From Substrate CAR Algebra to Relativistic Fermion Fields in VERSF

Lorentz-Covariant Smearing, the Positivity–Covariance Bridge, and the Emergent Free Dirac Field on the Coherent Substrate

Keith Taylor *VERSF Theoretical Physics Programme Part VI of the Matter-Sector Programme Companion paper to Part V: Fock-Space Construction and the Full CAR Algebra in VERSF*

General Reader Summary

This paper is the sixth in a series building up the theory of matter inside VERSF. To make sense of what it does, the larger story so far helps.

The story so far. The matter-sector programme is deriving — rather than postulating — the structures that physics uses to describe fermions: electrons, quarks, and the other particles of ordinary matter. Part III established that spin-½ structure emerges from the substrate. Part IV established that two such particles must exchange antisymmetrically (the Pauli principle). Part V completed the algebraic side: it built the substrate-level fermionic Fock space with the full canonical anticommutation relations (CAR algebra), strong Pauli exclusion, and a unique cyclic vacuum. After Part V, the substrate has all the standard algebraic machinery of free fermion theory.

But Part V stopped at a very specific level. The creation and annihilation operators it constructed were labelled by *substrate spinorial loops*:

$a(\mathcal{C})$, $a^\dagger(\mathcal{C})$, where $\mathcal{C} \in P_{\text{spin}}$.

Modern physics, however, does not usually describe matter by such loop-labelled operators. It describes matter by *quantum fields over spacetime*:

$\psi(x)$, $\bar{\psi}(x)$, where x is a spacetime point.

The Dirac field is the central such object — it is what physicists actually compute with when they describe electrons, when they couple to electromagnetism, when they do perturbative QED. So the matter-sector programme has an unavoidable next question: *how do the substrate-level CAR modes become spacetime-labelled relativistic fermion fields?*

That is what this paper addresses.

An important conceptual correction. A naïve reading of the question would be: "smear the substrate operators against spacetime test functions, get $\psi(x)$, done." This is roughly the textbook construction in free Dirac quantisation. But it cannot be done that way in VERSF, for a fundamental reason: *spacetime is not primitive in VERSF, and more strongly, time and space are not primitive either*. They emerge separately. *Time* arises from substrate-level sequential interface transport — the irreversible commitment-event ordering on the void substrate — together with σ-duality on K=7 wheel structures, which supply the substrate analogue of "time evolution" under admissible coarse-graining. *Space* arises from the K=7 architecture's spatial degrees of freedom plus coarse-graining of substrate configurations on scales much larger than the coherence scale ξ. *Lorentz invariance* — the structure that *unifies* emergent time and emergent space into a single 4-dimensional arena with light-cone causal structure — is then a further substrate-level emergence, derived from coarse-graining symmetries of the substrate dynamics. None of these three (time, space, the Lorentz structure unifying them) is primitive; each is an emergent feature with its own substrate-physics derivation, and together they constitute "emergent spacetime M" as a composite object. There is no pre-existing spacetime against which to smear; the unified 4-dimensional spacetime is the large-scale coarse-grained phenomenology that emerges from the substrate when one looks at it from far enough away.

This changes the structure of the question. The right question is not "construct $\psi(x)$ on a given spacetime" but: *in the coarse-graining limit where emergent spacetime is the appropriate description, does the substrate CAR algebra reproduce the textbook Dirac field algebra?* The paper is not constructing fields and laying them onto spacetime; it is establishing an *emergence theorem* — that the substrate-level fermionic structure, under appropriate coarse-graining, has the textbook Dirac field theory as its emergent large-scale description.

What this paper does. Subject to inheritances from the broader VERSF programme (in particular, that the emergent-spacetime track supplies a sufficiently rich substrate-to-emergent-spacetime correspondence), this paper:

1. Constructs the rigged-Hilbert-space apparatus on the substrate side that allows substrate CAR operators to be smeared against admissible test functions.
2. Defines the admissibility conditions for "substrate-to-spacetime mode functions" — the bridge objects U_α(𝐶; x), V_α(𝐶; x) that mediate between substrate loop labels 𝐶 and emergent spacetime points x. These conditions are derived from the substrate-physics inheritances rather than postulated.
3. Constructs candidate emergent fermion field operators $\psi(x)$ as smeared combinations of substrate CAR operators against the mode functions.
4. Derives the field-level CAR algebra from the substrate CAR algebra, identifying explicitly which textbook form is recovered in which limits.
5. Constructs the positivity–covariance bridge: shows how the positive-definite Hilbert structure of substrate Fock space can support a unitary representation of the emergent Lorentz group on Fock space, with the spacetime fields transforming covariantly. The construction is conditional and the conditions are explicit.
6. Establishes the emergent free Dirac equation as a coarse-graining limit, inheriting from Part III's first-order spinorial flow structure.
7. Addresses microcausality at the level it can be addressed without further structural input.

What this paper does not do. Even with all of the above, the paper does not establish:

- Interacting quantum field theory (deferred to Part VII).
- Renormalisation (Part VII).
- Antiparticles and charge conjugation (Part VIII).
- Species decomposition into Standard Model fermions (Part X).
- Coupling to the electroweak sector (Part XI).
- The full axiomatic QFT spin-statistics theorem.

Each of these is a separate deliverable for the matter-sector programme's continuation.

The bottom line. What Part VI establishes is the *emergence theorem* for the free Dirac field algebra on the coherent substrate: that substrate CAR + admissible coarse-graining inheritances + emergent-Lorentz track inheritances → free Dirac field theory on emergent spacetime. The construction is honest about its dependencies — it does not assume spacetime as primitive, it does not assume a Lorentz unitary representation exists, and it does not assume the mode functions are given. Each of these is supplied either by named inheritance from another VERSF track or by construction within the paper, with all conditional dependencies explicitly listed.

Abstract

Part V established the substrate-level antisymmetric Fock space $\mathscr{F}_A$, the one-particle Hilbert space $\mathscr{H}_1$, the full canonical anticommutation relations (CAR), strong Pauli exclusion, and vacuum cyclicity on the persistent spinorial sector P_{spin}. The creation and annihilation operators of Part V were labelled by substrate spinorial loops $\mathcal{C} \in P_{\text{spin}}$ rather than by emergent-spacetime points $x \in M$, and the construction of Lorentz-covariant spacetime-labelled fermion fields was explicitly deferred to the present paper, together with the *positivity–covariance bridge* — the reconciliation between the positive-definite Hilbert norm on $\mathscr{H}_1$ and Lorentz-covariant field transformation laws.

This paper supplies the construction.

The construction is framed as an *emergence theorem*: in the coarse-graining regime where emergent spacetime is the appropriate description, the substrate CAR algebra reproduces the standard free Dirac field algebra on emergent spacetime. The "emergent spacetime" of the present paper is a composite of three separate substrate-level emergences inherited from the broader VERSF programme: *emergent time* from sequential interface transport and σ-duality on K=7 wheel structures (commitment-event ordering on the void substrate); *emergent space* from K=7 architecture's spatial degrees of freedom plus admissible coarse-graining; and *emergent Lorentz invariance* unifying emergent time and emergent space into a 4-dimensional Lorentzian manifold. None of these three is primitive; each is a separate substrate-physics deliverable, and the present paper inherits the composite as a working object (I1–I3) without re-deriving the components.

Substrate-to-spacetime mode functions $U_\alpha(\mathcal{C}; x)$, $V_\alpha(\mathcal{C}; x)$ mediate the construction; their admissibility conditions are derived from substrate-physics inheritances, not postulated. The candidate emergent fermion field is

$$\psi_\alpha(x) = \int U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C}) + \int V_\alpha(\mathcal{C}; x)\, a^\dagger(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C}),$$

defined initially as an operator-valued distribution on $\mathcal{F}_A$ in the rigged-Hilbert-space topology.

The construction is organised in the two-layer architecture inherited from Parts IV and V.

Topological-core results:

- **Theorem 1 (Existence of smeared field operators).** Conditional on substrate inheritances I1–I3 and mode-function admissibility conditions MA1–MA6, the candidate emergent field operator $\psi_\alpha(x)$ is a well-defined operator-valued distribution on $\mathcal{F}_A$ in the rigged-Hilbert topology, satisfying the standard test-function smearing properties.
- **Theorem 2 (Field CAR from substrate CAR).** The field-level anticommutation relations are induced directly from the substrate CAR algebra of Part V via the mode-function construction. The local continuum limit reproduces the textbook free-Dirac field CAR:

  $$\{\psi_\alpha(t, \mathbf{x}), \psi_\beta^\dagger(t, \mathbf{y})\} = \delta_{\alpha\beta} \cdot \delta^3(\mathbf{x} - \mathbf{y}),\quad \{\psi_\alpha(x), \psi_\beta(y)\} = \{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)\} = 0,$$

  under the mode-function completeness condition MA4.

- **Theorem 3 (Positivity–covariance bridge).** Conditional on inheritances EL1, EL2, EL3a, EL3b, EL4 from the emergent-Lorentz track (with EL3b — the substrate-level analogue of mass-shell $1/(2E_p)$ Lorentz-normalisation — and EL4 — Gram-kernel Lorentz invariance — identified as the load-bearing structural commitments), the positive-definite Hilbert structure of $\mathcal{F}_A$ supports a unitary representation $U(\Lambda)$ of the emergent Lorentz group on Fock space, with the emergent fermion field transforming covariantly:

  $$U(\Lambda)\, \psi_\alpha(x)\, U(\Lambda)^{-1} = S(\Lambda)^{-1}{}_\alpha{}^\beta \cdot \psi_\beta(\Lambda x),$$

  where $S(\Lambda)$ is the standard Dirac spinor representation. The construction does not adopt the indefinite Lorentz bilinear $\bar{\psi}\psi$ as a Hilbert-space norm; positivity is preserved.

- **Theorem 4 (Emergent positive-frequency free Dirac equation).** Under the additional inheritances CG1–CG2 from the admissible coarse-graining track, mass-shell admissibility MA6, and the Part III first-order spinorial flow structure, the coarse-grained *positive-frequency* part of the emergent field satisfies the free Dirac equation

  $$(i\gamma^\mu \partial_\mu - m)\, \psi^{(+)}(x) = 0$$

in the regime where emergent spacetime is the appropriate description. The Dirac equation is not postulated; it emerges as the coarse-grained spacetime expression of coherent substrate spinorial transport, with the mass m fixed by the substrate κ-field structure inherited via Part III Theorem 1. The full Dirac equation $(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$ for the combined field $\psi = \psi^{(+)} + \psi^{(-)}$ is *conditional* on the Part VIII charge-conjugation structure supplying $V_\alpha = C\,\bar{U}_\alpha^T$; Part VI establishes the positive-frequency form directly.

Physical-realisation result:

- **Corollary 4' (Physical realisation of emergent fermion fields).** Under the coherence condition of Part IV §4A.3 (spatial extent $\lesssim \xi$, timescale $\lesssim \tau_s$, inherited from the entanglement-lattice paper under current scale-estimate calibrations), the emergent fermion field operator $\psi(x)$ is physically manifest as the observable relativistic fermion field on emergent spacetime in the regime of standard fermionic-matter phenomenology.

Explicit clarification of scope. The present paper supplies the substrate-to-emergent-Dirac-field bridge: rigged-Hilbert-space smearing apparatus (§4), substrate-derived mode-function admissibility conditions (§5), the existence of smeared field operators (Theorem 1), the induced field CAR algebra (Theorem 2), the positivity–covariance bridge (Theorem 3), the emergent positive-frequency free Dirac equation (Theorem 4), substrate-coherence-scale direction-independent suppression of the field cross-anticommutator (§13.3; *not* light-cone-respecting microcausality, which is not claimed even partially), and the physical-realisation reading on the coherent substrate (Corollary 4'). It does *not* derive: interacting fermionic QFT (Part VII); renormalisation (Part VII); antiparticle structure and charge conjugation (Part VIII); substrate-derived construction of the mode functions themselves (deferred to Part IX, the substrate-to-emergent-spacetime correspondence paper, which the present paper inherits from); species decomposition into Standard Model fermions (Part X); substrate-to-electroweak bridge (Part XI); the full axiomatic-QFT spin-statistics theorem; or light-cone-respecting microcausality $\{\psi_\alpha(x), \psi_\beta(y)\} = 0$ at space-like separation, which requires structural input beyond MA3 — either a sharpening of mode-function support or substrate-derived measure reduction to the mass-shell — and is identified as §18 item 6.

Contents

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1. Setting: The Part VI Bridge Inherited from Part V

Part V of the matter-sector strand completed the substrate-level free fermionic Fock construction. The full CAR algebra holds on $\mathscr{F}_A$ at substrate level:

$\{a(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = \langle \mathcal{C}_i \mid \mathcal{C}_j \rangle \cdot \mathbb{1}_{\mathscr{F}_A}$, $\{a(\mathcal{C}_i), a(\mathcal{C}_j)\} = 0$, $\{a^\dagger(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = 0$,

together with strong Pauli exclusion (Corollary 4 of Part V), vacuum uniqueness and cyclicity (Theorem 5 of Part V), and the CAR C*-algebra structure inside $B(\mathscr{F}_A)$ (§9.3 of Part V). The construction is purely algebraic and labels its operators by substrate spinorial loops $\mathcal{C} \in P_{spin}$.

The dominant remaining matter-sector deliverable identified by Part V is the *promotion to relativistic quantum field theory*: the construction of spacetime-labelled fermion field operators $\psi_\alpha(x)$ on $\mathscr{F}_A$, satisfying the textbook Dirac field commutation relations, transforming covariantly under an appropriate unitary representation of the emergent Lorentz group, and admitting the free Dirac equation as their equation of motion. Part V identified this as a linked deliverable with two components (§17 items 1 and 10 of Part V):

- **Item 1:** Lorentz-covariant smearing of CAR operators into $\psi(x)$, $\bar{\psi}(x)$.
- **Item 10:** Positivity–covariance bridge — reconciliation between the positive-definite Hilbert norm on $\mathscr{H}_1$ and the Lorentz-covariant field transformation laws.

The present paper supplies both, as a single linked construction.

1.1 The conceptual correction inherited from the broader VERSF programme

A naïve reading of the construction would treat spacetime as a pre-existing arena on which the smearing is performed. This is the textbook approach in free Dirac quantisation, where spacetime is a Minkowski background and the field operators are constructed by Fourier transform against momentum modes.

This approach is structurally incompatible with the VERSF foundations. *Spacetime in VERSF is emergent*, and more strongly, *its components emerge separately*:

- **Emergent time** arises from substrate-level sequential interface transport — the irreversible commitment-event ordering on the void substrate — together with σ-duality on $K=7$ wheel structures, which supply the substrate analogue of time evolution under admissible coarse-graining of substrate dynamics.
- **Emergent space** arises from the $K=7$ architecture's spatial degrees of freedom combined with admissible coarse-graining of substrate configurations on scales much larger than the substrate coherence scale ξ inherited from Part IV §3.9.
- **Emergent Lorentz invariance** — the structure that *unifies* emergent time and emergent space into a single 4-dimensional manifold with light-cone causal structure — is a further substrate-level emergence, derived from coarse-graining symmetries of the substrate dynamics.

Each of these three emergent features has its own substrate-physics derivation in the broader VERSF programme. The "emergent spacetime M" of the present paper is the composite of these three: a 4-dimensional manifold with a Lorentzian metric η, in which the time coordinate inherits its substrate origin from sequential interface transport, the space coordinates inherit their substrate origin from the $K=7$ spatial structure, and the metric η unifying them inherits its substrate origin from emergent-Lorentz coarse-graining symmetries. Spacetime is not the arena; it is the coarse-grained large-scale phenomenology of three separate substrate-level emergences acting in coordination.

The Part VI construction must therefore be framed as an *emergence theorem*: in the coarse-graining regime where emergent spacetime is the appropriate description, the substrate CAR algebra reproduces the textbook free Dirac field algebra on emergent spacetime, with all the standard structures (CAR cross-relation, Lorentz covariance, the Dirac equation) holding as derived consequences rather than as postulates.

This reframing has structural consequences throughout the paper:

- Mode functions $U_\alpha(\mathcal{C}; x)$, $V_\alpha(\mathcal{C}; x)$ are not free choices but must be derived from substrate-physics inheritances. Their admissibility conditions are inherited from the substrate-to-emergent-spacetime track, which is itself a composite of substrate-time-emergence + substrate-space-emergence + emergent-Lorentz-unification.
- The Lorentz group action on emergent spacetime is itself a derived structure, inherited from the emergent-Lorentz programme of the broader VERSF papers. The Lorentz group is what mixes time and space — its emergence is therefore structurally distinct from (and depends on) the emergence of time and space individually.
- The free Dirac equation is not the starting point but the coarse-graining limit of substrate spinorial flow, inheriting from Part III Theorem 1's Clifford-internal structure and from admissible coarse-graining of substrate dynamics. The temporal derivative ∂_0 in $i\gamma^\mu \partial_\mu$ inherits substrate-time-emergence; the spatial derivatives ∂_i inherit substrate-space-emergence; the γ^μ structure connecting them is the substrate-level Clifford analogue of the emergent-Lorentz structure.

1.2 The two-layer architecture inherited

The present construction inherits the two-layer architecture of Parts IV and V. The topological-core layer consists of the algebraic-and-distributional construction: the rigged-Hilbert-space apparatus on $\mathscr{H}_1$, the candidate field operators ψ_α(x) as operator-valued distributions on 𝓕_A, the induced field CAR algebra, the unitary Lorentz representation on Fock space (conditional on EL1–EL4 — a compact range denoting the five inheritances EL1, EL2, EL3a, EL3b, EL4 defined in §3.5. EL3 is split into EL3a (measure-pullback invariance) and EL3b (Lorentz-normalisation structure); EL4 (Gram-kernel Lorentz invariance) is a separate independent inheritance, *not* a sub-letter of EL3), and the emergent Dirac equation (conditional on CG1–CG2). These are mathematical constructions on the abstract substrate Fock space; they do not depend on physical observability at substrate scales.

The physical-realisation layer consists of the substrate-physical reading of these constructions: under the coherence condition inherited from Part IV §4A.3, the emergent field ψ(x) is physically manifest as the observable relativistic fermion field on emergent spacetime, in the regime of standard fermionic-matter phenomenology. The physical-realisation reading inherits the coherence condition throughout.

The four numbered theorems (Theorems 1, 2, 3, 4) are topological-core; Corollary 4' is the sole physical-realisation result.

1.3 Source-carrier discipline preserved

Following Parts III–V, "substrate spinorial mode" denotes the substrate-level loop 𝒞 ∈ P_spin with the algebraic structure derived in Part V, not "physical electron" or "Standard Model fermion." The emergent field ψ(x) constructed in the present paper is the *uniform spinorial sector* field — the substrate-level analogue of a single uniform Dirac field, prior to species decomposition. Promotion to physical fermions (electron, muon, quarks) still requires species decomposition (Part X) and coupling to the electroweak sector (Part XI). The Dirac mass m appearing in Theorem 4 is the substrate-level uniform mass parameter inherited from Part III Theorem 1; species-specific masses and the Yukawa structure that distinguishes them are downstream of Part X.

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2. Notation and Conventions

Conventions of Parts III, IV, V apply throughout, extended for the field-construction objects introduced here.

Substrate spinorial sector. P_spin, the persistent transport submanifold of loops in the U(2π) = −𝟙 component (Part III Theorem 4), with substrate-derived measure dμ_spin inherited from the entanglement-lattice substrate via Part IV §3.9. Used throughout Part VI in its continuum-parametrisation reading (see Part V §5.2, distributional reading).

Substrate one-particle Hilbert space. $\mathscr{H}_1$, the positive-definite separable Hilbert completion of the pre-Hilbert space of single-loop spinorial states (Part V Theorem 1).

Substrate antisymmetric Fock space. $\mathscr{F}_A = \bigoplus_n \Lambda^n \mathscr{H}_1$ (Part V Theorem 2).

Substrate CAR operators. $a(\mathcal{C})$, $a^\dagger(\mathcal{C})$ for $\mathcal{C} \in P_{spin}$, the bounded creation/annihilation operators on $\mathscr{F}_A$ satisfying the full CAR algebra (Part V Theorem 3). In the distributional reading of P_{spin}, the bounded operators are the smeared $a(f)$, $a^\dagger(f)$ for test functions $f \in \mathscr{H}_1$.

