Hodge Decomposition of the σ-Sector and the Cohomological Structure of Persistent Transport

A Framework for Master-Action Unification

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General Reader Summary

The previous VERSF papers developed two transport sectors that appear opposite in character. The σ-sector governs how the substrate restores admissibility from one update to the next — a process that smooths things out and damps disturbances over time, like heat diffusing through a metal or a magnet settling into equilibrium. The persistent transport sector identified elsewhere in the programme has the opposite character: it preserves things rather than smoothing them out. It is associated with the kind of conserved quantities that appear in electromagnetism and other gauge theories — quantities that survive change rather than relaxing away under it.

The natural question is whether these two sectors are actually related to each other, and if so how. They look like opposites. The σ-sector continuum-limit paper made this question precise by identifying four specific ways in which the two sectors differ. The σ-sector smooths disturbances; the persistent sector preserves them. The σ-sector loses energy over time; the persistent sector conserves it. The σ-sector lives on a one-dimensional rim — the outer edge of the K = 7 wheel; the persistent sector seems to live in the higher-dimensional setting of ordinary spacetime. And the σ-sector's surviving direction (the alternating pattern that resists smoothing) seems to have a different mathematical character from the persistent sector's surviving direction (the conserved topological loop).

This paper takes the fourth of these differences and dissolves it, while addressing the first two in a limited way and leaving the third open.

The central technical idea is best understood by analogy. Imagine a vibrating drumhead. The drumhead's motion can be decomposed into different modes — some that fade quickly (the highest-pitched vibrations) and some that persist longer (the lowest-pitched ones). In an idealised drumhead, the absolutely lowest mode — uniform displacement — doesn't fade at all; it just sits there. That uniform mode is special: it represents something the drumhead's restoring forces cannot push back against, because there's nothing to push back to.

The σ-sector dynamics turns out to work the same way. When you write it in the right mathematical setting, it is a *standard* smoothing dynamics — the same kind of process that governs heat diffusion and a great many other physical relaxations. The surviving direction is the

analog of the drumhead's uniform mode: the smoothing process cannot eliminate it because there is nothing further to smooth toward.

Now here is the key move. The σ-sector originally lives on the vertices of the rim — six numbers, one per vertex. But there is a *related* setting that lives on the edges of the rim — six numbers, one per edge. The mathematical relationship between vertices and edges is one of the most studied in modern geometry, going back to Hodge in the 1940s. In the edge setting, the same smoothing dynamics has its *own* surviving direction — and that surviving direction is exactly the kind of conserved loop quantity that the persistent gauge sector is built around. It is, in standard terminology, a Wilson loop: a quantity you compute by integrating around the closed rim, which depends only on the topology of the loop and not on the details of any particular configuration.

This is the unification the paper achieves at the *kinematic* level. The σ-sector lives on the vertices; the persistent gauge sector lives on the edges of the same rim; they are not opposite sectors but complementary sectors of one master configuration space. The fourth structural difference — the apparent difference between the σ-sector's surviving direction and the persistent sector's surviving direction — dissolves, because in this setting "kernel of a smoothing operator" and "conserved topological loop" turn out to be literally the same thing under two different descriptions. The same one-dimensional space, viewed two ways.

The first two structural differences (the σ-sector smooths; the persistent sector conserves) are addressed in a more limited way. The persistent direction in the master configuration space is indeed *invariant* under the smoothing dynamics — it doesn't decay. But invariance under smoothing is not the same as having the kind of conservative, wave-propagating dynamics that the persistent sector is supposed to have. The persistent direction here just sits there; it doesn't move at all. To get genuine wave-like dynamics on it, additional structure beyond the smoothing process is needed — something the paper flags as the next major open question but does not resolve here.

The third structural difference (the σ-sector is one-dimensional; the persistent sector lives in higher-dimensional spacetime) is not addressed. The construction here is built on the rim of the K = 7 wheel, which is a one-dimensional loop. Higher-dimensional generalisations require structural ingredients that the σ-sector alone does not provide.

What this paper therefore establishes is not the full unification of the σ-sector with the persistent gauge sector. It establishes the *framework* — the mathematical setting in which the unification can be made precise, where the surface differences between the two sectors can be examined individually, and where one of them fully dissolves on inspection. The result is a framework paper that opens the unification programme rather than closing it, but in opening it cleanly it provides the technical setting that subsequent papers can build on.

The paper does not claim to have derived electromagnetism, Maxwell's equations, or relativistic gauge theory. It claims something narrower: the σ-sector and a persistent loop quantity can be rigorously identified as complementary parts of one mathematical structure on the K = 7 architecture's rim, with explicit acknowledgement of what this identification does and does not yield. The remaining differences between the σ-sector and the broader persistent gauge sector are now precisely posable as open problems, ready for further structural work.

Abstract

The σ-sector papers established the discrete sequential transport sector of the $K = 7$ closure architecture as a dissipative parabolic gradient flow with the alternating spoke mode as its persistent direction. The persistent cohomological transport sector identified elsewhere in the VERSF programme has the opposite character: conservative, hyperbolic, gauge-structured. The σ-sector continuum-limit paper identified four structural obstacles to the identification of these sectors: parabolic vs hyperbolic dynamics; dissipative vs conservative character; one-dimensional vs higher-dimensional spatial domain; kernel-of-Laplacian vs cohomology-class kernel direction.

This paper addresses the fourth obstacle rigorously and the first two obstacles partially, by constructing the master-action unification *framework* on a cochain-complex setting.

We construct the discrete cochain complex on the rim cycle C_6 of W_7:

$C^0(C_6) \longrightarrow^\wedge \delta \; C^1(C_6)$,

with $C^0(C_6) \cong \mathbb{R}^6$ the 0-cochains on rim vertices, $C^1(C_6) \cong \mathbb{R}^6$ the 1-cochains on rim edges, and δ the simplicial coboundary operator. The Hodge Laplacian on 0-cochains is $\Delta^0 = \delta\delta$*; on 1-cochains it is* $\Delta^1 = \delta\delta$. Both have the spectrum of the standard cycle-graph Laplacian L.

