that the persistent spinorial sector admits the standard positive-definite Hilbert completion using the Clifford-internal ψ†ψ pairing. The Lorentz-vs-positivity disclaimer is scoped carefully in §6.1: algebraic positivity of ψ†ψ + Gram-kernel positivity are standard inputs, not substrate-level derivations; SA1–SA5 supplies the gauge-non-redundancy that excludes the Krein-space alternative, with the gauge-quotient performed at the loop-label level (SA5) rather than the Hilbert-space level (per §6.4 note). Theorem 1†'s "topological-core" label is conditional on the implicit Part IX inheritance of SA1–SA5's 4-vector notation (per §6.5); cross-paper reformulation of SA1–SA5 in pre-spacetime terms is a desirable deliverable that would render this conditioning unnecessary.

Theorem 2† establishes the antisymmetric Fock-space construction with vacuum-iterated density, promoting Part IV's anticommuting-half result to operator identity on $\mathscr{F}$_A.

Theorem 3† supplies the full CAR algebra. Both the i = j and i ≠ j cases of the cross-relation are derived explicitly in §10.2; the cancellation runs identically in both via (−1)^k + (−1)^{k−1} = 0 cross-term pairing.

Corollary 4† strengthens Part IV Corollary 6 to $N(\mathcal{C}) \in \{0, 1\}$.

Theorem 5† closes the Fock-construction loop via ONB-based uniqueness. The scope is on the specific Fock space $\mathscr{F}\_A$ constructed here; uniqueness across all Fock representations of the CAR algebra over $\mathscr{H}_1$ is a different question (the Bogoliubov-implementability issue of §13.4) and is *expected* to fail in the interacting and antiparticle settings.

Theorem 6† establishes representation-independence under the second-quantisation functor Γ(U) for unitaries $U : \mathscr{H}_1 \to \mathscr{H}_1'$. Different substrate parametrisations producing the same Hilbert dimension yield *-isomorphic CAR algebras. The §13.4 note carefully scopes Theorem 6† to the Γ(U) subclass and defers general Bogoliubov implementability ($\beta \neq 0$ transformations mixing creation and annihilation) to Shale–Stinespring analysis in downstream papers: Part VI (Lorentz boosts), Part VII (Haag-type interacting-vacuum inequivalences), Part VIII (charge conjugation).

Corollary 5'† supplies the physical-realisation reading under the coherence condition set up cleanly in §14.1 (with §3.6 emergent-time framing). The substrate-level CAR algebra is physically manifest as observable creation and annihilation of coherent spinorial transport modes; topological-core results hold as algebraic identities regardless of scale-estimate calibrations.

The two-layer architecture of Part IV is preserved throughout. Theorems 1†–6† are topological-core†; Corollary 5'† is the sole physical-realisation result.

Discipline notes on emergent-spacetime conventions (§3.5, §3.6, §6.5, §14.1, §17.2) flag that the 4-vector notation u^μ, J^μ inherited from the Microscopic Origin paper is the emergent-spacetime description of pre-spacetime substrate objects (acquired under the Part IX bridge), and that the τ_s "timescale" terminology is the emergent-time description of substrate-level sequential-interface-transport counts. The "equal-time" character of the present construction is the emergent description of substrate-level simultaneous-fold configurations; full microcausality requires Part VI's smearing and is downstream.

The synthesis statement:

The full canonical anticommutation relations of substrate-level fermionic quantisation — including the cross-relation deferred from Part IV — are forced at the topological-core level by the standard antisymmetric Fock-space construction on the one-particle Hilbert space $\mathscr{H}_1$ derived from the substrate-level Clifford-internal ψ†ψ pairing. Admissibility is established via gauge-non-redundancy of the persistent spinorial sector at the loop-label level via SA1–SA5 (Theorem 1†, modulo §6.5 implicit Part IX inheritance and §6.4 substrate gauge group specification). The antisymmetric Fock space exists with vacuum-iterated density (Theorem 2†); the full CAR algebra holds with both $i = j$ and $i \neq j$ cases explicit (Theorem 3†); strong Pauli exclusion follows (Corollary 4†); vacuum uniqueness on $\mathscr{F}\_A$ and cyclicity close the construction (Theorem 5†); CAR representation-independence under the second-quantisation functor is established (Theorem 6†, with §13.4 scoping to the Γ(U) subclass and deferring general Bogoliubov implementability to Parts VI–VIII). Physical realisation on the coherent entanglement substrate under the coherence condition