Emergent spacetime. M, the four-dimensional manifold supplied as an inheritance by the substrate-to-emergent-spacetime track (see §4). The Part VI construction is parametric in M; the explicit construction of M from substrate data is deferred to Part IX. M is equipped with the emergent Lorentz group action $\Lambda : M \to M$, inherited from the emergent-Lorentz track (see §4). *Note on terminology:* "emergent spacetime" throughout this paper denotes the *composite* of three substrate-level emergences — emergent time (from sequential interface transport / σ-duality / commitment-event ordering), emergent space (from K=7 architecture + admissible coarse-graining), and the emergent-Lorentz structure that unifies them. The Part VI construction takes the composite M as a working inheritance; where the distinction between the three components matters (e.g., in §6.3, §13 microcausality), it is made explicit.

Spacetime point. $x \in M$. Throughout this paper x is an emergent spacetime point in M, not a substrate-level coordinate.

Spinor index. $\alpha, \beta \in \{1, 2, 3, 4\}$ on the Clifford-internal $\mathbb{C}^4$ carried by each spinorial loop (Part III Theorem 1).

Gamma matrices. γ^μ, $\mu \in \{0, 1, 2, 3\}$, the standard Dirac gamma matrices on $\mathbb{C}^4$, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbb{1}_{\mathbb{C}^4}$ for emergent Minkowski metric $\eta = \mathrm{diag}(+1, -1, -1, -1)$.

Standard Dirac spinor representation. $S(\Lambda) \in GL(\mathbb{C}^4)$ for emergent Lorentz transformation Λ; the standard textbook $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation lifted to the emergent-Lorentz double cover, inherited from the emergent-Lorentz track (EL1).

Substrate-to-spacetime mode functions. $U_\alpha(\mathcal{C}; x)$, $V_\alpha(\mathcal{C}; x)$: admissible mappings $P_{spin} \times M \to \mathbb{C}^4$ defined in §6, with admissibility conditions MA1–MA6.

Emergent fermion field operator. $\psi_\alpha(x)$, the candidate emergent field defined in §7. The Dirac adjoint is $\bar{\psi}_\alpha(x) = \psi^\dagger(x)\, \gamma^0$ when needed.

Smeared field operator. For a test function f on M, $\psi_\alpha(f) = \int_M f(x)\, \psi_\alpha(x)\, d^4x$, defined as an operator-valued distribution on $\mathscr{F}_A$.

Field-level Gram kernel. $K_{\alpha\beta}(x, y)$, the spacetime-labelled Gram kernel induced from $K(\mathcal{C}, \mathcal{C}')$ via the mode-function construction (defined explicitly in Theorem 2).

Unitary Lorentz representation on Fock space. $U(\Lambda) : \mathscr{F}_A \to \mathscr{F}_A$, the candidate unitary representation of the emergent Lorentz group on substrate Fock space, constructed in §10 and §11 conditional on EL1–EL4.

Coherence condition. Inherited from Part IV §4A.3: spatial extent $\lesssim \xi$, exchange/observation timescale $\lesssim \tau_s$, with ξ, τ_s supplied by the entanglement-lattice paper under current scale-estimate calibrations. Used throughout the physical-realisation layer (§14).

3. Structural Dependencies

This section lists the prior results on which the Part VI construction depends. The dependencies span three categories: substrate-CAR inheritances from Part V (items 3.1–3.3), emergent-spacetime / emergent-Lorentz / coarse-graining inheritances from the broader VERSF programme (items 3.4–3.6), and substrate-physics inheritances from the physical-realisation layer of Part IV (item 3.7).

3.1 Substrate CAR algebra (Part V Theorem 3)

The full CAR algebra holds on the substrate Fock space $\mathscr{F}_A$:

$$\{a(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = \langle \mathcal{C}_i \mid \mathcal{C}_j \rangle \cdot \mathbb{1}_{\mathscr{F}_A}, \quad \{a(\mathcal{C}_i), a(\mathcal{C}_j)\} = 0, \quad \{a^\dagger(\mathcal{C}_i), a^\dagger(\mathcal{C}_j)\} = 0.$$

The bounded-operator form holds directly in the discrete reading of P_{spin}; in the distributional reading, the bounded operators are $a(f)$, $a^\dagger(f)$ for test functions $f \in \mathscr{H}_1$ (Part V §10.1). The present paper works throughout in the distributional reading; the bounded smeared operators $a(f)$, $a^\dagger(f)$ are the primary substrate input.

3.2 Positive-definite Hilbert completion (Part V Theorem 1)

The persistent spinorial sector admits the standard positive-definite Hilbert completion using the Clifford-internal $\psi^\dagger\psi$ pairing, with no Krein-space modification required. Source-admissibility (SA1–SA5) supplies gauge-non-redundancy at the loop-label level. The present paper's positivity–covariance bridge (Theorem 3) is the Part VI deliverable that Part V identified as the dominant prerequisite of Lorentz-covariant smearing.

Inherited implicit Part IX dependency. Part V §6.5 explicitly flags that the 4-vector notation in SA1–SA5 (u^μ, J^μ) is an emergent-spacetime description acquired under the Part IX substrate-to-spacetime bridge — Part V Theorem 1's "topological-core" status is therefore labelled "modulo §6.5 implicit Part IX inheritance." The present paper inherits Part V Theorem 1, and with it this same implicit Part IX dependency. This does not introduce *new* conditional structure: the Part IX inheritance Part V §6.5 flags is the same Part IX bridge that the present paper itself depends on through I1–I3 (§3.4). The chain of dependencies is: Part V Theorem 1 (positive-definite $\mathscr{H}_1$) inherits implicit Part IX bridge → present paper inherits Part V Theorem 1 with that implicit dependency → present paper also inherits Part IX bridge explicitly through I1–I3. The two strands of Part IX dependency converge and are jointly conditional on the Part IX deliverable, with no double-counting.

3.3 Vacuum and cyclicity (Part V Theorem 5)

The vacuum $|0\rangle \in \mathscr{F}\_A$ is the unique unit vector annihilated by every $a(\mathcal{C})$, and the substrate CAR operators generate a dense subspace of $\mathscr{F}\_A$ from $|0\rangle$. The present paper's unitary Lorentz representation (Theorem 3) is constructed using this cyclicity property to determine $U(\Lambda)$ up to phase on the dense subspace, then extend by continuity.

3.4 Substrate-to-emergent-spacetime correspondence (Part IX inheritance — present paper inherits from)

The four-dimensional emergent spacetime manifold M, with its differentiable structure, emergent Lorentz metric η, and the substrate-to-M correspondence (the map associating substrate-level local configurations to spacetime regions), is supplied as an inheritance by the substrate-to-emergent-spacetime correspondence — formally a Part IX deliverable, but inherited as a working input for the present paper. The relevant inheritances are catalogued in §4 as I1, I2, I3:

- **I1:** Existence of emergent spacetime M as a four-dimensional differentiable manifold derivable from substrate sequential interface transport, with substrate-to-M map ι : (substrate configurations) → (spacetime regions).
- **I2:** Existence of emergent Lorentzian metric η on M, induced by substrate light-cone structure (substrate causality), with signature (+, −, −, −) under the convention of §2.
- **I3:** Existence of substrate-derived measure dμ_spin on P_spin and its compatibility with the substrate-to-M map, in the sense that integrals against dμ_spin transform consistently under the inherited emergent Lorentz action.

These inheritances are the present paper's largest external dependencies. The Part VI construction is conditional on them; their detailed substrate-level derivation is the Part IX deliverable.

3.5 Emergent Lorentz group action (Emergent-Lorentz track inheritance)

The emergent Lorentz group $SO^+(1, 3)$ acts on emergent spacetime M, with this action arising from substrate-level coarse-graining symmetries inherited from the broader VERSF emergent-Lorentz programme. The relevant inheritances are catalogued in §10 as EL1, EL2, EL3a, EL3b, EL4:

- **EL1:** Existence of the standard Dirac spinor representation $S(\Lambda) \in GL(\mathbb{C}^4)$ of the emergent Lorentz double cover Spin(1, 3), acting on the Clifford-internal $\mathbb{C}^4$ of substrate spinorial loops in a manner compatible with the Part III Theorem 1 Clifford structure.
- **EL2:** Existence of an action $\Lambda \mapsto \Lambda$ of the emergent Lorentz group on the substrate spinorial sector P_spin, intertwined with the action on emergent-spacetime regions via the substrate-to-M map ι: $\iota(\Lambda\mathcal{C}) = \Lambda \cdot \iota(\mathcal{C})$ for every substrate configuration $\mathcal{C}$, with the

intertwining understood up to coherence-scale localisation. (The verb "intertwines" replaces an earlier weaker "covers"; the intertwining statement is the load-bearing one.)
- **EL3a (Measure-pullback invariance):** The substrate-derived measure $d\mu\_spin$ is invariant under the EL2 action: $\int_{P\_spin} F(\Lambda\mathcal{C})\, d\mu\_spin(\mathcal{C}) = \int_{P\_spin} F(\mathcal{C})\, d\mu\_spin(\mathcal{C})$ for measurable F on P_spin. Equivalently, $d\mu\_spin(\Lambda\mathcal{C}) = d\mu\_spin(\mathcal{C})$.
- **EL3b (Lorentz-normalisation structure):** The substrate-derived measure $d\mu\_spin$ encodes a Lorentz-normalisation structure functionally equivalent, in the coarse-graining limit, to the *full* mass-shell measure $d^3p / [(2\pi)^3 \cdot 2E\_p]$ — not just the $1/(2E\_p)$ weighting in isolation, but the combined energy-weighting-plus-boost-Jacobian structure that makes the measure Lorentz-invariant on the mass shell. Concretely: in the coarse-graining limit where substrate spinorial modes $\mathcal{C}$ reduce to momentum-space modes, $d\mu\_spin$ reduces to $d^3p / [(2\pi)^3 \cdot 2E\_p]$. Pre-coarse-graining, $d\mu\_spin$ carries the substrate-level analogue of this combined structure — the structural property that makes $\psi^\dagger\psi$ invariant under the combined ($S(\Lambda)$, measure-pullback) action despite $S(\Lambda)$ being non-unitary in the bare Clifford-internal pairing. EL3b is the load-bearing content of the EL3 family; it is *not* a consequence of EL3a alone, and the present paper makes its load-bearing role explicit (see §11.2 Step 2 for the full textbook analogy).
- **EL4 (Gram-kernel Lorentz invariance):** The substrate Gram kernel $K(\mathcal{C}, \mathcal{C}')$ is invariant under the EL2 action on both arguments: $K(\Lambda\mathcal{C}, \Lambda\mathcal{C}') = K(\mathcal{C}, \mathcal{C}')$. This is an additional structural requirement on the Gram kernel beyond EL2/EL3 measure compatibility, and is named separately to make the structural commitment explicit.

These inheritances supply the structural input required for Theorem 3 (positivity–covariance bridge) and Theorem 4 (emergent Dirac equation). Their detailed substrate-level derivation is the deliverable of the emergent-Lorentz programme. EL3b in particular is the dominant bet on Part IX: it requires the substrate-derived measure to carry the relativistic-normalisation structure of the textbook mass-shell measure, which is a substantive structural property and not a trivial consequence of measure invariance alone.

3.6 Admissible coarse-graining (Coarse-graining track inheritance)

Admissible coarse-graining of substrate dynamics produces, in the appropriate limit, continuum spacetime field equations from substrate-level discrete update structure. The relevant inheritances are catalogued in §12 as CG1–CG2:

- **CG1:** Existence of the coarse-graining limit in which substrate spinorial flow (Part III Theorem 1's first-order spinorial flow structure) is described by the standard Dirac operator $i\gamma^\mu \partial\_\mu - m$ on emergent spacetime M.
- **CG2:** The substrate κ-field uniqueness programme fixes the substrate-level mass parameter m appearing in CG1, supplying the inherited value of m as the uniform spinorial-sector mass prior to species decomposition.

These inheritances supply the structural input required for Theorem 4. The detailed substrate-level derivation of the coarse-graining limit is the deliverable of the admissible-coarse-graining and continuum-emergence programme.

3.7 Substrate-physics inheritances (Part IV §3.9–§3.11)

For the physical-realisation layer only:

- **Coherent entanglement substrate.** Coherence scale ξ, spinorial response time τ_s, effective inertial scale m_s — inherited from the entanglement-lattice paper via Part IV §3.9, under current scale-estimate calibrations.
- **Superfluid persistent transport stability.** Substrate persistent loops are dynamically stable substrate modes on timescales $\gg \tau_s$ — Part IV §3.10.
- **Effective-medium framing.** Substrate is an effective coherence medium, not a preferred-frame mechanical ether — Part IV §3.11.

These enter Corollary 4' (§14) and do *not* enter Theorems 1, 2, 3, 4, which are topological-core results.

3.8 Standard mathematical background

The present paper inherits standard mathematical background for the rigged Hilbert space construction (Gelfand triples), operator-valued distributions (Wightman framework), and the standard textbook construction of free Dirac field operators from creation/annihilation modes (any standard QFT text; e.g., Streater–Wightman, Reed–Simon Vol. II–IV).

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4. The Substrate-to-Emergent-Spacetime Correspondence: What Is Inherited

The single largest structural input to the Part VI construction is the substrate-to-emergent-spacetime correspondence, formally a Part IX deliverable but inherited as working input throughout the present paper. This section catalogues the inheritances explicitly and identifies the boundary between what is inherited and what the present paper constructs.

4.1 The inheritance I1: emergent spacetime as a four-dimensional manifold

Inheritance I1. There exists a four-dimensional differentiable manifold M, the *emergent spacetime*, derivable from substrate sequential interface transport under admissible coarse-graining. There exists a substrate-to-M correspondence

$\iota : \mathcal{P}(\text{substrate}) \to \mathcal{P}(M)$,

where $\mathcal{P}$(substrate) is the family of substrate local configurations and $\mathcal{P}(M)$ is the family of spacetime regions, such that:

- Substrate-local configurations correspond to localised spacetime regions, with the localisation scale set by the coherence scale ξ inherited from Part IV §3.9.

- Substrate causal structure (the substrate-level commitment-event ordering inherited from sequential interface transport) induces the emergent light-cone structure on M.
- Coarse-graining of substrate configurations on scales much larger than ξ produces continuum descriptions on M.

Composite structure of M. The emergent spacetime M is *not* a primitive 4-dimensional arena; it is a composite of three separate substrate-level emergences:

- The *time direction* of M (the timelike component) inherits its substrate origin from sequential interface transport and σ-duality on K=7 wheel structures, which together supply substrate-level commitment-event ordering. Emergent-time translations on M correspond, at substrate level, to evolution along the sequential-transport direction.
- The *spatial directions* of M (the spacelike components) inherit their substrate origin from the K=7 architecture's spatial degrees of freedom plus admissible coarse-graining of substrate configurations on scales much larger than ξ.
- The *4-dimensional structure that combines them* — the unified arena in which time and space sit on a common footing under boosts — is the substrate-emergent-Lorentz structure, deferred to inheritance I2 and the EL family (§3.5).

The inheritance I1 supplies all three together as the working object M of Part VI, but they are conceptually and structurally distinct substrate-level emergences. A reader should not assume that a substrate-level construction giving rise to (say) emergent time automatically gives rise to a coordinated emergent space, or that the two together automatically give rise to a Lorentz-invariant arena: each step is a separate substrate-physics deliverable.

What I1 supplies for Part VI. The existence of a four-dimensional spacetime arena M on which spacetime-labelled fermion fields ψ(x) can be defined, with x ∈ M an emergent-spacetime point. The substrate-to-M correspondence ι is what allows the substrate-level smearing construction to be re-expressed as a spacetime-level construction.

What I1 does not address. The detailed substrate-level construction of M, the relative coordination of the time-emergence and space-emergence components, its global topology, its causal-set structure, the precise form of the coarse-graining map, and the boundary conditions at which the emergent description breaks down. These are Part IX deliverables.

4.2 The inheritance I2: emergent Lorentzian metric

Inheritance I2. The emergent spacetime M carries an emergent Lorentzian metric η of signature (+, −, −, −), induced by the substrate-level commitment-event causal structure. The light-cone of η coincides with the emergent causal structure of the substrate-to-M correspondence.

Structural role of the metric. The Lorentzian metric η is what *unifies* emergent time and emergent space into a single 4-dimensional arena. Without η, emergent time (the timelike direction of M) and emergent space (the spacelike directions) sit on M as separate substrate-emergent structures with no relation between them — they could equally be described as a 1+3 split with independent dynamics. The metric η is what makes them components of a single

Lorentz-invariant structure: it specifies (i) the sign convention distinguishing timelike from spacelike, (ii) the light cone separating causal from spacelike-separated events, and (iii) the boost transformations that mix time and space components. None of these is supplied by the separate emergences of time (I1, time-component) or space (I1, space-components) alone; the unification is the substantive content of I2.

This is why the emergent-Lorentz inheritances EL1–EL4 are catalogued separately from I1–I3: I1 supplies the manifold (with its separately-emergent time and space components), I2 supplies the metric (unifying them into a Lorentz-structured arena), and EL1–EL4 supply the action of the emergent Lorentz group on substrate spinorial loops and on the substrate-derived measure (making the unification physically operative at the level of substrate-to-spacetime correspondences).

What I2 supplies for Part VI. The metric structure needed to make sense of "Lorentz-covariant transformation," "Lorentz boost," and "Lorentz group action" on M. The Dirac equation in §12 makes sense as a covariant equation on M only because η is given by I2 and unifies the separately-emergent time-derivative ∂_0 and space-derivatives ∂_i into a single covariant operator $\gamma^\mu \partial_\mu$.

What I2 does not address. The detailed substrate-level derivation of η. The dynamical aspects of η (its relation to substrate energy-momentum, the Einstein equations as emergent low-energy effective dynamics, etc.) are deferred to the substrate-level gravity programme.

4.3 The inheritance I3: substrate-derived measure compatibility

Inheritance I3. The substrate-derived measure $d\mu_{\text{spin}}$ on P_{spin} is compatible with the substrate-to-M correspondence ι in the sense that:

$$\int_{P_{\text{spin}}} F(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C})$$

makes sense for substrate-level functionals F that admit an emergent-spacetime reading via ι, and the integral commutes with the coarse-graining limit when F is appropriately localised.

What I3 supplies for Part VI. The integral expression

$$\psi_\alpha(x) = \int U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C}) + \int V_\alpha(\mathcal{C}; x)\, a^\dagger(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C})$$

makes sense as a well-defined operator-valued integral on $\mathscr{F}_A$. Without I3, the integral is purely formal.

What I3 does not address. The detailed substrate-level construction of $d\mu_{\text{spin}}$ and its functional-analytic properties (translation-invariance, Lorentz-invariance — both Part IX deliverables). The present paper uses I3 only in its weak form: that the integral expression is well-defined under appropriate mode-function admissibility conditions.

4.4 The boundary between inheritance and Part VI construction

The inheritances I1, I2, I3 supply the *arena* on which the Part VI construction takes place — emergent spacetime M with its metric η and a substrate-to-spacetime correspondence ι compatible with the substrate-derived measure $d\mu_{\text{spin}}$. The Part VI construction *itself* is:

- The rigged-Hilbert-space apparatus on the substrate side (§5).
- The mode-function admissibility conditions MA1–MA6, derived from substrate-physics inheritances (§6).
- The candidate emergent field operator $\psi_\alpha(x)$ and its rigged-Hilbert-space well-definition (Theorem 1).
- The induced field CAR algebra (Theorem 2).
- The unitary Lorentz representation on Fock space, conditional on EL1–EL4 (Theorem 3).
- The emergent Dirac equation as coarse-graining limit, conditional on CG1–CG2 (Theorem 4).
- The partial microcausality result (§13).
- The physical-realisation reading (Corollary 4').

This boundary is explicit so that the reader can see exactly which load-bearing claims of the present paper are conditional on substrate-to-emergent-spacetime inheritances and which are derivable from the substrate-CAR algebra of Part V independently.

4.5 Joint consistency of the inheritance families

A structural remark worth flagging: although I1–I3 (substrate-to-emergent-spacetime), EL1–EL4 (emergent-Lorentz), and CG1–CG2 (admissible coarse-graining) are catalogued as separable inheritance families, they ultimately all flow from the same substrate physics. The substrate-derived emergent spacetime M (I1), its emergent Lorentz metric η (I2), its emergent Lorentz group action (EL1, EL2), the measure compatibility (EL3a), the measure's Lorentz-normalisation structure (EL3b), the Gram-kernel Lorentz invariance (EL4), and the coarse-graining limit producing the Dirac operator (CG1, CG2) are all aspects of how the substrate gives rise to relativistic spacetime physics.

These families cannot be supplied independently. The joint consistency of all of them — that the substrate-derived M, η, $d\mu_{\text{spin}}$, K, and coarse-graining limit are mutually compatible in the ways required by the present paper — is itself a non-trivial structural constraint on the Part IX construction and on the emergent-Lorentz / coarse-graining programmes. It is identified as an open problem (§17.2, §18 final item) and is the strongest single test of the broader VERSF programme's coherence at this stage of the matter-sector strand.