The σ-sector envelope dynamics (from the continuum-limit paper, in envelope variables $\psi_i = (-1)^i \lambda_i$) is identified, after temporal rescaling, with the Hodge heat flow on 0-cochains:

$\partial\_\tilde{\tau} \, \psi = - \Delta^0 \psi$ on $C^0(C_6)$.

The natural extension to 1-cochains is the Hodge heat flow on $C^1(C_6)$:

$\partial\_\tilde{\tau} \, \alpha = - \Delta^1 \alpha$ on $C^1(C_6)$.

The Hodge decomposition of $C^1(C_6)$ is:

$C^1(C_6) = \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\delta^*)$ = (exact 1-cochains) $\oplus$ (harmonic 1-cochains),

with $\dim(\mathrm{Im}(\delta)) = 5$ and $\dim(\mathrm{Ker}(\delta^*)) = \dim(H^1(C_6)) = 1$.

We prove four results in this setting.

Theorem 1 (Dissipative annihilation of exact modes). *Under the master Hodge heat flow on* $C^1(C_6)$*, the exact 1-cochain component* $\alpha\_exact \in \mathrm{Im}(\delta)$ *decays exponentially with rate bounded*

below by the smallest non-zero eigenvalue of Δ^0 (= 1 for C_6). The exact 1-cochain component is in correspondence with the σ-sector envelope dynamics via α_exact = δψ; the decay of α_exact is the σ-sector relaxation of ψ toward its kernel direction (constant ψ).

Theorem 2 (Invariance of harmonic modes). *Under the master Hodge heat flow on $C^1(C_6)$, the harmonic 1-cochain component α_harm ∈ Ker(δ)* is exactly invariant:*

$\partial_{\tilde{\tau}}$ α_harm = 0.

The harmonic component is one-dimensional, isomorphic to $H^1(C_6) \cong \mathbb{R}$, and is generated by the constant 1-cochain $\alpha\_harm(e_i) = c$ for all $i \in \mathbb{Z}/6$, $c \in \mathbb{R}$. This element is the Wilson-loop generator: it integrates non-trivially around the rim cycle (giving 6c rather than zero) while not being expressible as a coboundary from C^0.

Theorem 3 (Master decomposition). *The master configuration space $C^1(C_6)$ decomposes orthogonally as*

$C^1(C_6)$ = (σ-sector) ⊕ (persistent gauge sector),

where the σ-sector is identified with Im(δ) ⊂ C^1 (5-dimensional, isomorphic to the σ-sector envelope space $C^0(C_6)$ modulo constants) and the persistent gauge sector is identified with Ker(δ) ⊂ C^1 (1-dimensional, isomorphic to $H^1(C_6)$). The master Hodge heat flow restricts to (i) the σ-sector envelope dynamics on Im(δ) (via the correspondence α_exact = δψ); and (ii) the trivial invariant dynamics on Ker(δ*). The σ-sector and persistent gauge sector are complementary subspaces of the same master configuration space, with mutually orthogonal dynamical behaviour.*

Theorem 4 (Emergent asymptotic gauge equivalence). *Two master configurations α, α' ∈ $C^1(C_6)$ differing by an exact 1-cochain — i.e., α' = α + δχ for some $\chi \in C^0(C_6)$ — become equivalent under the master flow in the asymptotic limit $\tilde{\tau} \to \infty$:*

$\|\alpha(\tilde{\tau}) - \alpha'(\tilde{\tau})\| = \|e^{(-\tilde{\tau}\Delta^1)}\, \delta\chi\| \leq \|\delta\chi\| \cdot e^{(-\tilde{\tau})}$ for all $\tilde{\tau} \geq 0$,

so that the asymptotic configurations differ by zero in the limit. This is an asymptotic equivalence under the dissipative flow — not an exact gauge equivalence of a fundamental dynamics. We call it an emergent asymptotic gauge equivalence and distinguish it from fundamental gauge redundancy.

The framework therefore establishes the cohomological structure of the σ-sector / persistent transport unification on the K = 7 architecture's rim cycle. **What this paper establishes is the *kinematic* structure of the master configuration space — the cochain-complex setting, the Hodge decomposition, the identification of subspaces — not the dynamics of the gauge sector, which requires the Hamiltonian upgrade flagged in P1.** The identification of the persistent direction simultaneously as the kernel of the Hodge Laplacian and as the first cohomology class $H^1(C_6)$ resolves the fourth structural obstacle from the continuum-limit paper at the kinematic level. The dissipative annihilation of exact modes plus the invariance of harmonic modes partially address the first two obstacles — by exhibiting a sector whose dynamics is invariant under dissipative flow — but do not deliver conservative Hamiltonian dynamics on the persistent sector. The third structural obstacle (dimensional restriction) is not addressed.

Epistemic status. *Proven (within this paper):* Theorems 1–4 above; the identification of the σ-sector envelope operator with the Hodge Laplacian Δ^0 on $C^0(C_6)$; the Hodge decomposition of $C^1(C_6)$; the spectral structure of Δ^0 and Δ^1. *Structural inputs (not derived here):* the K = 7 architecture with $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$ from the architecture papers; the σ-sector envelope dynamics from the continuum-limit paper; the carrier–envelope decomposition $\lambda_i = (-1)^i \psi_i$; the natural extension of the master configuration space from $C^0(C_6)$ (the σ-sector) to $C^1(C_6)$ (master). The master-action variational principle generating the Hodge heat flow on $C^1(C_6)$ is *postulated*, not derived from the K = 7 constraint catalogue. *Open:* derivation of conservative Hamiltonian dynamics on the persistent sector (to fully address obstacles 1 and 2); higher-dimensional lifting (obstacle 3); the master-action's relation to the K = 7 constraint catalogue; the extension to non-Abelian gauge structure; the identification with the broader programme's persistent cohomological transport sector elsewhere.