(§14.1 setup, §3.6 emergent-time framing) manifests the substrate-level CAR algebra as observable creation and annihilation (Corollary 5'†). Remaining matter-sector work: Lorentz-invariant fermionic QFT with positivity–covariance bridge and microcausality (Part VI, §18 items 1 ↔ 10); renormalised interactions with Haag-type Bogoliubov obstructions (Part VII); antiparticle structure with charge-conjugation Bogoliubov mixing (Part VIII); substrate-to-spacetime bridge supplying the emergent-spacetime semantics implicit throughout this paper (Part IX); species decomposition (Part X); substrate-to-electroweak bridge possibly supplying substrate gauge group (Part XI).

The matter-sector programme now possesses:

- Substrate ontology for persistent current (Microscopic Origin paper).
- Admissibility framework for gauge coupling (Matter Coupling paper).
- Algebraic-geometric origin of spin-½ structure (Part III).
- Substrate-level forcing of antisymmetric exchange, anticommutation (partial), Pauli exclusion at topological level (Part IV).
- Explicit physical realisation under coherence condition (Part IV physical-realisation layer).
- **Complete free fermionic Fock-space construction with full CAR algebra, strong Pauli exclusion, vacuum uniqueness/cyclicity on $\mathscr{F}$_A, representation-independence under second quantisation, and rigged-Hilbert-space formulation with explicit reproducing-kernel structure (the present paper, Part V).**
- *CAR C-algebra structure for operator-algebraic formulation of substrate-level fermionic QFT (§9.3).**

Remaining deliverables: Lorentz-covariant smearing + positivity–covariance bridge + microcausality + Lorentz-boost Bogoliubov implementability (Part VI); renormalised QFT (Part VII, including Haag-type obstructions per §13.4); antiparticles with charge-conjugation Bogoliubov (Part VIII); substrate-to-spacetime bridge (Part IX, supplying emergent-spacetime semantics); species decomposition (Part X); electroweak bridge (Part XI, possibly supplying substrate gauge group); pre-spacetime reformulation of SA1–SA5 (cross-paper, §6.5); substrate gauge group specification (cross-paper, §6.4 note); general Bogoliubov implementability programme (cross-paper, §13.4); P_spin topology specification (cross-paper, §5.2A).

Strongest framing. Given (i) Part III's spinorial-sector structure (Clifford forcing + SU(2) lift + U(2π) = −𝟙 holonomy), (ii) Part IV's F–R-based antisymmetric multi-loop forcing (Theorem 2 there), (iii) Part IV Proposition 4's anticommuting half, and (iv) SA1–SA5 supplying gauge-non-redundancy at the loop-label level (with §6.5 implicit Part IX inheritance acknowledged), the substrate-level antisymmetric Fock-space construction with full CAR algebra and second-quantisation representation-independence is forced at the topological-core level. $\mathscr{H}_1$ admits standard positive-definite Hilbert completion (Theorem 1†); $\mathscr{F}$_A exists with vacuum-iterated density (Theorem 2†); full CAR algebra holds (Theorem 3†); strong Pauli exclusion follows (Corollary 4†); vacuum uniqueness on $\mathscr{F}$_A and cyclicity close the construction (Theorem 5†); CAR Γ(U)-representation-independence is established (Theorem 6†, with §13.4 deferring general Bogoliubov to Parts VI–VIII); rigged-Hilbert-space formulation legitimises the distributional reading (§5.2A). Physical realisation on the coherent substrate under the coherence condition manifests the algebra observationally (Corollary 5'†). The matter-sector programme is algebraically closed at the free-Fock-construction level; remaining work is the promotion to Lorentz-invariant QFT, interactions, antiparticles, and species decomposition through Parts VI–XI, with the positivity–covariance bridge (§18 item 10, paired with item 1; microcausality downstream) as the dominant substructural prerequisite of Part VI.