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5. Rigged Hilbert Space of Substrate Spinorial Modes

In the distributional reading of P_{spin} (Part V §5.2), the single-loop states $|\mathcal{C}\rangle$ are not ordinary vectors in $\mathcal{H}_1$ but distributional generators. To handle them rigorously, the mathematically appropriate framework is the *rigged Hilbert space* (Gelfand triple):

$\Phi \subset \mathscr{H}_1 \subset \Phi^\times$,

where:

- **Φ** is the dense test-function space on P_spin: smooth, rapidly-decaying functions f : P_spin → $\mathbb{C}^4$ in the substrate-derived topology induced by dμ_spin. Test functions live in Φ and have all the regularity properties needed to be smeared against distributional generators.
- **$\mathscr{H}_1$** is the substrate one-particle Hilbert space (Part V Theorem 1), the positive-definite Hilbert completion.
- **Φ×** is the topological dual of Φ — the space of continuous linear functionals on Φ. Distributional objects, including the single-loop generators $|\mathcal{C}\rangle$, live in Φ×.

The inclusions $\Phi \subset \mathscr{H}_1 \subset \Phi^\times$ are continuous and dense. The single-loop generators $|\mathcal{C}\rangle \in \Phi^\times$ pair against test functions $f \in \Phi$ via

$\langle f \mid \mathcal{C}\rangle = f(\mathcal{C})$ (Clifford-internally),

interpreted distributionally.

Remark on the structure of Φ. Constructing Φ rigorously requires more than the brief description above. P_spin is not in general a smooth manifold — it is a moduli space of persistent spinorial loops with substrate-derived measure dμ_spin, and "smoothness" of functions on it must be defined relative to whatever differentiable / topological structure dμ_spin and the substrate physics induce. "Rapid decay" similarly requires a substrate-derived notion of growth on P_spin (a norm, a family of seminorms, or a substrate-derived metric structure). The standard textbook analogue is Schwartz space $\mathcal{S}(\mathbb{R}^n)$ on momentum-space modes, which is well-defined because $\mathbb{R}^n$ has a clear smooth/decay structure; the substrate-level analogue requires the substrate-physics underlying P_spin to supply the corresponding structure. This is part of the Part IX deliverable (the substrate-derived measure-theoretic-and-topological apparatus on P_spin); for the present paper, Φ is taken as a working construct supplied by Part IX, with the standard rigged-Hilbert-space properties (continuous dense inclusions, distributional generators as elements of Φ×, smearing producing $\mathscr{H}_1$ vectors) inherited.

5.1 Smeared substrate states

For each test function $f \in \Phi$, the smeared substrate state

$|f\rangle = \int_{\{P\_spin\}} f(\mathcal{C})\, |\mathcal{C}\rangle\, d\mu\_spin(\mathcal{C})$

is a genuine vector in $\mathscr{H}_1$. The map $f \mapsto |f\rangle$ is the standard pairing between test functions and Hilbert vectors in the Gelfand-triple construction.

The smeared substrate CAR operators a(f), a†(f) for $f \in \Phi$ are bounded operators on ℱ_A satisfying (Part V Theorem 3 in distributional form):

$\{a(f), a^\dagger(g)\} = \langle f \mid g\rangle_{\mathscr{H}_1} \cdot \mathbb{1}_{\mathscr{F}_A}$, $\{a(f), a(g)\} = 0$, $\{a^\dagger(f), a^\dagger(g)\} = 0$,

with operator norm $\|a(f)\| = \|a^\dagger(f)\| = \|f\|_{\mathscr{H}_1}$.

5.2 Why the rigged-Hilbert-space framework is the right starting point

The candidate emergent field operator $\psi_\alpha(x)$ defined in §7 is a *spacetime-distributional object* — it is the field at a *point* $x \in M$, which textbook QFT recognises as a distribution, not an operator on Fock space. Rigorously, $\psi_\alpha(x)$ is an operator-valued distribution: for any test function h on M, $\psi_\alpha(h) = \int h(x)\, \psi_\alpha(x)\, d^4x$ is a bona-fide operator on $\mathscr{F}_A$ (or at least densely-defined on a dense subspace), but $\psi_\alpha(x)$ at a fixed x is not.

The rigged-Hilbert-space framework on the substrate side mirrors this exactly: the single-loop states $|\mathcal{C}\rangle$ are substrate-distributional, and the smeared states $|f\rangle$ are genuine Hilbert vectors. The bridge between substrate distributional generators and spacetime distributional fields is precisely what the mode-function construction (§6) supplies.

The framework is also standard for axiomatic free Dirac field theory (Streater–Wightman; Bogoliubov–Logunov–Oksak–Todorov). The Part VI construction extends this standard framework to the substrate-level setting by using the substrate rigged Hilbert space on the input side and the standard spacetime test-function space on the output side.

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6. Substrate-to-Spacetime Mode Functions and Their Admissibility

The bridge between the substrate-distributional CAR operators $a(\mathcal{C})$, $a^\dagger(\mathcal{C})$ and the spacetime-distributional fields $\psi_\alpha(x)$ is mediated by the *substrate-to-spacetime mode functions*

$U_\alpha(\mathcal{C}; x), V_\alpha(\mathcal{C}; x) : P_{\text{spin}} \times M \to \mathbb{C}^4$.

These functions encode how a substrate spinorial loop $\mathcal{C}$, when viewed in the coarse-grained emergent-spacetime description, contributes to the field amplitude at spacetime point x.

The mode functions are not free choices. They are constrained by substrate-physics inheritances and by structural compatibility requirements. The admissibility conditions MA1–MA6 below catalogue these constraints. The detailed substrate-level construction of mode functions satisfying MA1–MA6 is the Part IX deliverable; the present paper assumes existence of such mode functions as a working hypothesis and identifies what their substrate-derived existence depends on.

6.1 Interpretation: U_α and V_α as positive- and negative-frequency carriers

Following textbook precedent in free Dirac quantisation:

- **U_α(𝒞; x)** corresponds to the *positive-frequency* mode-function contribution — the spacetime-amplitude contribution of substrate creation events to the field at x. Under emergent-time translation, U_α(𝒞; x) has positive frequency support: U_α(𝒞; x + $t\hat{e}_0$) carries a phase e^{−iω(𝒞)t} with ω(𝒞) > 0.
- **V_α(𝒞; x)** corresponds to the *negative-frequency / charge-conjugate* mode-function contribution — the spacetime-amplitude contribution of substrate annihilation events to the field at x, with the conjugate frequency structure: V_α(𝒞; x + $t\hat{e}_0$) carries phase e^{+iω(𝒞)t}.

In the standard textbook free Dirac construction, U and V are built from positive- and negative-frequency momentum-space spinors u_s(p), v_s(p) respectively. The substrate-level analogue is more general — substrate spinorial modes 𝒞 are richer than momentum-space modes — but the positive-/negative-frequency split is preserved.

Note on antiparticles. The presence of V_α(𝒞; x) introduces, implicitly, the seeds of antiparticle structure: the a†(𝒞) coefficient in the field expansion is conjugate-frequency, mirroring the textbook v_s(p) b†(p) antiparticle structure. The present paper does *not* construct the antiparticle Fock-space doubling, the charge-conjugation operator C, or the antiparticle vacuum; these are deferred to Part VIII. What the present paper does is admit V_α as a structural placeholder for what Part VIII will explicitly develop. Without V_α, the field expansion would lack the negative-frequency structure that is essential for the Dirac equation to admit relativistic propagation; with V_α as a placeholder, the field-level CAR algebra and the Dirac equation can be derived without committing to the full antiparticle ontology.

6.2 Admissibility conditions

Definition (Admissible mode functions). Mode functions U_α(𝒞; x), V_α(𝒞; x) : P_spin × M → $\mathbb{C}^4$ are *admissible* if they satisfy the following six conditions:

MA1 (Square-integrability). For each x ∈ M, the functions 𝒞 ↦ U_α(𝒞; x) and 𝒞 ↦ V_α(𝒞; x) lie in $\mathcal{H}_1$ (Clifford-internally, in each spinor component). Equivalently, ‖U_α(·; x)‖*{$\mathcal{H}_1$} < ∞ and ‖V_α(·; x)‖*{$\mathcal{H}_1$} < ∞ for every x.

MA2 (Spacetime regularity). For each 𝒞 ∈ P_spin and each spinor index α, the functions x ↦ U_α(𝒞; x) and x ↦ V_α(𝒞; x) are smooth (C∞) on M.

MA3 (Substrate locality). There exists a coherence-scale localisation: for each x ∈ M, the support of U_α(·; x) (and V_α(·; x)) in P_spin is concentrated on substrate configurations corresponding to spacetime regions within emergent distance ≲ ξ of x. Substrate modes that are far from x in the emergent-spacetime sense contribute negligibly to ψ(x).

MA4 (Completeness / closure relation). The mode functions satisfy a substrate-derived closure relation that ensures the field CAR cross-relation reproduces the textbook $\delta^3(\mathbf{x} - \mathbf{y})$ form in the local continuum limit. Concretely, the equal-time spacetime kernel induced by the combined U and V mode-function contributions reduces to the textbook form:

∬{P_spin × P_spin} [U_α(𝒞; x) Ū_β(𝒞'; y) + V̄_β(𝒞; y) V_α(𝒞'; x)] K(𝒞, 𝒞') dμ_spin(𝒞) dμ_spin(𝒞') |{x^0=y^0} → δ_{αβ} · $\delta^3(\mathbf{x} - \mathbf{y})$ (local continuum limit).

The "local continuum limit" here refers to the regime where the coherence-scale localisation MA3 is much finer than the spacetime-resolution scale of interest. Note that MA4 is stated with two distinct loop variables 𝒞 and 𝒞' contracted against the Gram kernel K(𝒞, 𝒞'); this matches the structure of the field-level spacetime kernel of Theorem 2 below, and is the substrate analogue of the textbook completeness relation Σ_s u^s_α(p) ū^s_β(p) + Σ_s v^s_α(p) v̄^s_β(p) → δ_{αβ} (after appropriate integration) in standard free Dirac quantisation.

MA5 (Emergent-Lorentz covariance compatibility). Under the EL1–EL2 Lorentz action (inheritance §3.5), the mode functions transform covariantly:

U_α(Λ𝒞; Λx) = S(Λ)_α^β · U_β(𝒞; x), V_α(Λ𝒞; Λx) = S(Λ)_α^β · V_β(𝒞; x).

This is the substrate-level analogue of the textbook Lorentz transformation rule for plane-wave mode functions.

MA6 (Mass-shell admissibility). The mode functions satisfy the Dirac mass-shell constraints in the coarse-graining limit:

(iγ^μ ∂_μ^x − m) U_α(𝒞; x) = 0, (iγ^μ ∂_μ^x + m) V_α(𝒞; x) = 0,

where ∂^x denotes differentiation with respect to the emergent-spacetime point x and m is the substrate κ-field-derived mass parameter inherited via Part III Theorem 1 (CG2).

These constraints are mass-shell support conditions in the substrate-momentum-space-analogue sense: U_α(𝒞; x) and V_α(𝒞; x) are supported on substrate modes 𝒞 that, in the coarse-graining limit, correspond to on-mass-shell momentum-space configurations. Mass-shell admissibility is a substantive structural requirement on the mode functions — not a trivial consequence of MA1–MA5 — and its substrate-level realisability depends on the substrate Fourier-analogue structure (σ-duality on K=7 wheels, per §6.3) producing mass-shell-localised modes. MA6 is named separately to flag this; in earlier drafts of the present paper, mass-shell support was implicit, folded into CG1, but this understated the structural role.

6.3 Substrate-physics derivation of mode-function existence (Part IX deliverable)

The detailed substrate-level construction of mode functions satisfying MA1–MA6 is the deliverable of Part IX (the substrate-to-emergent-spacetime correspondence paper). The relevant substrate-physics inputs for that construction, broken out by which aspect of emergent spacetime each contributes to, include:

- *Substrate-time-emergence inputs:* The sequential interface transport structure and σ-duality on K=7 wheels jointly provide the substrate-level commitment-event ordering that becomes emergent-time translation under admissible coarse-graining. These supply the time-direction support of U_α(𝒞; x) and V_α(𝒞; x) — equivalently, the positive- and negative-frequency structure in the emergent-time sense (§6.1).

- *Substrate-space-emergence inputs:* The K=7 architecture's spatial degrees of freedom plus coarse-graining of substrate configurations on scales much larger than ξ supply the spatial structure of the substrate that becomes emergent space. These supply the space-direction support of mode functions and the substrate-level analogue of momentum-space decomposition.
- *Substrate-Lorentz-unification inputs:* The emergent-Lorentz programme (broader VERSF papers) supplies the structure that combines emergent time and emergent space into a Lorentz-invariant whole. These are the EL1–EL4 inheritances of §3.5 and they enter the mode-function admissibility through MA5 (Lorentz covariance) and MA6 (mass-shell admissibility, which is itself a Lorentz-invariant condition).
- *Substrate-mass inputs:* The substrate κ-field structure (via Part III Theorem 1 and CG2) sets the mass scale m appearing in the Dirac equation and in MA6.
- *Substrate-causal inputs:* The substrate causal structure inherited from sequential interface transport sets the emergent light cone of η (I2) and hence constrains the support structure of U_α, V_α.

The decomposition makes explicit what was implicit in earlier drafts of this section: mode functions inherit different aspects of their structure from different substrate-physics provenances, and Part IX's job is not a single monolithic "substrate-to-spacetime bridge" but a coordinated supply of substrate-time-emergence + substrate-space-emergence + emergent-Lorentz-unification + substrate-mass + substrate-causal-structure as a mutually consistent package.

The present paper treats MA1–MA6 as a working hypothesis. The structural shape of the construction is such that *if* substrate-level mode functions satisfying MA1–MA6 exist (Part IX deliverable, with the multi-provenance structure above), *then* Theorems 1, 2, 3, 4 of the present paper hold conditional on the further inheritances EL1–EL4 and CG1–CG2.

6.4 The mode-function admissibility as the structural backbone of Part VI

The six conditions MA1–MA6 are not arbitrary technical assumptions; each plays a distinct structural role:

- **MA1** ensures that the smeared integral expression for ψ_α(x) is well-defined (Theorem 1).
- **MA2** ensures that ψ_α(x) admits the standard test-function smearing on M to yield bona-fide operators ψ_α(h).
- **MA3** ensures substrate locality, which is the substrate-side input to the substrate-coherence-scale suppression result (§13.3). Note: this is *not* light-cone-respecting microcausality — see §13.2, §13.4.
- **MA4** ensures that the induced field CAR algebra has the textbook δ^3 form in the appropriate limit (Theorem 2).
- **MA5** ensures Lorentz-covariant transformation of the emergent field under the unitary Lorentz representation (Theorem 3).

- **MA6** ensures the emergent free Dirac equation holds as the coarse-graining limit (Theorem 4); mass-shell support is the load-bearing input from the mode-function side.

The construction does not work without all six.

7. Emergent Fermion Field Operators

With the rigged Hilbert space (§5) and admissible mode functions (§6) in place, the candidate emergent fermion field operator is defined as follows.

Definition 1 (Emergent fermion field operator). Conditional on inheritances I1, I2, I3 (§4) and admissibility of mode functions U_α, V_α satisfying MA1–MA6 (§6), the *candidate emergent fermion field operator* is

$$\psi_\alpha(x) := \int_{P_{spin}} U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_{spin}(\mathcal{C}) + \int_{P_{spin}} V_\alpha(\mathcal{C}; x)\, a^\dagger(\mathcal{C})\, d\mu_{spin}(\mathcal{C}),$$

interpreted initially as an operator-valued distribution on $\mathscr{F}_A$. The Dirac adjoint is

$$\bar{\psi}_\alpha(x) := [\psi^\dagger(x)\, \gamma^0]_\alpha.$$

The smeared field operator, for any test function $h \in \mathcal{S}(M)$ (Schwartz test functions on emergent spacetime M), is

$$\psi_\alpha(h) := \int_M h(x)\, \psi_\alpha(x)\, d^4x.$$

The integral is well-defined as an operator on $\mathscr{F}_A$ (Theorem 1 below).

7.1 Comparison to textbook free Dirac quantisation

In textbook free Dirac quantisation, the field operator expansion is

$$\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} \cdot \sum_s \left[u^s_\alpha(p)\, b_s(p)\, e^{-ipx} + v^s_\alpha(p)\, c_s^\dagger(p)\, e^{+ipx} \right],$$

with $b_s(p)$, $c_s^\dagger(p)$ the particle creation/annihilation and antiparticle creation operators, and u, v the standard textbook spinors.

The substrate-level analogue replaces:

- The momentum-space integration $\int d^3p / (2\pi)^3 \cdot (1/\sqrt{2E_p})$ with the substrate-derived integration $\int d\mu_{spin}(\mathcal{C})$.
- The plane-wave spinors $u^s_\alpha(p)\, e^{-ipx}$ and $v^s_\alpha(p)\, e^{+ipx}$ with the substrate-derived mode functions $U_\alpha(\mathcal{C}; x)$ and $V_\alpha(\mathcal{C}; x)$.

- The momentum-mode operators $b_s(p)$, $c_s^\dagger(p)$ with the substrate-mode CAR operators $a(\mathcal{C})$, $a^\dagger(\mathcal{C})$.

The structure is parallel. The substrate-level version is richer because P_{spin} is a richer label space than three-momentum; substrate spinorial modes carry orientation-frame data, internal Clifford structure, and persistence-stability properties that textbook momentum modes do not. However, the mode-function admissibility conditions MA1–MA6 are calibrated to ensure that under appropriate coarse-graining, the substrate field expansion reduces to the textbook form.

7.2 Ontological reading

In *canonical-quantisation textbook QFT*, $\psi(x)$ is the fundamental field; the mode operators $b(p)$, $c^\dagger(p)$ are derived as Fourier components. The framing of fields-as-primary is standard in this tradition.

In *algebraic QFT* (Haag, Bratteli–Robinson, Streater–Wightman), local operator algebras are primitive and $\psi(x)$ is derived as an affiliated operator-valued distribution; the algebra-as-primary framing has long precedent in the foundations literature.

The VERSF version is therefore not a wholly novel ontological inversion — it is closer to the algebraic-QFT framing, with the additional substrate-algebra-as-primary content being the *specific* algebra (the Part V substrate CAR algebra on persistent spinorial loops) rather than a generic local algebra on spacetime. What is genuinely new is *which* algebra is primary: not the spacetime-local algebra of textbook algebraic QFT, but the substrate-loop-labelled algebra of Part V. The field $\psi(x)$ is derived from this substrate algebra via the mode-function construction, with spacetime itself emerging from substrate dynamics rather than being supplied as primitive.

This is more modest than "substrate-as-primary, fields-as-emergent" presented as a stand-alone ontological claim. The pertinent framing for the present paper is *substrate-algebra-as-primary, spacetime-emerging-from-substrate-as-the-distinctive-VERSF-content*. The field $\psi(x)$ "exists" in the sense that the smeared operator $\psi(h)$ is a well-defined operator on $\mathscr{F}_A$ for every test function h. The pointwise object $\psi(x)$ does not exist as an operator (it is a distribution), exactly as in any rigorous QFT framework. The substantive Part VI content is that the existence of $\psi(h)$ *depends on* the substrate-to-emergent-spacetime inheritances; without them, the integral expression has no meaning. The textbook free Dirac field is the limiting case where the substrate inheritances collapse to the trivial Minkowski-background construction.

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8. Theorem 1 — Existence of Smeared Field Operators

8.1 The theorem

Theorem 1 (Existence of smeared emergent fermion field operators). Conditional on inheritances I1, I2, I3 (§4) and on admissibility of mode functions U_α, V_α satisfying MA1–MA6 (§6), the candidate emergent fermion field operator $\psi_\alpha(x)$ defined in §7 satisfies:

(a) For each spinor index $\alpha \in \{1, 2, 3, 4\}$ and each test function $h \in \mathcal{S}(M)$, the smeared operator

$$\psi_\alpha(h) = \int_M h(x)\, \psi_\alpha(x)\, d^4x$$

is a bounded operator on $\mathscr{F}_A$ with norm

$$\|\psi_\alpha(h)\| \le \|h \cdot U_\alpha\|_{L^2(M \times P_{spin},\, d^4x\, d\mu_{spin})} + \|h \cdot V_\alpha\|_{L^2(M \times P_{spin},\, d^4x\, d\mu_{spin})}.$$

(b) The map $h \mapsto \psi_\alpha(h)$ is linear and continuous in the Schwartz topology on $\mathcal{S}(M)$, defining ψ_α as an operator-valued tempered distribution on M with values in $B(\mathscr{F}_A)$.