Contents

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1. Introduction

The VERSF transport architecture currently exhibits two apparently distinct sectors. The σ-sector — derived in the preceding three papers as the substrate's constitutive admissibility-restoring response on the K = 7 closure wheel — is dissipative, parabolic, one-dimensional, and has the alternating spoke mode as its persistent direction. The persistent cohomological transport sector identified elsewhere in the VERSF programme is conservative, hyperbolic, gauge-structured, and has Wilson-loop-like cohomology classes as its persistent direction.

The σ-sector continuum-limit paper made the question of their relation precise. Four structural obstacles to the identification of these sectors were identified — each a specific way in which the two sectors differ at the level of mathematical character. These obstacles are not deficiencies of the σ-sector; they are correct features of what the σ-sector is. But for any unification with the persistent gauge sector to be coherent, each obstacle has to be either resolved or accommodated by additional structure.

The purpose of this paper is to address as many of the four obstacles as can be addressed within the present framework, and to set up the cochain-complex setting in which the full unification can be precisely formulated. The result is *not* a full unification of the σ-sector with the persistent gauge sector — that is the subject of subsequent work. It is the framework paper for the unification: the technical setting in which the unification programme acquires precise statements, where the master-action variational structure can be built, and where the four obstacles can be addressed individually.

A central observation drives the construction. The σ-sector envelope dynamics, after the carrier–envelope decomposition of the continuum-limit paper, is governed by the standard graph Laplacian L on the cycle C_6. This Laplacian is, in the language of discrete differential geometry, precisely the Hodge Laplacian $\Delta^0 = \delta^*\delta$ on 0-cochains of C_6. The σ-sector dynamics is therefore not an exotic dissipative flow but a standard Hodge heat flow — on 0-cochains.

The natural extension to 1-cochains gives the Hodge Laplacian $\Delta^1 = \delta\delta^*$ on $C^1(C_6)$, and the Hodge decomposition of 1-cochains:

$$C^1(C_6) = \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\delta^*),$$

splits into exact 1-cochains (5-dimensional on C_6) and harmonic 1-cochains (1-dimensional, isomorphic to $H^1(C_6)$). The exact part carries the σ-sector envelope dynamics; the harmonic part is invariant under the flow. The harmonic part is the Wilson-loop direction — and it has the *simultaneous* structure of a Hodge Laplacian kernel and a cohomology class, by the Hodge theorem. The fourth structural obstacle of the continuum-limit paper — that the σ-sector's kernel direction is a Laplacian kernel while the persistent sector's kernel direction is a cohomology class — dissolves: in this setting, the two are the same one-dimensional subspace under two equivalent descriptions.

The paper develops this framework in §§3–5, establishes the four theorems in §§6–9, identifies which obstacles are addressed and which remain in §10, and sketches the open programme in §13.

A guiding principle. The σ-family papers have been mathematically clean — each paper establishes specific theorems with explicit structural inputs and epistemic discipline. This paper is methodologically continuous: it does not claim to do the full unification. It does claim, rigorously, that the cochain-complex setting is the right one for the unification programme to proceed in, and that within that setting the four obstacles become individually addressable.

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2. The Four Structural Obstacles

The σ-sector continuum-limit paper (§12 of that paper, restated here for self-containment) identified four structural obstacles to the identification of the σ-sector with the persistent cohomological transport sector:

- **Obstacle 1 (Parabolic vs hyperbolic).** The σ-sector continuum equation $\partial_{\tilde{\tau}} \phi = D\, \partial^2_x \phi$ is parabolic (first-order in $\tilde{\tau}$, second-order in x). The persistent gauge sector in any conventional VERSF formulation is hyperbolic (second-order in both temporal and spatial coordinates, Maxwell-like). The two have different mathematical character.
- **Obstacle 2 (Dissipative vs conservative).** The σ-sector dynamics is intrinsically dissipative: the effective action A_{eff} decreases monotonically along the flow except on the kernel direction, and the dynamics has no conserved kinetic quantity. The persistent gauge sector is conservative: it has Hamiltonian structure, conserves energy, and supports propagating solutions.
- **Obstacle 3 (One-dimensional vs higher-dimensional).** The σ-sector continuum lives on S^1, the rim circle of the wheel — a one-dimensional spatial domain. The persistent gauge sector lives in higher-dimensional spacetime. The two have different spatial dimensions.
- **Obstacle 4 (Kernel of Laplacian vs cohomology class).** The σ-sector's persistent direction is the kernel of the continuum operator ∂^2_x (the constant envelope, in continuum variables; the constant ψ, in discrete envelope variables). The persistent gauge sector's persistent direction is a cohomology class — a Wilson loop, a class in H^1 of some appropriate complex. The two have different structural origins: an analytic kernel versus a topological class.

This paper addresses obstacles 1 and 2 partially, addresses obstacle 4 fully, and does not address obstacle 3.

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3. The Cochain Complex on the Rim Cycle

3.1 The rim cycle as a 1-complex

The K = 7 architecture W_7 has, as a sub-complex, the rim cycle C_6: the 1-complex with 6 vertices $v_0, \ldots, v_5$ arranged cyclically and 6 edges $e_0, \ldots, e_5$, with each edge e_i connecting v_i to v_{i+1} (indices mod 6). The rim cycle is the boundary structure that carries the homological persistence of the K = 7 architecture: $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$ where $C = e_0 + e_1 + \cdots + e_5$ is the primitive rim 1-cycle (preceding papers, §3.1).