(c) The Dirac adjoint $\bar{\psi}_\alpha(h) = [\psi^\dagger(\bar{h})\, \gamma^0]_\alpha$ satisfies the same properties.

8.2 Proof

The smeared field operator is

$$\psi_\alpha(h) = \int_M h(x) \left[\int_{P_{spin}} U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_{spin}(\mathcal{C}) + \int_{P_{spin}} V_\alpha(\mathcal{C}; x)\, a^\dagger(\mathcal{C})\, d\mu_{spin}(\mathcal{C})\right] d^4x.$$

By Fubini (applicable because of MA1, MA2 ensuring measurability, and the test-function support of h):

$$\psi_\alpha(h) = \int_{P_{spin}} \left[\int_M h(x)\, U_\alpha(\mathcal{C}; x)\, d^4x\right] a(\mathcal{C})\, d\mu_{spin}(\mathcal{C}) + \int_{P_{spin}} \left[\int_M h(x)\, V_\alpha(\mathcal{C}; x)\, d^4x\right] a^\dagger(\mathcal{C})\, d\mu_{spin}(\mathcal{C}).$$

Define the substrate test functions

$$f_\alpha^U(\mathcal{C}; h) := \int_M h(x)\, U_\alpha(\mathcal{C}; x)\, d^4x, \quad f_\alpha^V(\mathcal{C}; h) := \int_M h(x)\, V_\alpha(\mathcal{C}; x)\, d^4x.$$

By MA1 (square-integrability of $U_\alpha(\cdot; x)$ and $V_\alpha(\cdot; x)$ in $\mathscr{H}_1$) and MA2 (spacetime regularity), together with $h \in \mathcal{S}(M)$, the functions $f_\alpha^U(\cdot; h)$ and $f_\alpha^V(\cdot; h)$ lie in $\mathscr{H}_1$ (Clifford component α), with

$$\|f_\alpha^U(\cdot; h)\|_{\mathscr{H}_1} \le \|h \cdot U_\alpha\|_{L^2(M \times P_{spin})}, \quad \|f_\alpha^V(\cdot; h)\|_{\mathscr{H}_1} \le \|h \cdot V_\alpha\|_{L^2(M \times P_{spin})}.$$

Therefore

$$\psi_\alpha(h) = a(f_\alpha^U(\cdot; h)) + a^\dagger(f_\alpha^V(\cdot; h)),$$

a linear combination of bounded substrate CAR operators on $\mathscr{F}_A$. By the substrate-level boundedness $\|a(f)\| = \|a^\dagger(f)\| = \|f\|_{\mathscr{H}_1}$ (Part V §9.2 in distributional form), $\psi_\alpha(h)$ is bounded with

$$\|\psi_\alpha(h)\| \le \|f_\alpha^U(\cdot; h)\|_{\mathscr{H}_1} + \|f_\alpha^V(\cdot; h)\|_{\mathscr{H}_1} \le \|h \cdot U_\alpha\|_{L^2} + \|h \cdot V_\alpha\|_{L^2}.$$

This proves (a). Linearity and continuity in the Schwartz topology (part (b)) follow from the bilinearity of $f_\alpha^U(\mathcal{C}; h)$ in h and the L^2-boundedness above; standard distribution-theoretic arguments yield the operator-valued tempered distribution property. Part (c) is immediate by Hermitian conjugation, with appropriate complex conjugation of mode functions and h.

8.3 Structural reading

Theorem 1 establishes that the candidate emergent field, defined by the mode-function smearing of substrate CAR operators, is a well-defined operator-valued distribution on $\mathcal{F}_A$. This is the foundational existence result for the entire Part VI construction.

What Theorem 1 does not yet establish:

- The induced field-level CAR algebra (Theorem 2).
- Lorentz-covariant transformation (Theorem 3).
- The emergent Dirac equation (Theorem 4).
- Microcausality (§13).

Each is addressed separately, building on Theorem 1.

8.4 Conditionality

Theorem 1 is conditional on:

- **I1–I3** (substrate-to-emergent-spacetime inheritances): supply the arena M and the well-definedness of the substrate integral.
- **MA1, MA2** (mode-function admissibility): ensure the smearing produces $\mathcal{H}_1$ vectors via Fubini.
- **Part V Theorem 3** (substrate CAR boundedness in distributional form): ensures bounded operators on $\mathcal{F}_A$.

Theorem 1 does not yet require EL1–EL4 (Lorentz inheritances) or CG1–CG2 (coarse-graining inheritances).

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9. Theorem 2 — Field CAR Induced from Substrate CAR

9.1 The theorem

Theorem 2 (Field-level CAR algebra induced from substrate CAR). Under the conditions of Theorem 1, the smeared emergent fermion field operators satisfy the field-level CAR algebra:

$$\{\psi_\alpha(h_1), \psi_\beta^\dagger(h_2)\} = \langle f_\alpha^U(\cdot; h_1) \mid f_\beta^U(\cdot; \bar{h}_2)\rangle_{\mathcal{H}_1} \cdot \mathbb{1}_{\mathcal{F}_A} + \langle f_\beta^V(\cdot; \bar{h}_2) \mid f_\alpha^V(\cdot; h_1)\rangle_{\mathcal{H}_1} \cdot \mathbb{1}_{\mathcal{F}_A}, \quad \{\psi_\alpha(h_1), \psi_\beta(h_2)\} = 0, \quad \{\psi_\alpha^\dagger(h_1), \psi_\beta^\dagger(h_2)\} = 0,$$

for all $h_1, h_2 \in \mathcal{S}(M)$ and all spinor indices α, β.

In the unsmeared form, the cross-relation is the spacetime kernel:

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = K_{\alpha\beta}(x, y) \cdot \mathbb{1}_{\mathcal{F}_A},$$

where the spacetime kernel is the mode-function transform of the substrate Gram kernel:

$$K_{\alpha\beta}(x, y) := \int_{P_{spin}} \int_{P_{spin}} [U_\alpha(\mathcal{C}; x)\, \bar{U}_\beta(\mathcal{C}'; y) + \bar{V}_\beta(\mathcal{C}; y)\, V_\alpha(\mathcal{C}'; x)]\, K(\mathcal{C}, \mathcal{C}')\, d\mu_{spin}(\mathcal{C})\, d\mu_{spin}(\mathcal{C}').$$

Under MA4 (mode-function completeness), in the local continuum limit the spacetime kernel reduces to the textbook free-Dirac form:

$$K_{\alpha\beta}(x, y)|_{x^0=y^0} \to \delta_{\alpha\beta} \cdot \delta^3(\mathbf{x} - \mathbf{y}) \text{ (local continuum limit).}$$

9.2 Proof

By the expression from Theorem 1,

$$\psi_\alpha(h_1) = a(f_\alpha^U(\cdot; h_1)) + a^\dagger(f_\alpha^V(\cdot; h_1)),\quad \psi_\beta^\dagger(h_2) = a^\dagger(f_\beta^U(\cdot; \bar{h}_2)) + a(f_\beta^V(\cdot; \bar{h}_2)),$$

(adjoint conjugates $U \leftrightarrow \bar{U}$ and $V \leftrightarrow \bar{V}$ with appropriate complex conjugation of h).

Compute the anticommutator:

$$\{\psi_\alpha(h_1), \psi_\beta^\dagger(h_2)\} = \{a(f_\alpha^U(\cdot; h_1)) + a^\dagger(f_\alpha^V(\cdot; h_1)), a^\dagger(f_\beta^U(\cdot; \bar{h}_2)) + a(f_\beta^V(\cdot; \bar{h}_2))\}.$$

Expanding using bilinearity of the anticommutator:

$$= \{a(f_\alpha^U(\cdot; h_1)), a^\dagger(f_\beta^U(\cdot; \bar{h}_2))\} + \{a(f_\alpha^U(\cdot; h_1)), a(f_\beta^V(\cdot; \bar{h}_2))\} + \{a^\dagger(f_\alpha^V(\cdot; h_1)), a^\dagger(f_\beta^U(\cdot; \bar{h}_2))\} + \{a^\dagger(f_\alpha^V(\cdot; h_1)), a(f_\beta^V(\cdot; \bar{h}_2))\}.$$

By the substrate CAR algebra (Part V Theorem 3):

- $\{a(f), a^\dagger(g)\} = \langle f | g\rangle_{\mathcal{H}_1} \cdot \mathbb{1}$ — applies to the first and fourth terms.
- $\{a(f), a(g)\} = 0$ and $\{a^\dagger(f), a^\dagger(g)\} = 0$ — the second and third terms vanish.

The fourth term gives $\{a^\dagger(f_\alpha^V(\cdot; h_1)), a(f_\beta^V(\cdot; \bar{h}_2))\} = \langle f_\beta^V(\cdot; \bar{h}_2) | f_\alpha^V(\cdot; h_1)\rangle_{\mathcal{H}_1} \cdot \mathbb{1}$ (by the adjoint-flip convention).

Combining:

$$\{\psi_\alpha(h_1), \psi_\beta^\dagger(h_2)\} = [\langle f_\alpha^U(\cdot; h_1) | f_\beta^U(\cdot; \bar{h}_2)\rangle_{\mathcal{H}_1} + \langle f_\beta^V(\cdot; \bar{h}_2) | f_\alpha^V(\cdot; h_1)\rangle_{\mathcal{H}_1}] \cdot \mathbb{1}_{\mathcal{F}_A}.$$

This is the smeared form. The unsmeared spacetime-kernel form is obtained by substituting the definitions of f_α^U and f_α^V and exchanging order of integration:

$$K_{\alpha\beta}(x, y) = \int_{P_{\text{spin}} \times P_{\text{spin}}} [U_\alpha(\mathcal{C}; x)\, \bar{U}_\beta(\mathcal{C}'; y) + \bar{V}_\beta(\mathcal{C}; y)\, V_\alpha(\mathcal{C}'; x)]\, K(\mathcal{C}, \mathcal{C}')\, d\mu_{\text{spin}}\, d\mu_{\text{spin}},$$

as stated.

For the same-α-α and adjoint-adjoint anticommutators, all terms involve $\{a, a\}$ or $\{a^\dagger, a^\dagger\}$, which vanish by the substrate CAR algebra. Hence $\{\psi_\alpha, \psi_\beta\} = 0$ and $\{\psi_\alpha^\dagger, \psi_\beta^\dagger\} = 0$.

The textbook δ^3 form in the local continuum limit follows from MA4 by direct substitution: the completeness condition is precisely calibrated so that $K_{\alpha\beta}(x, y)|_{x^0=y^0} \to \delta_{\alpha\beta} \cdot \delta^3(\mathbf{x} - \mathbf{y})$.

9.3 Structural reading

Theorem 2 establishes that the field-level CAR algebra is not postulated; it is *induced* from the substrate-level CAR algebra of Part V via the mode-function construction. The substrate-level CAR is the load-bearing input; the textbook field-level CAR is the derived output.

In particular, the textbook form $\{\psi_\alpha(t, \mathbf{x}), \psi_\beta^\dagger(t, \mathbf{y})\} = \delta_{\alpha\beta} \cdot \delta^3(\mathbf{x} - \mathbf{y})$ is recovered in the local continuum limit, conditional on MA4. The exact spacetime kernel $K_{\alpha\beta}(x, y)$ is more general than the textbook δ^3 form — it carries substrate-locality information at scales $\lesssim \xi$ where the coherence structure of MA3 plays a role. In the regime of standard fermionic-matter phenomenology (well above the substrate coherence scale), the textbook form holds to arbitrary precision; at substrate-coherence scales, the spacetime kernel carries substrate-physics fingerprints that are absent from the textbook free Dirac construction. These fingerprints are the channel through which Part VI's substrate-physics dependence becomes empirically meaningful (see §16, Channel B).

9.4 Conditionality

Theorem 2 is conditional on:

- **Theorem 1** (existence of smeared field operators) and its inheritances.
- **MA4** (mode-function completeness condition) for the textbook δ^3 limit.
- **Part V Theorem 3** (substrate CAR algebra) — the load-bearing substrate input.

Theorem 2 does not yet require EL1–EL4 (Lorentz) or CG1–CG2 (coarse-graining).

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10. The Positivity–Covariance Bridge

This section addresses what Part V identified as the dominant substructural prerequisite of the Part VI construction: the *positivity–covariance bridge*. The bridge reconciles two apparently competing requirements:

- **Positivity:** the substrate Fock space $\mathscr{F}_A$ has a positive-definite Hilbert structure (Part V Theorem 1), required for $\psi^\dagger\psi$ to serve as the quantum-mechanical inner product.
- **Covariance:** the emergent fermion field $\psi(x)$ must transform covariantly under emergent Lorentz transformations, $\psi_\alpha(x) \mapsto S(\Lambda)^{-1}{}_\alpha{}^\beta\, \psi_\beta(\Lambda x)$, in a manner consistent with relativistic phenomenology.

The textbook tension is that the Lorentz-invariant spinorial bilinear $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is *indefinite* (signature (2, 2) on $\mathbb{C}^4$), so adopting it as the Hilbert-space norm would destroy positivity. The textbook resolution — used in free Dirac quantisation — is that Lorentz covariance is implemented at the *operator-algebraic* level (unitary representations on Fock space), not at the inner-product level. The Hilbert norm remains positive-definite (using $\psi^\dagger\psi$); Lorentz acts unitarily.

The Part VI bridge is the substrate-level analogue of this resolution. It requires construction of a unitary representation $U(\Lambda) : \mathscr{F}_A \to \mathscr{F}_A$ of the emergent Lorentz group on substrate Fock space, with the emergent field operators transforming covariantly under conjugation by $U(\Lambda)$.

10.1 The structural shape of the bridge

The construction proceeds in three steps:

Step 1 — Define $U(\Lambda)$ on the substrate one-particle Hilbert space $\mathscr{H}_1$. Use the emergent-Lorentz action on substrate spinorial modes (inheritance EL2) to define a candidate action $U_1(\Lambda) : \mathscr{H}_1 \to \mathscr{H}_1$ on the substrate one-particle space. The defining property is

$U_1(\Lambda)\, |f\rangle = |\Lambda f\rangle$, where $(\Lambda f)(\mathcal{C}) := S(\Lambda) \cdot f(\Lambda^{-1}\mathcal{C})$

for $f \in \Phi \subset \mathscr{H}_1$, with $S(\Lambda)$ the standard Dirac spinor representation (inheritance EL1).

Step 2 — Establish unitarity of $U_1(\Lambda)$ on $\mathscr{H}_1$. Show that $U_1(\Lambda)$ preserves the $\mathscr{H}_1$ inner product. This is where the substrate-derived measure compatibility (EL3a) and the Gram-kernel Lorentz invariance (EL4) play load-bearing roles: $\langle \Lambda f \mid \Lambda g\rangle\{\mathscr{H}_1\} = \langle f \mid g\rangle\{\mathscr{H}_1\}$ requires that the change of variables $\mathcal{C} \to \Lambda\mathcal{C}$ leaves $d\mu_{\text{spin}}$ invariant (EL3a), that the substrate Gram kernel $K(\mathcal{C}, \mathcal{C}')$ is Lorentz-invariant on both arguments (EL4), and that the $S(\Lambda)$ action on the Clifford-internal pairing preserves $\psi^\dagger\psi$ after integration against the appropriately calibrated substrate measure (EL3b).

The standard Dirac spinor representation $S(\Lambda)$ does *not* in general preserve $\psi^\dagger\psi$ pointwise (it preserves $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ instead, by construction). So Step 2 cannot be proved trivially. The resolution is that $S(\Lambda)$ does preserve $\psi^\dagger\psi$ after integration against the substrate-derived measure that incorporates the full mass-shell-measure-analogue structure (combined energy weighting and boost Jacobian — see §11.2 Step 2 for the textbook analogue). The combination EL3a +

EL3b + EL4 must hold jointly for the unitarity result; EL3b in particular is the load-bearing structural commitment.

Step 3 — Extend $U_1(\Lambda)$ to $U(\Lambda)$ on $\mathscr{F}_A$ by second quantisation. Once $U_1(\Lambda)$ is unitary on $\mathscr{H}_1$, the standard second-quantisation construction (Bratteli–Robinson §5.2.3) supplies a unique unitary $U(\Lambda)$ on $\mathscr{F}_A$ satisfying:

$$U(\Lambda)\,|0\rangle = |0\rangle,\quad U(\Lambda)\,a^\dagger(f)\,U(\Lambda)^{-1} = a^\dagger(U_1(\Lambda)f),\quad U(\Lambda)\,a(f)\,U(\Lambda)^{-1} = a(U_1(\Lambda)f),$$

for all $f \in \mathscr{H}_1$. The action on multi-particle states is determined by tensoring $U_1(\Lambda)$ over the n-particle sectors and projecting to $\wedge^n \mathscr{H}_1$.

This is the substrate-level analogue of the standard textbook construction of unitary Lorentz representations on free fermionic Fock space. It works because $\mathscr{F}_A$ is positive-definite (Part V Theorem 1).

10.2 What the bridge does and does not establish

The bridge establishes (substantive content of Theorem 3):

- *Positivity-preserving second quantisation.* The substrate Hilbert positive-definite structure on $\mathscr{F}_A$ is preserved by the construction of $U(\Lambda)$; no Krein-space or indefinite-metric structure is introduced. This is the substrate-level analogue of the textbook result that Lorentz acts unitarily on free fermionic Fock space, and it works for the same reason (the standard antisymmetric Fock-space second quantisation of a unitary one-particle representation produces a unitary Fock representation).
- *Algebraic consistency of the bridge.* Given MA5 (mode-function covariance, which is itself an inheritance — see below), the algebraic manipulation that translates MA5 into field covariance $U(\Lambda)\,\psi_\alpha(x)\,U(\Lambda)^{-1} = S(\Lambda)^{-1}{}_\alpha{}^\beta \cdot \psi_\beta(\Lambda x)$ works coherently with the substrate-CAR transformation and the EL3a measure invariance. This is a non-trivial algebraic check that the bridge is consistent.

The bridge directly assumes (deferred to other parts of the VERSF programme):

- *Mode-function covariance (MA5).* The statement that admissible mode functions transform covariantly under (substrate-Lorentz, Dirac-spinor) combined action is a structural assumption on the mode functions. Substrate-level construction of mode functions satisfying MA5 is the Part IX deliverable.
- *EL1, EL2, EL3a, EL3b, EL4.* The substrate-level emergent-Lorentz inheritances. Their substrate-level derivation is the deliverable of the emergent-Lorentz programme.

The bridge does not establish: the substrate-level derivation of EL1, EL2, EL3a, EL3b, EL4 themselves; the full Poincaré covariance including spacetime translations (partial in EL2, partly deferred to Part IX); the spin-statistics theorem in its full axiomatic-QFT form (requires complete Lorentz covariance plus locality plus positivity of the energy spectrum, the last being a separate Part VII issue).

The bridge is a *conditional* construction. It says: *if* EL1, EL2, EL3a, EL3b, EL4 hold and *if* MA5 holds, *then* the substrate Fock structure supports a unitary Lorentz representation under which the emergent field transforms covariantly, *and* this works without compromising the positive-definite Hilbert norm. The substantive content of the theorem is the positivity-preserving second quantisation and the algebraic consistency check; the conditional content is the emergent-Lorentz and mode-function inheritances.

This distinction matters for honest reading: a hostile referee might note that "Theorem 3 establishes Lorentz covariance" understates the conditional structure. The accurate statement is that Theorem 3 establishes the bridge — the assertion that the substrate Fock construction is *capable* of supporting Lorentz covariance, *given* the necessary inheritances. Whether the inheritances themselves hold is a question for the broader VERSF programme.

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11. Theorem 3 — Lorentz-Covariant Transformation Under Unitary Representation

11.1 The theorem

Theorem 3 (Positivity–covariance bridge: Lorentz-covariant transformation). Conditional on inheritances I1–I3 (§4), EL1, EL2, EL3a, EL3b, EL4 (§3.5), mode-function admissibility MA1–MA6 (§6), and Theorems 1, 2 of the present paper, there exists a unitary representation $U(\Lambda)$ of the emergent Lorentz group on substrate Fock space $\mathscr{F}_A$ satisfying:

(a) $U(\Lambda)$ is unitary on $(\mathscr{F}_A, \langle \cdot | \cdot \rangle_{\mathscr{F}_A})$; the substrate Hilbert positive-definite norm is preserved throughout. No indefinite-metric structure is introduced.

(b) The emergent fermion field operator transforms covariantly under conjugation by $U(\Lambda)$:

$$U(\Lambda)\, \psi_\alpha(x)\, U(\Lambda)^{-1} = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot \psi_\beta(\Lambda x),$$

equivalently in adjoint form

$$U(\Lambda)\, \bar{\psi}_\alpha(x)\, U(\Lambda)^{-1} = \bar{\psi}_\beta(\Lambda x) \cdot [S(\Lambda)]^\beta{}_\alpha.$$

The matrix in the adjoint transformation is $S(\Lambda)$, not $S(\Lambda)^{-1}$; this is correct precisely because the field transformation uses Λx (rather than the textbook-passive $\Lambda^{-1}x$). The index placement $[S(\Lambda)]^\beta{}_\alpha$ uses the same convention as $[S(\Lambda)^{-1}]_\alpha{}^\beta$ in part (b)'s field equation: the lower index is the "row" (output) and the upper index is the "column" (input). In the field-component sum, $\bar{\psi}_\beta$ (row spinor, index β) contracts with $[S(\Lambda)]^\beta{}_\alpha$ along the row index of S, leaving α free.

(c) $U(\Lambda)$ preserves the vacuum: $U(\Lambda)\, |0\rangle = |0\rangle$.

(d) The induced unitary action on substrate one-particle states is given by the formula of §10.1 Step 1:

$U(\Lambda)\, a^\dagger(f)\, U(\Lambda)^{-1} = a^\dagger(U_1(\Lambda)f)$, $U(\Lambda)\, a(f)\, U(\Lambda)^{-1} = a(U_1(\Lambda)f)$.

11.2 Proof

Step 1 — Define $U_1(\Lambda)$ on $\mathscr{H}_1$. For Λ in the emergent Lorentz group (acting on M by I2 and on P_{spin} by EL2), define the candidate operator $U_1(\Lambda)$ on $\mathscr{H}_1$ by

$U_1(\Lambda)\, |f\rangle = |\Lambda f\rangle$, where $(\Lambda f)(\mathcal{C}) := S(\Lambda) \cdot f(\Lambda^{-1}\mathcal{C})$,

for test functions $f \in \Phi$. The $S(\Lambda)$ action is the standard Dirac spinor representation (EL1).

Step 2 — $U_1(\Lambda)$ is unitary on $\mathscr{H}_1$. Compute the inner product:

$\langle \Lambda f \mid \Lambda g\rangle_{\mathscr{H}_1} = \iint K(\mathcal{C}, \mathcal{C}') \cdot [S(\Lambda)\, f(\Lambda^{-1}\mathcal{C})]^\dagger\, [S(\Lambda)\, g(\Lambda^{-1}\mathcal{C}')]\, d\mu_{\text{spin}}(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C}')$.

Substitute $\mathcal{C}' = \Lambda\mathcal{C}''$, $\mathcal{C} = \Lambda\mathcal{C}'''$ (change of integration variables):

$= \iint K(\Lambda\mathcal{C}''', \Lambda\mathcal{C}'') \cdot [S(\Lambda)\, f(\mathcal{C}''')]^\dagger\, [S(\Lambda)\, g(\mathcal{C}'')]\, d\mu_{\text{spin}}(\Lambda\mathcal{C}''')\, d\mu_{\text{spin}}(\Lambda\mathcal{C}'')$.

By **EL3a** (measure-pullback invariance), $d\mu_{\text{spin}}(\Lambda\mathcal{C}) = d\mu_{\text{spin}}(\mathcal{C})$, so the Jacobian is trivial.

By **EL4** (Gram-kernel Lorentz invariance), $K(\Lambda\mathcal{C}''', \Lambda\mathcal{C}'') = K(\mathcal{C}''', \mathcal{C}'')$. Note: this is an *independent* inheritance, not a consequence of EL2 + EL3a; it is named separately for that reason (see §3.5).

After both substitutions:

$\langle \Lambda f \mid \Lambda g\rangle_{\mathscr{H}_1} = \iint K(\mathcal{C}''', \mathcal{C}'') \cdot [S(\Lambda)\, f(\mathcal{C}''')]^\dagger\, [S(\Lambda)\, g(\mathcal{C}'')]\, d\mu_{\text{spin}}(\mathcal{C}''')\, d\mu_{\text{spin}}(\mathcal{C}'')$.

The remaining Clifford-internal pairing is

$[S(\Lambda)\, f(\mathcal{C}''')]^\dagger\, [S(\Lambda)\, g(\mathcal{C}'')] = f(\mathcal{C}''')^\dagger\, S(\Lambda)^\dagger\, S(\Lambda)\, g(\mathcal{C}'')$.

Standard Dirac-spinor algebra gives $S(\Lambda)^\dagger\, \gamma^0\, S(\Lambda) = \gamma^0$ (this is the *defining* property of the Lorentz-covariant Dirac adjoint), equivalently $S(\Lambda)^\dagger = \gamma^0\, S(\Lambda)^{-1}\, \gamma^0$. Hence $S(\Lambda)^\dagger\, S(\Lambda) = \gamma^0\, S(\Lambda)^{-1}\, \gamma^0\, S(\Lambda)$, which is **not** the identity for general Λ — this is the well-known textbook fact that $S(\Lambda)$ is not unitary in the bare $\psi^\dagger\psi$ pairing on $\mathbb{C}^4$.

This is where the bridge either works or quietly fails. The resolution is the content of **EL3b** (substrate Lorentz-normalisation structure). Specifically:

In the textbook free Dirac case, the analogous would-be-failure is resolved because the momentum-space integration uses the Lorentz-invariant mass-shell measure $d^3p \,/\, [(2\pi)^3 \cdot 2E_p]$.

The cancellation does *not* come from the 1/(2E_p) factor in isolation — it comes from the *combined* effect of (i) the 1/(2E_p) energy weighting and (ii) the boost-transformation Jacobian of d^3p (which carries the factor E_p'/E_p relating boosted to unboosted momentum-volume elements). Together, these produce the Lorentz-invariant measure $d^3p / [(2\pi)^3 \cdot 2E_p]$ on the mass shell. The energy-weighting-and-Jacobian combination is what cancels the non-unitarity of S(Λ) in the $\psi^\dagger\psi$ pairing; neither piece alone suffices.

Concretely, the textbook calculation runs:

$$\int \psi^\dagger(p)\,[\gamma^0 S(\Lambda)^{-1} \gamma^0 S(\Lambda)]\,\psi(p) \cdot d^3p/[(2\pi)^3 \cdot 2E_p]$$

reduces — after the Lorentz transformation of momenta (which produces the E_p'/E_p Jacobian) and the 1/(2E_p) weighting (which cancels the new E_p' in the denominator) combine — to the unboosted inner product. The Lorentz-invariance of the full mass-shell measure is the load-bearing fact; the 1/(2E_p) factor is one of two structural components.

The substrate-level statement (EL3b) is the assertion that the substrate-derived measure dμ_spin carries a *functionally equivalent* Lorentz-normalisation structure: that on substrate spinorial modes, dμ_spin has the combined energy-weighting-plus-boost-Jacobian structure that makes it Lorentz-invariant in the coarse-graining limit, in the same way $d^3p / [(2\pi)^3 \cdot 2E_p]$ is Lorentz-invariant on the mass shell. The cancellation works for substrate modes because EL3b is *defined* to make it work — it is the structural requirement on dμ_spin that the present paper imposes on the emergent-Lorentz programme.

It is honest to note that EL3b is a substantive structural commitment. It is not a consequence of EL3a (measure invariance alone), nor of EL4 (Gram-kernel invariance alone). The textbook analogy is the *only* warrant the present paper has that the substrate construction will admit such a measure; the Part IX construction must explicitly verify that the substrate-derived dμ_spin satisfies EL3b, not merely EL3a. In particular, EL3b is *not* the weaker statement that some Lorentz-normalisation factor exists; it is the stronger statement that the substrate measure has the full mass-shell-measure-analogue structure (energy weighting plus boost Jacobian combined into a Lorentz-invariant whole).

Conditional on EL3a, EL3b, and EL4, $\langle \Lambda f \mid \Lambda g\rangle\{\mathscr{H}_1\} = \langle f \mid g\rangle\{\mathscr{H}_1\}$, and $U_1(\Lambda)$ is unitary on $\mathscr{H}_1$.

Step 3 — Lift to U(Λ) on 𝓕_A by second quantisation. The standard antisymmetric Fock-space second quantisation (Bratteli–Robinson §5.2.3) supplies a unique unitary U(Λ) on 𝓕_A from the unitary $U_1(\Lambda)$ on $\mathscr{H}_1$:

$U(\Lambda)\,|0\rangle = |0\rangle$, $U(\Lambda)\,(f_1 \wedge \cdots \wedge f_n) = (U_1(\Lambda)f_1) \wedge \cdots \wedge (U_1(\Lambda)f_n)$, on $\wedge^n \mathscr{H}_1$, $U(\Lambda)\,a^\dagger(f)\,U(\Lambda)^{-1} = a^\dagger(U_1(\Lambda)f)$, $U(\Lambda)\,a(f)\,U(\Lambda)^{-1} = a(U_1(\Lambda)f)$.

The group homomorphism property $U(\Lambda_1\Lambda_2) = U(\Lambda_1)\,U(\Lambda_2)$ follows from the same property on $\mathscr{H}_1$ via the universal property of second quantisation.

Step 4 — Covariant transformation of ψ_α(x). Compute:

$U(\Lambda)\,\psi_\alpha(x)\,U(\Lambda)^{-1} = U(\Lambda)\,[\int U_\alpha(\mathcal{C}; x)\,a(\mathcal{C})\,d\mu_{\text{spin}} + \int V_\alpha(\mathcal{C}; x)\,a^\dagger(\mathcal{C})\,d\mu_{\text{spin}}]\,U(\Lambda)^{-1} = \int U_\alpha(\mathcal{C}; x)\,[U(\Lambda)\,a(\mathcal{C})\,U(\Lambda)^{-1}]\,d\mu_{\text{spin}} + \int V_\alpha(\mathcal{C}; x)\,[U(\Lambda)\,a^\dagger(\mathcal{C})\,U(\Lambda)^{-1}]\,d\mu_{\text{spin}} = \int U_\alpha(\mathcal{C}; x)\,a(\Lambda\mathcal{C})\,d\mu_{\text{spin}} + \int V_\alpha(\mathcal{C}; x)\,a^\dagger(\Lambda\mathcal{C})\,d\mu_{\text{spin}}$ [by Step 3 transformation, applied distributionally] $= \int U_\alpha(\Lambda^{-1}\mathcal{C}'; x)\,a(\mathcal{C}')\,d\mu_{\text{spin}}(\Lambda^{-1}\mathcal{C}') + \int V_\alpha(\Lambda^{-1}\mathcal{C}'; x)\,a^\dagger(\mathcal{C}')\,d\mu_{\text{spin}}(\Lambda^{-1}\mathcal{C}')$ [$\mathcal{C}' = \Lambda\mathcal{C}$] $= \int U_\alpha(\Lambda^{-1}\mathcal{C}'; x)\,a(\mathcal{C}')\,d\mu_{\text{spin}}(\mathcal{C}') + \int V_\alpha(\Lambda^{-1}\mathcal{C}'; x)\,a^\dagger(\mathcal{C}')\,d\mu_{\text{spin}}(\mathcal{C}')$ [by EL3a].

Now invoke MA5 in the appropriate form. From MA5 as stated in §6.2:

$U_\alpha(\Lambda\mathcal{C}; \Lambda x) = S(\Lambda)_\alpha{}^\beta \cdot U_\beta(\mathcal{C}; x)$.

Solve for $U_\beta(\mathcal{C}; x)$ by inverting:

$U_\alpha(\mathcal{C}; x) = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot U_\beta(\Lambda\mathcal{C}; \Lambda x)$,

with α now the contracted spinor index on the LHS (free index β has been renamed and the relation rearranged). Setting $\mathcal{C} \to \Lambda^{-1}\mathcal{C}'$ (so $\Lambda\mathcal{C} = \mathcal{C}'$), this becomes:

$U_\alpha(\Lambda^{-1}\mathcal{C}'; x) = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot U_\beta(\mathcal{C}'; \Lambda x)$,

and similarly $V_\alpha(\Lambda^{-1}\mathcal{C}'; x) = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot V_\beta(\mathcal{C}'; \Lambda x)$. Substituting into the integrals above:

$U(\Lambda)\,\psi_\alpha(x)\,U(\Lambda)^{-1} = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot [\int U_\beta(\mathcal{C}'; \Lambda x)\,a(\mathcal{C}')\,d\mu_{\text{spin}}(\mathcal{C}') + \int V_\beta(\mathcal{C}'; \Lambda x)\,a^\dagger(\mathcal{C}')\,d\mu_{\text{spin}}(\mathcal{C}')] = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot \psi_\beta(\Lambda x)$.

This is the claimed covariant transformation rule. The free index α on the LHS is contracted with the matrix $[S(\Lambda)^{-1}]_\alpha{}^\beta$ on the RHS, which carries the substrate-derived Lorentz transformation to the spinor index β of the emergent field at Λx.

Step 5 — Vacuum invariance. $U(\Lambda)\,|0\rangle = |0\rangle$ by Step 3, completing part (c).

11.3 Structural reading

Theorem 3 supplies the positivity–covariance bridge. To be honest about *what* the theorem establishes and what is assumed:

What MA5 directly assumes (deferred to Part IX). MA5 states that admissible mode functions transform covariantly under combined emergent-Lorentz action on substrate and Dirac-spinor action on Clifford-internal data. This is a structural assumption on the mode functions themselves, and the substrate-level construction of mode functions satisfying MA5 is deferred to Part IX.

What Theorem 3 substantively establishes (the work done by Part VI). Given MA5, Theorem 3 establishes two non-trivial results:

(i) *The positivity-preserving second quantisation works.* The substrate Hilbert positive-definite structure on $\mathscr{F}_A$ (Part V Theorem 1) survives the second-quantisation construction in §10.1

Step 3, producing a unitary $U(\Lambda)$ on $\mathscr{F}_A$ that preserves the substrate Hilbert norm. No indefinite-metric structure is introduced at any stage. This is the substrate-level analogue of the textbook free-Dirac result that Lorentz acts unitarily on Fock space despite $S(\Lambda)$ being non-unitary on $\mathbb{C}^4$ — and it works for the same structural reason: the second quantisation lifts the unitary one-particle action to a unitary Fock action via standard second-quantisation machinery (Bratteli–Robinson §5.2.3).

(ii) *The mode-function covariance of MA5 lifts coherently to field covariance.* The algebraic manipulation of Step 4 shows that mode-function covariance, combined with the substrate-CAR transformation property of Step 3 and the EL3a measure invariance, produces field-level covariance $\psi_\alpha(x) \mapsto S(\Lambda)^{-1}{}_\alpha{}^\beta \cdot \psi_\beta(\Lambda x)$. This is not automatic — it requires the substrate measure to be Lorentz-invariant (EL3a), the substrate CAR transformation to commute with the integration (Step 3), and the mode-function covariance (MA5) to combine in the right way. Step 4 verifies this combination is consistent.

What Theorem 3 does *not* establish. The substrate-level derivation of EL1, EL2, EL3a, EL3b, EL4 themselves; these are inheritances from the emergent-Lorentz programme. The full Poincaré covariance including spacetime translations (which requires the emergent-spacetime translation group, partially in EL2 and partially deferred to Part IX). The spin-statistics theorem in its full axiomatic-QFT form (which requires complete Lorentz covariance plus locality plus positivity of the energy spectrum, the last of which is a separate Part VII issue).

The bridge is a *conditional* construction. It says: *if* EL1, EL2, EL3a, EL3b, EL4 hold and *if* MA5 holds, *then* the positivity–covariance reconciliation works as in the textbook free Dirac case via positivity-preserving second quantisation. The conditional nature is the structurally honest position; the present paper does not claim to derive emergent Lorentz invariance from first principles within Part VI itself (that is the emergent-Lorentz programme). What it does claim is that the second-quantisation step and the algebraic consistency of the bridge are sound — i.e., that *if* the inheritances are supplied, the bridge will work.

The textbook free Dirac theory's positivity–covariance reconciliation is the special case of Theorem 3 in which substrate spinorial modes are identified with momentum-space modes and $d\mu_{spin}$ reduces to the standard Lorentz-invariant mass-shell measure $d^3p / [(2\pi)^3 \cdot 2E_p]$ (the EL3b commitment becomes the standard $1/(2E_p)$ weighting, automatic in the textbook setting).

Bogoliubov-implementability of the substrate Lorentz action. Part V §13.4 flagged the general Bogoliubov-implementability question — when does a one-particle transformation lift to a unitary on Fock space — as a Part VI deliverable, with Lorentz boosts identified as the first downstream context. The present construction resolves this in the negative for the substrate-level emergent-Lorentz action: $U_1(\Lambda)$ is unitary on $\mathscr{H}_1$ (Step 2 of §11.2) and the lift to $\mathscr{F}_A$ proceeds via the standard $\Gamma(U)$ second quantisation (Step 3), which is the Bratteli–Robinson §5.2.3 construction covered by Part V Theorem 6. No Bogoliubov mixing ($\beta \neq 0$) is introduced; the action remains in the $\Gamma(U)$ subclass throughout.

The non-trivial Bogoliubov mixing that Part V §13.4 anticipated arises only under the *antiparticle reinterpretation* of Part VIII, where the Dirac field expansion mixes positive- and negative-frequency components in the spacetime-Fourier sense ($b(p)$, $c^\dagger(p)$ creation/annihilation operators with mass-shell support split by energy sign). Part VI's substrate-level construction is a layer below this: substrate spinorial modes $\mathcal{C} \in P_{spin}$ are not yet split into particle/antiparticle sectors (Part VIII deliverable), and the emergent Lorentz boosts act on the substrate one-particle space without mixing sectors that have not yet been defined. The Bogoliubov question Part V flagged for "Lorentz boosts" is therefore a *downstream-of-Part-VIII* question, accessed only after the antiparticle reinterpretation supplies the b/c operator decomposition. Part VI's substrate-level Lorentz action sidesteps it.

11.4 Conditionality

Theorem 3 inherits:

- **I1–I3** (substrate-to-emergent-spacetime).
- **EL1, EL2, EL3a, EL3b, EL4** (emergent-Lorentz programme inheritances). Load-bearing throughout, with EL3b (substrate analogue of mass-shell $1/(2E_p)$ Lorentz-normalisation) and EL4 (Gram-kernel Lorentz invariance) flagged as the load-bearing structural commitments — see §11.2 Step 2.
- **MA1–MA6** (mode-function admissibility), in particular MA5 for covariance.
- **Theorems 1, 2** of the present paper.
- **Part V Theorem 1** (positive-definite Hilbert structure on $\mathscr{H}_1$).
- **Part V Theorem 5** (vacuum uniqueness and cyclicity).
- **Standard second-quantisation construction** (Bratteli–Robinson §5.2.3).

Theorem 3 is purely topological-core.

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12. Theorem 4 — Emergence of the Free Dirac Equation

12.1 The theorem

Theorem 4 (Emergence of the positive-frequency free Dirac equation as coarse-graining limit). Conditional on inheritances I1–I3, EL1, EL2, EL3a, EL3b, EL4, mode-function admissibility MA1–MA6 (with MA6 — mass-shell admissibility — directly load-bearing for the Dirac equation structure), the additional admissible-coarse-graining inheritances CG1–CG2 (§3.6), and the Part III Theorem 1 first-order spinorial flow structure, the positive-frequency emergent fermion field

$$\psi^{(+)}\alpha(x) := \int\{P_{spin}\}\ U_\alpha(\mathcal{C}; x)\ a(\mathcal{C})\ d\mu_{spin}(\mathcal{C})$$

satisfies the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\ \psi^{(+)}(x) = 0$$

in the coarse-grained regime where emergent spacetime M is the appropriate description. The mass parameter m is the substrate κ-field-derived uniform spinorial-sector mass inherited via Part III Theorem 1 and fixed by CG2.

The full field ψ(x) = ψ^(+)(x) + ψ^(−)(x), where

ψ^(−)*α(x) :=* ∫{P_spin} V_α(𝐶; x) a†(𝐶) dμ_spin(𝐶)

is the negative-frequency / antiparticle-placeholder contribution, satisfies the same Dirac equation *conditional* on the Part VIII charge-conjugation structure giving V_α the standard relation V_α = C Ū_α^T (where C is the charge-conjugation matrix). Within Part VI's free-construction scope, only the positive-frequency form is established directly; the full form awaits Part VIII.

12.2 Argument

The argument proceeds by combining (i) the Part III first-order spinorial flow structure on substrate spinorial loops; (ii) the CG1 coarse-graining inheritance that translates substrate dynamics into emergent-spacetime field equations; and (iii) the mode-function structure of §6.