The carrier–envelope decomposition of the continuum-limit paper maps the σ-sector envelope variable ψ_i to functions on the rim vertex set: $\psi_i \leftrightarrow \psi(v_i)$ for $i \in \mathbb{Z}/6$. We take this identification as primitive in the present paper and work with functions on the rim cycle structure throughout.

3.2 Cochain spaces and the coboundary operator

We define the cochain spaces of C_6 as follows:

- **0-cochains** $C^0(C_6) := \{ \psi : V(C_6) \to \mathbb{R} \} \cong \mathbb{R}^6$, with $V(C_6) = \{v_0, \ldots, v_5\}$ the vertex set.
- **1-cochains** $C^1(C_6) := \{ \alpha : E(C_6) \to \mathbb{R} \} \cong \mathbb{R}^6$, with $E(C_6) = \{e_0, \ldots, e_5\}$ the edge set.
- **2-cochains** $C^2(C_6) := 0$, since C_6 has no 2-cells.

We orient each edge e_i from v_i to v_{i+1}. The simplicial coboundary operator $\delta : C^0(C_6) \to C^1(C_6)$ is defined by

$$(\delta\psi)(e_i) := \psi(v_{i+1}) - \psi(v_i).$$

The next coboundary $\delta^1 : C^1(C_6) \to C^2(C_6)$ is zero (since $C^2(C_6) = 0$), so every 1-cochain is closed:

$$Ker(\delta^1) = C^1(C_6),\ dim = 6.$$

The exact 1-cochains form the image of δ:

$$Im(\delta) \subset C^1(C_6),\ dim = 5$$

(the constants $\psi(v_i) = c$ are in the kernel of δ, so $\dim(Im(\delta)) = \dim(C^0) - \dim(Ker\ \delta) = 6 - 1 = 5$).

The first cohomology of C_6 is:

$$H^1(C_6) := Ker(\delta^1) / Im(\delta) = C^1(C_6) / Im(\delta) \cong \mathbb{R},$$

a one-dimensional space generated by, e.g., the constant 1-cochain $\alpha_0(e_i) = 1$ for all i.

Lemma 3.1 (Well-definedness of circulation on $H^1(C_6)$). *The circulation map*

$$\oint\_C : C^1(C_6) \longrightarrow \mathbb{R},\ \alpha \mapsto \Sigma_i\, \alpha(e_i)$$

is well-defined on cohomology classes: $\oint\_C (\alpha + \delta\chi) = \oint\_C \alpha$ *for any* $\chi \in C^0(C_6)$. *The induced map* $\oint\_C : H^1(C_6) \to \mathbb{R}$ *is an isomorphism.*

Proof. For any $\chi \in C^0(C_6)$,

$\oint C\,(\delta\chi) = \Sigma_i\,(\delta\chi)(e_i) = \Sigma_i\,(\chi(v\{i+1\}) - \chi(v_i)) = 0$,

by telescoping around the cycle (each $\chi(v_j)$ appears once with + and once with −). So $\oint\_C\,(\alpha + \delta\chi) = \oint\_C\,\alpha + \oint\_C\,(\delta\chi) = \oint\_C\,\alpha$.

For the isomorphism: $H^1(C_6)$ is one-dimensional, generated by $[\alpha_0]$ with $\oint\_C\,\alpha_0 = 6 \neq 0$. So the induced map is non-zero on the generator, hence an isomorphism of one-dimensional spaces. □

The generator α_0 has *circulation* 6 around the cycle, and any 1-cochain in the image of δ has circulation 0. The Lemma confirms that "circulation around the rim" is a well-defined cohomology invariant — the basic object that the §5.3 Wilson-loop interpretation is built on.

3.3 The Hodge Laplacian on 0-cochains and 1-cochains

We equip $C^0(C_6)$ with the standard inner product $\langle\psi, \psi'\rangle := \Sigma_i\,\psi(v_i)\,\psi'(v_i)$ and $C^1(C_6)$ with the analogous inner product on edges. The adjoint of δ is $\delta^* : C^1(C_6) \to C^0(C_6)$, defined by $\langle\delta\psi, \alpha\rangle = \langle\psi, \delta^*\alpha\rangle$, which gives

$(\delta\alpha)(v_i) = \alpha(e\_\{i-1\}) - \alpha(e_i)$.*

The Hodge Laplacian on 0-cochains is:

$\Delta^0 := \delta\delta : C^0(C_6) \to C^0(C_6)$,*

with explicit action

$(\Delta^0\,\psi)(v_i) = 2\,\psi(v_i) - \psi(v\_\{i-1\}) - \psi(v\_\{i+1\})$.

This is the standard graph Laplacian L of the cycle C_6.

The Hodge Laplacian on 1-cochains is:

$\Delta^1 := \delta\delta + \delta^1\delta^1 = \delta\delta$ (since $\delta^1 = 0$),*

with explicit action

$(\Delta^1\,\alpha)(e_i) = 2\,\alpha(e_i) - \alpha(e\_\{i-1\}) - \alpha(e\_\{i+1\})$.

Also the standard graph Laplacian structure, but acting on edge functions. The spectra of Δ^0 and Δ^1 are identical:

$spec(\Delta^0) = spec(\Delta^1) = \{\,2 - 2\cos(2\pi k/6) : k = 0, 1, 2, 3, 4, 5\,\} = \{\,0, 1, 3, 4, 3, 1\,\} = \{\,0, 1, 1, 3, 3, 4\,\}$.

Both operators have one-dimensional kernels:

$Ker(\Delta^0) = \{ \psi : \psi(v_i) = c$ for all i, $c \in \mathbb{R} \} \cong \mathbb{R}$ (the constants); $Ker(\Delta^1) = \{ \alpha : \alpha(e_i) = c$ for all i, $c \in \mathbb{R} \} \cong \mathbb{R}$ (the constant 1-cochains).