On the mass scale m. Before proceeding, a note on where m sits in the substrate hierarchy. The κ-field uniqueness programme inherited via Part III Theorem 1 fixes m as the substrate-level uniform spinorial-sector mass *prior to species decomposition*. This is not a Planck-scale mass and not any of the individual physical fermion masses (electron, muon, etc.). The natural reading is that m sits at the electroweak coherence-selection scale identified by the broader VERSF electroweak coherence programme — roughly the geometric-mean scale between the substrate UV and the species-mass spread that Part X will produce. Pre-decomposition, m is a single parameter; Part X will explain how species-specific masses (with their observed Yukawa-hierarchy spread) arise from a single substrate-level m through decomposition mechanics.

Step 1 — Substrate spinorial flow. By Part III Theorem 1, each persistent spinorial loop 𝐶 ∈ P_spin carries a Clifford-internal spinor state ψ_𝐶 ∈ $\mathbb{C}^4$ on which the standard Dirac operator iγ^μ ∂_μ − m acts. This is the substrate-level first-order spinorial flow inheritance.

Step 2 — Mode-function mass-shell constraints (MA6). The admissibility condition MA6 (mass-shell admissibility, §6.2) supplies the standard Dirac mode-function constraints directly:

(iγ^μ ∂_μ^x − m) U_α(𝐶; x) = 0, (iγ^μ ∂_μ^x + m) V_α(𝐶; x) = 0,

where ∂^x denotes differentiation with respect to the spacetime point x. These are the standard textbook plane-wave-spinor constraints transferred to the substrate setting. MA6 is the dedicated admissibility condition for mass-shell support; CG1 provides the broader coarse-graining limit that makes the Dirac operator the natural object on emergent spacetime, with MA6 supplying the mode-function-side input. The combination ensures that admissible mode functions automatically satisfy the Dirac mass-shell constraints in the coarse-graining limit.

Step 3 — Apply $i\gamma^\mu \partial_\mu^x$ to the positive-frequency field. Compute:

$$i\gamma^\mu \partial_\mu^x \psi^{(+)}\alpha(x) = \int_{P_\text{spin}} [i\gamma^\mu \partial_\mu^x U_\alpha(\mathcal{C}; x)]\, a(\mathcal{C})\, d\mu_\text{spin}(\mathcal{C}).$$

By Step 2:

$$i\gamma^\mu \partial_\mu^x U_\alpha(\mathcal{C}; x) = m \cdot U_\alpha(\mathcal{C}; x).$$

Substituting:

$$i\gamma^\mu \partial_\mu^x \psi^{(+)}\alpha(x) = m \cdot \int_{P_\text{spin}} U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_\text{spin}(\mathcal{C}) = m \cdot \psi^{(+)}_\alpha(x).$$

Hence $(i\gamma^\mu \partial_\mu - m)\, \psi^{(+)}(x) = 0$, establishing the positive-frequency Dirac equation as stated in the theorem.

Step 4 — Status of the full field. Apply the same calculation to the full field $\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$:

$$i\gamma^\mu \partial\mu^x \psi_\alpha(x) = m \cdot \int_{P_\text{spin}} U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_\text{spin}(\mathcal{C}) - m \cdot \int_{P_\text{spin}} V_\alpha(\mathcal{C}; x)\, a^\dagger(\mathcal{C})\, d\mu_\text{spin}(\mathcal{C}).$$

The negative-frequency term enters with a relative minus sign (the V_α-constraint of Step 2 has the $+m$ sign, opposite to the U_α-constraint). The full Dirac equation $(i\gamma^\mu \partial_\mu - m)\, \psi = 0$ holds for ψ if and only if the V_α term is consistently interpreted as the charge-conjugate of U_α — specifically, if $V_\alpha = C\, \bar{U}_\alpha^T$ where C is the standard charge-conjugation matrix satisfying $C\, \gamma^\mu C^{-1} = -(\gamma^\mu)^T$. This is the textbook structure of the free Dirac field expansion in standard QFT.

In standard textbook free Dirac quantisation, this consistency is built in by construction: V is defined to be $C\, \bar{U}^T$ precisely so that the full field expansion satisfies the Dirac equation. In the substrate-level setting, the analogous relation must arise as a consequence of the substrate-level charge-conjugation symmetry — which is the Part VIII deliverable. Within Part VI's free-construction scope, V_α is a structural placeholder for what Part VIII will explicitly construct; the full Dirac equation including the V_α term holds conditional on Part VIII supplying the charge-conjugation structure.

The Theorem 4 statement is therefore the positive-frequency form (Step 3) directly; the full form is flagged in the theorem as conditional on Part VIII (Step 4).

12.3 Structural reading

Theorem 4 establishes that the positive-frequency form of the free Dirac equation is not postulated for the emergent fermion field; it emerges as the coarse-grained spacetime expression of:

- Substrate first-order spinorial flow (Part III Theorem 1).

- Admissible coarse-graining of substrate dynamics (CG1).
- Substrate κ-field mass (CG2).
- Mode-function admissibility (MA1–MA6).

The mass m is fixed by substrate physics, not by external calibration. This is the structural origin of the Dirac mass at substrate level. The location of m relative to other substrate scales is sketched in the opening remark of §12.2.

The Part VI free-construction scope establishes the positive-frequency form of the Dirac equation directly; the full Dirac equation including antiparticles is conditional on the Part VIII charge-conjugation structure, which will supply the $V_\alpha = C \bar{U}_\alpha^T$ relation that makes the full Dirac equation $(i\gamma^\mu \partial_\mu - m) \psi = 0$ hold for the combined field $\psi = \psi^{(+)} + \psi^{(-)}$.

12.4 Conditionality

Theorem 4 (positive-frequency form) inherits:

- **I1–I3, EL1, EL2, EL3a, EL3b, EL4, MA1–MA6** (all prior Part VI inheritances).
- **CG1, CG2** (admissible coarse-graining inheritances). Load-bearing for the coarse-graining limit.
- **Part III Theorem 1** (substrate first-order spinorial flow with Dirac structure). Load-bearing for the mass and spinor-flow structure.
- **Substrate κ-field uniqueness programme** (via Part III Theorem 1) for the mass parameter.

Full Dirac equation including antiparticle structure is conditional on Part VIII (charge conjugation supplying $V_\alpha = C \bar{U}_\alpha^T$).

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13. Microcausality, Substrate Locality, and What Is Partially Established

The full axiomatic-QFT microcausality requirement is that the field anticommutators

$$\{\psi_\alpha(x), \psi_\beta(y)\}, \{\psi_\alpha(x), \psi_\beta^\dagger(y)\}$$

vanish for space-like separated x, y in emergent spacetime M. This is a load-bearing input to the full axiomatic-QFT spin-statistics theorem and to the unitarity of the emergent S-matrix.

The Part VI construction establishes considerably less than this. It is important to be honest about what is and is not established, and in particular about the *direction* of the suppression.

13.1 What Theorem 2 establishes directly

By Theorem 2, the field-level anticommutators satisfy:

$\{\psi_\alpha(x), \psi_\beta(y)\} = 0$ (exact, all x, y), $\{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)\} = 0$ (exact, all x, y), $\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = K_{\alpha\beta}(x, y) \cdot \mathbb{1}_{\mathscr{F}_A}$,

with the spacetime kernel

$K_{\alpha\beta}(x, y) = \iint [U_\alpha(\mathcal{C}; x)\, \bar{U}_\beta(\mathcal{C}'; y) + \bar{V}_\beta(\mathcal{C}; y)\, V_\alpha(\mathcal{C}'; x)]\, K(\mathcal{C}, \mathcal{C}')\, d\mu_{\mathrm{spin}}\, d\mu_{\mathrm{spin}}$.

The $\{\psi, \psi\}$ and $\{\psi^\dagger, \psi^\dagger\}$ vanishings are exact and hold for all x, y; they do not require any locality argument. The substantive question is the structure of the cross-anticommutator kernel $K_{\alpha\beta}(x, y)$.

13.2 What MA3 gives, and what it does not

MA3 (substrate locality of mode functions) ensures that mode-function support is concentrated within coherence distance ξ in the emergent-spacetime sense. This produces exponential suppression of $K_{\alpha\beta}(x, y)$ when $|x - y| \gg \xi$ — but **the suppression is direction-independent**: it applies equally to time-like, space-like, and light-like separations.

This is *not* microcausality. Microcausality, even in its weakest form, requires the suppression to *respect the light cone*: the field anticommutator must vanish (or near-vanish) at space-like separation specifically, while being free to be non-trivial at time-like separation. The textbook free Dirac microcausality result, $\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = 0$ at space-like separation, follows from the Lorentz-invariant mass-shell measure and contour-deformation arguments on the integrand; it gives *exact* vanishing at space-like separation, regardless of the magnitude of the separation, while permitting non-trivial structure at time-like separation.

The substrate-coherence-scale story does not capture even the qualitative direction-dependence of microcausality. It tells us that $K_{\alpha\beta}(x, y)$ is small whenever x and y are far apart in the substrate-coherence sense; it does not distinguish space-like from time-like far-apart.

13.3 Honest statement of what is established

Proposition (Substrate-coherence-scale suppression). Under the conditions of Theorem 2 and MA3:

$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)\} = 0$ (exact, all x, y), $\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = K_{\alpha\beta}(x, y) \cdot \mathbb{1}_{\mathscr{F}_A}$,

with $K_{\alpha\beta}(x, y) \to 0$ exponentially as $|x - y| \gg \xi$, where $|x - y|$ denotes any reasonable distance measure on M (substrate-coherence-scale Euclidean, geodesic, or analogous) and *the suppression holds for all relative orientations of x − y — time-like, space-like, and light-like alike*. The suppression is direction-independent and *does not respect the emergent light cone*.

This is *not* microcausality. It is a substrate-coherence-scale suppression result that is independent of light-cone structure.

13.4 The actual microcausality theorem as an open problem

Light-cone-respecting microcausality — vanishing of {ψ_α(x), ψ_β†(y)} at space-like separation, possibly with non-trivial behaviour at time-like separation — is *not* established by the present paper. It requires structural input that goes beyond MA3:

- Either a sharpening of MA3 that produces directionally-asymmetric mode-function support (suppression at space-like substrate-separations specifically), or
- The substrate-derived measure dμ_spin reducing to the exact Lorentz-invariant mass-shell measure in the coarse-graining limit, with the substrate mode functions reducing to textbook plane-wave spinor modes (whose contour-deformation properties produce exact microcausality).

Why this is structurally hard. The substrate-coherence-scale suppression of §13.3 is direction-independent because it inherits from MA3 (substrate-locality of mode functions) which has no built-in light-cone structure — substrate locality is a property of the mode-function support at substrate level, prior to and independent of the emergent-Lorentz unification of time and space. The light cone, by contrast, is the structural feature *that distinguishes time-like from space-like* in emergent spacetime — it is precisely what the emergent-Lorentz inheritance (I2 + EL1–EL4) supplies on top of separately-emergent time and space. For mode-function support to respect the light cone, the substrate-level mode-function construction must inherit from emergent-Lorentz structure (sharpening of MA3 in a Lorentz-aware way) or from the mass-shell measure (which is Lorentz-invariant by construction). MA3 alone, as a substrate-locality condition prior to the emergent-Lorentz inheritance, cannot produce light-cone-respecting suppression.

This is the structural reason light-cone-respecting microcausality is harder than substrate-coherence-scale suppression: it requires the substrate construction to feel the emergent-Lorentz unification at the mode-function-support level, not merely at the field-transformation level (which MA5 supplies). Part IX must address how mode-function support inherits the emergent-Lorentz structure beyond just MA5 covariance.

Both routes are deferred to Part IX. The present paper does not claim a partial-microcausality result. What it claims is a direction-independent coherence-scale suppression (§13.3), which is a structurally different statement.

This is identified as §18 item 6 (open) and is recognised as one of the load-bearing remaining deliverables before the full axiomatic-QFT spin-statistics theorem becomes accessible.

Note on Part V §17.2 framing. Part V §17.2 listed microcausality as "downstream of Part VI smearing," which a reader might naturally take to mean Part VI's smearing construction itself delivers some form of microcausality result. The present paper *honours that expectation in the negative*: the smearing construction supplies direction-independent substrate-coherence-scale suppression (§13.3), not light-cone-respecting microcausality. The latter is downstream of Part

VI smearing in the sense that *it cannot be addressed before the smearing exists*, but it requires further structural input (sharpening of MA3 or substrate-derived measure reduction to mass-shell) that the smearing construction itself does not supply. The conditional disposition is honest: Part V's wording is "downstream of Part VI smearing" rather than "established by Part VI smearing," which is consistent with the present paper's resolution, but the natural expectation a reader might form is honestly disappointed and the explanation is deferred to Part IX.

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14. Corollary 4' — Physical Realisation on the Coherent Substrate

14.1 The corollary

Corollary 4' (Physical realisation of emergent fermion fields). Under the coherence condition of Part IV §4A.3 (spatial extent $\lesssim \xi$, exchange/observation timescale $\lesssim \tau_s$, inherited from the entanglement-lattice paper via Part IV §3.9, under current scale-estimate calibrations), and given the conditions of Theorems 1–4 of the present paper, the emergent fermion field operator $\psi(x)$ is physically manifest as the observable relativistic fermion field on emergent spacetime M, in the regime of standard fermionic-matter phenomenology.

Specifically:

- The smeared field operator $\psi(h)$ for test functions h supported on emergent-spacetime regions corresponds to physically observable fermion-field excitations on the coherent substrate.
- The field-level CAR algebra (Theorem 2) is physically observable as the standard textbook free-Dirac field anticommutation structure in regimes where coherence is maintained.
- The Lorentz-covariant transformation (Theorem 3) is physically observable as the standard relativistic covariance of fermionic-matter observables.
- The free Dirac equation (Theorem 4) governs the dynamics of physical fermion-field excitations in the regime of standard fermionic-matter phenomenology, prior to coupling to electroweak gauge fields (Part XI) and species decomposition into specific fermion species (Part X).

14.2 Argument

By Theorems 1–4, the emergent fermion field $\psi(x)$ is constructed at the topological level on substrate Fock space $\mathscr{F}$_A with the standard textbook free-Dirac structure. By Part V Corollary 5' (physical realisation of substrate-level CAR observables under coherence condition), the substrate-level CAR operators $a(\mathcal{C})$, $a^\dagger(\mathcal{C})$ are physically manifest under the coherence condition.

The emergent field operator $\psi(x)$ is constructed from substrate CAR operators via the mode-function smearing (§7). The same coherence-condition argument that promotes Part V Corollary

5' (substrate CAR observability) promotes the emergent field operator to physical observability: within the coherence regime, the substrate supports the coherent multi-mode transport needed for the mode-function integrals to be physically meaningful, and the emergent field operator is observable.

For physical fermion-field systems at observable laboratory scales — electrons in atomic physics, condensed-matter physics, accelerator-energy phenomenology — the coherence condition is comfortably satisfied (Part IV §4A.3 footnote 1 inheritance to entanglement-lattice scale estimates, under current calibrations). The emergent fermion field of Theorems 1–4 is therefore physically manifest in all standard fermionic-matter regimes.

14.3 Structural reading

Corollary 4' is the physical-realisation companion to Theorems 1–4, paralleling Part V Corollary 5' as the physical-realisation companion to Part V Theorem 3. The emergent fermion field is a topological-core construction (it exists on $\mathscr{F}$_A independently of any substrate-physics observability requirements); its physical observability requires the coherence condition.

In the regime where the coherence condition holds, the emergent field ψ(x) is *the* substrate-level analogue of the textbook free Dirac field — the object that physicists working in standard fermionic-matter contexts implicitly compute with.

14.4 Conditionality

Corollary 4' inherits:

- **Theorems 1–4** (the topological-core results of the present paper).
- **Part V Corollary 5'** (physical-realisation pattern from substrate CAR).
- **Part IV §3.9** (entanglement-lattice coherence scales ξ, τ_s).
- **Part IV §3.10–§3.11** (substrate stability and effective-medium framing).
- *Specifically conditional on:* the entanglement-lattice scale-estimate calibrations as currently established. Scale-estimate revisions affect Corollary 4'-level observability without affecting the topological-core results of Theorems 1–4.

Corollary 4' is the only physical-realisation result of the present paper. All numbered theorems (Theorems 1, 2, 3, 4) are topological-core.

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15. Structural Consequences and the Strengthened Hierarchy Chain

15.1 Catalogue of structural consequences

15.1.1 Emergent fermion field operators exist as operator-valued distributions on $\mathscr{F}_A$. *Derived (Theorem 1).* Conditional on substrate-to-emergent-spacetime inheritances I1–I3 and mode-function admissibility MA1–MA6.

15.1.2 Field CAR algebra is induced from substrate CAR algebra. *Derived (Theorem 2).* The textbook free-Dirac field CAR is recovered in the local continuum limit under MA4 completeness.

15.1.3 Positivity–covariance bridge works. *Derived (Theorem 3).* Conditional on emergent-Lorentz inheritances EL1–EL4, the positive-definite substrate Hilbert structure supports a unitary representation of the emergent Lorentz group, with the field transforming covariantly.

15.1.4 Free Dirac equation emerges as coarse-graining limit. *Derived (Theorem 4).* Conditional on coarse-graining inheritances CG1–CG2 and Part III Theorem 1, the substrate-level spinorial flow becomes the standard textbook free Dirac equation in the coarse-grained spacetime description.

15.1.5 Substrate-coherence-scale suppression of $K_{\alpha\beta}(x, y)$ holds; light-cone-respecting microcausality does not. *Derived (§13.3) and explicitly limited (§13.4).* The cross-anticommutator is exponentially suppressed at substrate-separations $\gg \xi$ in *all* spacetime directions; this is not microcausality, which requires direction-dependent (light-cone-respecting) suppression. Light-cone-respecting microcausality is identified as an open problem (§18 item 6), with no partial result claimed.

15.1.6 Emergent fermion field physically observable under coherence condition. *Derived (Corollary 4').* The full topological-core construction is physically manifest in the regime of standard fermionic-matter phenomenology.

15.1.7 Substrate-level free Dirac field theory is established at the algebraic-and-distributional level. *Synthetic.* Theorems 1–4 collectively constitute the substrate-level analogue of textbook free Dirac field theory, with all the standard structures (field CAR, Lorentz covariance, Dirac equation) derived rather than postulated.

15.2 The strengthened hierarchy chain

With Theorems 1–4 supplying the substrate-to-spacetime-Dirac-field bridge, the matter-sector chain from void substrate through to relativistic fermion-field theory is now complete at the free-construction level. The chain is displayed below in indented schematic form.

Logical chain (topological-core throughout):

```
K=7 architecture (d ≥ 3)
  + κ-field uniqueness
  + Schrödinger→Dirac
        ↓
Clifford structure on spinorial loops
  (Part III Theorem 1)
```

```
        ↓
Closure-frame bundle SU(2) lift
  (Part III SI, conditional on C1–C3)
        ↓
Spinorial sector U(2π) = −𝟙
  (Part III Theorem 4)
        ↓
F–R exchange holonomy at topological level
  (Part IV Theorem 1)
        ↓
Antisymmetric multi-loop sector
  (Part IV Theorem 2)
        ↓
Spin-statistics correspondence
  (Part IV Theorem 3)
        ↓
Anticommuting half of CAR
  (Part IV Proposition 4
   + Part V Theorem 2 for density)
        ↓
Admissibility of positive-definite ℋ₁
  (Part V Theorem 1)
        ↓
Antisymmetric Fock space ℱ_A
  with vacuum-iterated density
  (Part V Theorem 2)
        ↓
Full substrate CAR algebra
  (Part V Theorem 3)
        ↓
Rigged-Hilbert-space apparatus
  + mode-function admissibility
  (this paper §5, §6)
        ↓
Emergent fermion field operators
  (this paper Theorem 1)
        ↓
Field-level CAR algebra
  (this paper Theorem 2)
        ↓                       ↓
Positivity-covariance       Emergent free Dirac equation
bridge (Theorem 3)          (Theorem 4)
        ↓                       ↓
        ↓ ← combined ← ←  ←  ↓
        ↓
Substrate-coherence-scale suppression of K_{αβ}(x, y)
  (this paper §13; not light-cone-respecting microcausality —
   that remains open, §18 item 6)
```

Inheritance chain (external dependencies for Part VI):

```
Substrate-to-emergent-spacetime correspondence
  (Part IX deliverable; inherited as I1–I3)
        ↓
Emergent-Lorentz group action
  (Emergent-Lorentz programme; inherited as EL1–EL4)
```

```
        ↓
Admissible coarse-graining
  (Coarse-graining programme; inherited as CG1–CG2)
        ↓
Part VI construction
  (this paper)
```

Physical-grounding chain (requires substrate-physics inheritances):

```
Void
  ↓
Entanglement lattice
  characteristic scales ξ, τₛ, mₛ
  (Part IV §3.9)
  ↓
Coherent persistent transport
  (Part IV §3.10, Proposition 5)
  ↓
Coherence condition
  (Part IV §4A.3)
  ↓
Substrate CAR observables physically manifest
  (Part V Corollary 5')
  ↓
Emergent fermion field physically manifest
  (this paper Corollary 4')
```

The logical chain is now closed at the free-fermionic-quantum-field-theory level, conditional on the external inheritances I1–I3, EL1–EL4, CG1–CG2 from the broader VERSF programme. The matter-sector programme has, at the free-construction level, derived everything from K=7 architecture through to relativistic fermion fields on emergent spacetime.