The first identification (constants ↔ $Ker(\Delta^0)$) is the standard correspondence between $H^0(C_6)$ and the kernel of the 0-Laplacian. The second identification (constant 1-cochains ↔ $Ker(\Delta^1)$) is the standard Hodge correspondence between $H^1(C_6)$ and the kernel of the 1-Laplacian. This is the Hodge theorem applied to the cycle graph: every cohomology class has a unique harmonic representative, and the harmonic representatives form the kernel of the appropriate Hodge Laplacian.

The kernel of Δ^1 is therefore *simultaneously* a Laplacian kernel and a cohomology class. This is the structural fact that resolves Obstacle 4 in the framework developed below.

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4. Identification of the σ-Sector with the Hodge Laplacian

The continuum-limit paper established that the σ-sector envelope variable $\psi_i = (-1)^i \lambda_i$ satisfies, in discrete form, the gradient flow

$\partial\_\tau \psi = -2 L\_N \psi$ on $C^0(C_6)$,

where L_N is the standard graph Laplacian of the cycle graph C_N (for N = 6 in the K = 7 case). After temporal rescaling $\tilde{\tau} := 2\tau$, this becomes

$\partial\tilde{\tau} \psi = - L \psi$, $L = L\{N=6\}$.

By §3.3, $L = \Delta^0$. The σ-sector envelope dynamics is therefore exactly the Hodge heat flow on 0-cochains of C_6:

$\partial\_\tilde{\tau} \psi = - \Delta^0 \psi$.

Note on temporal rescalings. The $\tilde{\tau}$ in this paper is a *discrete* rescaled time ($\tilde{\tau} = 2\tau$, with τ the discrete flow parameter of the σ-sector master-action variation). It is not the same as the $\tilde{\tau}$ of the continuum-limit paper, which was a *continuum* rescaled time ($\tilde{\tau} = \tau/a^2$, with a the lattice spacing in the K_N refinement). The two $\tilde{\tau}$s use the same symbol but rescale different time parameters: the present paper works at the discrete K = 7 (N = 6) level; the continuum-limit paper worked at the K_N → ∞ refinement level. Throughout this paper, $\tilde{\tau}$ denotes the discrete rescaled time of the K = 7 case, with all 1-cochain dynamics on $C^1(C_6)$ understood as discrete-time Hodge heat flow.

This identification is the technical bridge between the σ-sector and the Hodge-decomposition framework. The σ-sector papers' construction — derived from the master-action variation of A_cl = α A_circ + γ A_comp on the K = 7 architecture, then carrier-envelope-decomposed —

turns out to coincide, in envelope variables, with a standard mathematical object: the heat semigroup generated by the Hodge Laplacian on 0-cochains of the rim cycle.

This is not a triviality. It is what makes the present paper possible. The Hodge-decomposition framework is highly developed in pure mathematics; identifying the σ-sector dynamics within it gives access to standard tools (Hodge decomposition, kernel-cohomology correspondence, harmonic representatives) that are not available for arbitrary gradient flows.

The persistent direction of the σ-sector — constant ψ on rim vertices — is now identified with $H^0(C_6) \cong \mathbb{R}$: the connected-components cohomology of the cycle graph. This is the trivial topological invariant of the rim (which has one connected component), and is *not* the same as $H^1(C_6)$, the Wilson-loop cohomology. The σ-sector's persistent direction is therefore cohomologically real but is a *zeroth* cohomology class, not a first cohomology class. The Wilson-loop direction lives in $H^1(C_6)$, which is not accessible from $C^0(C_6)$ — it requires extending to 1-cochains.

This is the structural reason the extension to 1-cochains is needed.

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5. The Master Configuration Space and the Hodge Decomposition

5.1 The extended configuration space

We define the master configuration space as the 1-cochain space of the rim cycle:

$\mathbf{\Lambda\_master := C^1(C_6) \cong \mathbb{R}^6}$.

Each element $\alpha \in \Lambda\_master$ is a function $\alpha : E(C_6) \to \mathbb{R}$ on the rim edges. The σ-sector envelope variable $\psi \in C^0(C_6)$ maps into Λ_master via the coboundary:

$\psi \mapsto \delta\psi \in Im(\delta) \subset \Lambda\_master,$

so the σ-sector envelope dynamics induces a flow on the exact subspace Im(δ) of Λ_master, with

$\partial\_\tilde{\tau}\,(\delta\psi) = \delta\,(\partial\_\tilde{\tau}\,\psi) = \delta\,(-\Delta^0\,\psi) = -\Delta^1\,(\delta\psi),$

using the commutation relation $\delta\Delta^0 = \Delta^1\delta$ on the chain complex with adjoints. *Derivation:* $\Delta^0 = \delta\delta$ *on* C^0 *and* $\Delta^1 = \delta\delta$ on C^1 (since $\delta^1 = 0$). Then $\delta\Delta^0 = \delta(\delta\delta) = (\delta\delta)\delta = \Delta^1\delta$. □

The σ-sector flow on $C^0(C_6)$ therefore lifts to the Hodge heat flow on $Im(\delta) \subset C^1(C_6)$.

The extension to the full Λ_master postulates the natural Hodge heat flow on 1-cochains:

$\partial\_\tilde{\tau}\,\alpha = -\,\Delta^1\,\alpha$ *on* $\Lambda\_master.$

This is the master dynamics of the framework. It restricts to the σ-sector envelope dynamics on Im(δ) and extends it to the complementary subspace Ker(δ*) by the same Hodge Laplacian.

Epistemic note. The extension from the σ-sector envelope dynamics (on 0-cochains) to the master dynamics (on 1-cochains) involves two distinct postulates, both of which deserve to be flagged explicitly.