15.3 What remains for the matter-sector programme

With Theorems 1–4 in place, what remains is:

- **Interactions and renormalisation** (Part VII). Substrate-level fermionic interactions, gauge couplings, renormalisation, anomaly structure.
- **Antiparticle structure and charge conjugation** (Part VIII). Substrate-level C operator, particle/antiparticle Fock-space doubling, completion of the V_α antiparticle structure currently held as a placeholder in §6.1.
- **Substrate-to-emergent-spacetime correspondence** (Part IX). The detailed substrate-level construction supplying I1–I3, EL1–EL4, CG1–CG2 as derived results rather than as inheritances.
- **Species decomposition** (Part X). Decomposition of the uniform substrate spinorial sector into Standard Model fermion species (electron, muon, tau, quarks, neutrinos).
- **Substrate-to-electroweak bridge** (Part XI). Coupling to the electroweak gauge sector, completing the substrate-level fermionic Standard Model.

Each of these is a concrete next-paper target with explicit conditions identified.

16. Falsifiability Channels

16.1 Structural channels

Channel A: Mode-function admissibility fails (MA1–MA6). Tests the foundational mode-function admissibility. Would occur if no mode functions U_α, V_α satisfying MA1–MA6 can be constructed from substrate data — i.e., if the substrate-to-emergent-spacetime track (Part IX) fails to supply admissible mode functions in any concrete realisation. Observational signature: structural impossibility identified in the Part IX construction.

Channel B: Field CAR cross-relation deviates from textbook δ^3 form at observable scales. Tests Theorem 2 under MA4. Would occur if the substrate-coherence scale ξ is larger than current calibrations suggest, allowing the spacetime kernel $K_{\alpha\beta}(x, y)$ to deviate detectably from $\delta_{\alpha\beta} \cdot \delta^3(\mathbf{x} - \mathbf{y})$ in observable regimes. Observational signature: substrate-coherence-scale corrections to the textbook free-Dirac anticommutator structure in high-precision experiments.

Channel C: Positivity–covariance bridge fails (EL1–EL4 incompatible with substrate Hilbert positivity). Tests Theorem 3. Would occur if the emergent-Lorentz inheritances EL1–EL4 are inconsistent with the substrate $\psi^\dagger\psi$ Hilbert positivity in some structural way not captured by Part V Theorem 1. Observational signature: structural failure of the second-quantisation step in §10.1 Step 3 with concrete EL1–EL4 candidates from the emergent-Lorentz programme.

Channel D: Emergent Dirac equation deviates from the textbook form (CG1–CG2 inheritances fail). Tests Theorem 4. Would occur if the coarse-graining limit does not produce the textbook Dirac form. Observational signature: structural failure in the admissible coarse-graining programme to produce $i\gamma^\mu \partial_\mu - m$ as the coarse-grained spinorial flow, or production of a modified Dirac equation with terms not present in the textbook form.

Channel E: Substrate-coherence-scale suppression fails. Tests §13.3. Would occur if the spacetime kernel $K_{\alpha\beta}(x, y)$ does not decay exponentially at substrate-separations $\gg \xi$. Observational signature: structural failure of MA3 (mode-function substrate locality). Note: the *light-cone-respecting* microcausality result is *not* claimed by the present paper (see §13.4); its falsification is therefore not a falsification of Part VI, since Part VI does not claim it.

Channel F: Inheritance from Part V fails. Tests the substrate-CAR inheritance. If any of Part V Theorems 1, 2, 3 fail, the substrate input to Part VI fails and the entire construction loses its starting point.

Channel G: External inheritance fails (I1–I3, EL1–EL4, CG1–CG2 not derivable in their respective programmes). Tests the broader-programme inheritances. If the substrate-to-emergent-spacetime track, emergent-Lorentz programme, or admissible-coarse-graining programme fails to deliver their stated inheritances, the conditional Part VI results lose their conditions.

16.2 Empirical channels

Channel H: Coherence condition violated at scales where the emergent fermion field is empirically tested. Tests Corollary 4'. If the coherence condition fails at observable laboratory scales, the emergent fermion field of Part VI is not physically manifest in the regime of standard fermionic-matter phenomenology. The topological-core results (Theorems 1–4) hold as algebraic identities regardless.

Channel I: Substrate-physics fingerprints in high-energy / high-precision fermionic-matter experiments. Tests Channel B in a more specific empirical form. The spacetime kernel K_{αβ}(x, y) carries substrate-physics fingerprints (from the K(𝒞, 𝒞') Gram kernel and the mode-function support structure) at scales ≲ ξ. High-energy collider experiments at scales approaching the substrate coherence scale could in principle detect these fingerprints. Observational signature: anomalies in high-precision fermionic-matter scattering that fit the substrate-coherence-scale prediction.

16.3 General empirical posture

The structural channels A–G test load-bearing claims of the present paper and its inheritances. The empirical channels H–I test the substrate-physics inheritances and the regime of validity of the textbook reduction.

The present paper does not establish new directly-empirical predictions beyond:

- The substrate-coherence-scale corrections to the textbook free-Dirac CAR algebra (Channels B, I).
- The substrate-coherence-scale suppression structure of K_{αβ}(x, y) (Channel E). Note: this is a substrate-coherence-scale phenomenon, not a microcausality phenomenon — the suppression is direction-independent and does not respect the emergent light cone (§13.2, §13.4).

These are not testable in current fermionic-matter experiments (the coherence scale ξ is far below any direct experimental reach), but they identify the channels through which Part VI's substrate-physics dependence would become empirically meaningful at sufficiently high resolution.

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17. What This Paper Achieves, and What It Does Not

17.1 What is achieved

Topological-core results:

1. **Existence of smeared emergent fermion field operators on $\mathscr{F}_A$** (Theorem 1). Conditional on substrate-to-emergent-spacetime inheritances I1–I3 and mode-function admissibility MA1–MA6, the candidate field operator $\psi_\alpha(x)$ is a well-defined operator-valued tempered distribution on $\mathscr{F}_A$ with the standard test-function smearing properties.
2. **Field-level CAR algebra induced from substrate CAR** (Theorem 2). The field CAR is not postulated; it emerges as the mode-function transform of the substrate CAR. The textbook δ^3 form is recovered in the local continuum limit under MA4.
3. **Positivity–covariance bridge** (Theorem 3). Conditional on emergent-Lorentz inheritances EL1, EL2, EL3a, EL3b, EL4 (with EL3b and EL4 identified as the load-bearing structural commitments), the positive-definite substrate Hilbert structure on $\mathscr{F}_A$ supports a unitary representation $U(\Lambda)$ of the emergent Lorentz group, with the field operator transforming covariantly. The positivity of the Hilbert norm is preserved throughout; no indefinite-metric structure is introduced. The substantive content of the theorem is the positivity-preserving second quantisation and the algebraic consistency of the bridge; the mode-function covariance is supplied by MA5 as an inheritance (§11.3).
4. **Emergent positive-frequency free Dirac equation as coarse-graining limit** (Theorem 4). Conditional on coarse-graining inheritances CG1–CG2 and the Part III Theorem 1 first-order spinorial flow structure, the *positive-frequency* part $\psi^{(+)}$ of the emergent field satisfies the standard textbook free Dirac equation with mass m fixed by substrate κ-field structure. The full Dirac equation including the V_α antiparticle term is conditional on the Part VIII charge-conjugation structure supplying $V_\alpha = C\,\bar{U}_\alpha^T$.
5. **Substrate-coherence-scale suppression of $K_{\alpha\beta}(x, y)$** (§13.3). The field cross-anticommutator is exponentially suppressed at substrate-separations $\gg \xi$ in all directions. This is *not* light-cone-respecting microcausality, which requires directional (space-like vs time-like) asymmetry; that result is *not* claimed by the present paper and is identified as §18 item 6 (open).

Physical-realisation result:

6. **Emergent fermion field physically observable under coherence condition** (Corollary 4'). The emergent fermion field of Theorems 1–4 is physically manifest as the observable relativistic fermion field on emergent spacetime in the regime of standard fermionic-matter phenomenology.

17.2 What is not achieved

The present paper supplies the substrate-to-emergent-Dirac-field bridge at the free-construction level: rigged-Hilbert-space smearing apparatus, substrate-derived mode-function admissibility conditions, the existence of smeared field operators (Theorem 1), the induced field CAR algebra (Theorem 2), the positivity–covariance bridge (Theorem 3), the emergent positive-frequency free Dirac equation (Theorem 4), substrate-coherence-scale suppression of the field cross-anticommutator (§13.3, *not* microcausality), and the physical-realisation reading (Corollary 4'). It does not derive:

- **Interacting fermionic QFT** (§18 item 1, Part VII deliverable). Substrate-level fermionic interactions, perturbative expansion, S-matrix construction, renormalisation theory.

- **Antiparticle structure and charge conjugation** (§18 item 2, Part VIII deliverable). The full charge-conjugation operator C, particle/antiparticle Fock-space doubling, antiparticle vacuum structure. The V_α(𝐶; x) mode functions of §6.1 are structurally consistent with the future Part VIII construction but do not yet supply it. Theorem 4's restriction to the positive-frequency form is the immediate consequence.
- **Substrate-level derivation of I1–I3, EL1, EL2, EL3a, EL3b, EL4, CG1, CG2** (§18 item 3, Part IX deliverable and emergent-Lorentz/coarse-graining programmes). The substrate-to-emergent-spacetime correspondence, the emergent Lorentz group action (with all five EL sub-components), and the admissible coarse-graining are inherited by the present paper but not derived within it. EL3b in particular (substrate analogue of mass-shell 1/(2E_p) Lorentz-normalisation) is the dominant structural bet on Part IX.
- **Substrate-derived construction of mode functions satisfying MA1–MA6** (§18 item 4, Part IX deliverable). The present paper supplies the admissibility conditions; the explicit construction is the Part IX deliverable.
- **Species decomposition** (§18 item 5, Part X deliverable). The decomposition of the uniform spinorial sector into Standard Model fermion species.
- **Light-cone-respecting microcausality at any scale** (§18 item 6). Vanishing (exact or otherwise) of K_{αβ}(x, y) at space-like separation with non-trivial behaviour permitted at time-like separation. The present paper does *not* establish even a partial light-cone-respecting result; what it establishes is direction-independent substrate-coherence-scale suppression (§13.3), which is structurally different. Microcausality in any form requires further structural input (sharpening of MA3 or substrate-derived measure reduction to mass-shell), all deferred to Part IX.
- **Substrate-to-electroweak bridge** (§18 item 7, Part XI deliverable). Coupling to the electroweak gauge sector.
- **Full axiomatic-QFT spin-statistics theorem.** Requires full Lorentz covariance (item 6), interactions (item 1), positive-energy spectrum (Part VII), and locality at all scales.

Deliverable table:

Deliverable	Status
Rigged Hilbert space on substrate spinorial sector	**§5**
Mode-function admissibility conditions MA1–MA6	**§6**
Existence of smeared emergent field operators	**Theorem 1**
Field-level CAR algebra induced from substrate CAR	**Theorem 2**
Positivity–covariance bridge (positivity-preserving second quantisation + algebraic consistency)	**Theorem 3**
Emergent positive-frequency free Dirac equation	**Theorem 4**
Emergent full Dirac equation (including V_α term)	**Conditional on Part VIII**

Deliverable	Status
Substrate-coherence-scale suppression of K_{αβ}(x, y) (direction-independent)	**§13.3**
Light-cone-respecting microcausality (any scale)	**Open** (Part IX; not claimed by Part VI)
Physical realisation of emergent fermion field	**Corollary 4'** (under coherence condition + current scale calibrations)
Part V §17 items 1 ↔ 10 (linked deliverable)	**Closed (conditional on I1–I3, EL1, EL2, EL3a, EL3b, EL4, CG1, CG2)**
Substrate-derived mode functions satisfying MA1–MA6	**Open** (Part IX)
Substrate-derivation of I1–I3	**Open** (Part IX)
Substrate-derivation of EL1, EL2	**Open** (Emergent-Lorentz programme)
Substrate-derivation of EL3a (measure invariance)	**Open** (Emergent-Lorentz programme)
Substrate-derivation of EL3b (mass-shell-1/(2E_p)-analogue weighting)	**Open** (Emergent-Lorentz programme; load-bearing)
Substrate-derivation of EL4 (Gram-kernel Lorentz invariance)	**Open** (Emergent-Lorentz programme)
Substrate-derivation of CG1–CG2	**Open** (Coarse-graining programme)
Interacting fermionic QFT	**Open** (Part VII)
Antiparticle structure and charge conjugation	**Open** (Part VIII)
Species decomposition	**Open** (Part X)
Substrate-to-electroweak bridge	**Open** (Part XI)
Full axiomatic-QFT spin-statistics theorem	**Open** (requires Parts VII, VIII, IX)
Joint consistency of inheritance families (I1–I3 + EL1–EL4 + CG1–CG2)	**Open** (constraint across Part IX and emergent-Lorentz / coarse-graining programmes)

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18. Open Problems

1. Interacting fermionic QFT (Part VII deliverable). The dominant matter-sector deliverable post-Part-VI. Substrate-level fermionic interactions, gauge couplings, perturbative expansion, renormalisation, anomaly structure, S-matrix construction.

2. Antiparticle structure and charge conjugation (Part VIII deliverable). Substrate-level charge-conjugation operator C, particle/antiparticle Fock-space doubling, antiparticle vacuum structure. Completes the V_α(𝒞; x) antiparticle structure currently held as a placeholder in §6.1. Required for the full Dirac equation (Theorem 4 currently restricted to positive-frequency form) and for the full crossing symmetry in S-matrix elements.

3. Substrate-to-emergent-spacetime correspondence (Part IX deliverable). The detailed substrate-level construction supplying I1–I3 and the substrate-derived construction of mode functions satisfying MA1–MA6. The largest external dependency of the present paper.

4. Substrate-derived construction of mode functions (Part IX sub-deliverable). Concrete construction of U_α(𝒞; x), V_α(𝒞; x) satisfying MA1–MA6, derived from substrate dynamics. Sequential interface transport, σ-duality on K=7 wheels, and the substrate κ-field structure are the relevant substrate-physics inputs.

5. Species decomposition into Standard Model fermions (Part X deliverable). Decomposition of the uniform spinorial sector P_spin into substrate-level sub-sectors corresponding to electron, muon, tau, up-quark, down-quark, charm-quark, strange-quark, top-quark, bottom-quark, and the three neutrino species. The decomposition mechanism likely involves the K=7 architecture's internal symmetry generators.

6. Light-cone-respecting microcausality (Part IX deliverable). Vanishing of K_{αβ}(x, y) at space-like separation, with non-trivial behaviour permitted at time-like separation — the textbook microcausality condition required for the axiomatic-QFT spin-statistics theorem. The present paper establishes substrate-coherence-scale suppression in *all* spacetime directions (§13.3), which is structurally different and does *not* constitute even a weak microcausality result. Light-cone-respecting microcausality requires either (i) a sharpening of MA3 to produce directionally-asymmetric mode-function support, or (ii) substrate-derived measure reduction to the Lorentz-invariant mass-shell measure with corresponding mode-function reduction to textbook plane-wave spinor modes. Both routes are deferred to Part IX.

7. Substrate-to-electroweak bridge (Part XI deliverable). Coupling between the emergent fermion field of the present paper and the electroweak gauge sector. Requires the substrate-level electroweak structure from the broader VERSF programme.

8. Substrate-level energy positivity. The emergent fermion field of Theorem 4 inherits the textbook Dirac mass $m > 0$. Substrate-level energy positivity (the spectral condition of the unitary Lorentz representation) is a Part VII deliverable; the present paper does not address it.

9. Full axiomatic-QFT spin-statistics theorem. Requires items 1, 2, 3, 6, plus positive-energy spectrum (item 8). Becomes accessible once Part VII supplies interactions and Part IX supplies light-cone-respecting microcausality.

10. Confinement-fermion bridge. Inherited from Part V §17 item 8. Whether stable fermionic matter admits confinement-stabilised topological-transport reformulation, in the context of QCD substrate structure (post-Part X).

11. Joint consistency of inheritance families (Part IX + emergent-Lorentz / coarse-graining programmes). I1–I3, EL1–EL4, CG1–CG2 are catalogued as separable but all flow from the same substrate physics (§4.5). The joint consistency of the substrate-derived emergent spacetime, Lorentz structure, measure structure, Gram-kernel structure, and coarse-graining limit is itself a non-trivial structural constraint. Verification that all inheritance families can be simultaneously realised in a single substrate-physics construction is the strongest open structural challenge for the broader VERSF programme at the level of the matter-sector strand.

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19. Relation to Earlier VERSF Papers, and the Dependency Graph

Direct Part V inheritances (load-bearing throughout):

- Part V Theorem 1 (positive-definite Hilbert completion). Inherited by Theorem 3 of the present paper (positivity–covariance bridge): the positive-definite substrate Hilbert structure is what allows $U(\Lambda)$ to be unitary in the standard sense.
- Part V Theorem 2 (antisymmetric Fock-space construction with density). Inherited throughout.
- Part V Theorem 3 (full substrate CAR algebra). Load-bearing for Theorems 1, 2 of the present paper.
- Part V Theorem 5 (vacuum uniqueness and cyclicity). Inherited by Theorem 3 of the present paper (vacuum-invariance condition $U(\Lambda)|0\rangle = |0\rangle$).
- Part V Corollary 5' (physical realisation under coherence condition). Pattern inherited in Corollary 4' of the present paper.
- Part V's two-layer architecture (§1.1). Preserved throughout the present paper.
- Part V §5.2 (Gram-kernel / discrete-vs-distributional formulation). Used throughout; the present paper works in the distributional reading.

Part IV inheritances:

- Part IV §3.9 (entanglement-lattice coherence scales ξ, τ_s, m_s). Used in Corollary 4' and Part IX-deliverable mode-function-locality conditions.
- Part IV §3.10, §3.11 (substrate stability, effective-medium framing). Used in physical-realisation layer.

Part III inheritances:

- Part III Theorem 1 (Clifford-compatible spinorial structure). Load-bearing for Theorem 4 of the present paper (Dirac operator structure on substrate spinorial loops).
- Part III Theorem 4 (spinorial-sector decomposition). Inherited via Part V.

Other VERSF inheritances:

- κ-field uniqueness programme. Load-bearing for Theorem 4 mass structure (via CG2).
- Schrödinger→Dirac paper. Load-bearing for Theorem 4 (via Part III Theorem 1 + CG1).
- K=7 minimal fact architecture. Load-bearing throughout via Parts III, V inheritances.
- Microscopic Origin paper, especially Lemma 1 (source-admissibility SA1–SA5). Load-bearing for the substrate-CAR positivity (Part V Theorem 1).

- **Emergent-Lorentz programme** (broader VERSF papers). *Newly load-bearing in this paper* for EL1, EL2, EL3a, EL3b, EL4 inheritances. Theorem 3 is conditional on this programme's deliverables. EL3b (substrate analogue of mass-shell 1/(2E_p) Lorentz-normalisation) and EL4 (Gram-kernel Lorentz invariance) are the dominant load-bearing structural commitments on this programme.
- **Admissible coarse-graining and continuum emergence programme.** *Newly load-bearing in this paper* for CG1–CG2 inheritances. Theorem 4 is conditional on this programme's deliverables.
- **Substrate-to-emergent-spacetime track / sequential interface transport / σ-duality on K=7 wheels.** *Newly load-bearing in this paper* for I1–I3 inheritances and Part IX mode-function construction.

Substrate-physics inheritances (physical-realisation layer only):

- Entanglement-lattice papers (via Part IV §3.9).
- Superfluid-transport paper (via Part IV §3.10).
- *When Space Itself Has Mass* (via Part IV §3.11).

Standard mathematical background (not VERSF):

- Rigged Hilbert space / Gelfand triple framework (standard distribution theory).
- Operator-valued tempered distributions (Streater–Wightman; Reed–Simon Vol. II).
- Second-quantisation of unitary one-particle representations (Bratteli–Robinson §5.2.3).
- Standard textbook free Dirac field theory (any standard QFT text).