Postulate 1: choice of master configuration space. We take Λ_master = $C^1(C_6)$, the 1-cochains on the rim cycle, as the natural extension of the σ-sector envelope space $C^0(C_6)$. This is *one* choice; other extensions are conceivable (e.g., extending to cochains on the full W_7 complex including spokes and 2-cells; or to direct sums incorporating different sub-complexes). The choice of $C^1(C_6)$ is motivated by the fact that the Wilson-loop / H^1 structure lives there, but it is not forced by the K = 7 architecture papers — the architecture papers worked with the spoke variables only.

Postulate 2: choice of master dynamics on Λ_master. We take the master dynamics to be the Hodge heat flow $\partial\_\tilde{\tau}\ \alpha = -\Delta^1\ \alpha$ everywhere on $C^1(C_6)$ — including on the harmonic subspace Ker(δ*), where it gives the trivial invariant dynamics $\partial\_\tilde{\tau}$ α_harm = 0. This is *one* choice; a different choice — for instance, postulating a Hamiltonian operator on Ker(δ*) alongside the dissipative Hodge flow on Im(δ) — would give the same σ-sector dissipation but *non-trivial* gauge dynamics, and would resolve obstacles 1 and 2 fully rather than partially.

Both postulates are non-trivial. The σ-sector master-action variation of the K = 7 architecture (preceding papers) was derived for spoke variables and, in envelope form, for 0-cochains. The natural extension to 1-cochains via the same Hodge Laplacian is the simplest non-trivial extension respecting the cochain-complex structure, but it is *not* derived from the K = 7 constraint catalogue — it is a working hypothesis whose consequences are then derived rigorously in §§6–9.

The consequence: the "static gauge sector" of the present framework is partly an artefact of the specific choice of master dynamics (Postulate 2). A different operator on Ker(δ*) — postulated separately, or derived from an extended master-action principle — could give the gauge sector its own non-trivial dynamics. The framework is therefore *permissive* about the dynamics on the persistent sector: it accommodates a static dynamics (as in the present paper) or a Hamiltonian dynamics (as a future extension), depending on which postulate is chosen.

This reframes the load-bearing open question. It is not "we need a Hamiltonian upgrade to the present dynamics" but rather "the framework permits a Hamiltonian upgrade at the cost of postulating different dynamics on Ker(δ*); whether the K = 7 master action — suitably extended to 1-cochains — *forces* or merely *permits* such a Hamiltonian structure is the substantive open question." See P1 below for the sharpened version.

5.2 The Hodge decomposition of $C^1(C_6)$

The Hodge decomposition of 1-cochains on the cycle graph C_6 is:

$C^1(C_6) = Im(\delta) \oplus Ker(\delta),*$

an orthogonal decomposition with respect to the standard inner product. The two subspaces are:

- **Exact 1-cochains** $Im(\delta) = \{ \alpha : \alpha = \delta\psi \text{ for some } \psi \in C^0(C_6) \} \subset C^1(C_6)$, *dim = 5.*
- **Harmonic 1-cochains** $Ker(\delta) = Ker(\Delta^1) = \{ \alpha : \alpha(e_i) = c \text{ for all } i, c \in \mathbb{R} \} \subset C^1(C_6)$, dim = 1.*

The harmonic subspace is isomorphic to $H^1(C_6) \cong \mathbb{R}$ by the Hodge theorem: every cohomology class has a unique harmonic representative, and the harmonic representatives form $Ker(\Delta^1)$. For C_6, the harmonic representative of the generator of H^1 is the constant 1-cochain $\alpha\_harm(e_i) = c$.

Every $\alpha \in C^1(C_6)$ can be written uniquely as $\alpha = \alpha\_exact + \alpha\_harm$ with $\alpha\_exact \in Im(\delta)$ and $\alpha\_harm \in Ker(\delta^*)$.

5.3 $H^1(C_6)$ and the Wilson-loop interpretation

The harmonic representative of the generator of $H^1(C_6)$ is the constant 1-cochain $\alpha_0(e_i) = 1$ (or any non-zero scalar multiple). Its circulation around the rim is

$$\oint\_C \alpha_0 := \Sigma_i \alpha_0(e_i) = 6,$$

which is non-zero, confirming that α_0 is not in the image of δ.

This circulation has the structural form of a Wilson loop: the line integral of the gauge potential α around the closed loop C, giving a topologically protected quantity that depends only on the cohomology class $[\alpha] \in H^1(C_6)$, not on the specific representative α.

Two 1-cochains α and $\alpha + \delta\chi$ differ by an exact 1-cochain, which has zero circulation: $\oint C (\delta\chi) = \Sigma_i (\chi(v\{i+1\}) - \chi(v_i)) = 0$ by telescoping. So $\oint\_C \alpha = \oint\_C (\alpha + \delta\chi)$, confirming that the circulation depends only on the cohomology class.

The harmonic subspace $Ker(\delta^*) \cong H^1(C_6)$ is therefore the *Wilson-loop subspace* of the master configuration space. Its persistent direction (constant 1-cochain, generator of H^1) is the Wilson-loop generator — the topologically protected flux around the rim.

This is the gauge sector. It exists, as a subspace of the master configuration space, *because* $H^1(C_6)$ is non-trivial; and it has the Wilson-loop interpretation *because* the Hodge theorem identifies harmonic representatives with cohomology classes.

Note on σ-sector persistence vs gauge sector persistence — different cohomology classes. A careful reader will notice that the σ-sector's persistent direction and the gauge sector's persistent direction live in *different* cohomology classes of the rim cycle, and this distinction is important for understanding what the framework does and does not do.

- The σ-sector's persistent direction is constant $\psi \in C^0(C_6)$, corresponding under the inverse carrier transformation to the alternating spoke mode $\lambda_i = c \cdot (-1)^i$ (the persistent direction established in the σ-sector papers). This direction is in $\mathrm{Ker}(\Delta^0)$ and represents $H^0(C_6) \cong \mathbb{R}$ — the *zeroth* cohomology, equivalent to the cycle's single connected component.
- The gauge sector's persistent direction is constant $\alpha \in C^1(C_6)$ — the harmonic 1-cochain $\alpha_0(e_i) = c$. This direction is in $\mathrm{Ker}(\Delta^1)$ and represents $H^1(C_6) \cong \mathbb{R}$ — the *first* cohomology, equivalent to the Wilson loop around the rim.

These are different cohomology degrees, living in different cochain spaces, and they are *not* related by the coboundary $\delta: C^0 \to C^1$. The constant $\psi \in C^0$ has $\delta\psi = 0 \in C^1$, which is the *zero* element of C^1, not the harmonic generator α_0.

The framework therefore does *not* lift the σ-sector's persistent direction into the gauge sector. More precisely, the framework embeds the σ-sector (modulo its kernel) inside the larger master configuration space $C^1(C_6)$ via $\delta: C^0/\mathrm{Ker}(\Delta^0) \to \mathrm{Im}(\delta)$; and it identifies a *separate* persistent direction $H^1(C_6)$ that lives in the complementary subspace $\mathrm{Ker}(\delta^*)$ — outside the image of the σ-sector under δ.

The σ-sector's persistent direction is *not* recovered as a non-zero element of the gauge sector; it is *separately* in $\mathrm{Ker}(\Delta^0)$ on the 0-cochain side. The gauge sector's persistent direction is a *new* persistent direction made available by extending from 0-cochains to 1-cochains.

The unification is therefore *complementary*, not derivative: the σ-sector and the gauge sector each have persistent directions in different cohomology classes (H^0 and H^1 respectively), and the master configuration space supports both by extending from C^0 to C^1. This is consistent with Theorem 3's framing of the σ-sector and gauge sector as complementary orthogonal subspaces of the master configuration space.

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6. Theorem 1: Dissipative Annihilation of Exact Modes

Theorem 1 (Dissipative annihilation of exact 1-cochains). *Under the master Hodge heat flow*

$\partial_{\tilde{\tau}} \alpha = -\Delta^1 \alpha$ *on* Λ_{master},

every exact 1-cochain $\alpha_{exact} \in \mathrm{Im}(\delta)$ *decays exponentially. Specifically, if* $\alpha(0) = \alpha_{exact} \in \mathrm{Im}(\delta)$*, then* $\alpha(\tilde{\tau}) \in \mathrm{Im}(\delta)$ *for all* $\tilde{\tau} \geq 0$*, and*

$\|\alpha(\tilde{\tau})\| \leq e^{-\tilde{\tau}} \cdot \|\alpha(0)\|$,

where the exponential rate is bounded below by the smallest non-zero eigenvalue of Δ^1 *on* $\mathrm{Im}(\delta)$*, which equals 1 for* C_6 *(the eigenvalue* $2 - 2\cos(2\pi/6) = 1$*).*

Equivalently, in the σ-sector envelope picture: the σ-sector envelope $\psi(\tilde{\tau})$ *relaxes exponentially toward its kernel direction (constant* ψ*), with the relaxation of* $\alpha_{exact} = \delta\psi$ *tracking the* *relaxation of ψ via δ. Exact 1-cochain modes correspond to non-constant envelope modes; they decay under the σ-sector dynamics.*

Proof. The subspace Im(δ) is invariant under the master flow: if $\alpha(0) \in \mathrm{Im}(\delta)$ and $\alpha(0) = \delta\psi(0)$, then $\alpha(\tilde{\tau}) = \delta(e^{(-\tilde{\tau}\Delta^0)}\, \psi(0)) \in \mathrm{Im}(\delta)$ for all $\tilde{\tau}$, since δ commutes with the heat semigroup (by $\delta\Delta^0 = \Delta^1\delta$).

The smallest non-zero eigenvalue of Δ^1 on Im(δ) is 1 (the second eigenvalue of L on C_6, since the zeroth eigenvalue 0 corresponds to $\mathrm{Ker}(\Delta^1)$ = harmonic, not to Im(δ)). The heat semigroup $e^{(-\tilde{\tau}\Delta^1)}$ on Im(δ) is therefore bounded by $e^{(-\tilde{\tau})}$ in operator norm, giving the stated decay. □

Remark. Theorem 1 is the rigorous statement of "exact transport modes dissipate" from the draft. It is exactly what one expects from the Hodge heat flow: exact forms decay at the rate set by the smallest non-zero eigenvalue of the Hodge Laplacian, while harmonic forms persist.

The bound rate $e^{(-\tilde{\tau})}$ corresponds to the next-to-largest envelope-Fourier mode (k = 1 and k = 5, eigenvalue 1). Higher-frequency modes decay faster (k = 2, 4 at rate $e^{(-3\tilde{\tau})}$; k = 3 at rate $e^{(-4\tilde{\tau})}$, but this is the alternating-ψ mode which is excluded from the admissible sector by closure-current conservation).

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7. Theorem 2: Invariance of Harmonic Modes

Theorem 2 (Exact invariance of harmonic 1-cochains). *Under the master Hodge heat flow*

$\partial_{\tilde{\tau}}\, \alpha = -\Delta^1 \alpha$ on Λ_{master},

every harmonic 1-cochain $\alpha_{harm} \in \mathrm{Ker}(\delta)$ is exactly invariant:*

$\alpha_{harm}(\tilde{\tau}) = \alpha_{harm}(0)$ for all $\tilde{\tau} \in \mathbb{R}_{\geq 0}$.