Dependency graph (key results of the present paper):

Result	Depends on (Part VI inheritances)	Depends on (external inheritances)	Layer
Rigged Hilbert space (§5)	—	Distributional reading of Part V §5.2	Topological
Mode-function admissibility MA1–MA6 (§6)	—	Substrate-physics inheritances (Part IX)	Topological
Emergent field operator (Definition 1, §7)	§5, §6	Part V Theorem 3 (substrate CAR)	Topological
Theorem 1 (existence of smeared field operators)	§5, §6, §7	I1–I3, Part V Theorem 3, MA1, MA2	Topological
Theorem 2 (field CAR from substrate CAR)	Theorem 1	Part V Theorem 3, MA4 (for δ^3 limit)	Topological
Positivity–covariance bridge construction (§10)	Theorems 1, 2	EL1, EL2, EL3a, EL3b, EL4	Topological
Theorem 3 (Lorentz-covariant transformation)	§10, Theorems 1, 2	EL1, EL2, EL3a, EL3b, EL4, MA5, Part V Theorems 1, 5	Topological

Result	Depends on (Part VI inheritances)	Depends on (external inheritances)	Layer
Theorem 4 (emergent positive-frequency free Dirac equation)	Theorems 1, 2, 3	CG1–CG2, Part III Theorem 1, MA5	Topological
Theorem 4 extension to full Dirac equation	Theorem 4 + Part VIII (open)	Part VIII charge-conjugation structure	—
Substrate-coherence-scale suppression of K_{αβ}(x, y) (§13.3)	Theorem 2	MA3	Topological
Light-cone-respecting microcausality (open)	Theorem 2	Sharpening of MA3 or substrate-derived measure reduction (Part IX)	—
Corollary 4' (physical realisation)	Theorems 1–4	Part V Corollary 5', Part IV §3.9–§3.11	Physical-realisation

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20. Epistemic-Status Labelling and Status Table

20.1 Epistemic-status labelling

Derived at the topological level (under named inheritance).

- Theorem 1 (under I1–I3, MA1–MA2, Part V Theorem 3).
- Theorem 2 (under Theorem 1, MA4 for δ^3 limit, Part V Theorem 3).
- Theorem 3 (under EL1, EL2, EL3a, EL3b, EL4, MA5, Theorems 1–2, Part V Theorems 1 + 5). Substantive content: positivity-preserving second quantisation + algebraic consistency of bridge; covariance of mode functions (MA5) is inherited.
- Theorem 4 [positive-frequency form] (under CG1–CG2, Part III Theorem 1, MA5, Theorems 1–3). Full form including V_α conditional on Part VIII charge-conjugation structure.
- §13.3 substrate-coherence-scale suppression (under MA3); *not* light-cone-respecting microcausality.

Derived at the physical-realisation level. Corollary 4' (under Theorems 1–4, Part V Corollary 5', and coherence condition with current scale-estimate calibrations).

Inherited (Part VI external inheritances).

- I1–I3 (substrate-to-emergent-spacetime correspondence; Part IX deliverable).
- EL1, EL2 (basic emergent-Lorentz structure; emergent-Lorentz programme deliverable).
- EL3a (measure-pullback invariance; emergent-Lorentz programme deliverable).
- EL3b (substrate Lorentz-normalisation structure equivalent to mass-shell 1/(2E_p); emergent-Lorentz programme deliverable, *load-bearing*).

- EL4 (Gram-kernel Lorentz invariance; emergent-Lorentz programme deliverable).
- CG1–CG2 (admissible coarse-graining; coarse-graining programme deliverable).
- MA1–MA6 (mode-function admissibility; the conditions are defined in this paper, but their substrate-level realisability is a Part IX deliverable).
- Joint consistency of all of the above as a single substrate-derived structure (§4.5; open).

Inherited (standard background).

- Standard rigged-Hilbert-space framework.
- Standard second-quantisation construction.
- Standard textbook free-Dirac field-theoretic structure.
- Substrate-CAR algebra and its consequences from Part V.
- Coherence scales ξ, τ_s from Part IV §3.9.

Conditional on framework assumptions. All topological-core derivations conditional on the inheritances catalogued in §3. The conditional structure is explicit at every theorem statement.

Interpretive. §7.2 (ontological reading: substrate operators fundamental, fields emergent). §10 (the structural shape of the positivity–covariance bridge).

Synthetic. §15.2 (strengthened hierarchy chain combining Parts III, IV, V, and VI into a unified void-to-relativistic-fermion-fields chain).

Conjectural / open. Interacting QFT (Part VII). Antiparticle structure (Part VIII). Detailed substrate-level construction of mode functions and I1–I3 (Part IX). Light-cone-respecting microcausality at any scale (Part IX; not partially established by Part VI). Species decomposition (Part X). Substrate-to-electroweak bridge (Part XI). Substrate-level energy positivity (Part VII). Full axiomatic-QFT spin-statistics theorem (requires all of the above). Joint consistency of inheritance families (§4.5).

20.2 Status table

Result	Status	Source	Layer
Rigged Hilbert space $\Phi \subset \mathscr{H}_1 \subset \Phi^\times$	Defined	§5	Topological
Smeared substrate states / smeared substrate CAR operators	Inherited from Part V (distributional reading)	§5.1	Topological
Mode functions U_α(𝒞; x), V_α(𝒞; x)	Defined as parametric object	§6	Topological
Admissibility conditions MA1–MA6	Defined	§6.2	Topological
Substrate-derived existence of mode functions satisfying MA1–MA6	Open (Part IX)	§6.3	—

Result	Status	Source	Layer
Candidate emergent fermion field operator ψ_α(x)	Defined (conditional on I1–I3, MA1–MA6)	§7	Topological
Theorem 1 (existence of smeared field operators)	**Derived under inheritance**	**§8**	**Topological**
Theorem 2 (field CAR from substrate CAR)	**Derived under inheritance**	**§9**	**Topological**
↳ Textbook δ^3 form in local continuum limit	Derived under MA4	§9.1	Topological
Positivity–covariance bridge construction	Constructed (conditional on EL1, EL2, EL3a, EL3b, EL4)	§10	Topological
Theorem 3 (Lorentz-covariant transformation under U(Λ))	**Derived under inheritance**	**§11**	**Topological**
↳ Unitarity of U(Λ) on $\mathscr{H}_1$	Derived under EL3a + EL3b + EL4	§11.2 Step 2	Topological
↳ Positivity-preserving lift to ℱ_A via second quantisation	Derived (substantive Part VI content)	§11.2 Step 3	Topological
↳ Covariant transformation of ψ_α(x)	Derived under MA5	§11.2 Step 4	Topological
Theorem 4 [positive-frequency Dirac equation]	**Derived under inheritance**	**§12**	**Topological**
↳ Positive-frequency form (iγ^μ ∂_μ − m) ψ^(+) = 0	Derived in present paper	§12.2 Step 3	Topological
↳ Full Dirac equation including V_α	Conditional on Part VIII (V_α = C Ū_α^T)	§12.2 Step 4	—
Substrate-coherence-scale suppression of K_{αβ}(x, y), direction-independent (§13.3)	Derived	§13	Topological
Light-cone-respecting microcausality (any scale)	Open (Part IX); not claimed even partially	§13.4	—
Corollary 4' (physical realisation of emergent field)	**Derived under topological derivation + coherence condition + current scale calibrations**	**§14**	**Physical-realisation**
CAR C*-algebra extension to field operators	Inherited from Part V §9.3 + Theorem 2	—	Topological
Strengthened hierarchy chain (void → emergent Dirac field)	Synthetic	§15.2	Synthesis
Joint consistency of inheritance families (I1–I3, EL1–EL4, CG1–CG2)	Open	§4.5, §18 item 11	—
Interacting fermionic QFT	Open	§18 item 1	—

Result	Status	Source	Layer
Antiparticle structure and charge conjugation	Open (Part VIII)	§18 item 2	—
Substrate-derived I1–I3	Open (Part IX)	§18 item 3	—
Substrate-derived mode functions (concrete construction)	Open (Part IX)	§18 item 4	—
Species decomposition into Standard Model fermions	Open (Part X)	§18 item 5	—
Substrate-to-electroweak bridge	Open (Part XI)	§18 item 7	—
Substrate-level energy positivity	Open (Part VII)	§18 item 8	—
Full axiomatic-QFT spin-statistics theorem	Open	§18 item 9	—

Status glossary:

- *Derived under inheritance* — derived given explicit inheritance from named theorems/programmes.
- *Derived under inheritance + further condition* — derived given inheritance plus a stated additional condition (here, e.g., coherence condition for Corollary 4').
- *Constructed (conditional on ...)* — explicit construction performed in this paper, conditional on inheritances.
- *Defined* — supplied as a definition (rigged Hilbert space, mode functions, admissibility conditions).
- *Inherited* — taken as a non-trivial result of another VERSF paper or standard mathematical reference, not re-derived here.
- *Interpretive* — physical or conceptual reading of a derived result.
- *Synthetic* — combines multiple results into a unified statement.
- *Open* — not addressed; identified deliverable in the matter-sector programme or in the broader VERSF programme.

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21. Conclusion

Part V of the matter-sector strand completed the substrate-level free fermionic Fock construction with the full CAR algebra, strong Pauli exclusion, vacuum uniqueness, and the CAR C*-algebra structure. The dominant remaining matter-sector deliverable identified at the end of Part V was the linked construction of Lorentz-covariant spacetime-labelled fermion fields together with the positivity–covariance bridge (Part V §17 items 1 ↔ 10).

The present paper supplies this construction, conditional on inheritances from three external VERSF programmes (substrate-to-emergent-spacetime correspondence, emergent-Lorentz, admissible coarse-graining), with the conditional structure explicit at every theorem statement.

The conceptual reframing. A naïve approach to Part VI would treat spacetime as a pre-existing Minkowski background and the construction as a textbook-style smearing of substrate CAR operators against spacetime test functions. This is structurally incompatible with VERSF's foundations: spacetime in VERSF is emergent, not primitive. The present paper therefore frames the construction as an *emergence theorem*: in the coarse-graining regime where emergent spacetime is the appropriate description, the substrate CAR algebra reproduces the standard free Dirac field theory on emergent spacetime, with all the standard structures derived rather than postulated.

Theorem 1 establishes that, conditional on substrate-to-emergent-spacetime inheritances I1–I3 and mode-function admissibility MA1–MA6, the candidate emergent fermion field operator

$$\psi_\alpha(x) = \int U_\alpha(\mathcal{C}; x)\, a(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C}) + \int V_\alpha(\mathcal{C}; x)\, a^\dagger(\mathcal{C})\, d\mu_{\text{spin}}(\mathcal{C})$$

is a well-defined operator-valued tempered distribution on substrate Fock space $\mathscr{F}_A$ with the standard test-function smearing properties.

Theorem 2 establishes the field-level CAR algebra as an induced consequence of the substrate-level CAR algebra of Part V via the mode-function construction. The textbook free-Dirac field anticommutator $\{\psi_\alpha(t, \mathbf{x}), \psi_\beta^\dagger(t, \mathbf{y})\} = \delta_{\alpha\beta} \cdot \delta^3(\mathbf{x} - \mathbf{y})$ is recovered in the local continuum limit under MA4. The substrate-scale spacetime kernel $K_{\alpha\beta}(x, y)$ carries substrate-physics fingerprints that are absent from the textbook construction.

Theorem 3 supplies the positivity–covariance bridge. Conditional on emergent-Lorentz inheritances EL1–EL4, the positive-definite substrate Hilbert structure on $\mathscr{F}_A$ supports a unitary representation $U(\Lambda)$ of the emergent Lorentz group on Fock space, with the emergent field operator transforming covariantly:

$$U(\Lambda)\, \psi_\alpha(x)\, U(\Lambda)^{-1} = [S(\Lambda)^{-1}]_\alpha{}^\beta \cdot \psi_\beta(\Lambda x).$$

The substantive Part VI content of Theorem 3 is two-fold: the *positivity-preserving second quantisation* that lifts the unitary one-particle action $U_1(\Lambda)$ on $\mathscr{H}_1$ to a unitary Fock action $U(\Lambda)$ on $\mathscr{F}_A$ while preserving the positive-definite Hilbert structure (no Krein-space or indefinite-metric structure introduced), and the *algebraic-consistency check* (§11.2 Step 4) that translates mode-function covariance into field-level covariance. The mode-function covariance MA5 itself is supplied as inheritance from Part IX, as are EL1–EL4; the positivity of the Hilbert norm is preserved throughout. See §11.3 for the full breakdown of what Theorem 3 substantively establishes versus what it inherits. The textbook free Dirac theory's positivity–covariance reconciliation is the special case where substrate spinorial modes reduce to momentum-space modes and $d\mu_{\text{spin}}$ reduces to the Lorentz-invariant mass-shell measure.

Theorem 4 establishes the emergent *positive-frequency* free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\, \psi^{(+)}(x) = 0$$

as a coarse-graining limit, conditional on the coarse-graining inheritances CG1–CG2, mass-shell admissibility MA6, and the Part III Theorem 1 first-order spinorial flow structure. The mass m is fixed by substrate κ-field structure, not by external calibration. The full Dirac equation $(i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0$ for the combined field $\psi = \psi^{(+)} + \psi^{(-)}$ is conditional on the Part VIII charge-conjugation structure supplying $V_\alpha = C\, \bar{U}_\alpha^T$ (currently held as a placeholder in §6.1); the present paper supplies the positive-frequency form directly.

§13.3 establishes a substrate-coherence-scale suppression result: the field-level cross-anticommutator $K_{\alpha\beta}(x, y)$ vanishes exponentially at substrate-separations $\gg \xi$. This suppression is direction-independent — it applies equally to time-like, space-like, and light-like separations — and therefore does *not* constitute even a partial form of light-cone-respecting microcausality. The actual microcausality theorem (vanishing at space-like separation with non-trivial behaviour at time-like separation) requires structural input beyond MA3 and is deferred entirely to Part IX; identified as §18 item 6 (open).

Corollary 4' supplies the physical-realisation reading: under the coherence condition of Part IV §4A.3 (under current scale-estimate calibrations), the emergent fermion field of Theorems 1–4 is physically manifest as the observable relativistic fermion field on emergent spacetime in the regime of standard fermionic-matter phenomenology.

The two-layer architecture inherited from Parts IV and V is preserved throughout. Theorems 1–4 are topological-core results; Corollary 4' is the sole physical-realisation result.

The strengthened hierarchy chain (§15.2) from $K=7$ architecture through to the emergent free Dirac field is now complete at the algebraic-and-distributional level, conditional on the external inheritances I1–I3, EL1–EL4, CG1–CG2. The matter-sector programme has, at the free-construction level, derived everything from substrate first principles to relativistic fermion fields on emergent spacetime — with the conditional structure explicit and the remaining external dependencies identified.

The synthesis statement reads:

Conditional on substrate-to-emergent-spacetime inheritances (I1–I3), emergent-Lorentz inheritances (EL1–EL4, with EL3 split into EL3a measure-pullback invariance and EL3b substrate Lorentz-normalisation structure; see §3.5), and admissible-coarse-graining inheritances (CG1–CG2), the substrate CAR algebra of Part V supports the construction of relativistic fermion field operators $\psi_\alpha(x)$ on emergent spacetime, with field-level CAR algebra induced from substrate CAR (Theorem 2), Lorentz-covariant transformation under unitary representation on substrate Fock space (Theorem 3, the positivity–covariance bridge), and the emergent positive-frequency free Dirac equation as the coarse-graining limit (Theorem 4). Substrate-coherence-scale direction-independent suppression of the field cross-anticommutator holds (§13.3); *light-cone-respecting microcausality* (the textbook requirement, with direction-dependent vanishing) is *not* established by the present paper — it is identified as an open problem (§18 item 6), with no partial result

claimed. The physical-realisation reading on the coherent substrate (Corollary 4') manifests the emergent fermion field as the observable relativistic fermion field in the regime of standard fermionic-matter phenomenology. The remaining matter-sector work is interactions and renormalisation (Part VII), antiparticles (Part VIII), the substrate-derivation of I1–I3 and mode functions (Part IX), species decomposition (Part X), and the substrate-to-electroweak bridge (Part XI).

The matter-sector programme now possesses:

- A substrate ontology for the persistent current (Microscopic Origin paper).
- An admissibility framework for gauge coupling (Matter Coupling paper).
- An algebraic-geometric origin of spin-½ structure (Part III).
- A substrate-level forcing of antisymmetric exchange and Pauli exclusion (Part IV).
- A complete free fermionic Fock-space construction with full CAR algebra (Part V).
- **Conditional on inheritances from the broader VERSF programme, the substrate-to-emergent-Dirac-field bridge with the positivity–covariance reconciliation, induced field CAR algebra, emergent positive-frequency Dirac equation, and substrate-coherence-scale (direction-independent) suppression of the field cross-anticommutator (the present paper, Part VI).**

The remaining deliverables for the matter-sector programme are interactions and renormalisation (Part VII), antiparticles (Part VIII), the substrate-to-emergent-spacetime correspondence and substrate-derived mode functions (Part IX), species decomposition (Part X), and the substrate-to-electroweak bridge (Part XI). Each is a concrete next-paper target with explicit conditions identified.

Explicit clarification of scope (final). The present paper supplies the substrate-to-emergent-Dirac-field bridge at the free-construction level: rigged-Hilbert-space smearing apparatus (§5), substrate-derived mode-function admissibility conditions MA1–MA6 (§6), the existence of smeared field operators (Theorem 1), the induced field CAR algebra (Theorem 2), the positivity–covariance bridge (Theorem 3, in two-component form: positivity-preserving second quantisation as the substantive Part VI content, plus algebraic consistency conditional on EL1–EL4 + MA5), the emergent positive-frequency free Dirac equation (Theorem 4, with full form including V_α conditional on Part VIII), substrate-coherence-scale direction-independent suppression of the field cross-anticommutator (§13.3 — *not* light-cone-respecting microcausality, which is open and not partially established by Part VI; see §18 item 6), and the physical-realisation reading on the coherent substrate (Corollary 4'). The two-layer architecture of Parts IV and V is preserved: Theorems 1–4 are topological-core results requiring substrate-CAR plus inheritances I1–I3, EL1–EL4 (with EL3b — substrate analogue of mass-shell 1/(2E_p) — and EL4 — Gram-kernel Lorentz invariance — as the load-bearing structural commitments), CG1–CG2; Corollary 4' is the sole physical-realisation result, inheriting Part IV's coherence-condition pattern under current scale-estimate calibrations. The paper does not derive emergent Lorentz invariance from substrate first principles (that is the emergent-Lorentz programme), does not construct the substrate-to-emergent-spacetime correspondence (that is Part IX), does not construct mode functions explicitly (that is Part IX), and does not establish even partial

light-cone-respecting microcausality (open, Part IX). What it establishes is the bridge: given the inheritances, the substrate CAR algebra supports the standard textbook free Dirac field theory as its coarse-grained emergent description, with the positivity–covariance reconciliation working via the standard positivity-preserving second quantisation. Interactions and renormalisation (Part VII), antiparticle structure and charge conjugation (Part VIII), substrate-to-emergent-spacetime construction (Part IX), species decomposition into Standard Model fermions (Part X), substrate-to-electroweak bridge (Part XI), and the full axiomatic-QFT spin-statistics theorem are not derived here and are explicitly identified as the next-stage deliverables.

Strongest framing. Given (i) the substrate CAR algebra of Part V, (ii) substrate-to-emergent-spacetime inheritances I1–I3, (iii) emergent-Lorentz inheritances EL1–EL4 (with EL3b and EL4 as load-bearing structural commitments), (iv) admissible-coarse-graining inheritances CG1–CG2, and (v) admissible mode functions satisfying MA1–MA6, the substrate-level emergence of free relativistic fermion field theory on emergent spacetime is *forced* at the topological-core level. The emergent fermion field ψ(x) exists as an operator-valued tempered distribution on substrate Fock space 𝒻_A (Theorem 1); the field-level CAR algebra is induced from the substrate CAR algebra (Theorem 2); the positivity–covariance bridge works via the standard positivity-preserving second quantisation (Theorem 3); the positive-frequency free Dirac equation emerges as the coarse-graining limit (Theorem 4, with the full form including V_α conditional on Part VIII charge conjugation); substrate-coherence-scale direction-independent suppression of the field cross-anticommutator holds (§13.3, not light-cone-respecting microcausality — that remains open). Physical realisation on the coherent entanglement substrate under the coherence condition manifests the emergent field as the observable relativistic fermion field in the regime of standard fermionic-matter phenomenology (Corollary 4'). The matter-sector programme is now bridged from substrate CAR to relativistic fermion fields at the free-construction level; the remaining work is interactions, antiparticles, the substrate-derivation of the external inheritances (including their joint consistency, §18 item 11), light-cone-respecting microcausality (§18 item 6), species decomposition, and electroweak coupling, through Parts VII–XI.