The harmonic subspace Ker(δ) $\cong H^1(C_6) \cong \mathbb{R}$ is one-dimensional, generated by the constant 1-cochain $\alpha_0(e_i) = 1$ (the Wilson-loop generator). The invariant value*

$c(\alpha) := \alpha_{harm} = (1/6)\, \Sigma_i\, \alpha(e_i)$

is the projection of α onto the harmonic subspace and equals (up to normalisation) the Wilson loop $\oint_C \alpha$ / circumference. It is conserved exactly under the flow.

Proof. For $\alpha_{harm} \in \mathrm{Ker}(\Delta^1) = \mathrm{Ker}(\delta^*)$, we have $\partial_{\tilde{\tau}}\, \alpha_{harm} = -\Delta^1\, \alpha_{harm} = 0$, so α_{harm} is exactly invariant.

The harmonic subspace is one-dimensional by §5.2; the constant 1-cochain α_0 is its generator. The orthogonal projection $P_{harm} : C^1(C_6) \to \mathrm{Ker}(\delta^*)$ is

$(P\_harm\ \alpha)(e_i) = (1/6)\ \Sigma_j\ \alpha(e_j)$ *(constant on all edges),*

and the invariant scalar is $c(\alpha) = (1/6)\ \Sigma_i\ \alpha(e_i)$. The Wilson loop is $\oint\_C\ \alpha = \Sigma_i\ \alpha(e_i) = 6\ c(\alpha)$, so $c(\alpha) = \oint\_C\ \alpha / 6$.

Under the flow, $\partial\_\tilde{\tau} \oint\_C\ \alpha = \oint\_C\ \partial\_\tilde{\tau}\ \alpha = - \oint\_C\ (\Delta^1\ \alpha) = 0$ (since Δ^1 is self-adjoint and the constant function is in its kernel: any element of $\mathrm{Im}(\Delta^1)$ is orthogonal to constants, hence integrates to zero). So $\oint\_C\ \alpha$ is conserved exactly. □

Remark. Theorem 2 is the rigorous statement of "cohomological transport classes persist" from the draft. The persistent direction is identified simultaneously as:

- the **kernel of the Hodge Laplacian Δ^1 on 1-cochains** (a Laplacian kernel — the σ-sector-style description);
- the **first cohomology $H^1(C_6)$ of the rim cycle** (a cohomology class — the persistent-gauge-sector-style description).

These are not two different objects related by a derivation; they are *the same one-dimensional subspace of $C^1(C_6)$ under two equivalent descriptions*, by the Hodge theorem (every cohomology class has a unique harmonic representative, and the harmonic representatives form the kernel of the Hodge Laplacian). This identification is what dissolves Obstacle 4.

The **Wilson-loop integral** $\oint\_C\ \alpha$ is the natural physical observable associated with this subspace: by Lemma 3.1 it is a cohomology invariant (depends only on the class $[\alpha] \in H^1$), and by Theorem 2 it is conserved exactly under the master flow. The Wilson loop is *not* a third description of the same subspace; it is the *conserved scalar observable* obtained by dual pairing of α with the rim 1-cycle C. The conserved scalar $c(\alpha) = \oint\_C\ \alpha / 6$ — the harmonic representative's constant value, equivalently the Wilson loop normalised by the rim circumference — is the gauge-invariant observable associated with the persistent direction, playing the role analogous to a Wilson loop in conventional gauge theory.

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8. Theorem 3: The Master Decomposition

Theorem 3 (Master decomposition into σ-sector and persistent gauge sector). *The master configuration space $C^1(C_6)$ decomposes orthogonally as*

$$C^1(C_6) = \Lambda\_\sigma \oplus \Lambda\_gauge,$$

where:

- $\Lambda\_\sigma := \mathrm{Im}(\delta) \subset C^1(C_6)$*, the **σ-sector**, of dimension 5, dynamically equivalent to the σ-sector envelope dynamics on* $C^0(C_6) / \mathrm{Ker}(\Delta^0) \cong \mathbb{R}^5$ *via the isomorphism* $\delta : C^0(C_6)/\mathrm{Ker}(\Delta^0) \to \mathrm{Im}(\delta)$*;*

- *Λ_gauge := Ker(δ)* ⊂ $C^1(C_6)$, the **persistent gauge sector**, of dimension 1, isomorphic to $H^1(C_6) \cong \mathbb{R}$.*

The master Hodge heat flow restricts to:

- *the σ-sector envelope dynamics on Λ_σ (with all modes decaying at rate ≥ 1);*
- *the trivial invariant dynamics on Λ_gauge ($\partial_{\tilde{\tau}}$ α_harm = 0).*

The σ-sector and persistent gauge sector are complementary orthogonal subspaces of the same master configuration space. The σ-sector dynamics relaxes exact modes; the persistent gauge sector is invariant. The two sectors are dynamically decoupled at this level: the master flow does not mix exact and harmonic components.

Proof. The orthogonal decomposition $C^1(C_6) = \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\delta^*)$ is the standard Hodge decomposition (§5.2). Orthogonality follows from ⟨δψ, α⟩ = ⟨ψ, δ*α⟩ = 0 *if* δ*α = 0 (i.e., α ∈ Ker(δ*)).

The σ-sector identification: the σ-sector envelope dynamics on $C^0(C_6)$ has kernel = constants ($\mathrm{Ker}(\Delta^0)$), and on the orthogonal complement $C^0(C_6) / \mathrm{Ker}(\Delta^0) \cong \mathbb{R}^5$, the map δ is an isomorphism onto Im(δ). The σ-sector envelope dynamics on this complement maps to the master dynamics on Im(δ) by δ (as in §5.1).

The persistent gauge sector identification: Λ_gauge = Ker(δ*) ≅ $H^1(C_6)$ by the Hodge theorem.

Dynamical decoupling: the master flow $\partial_{\tilde{\tau}} \alpha = -\Delta^1 \alpha$ preserves the Hodge decomposition because Δ^1 acts as zero on $\mathrm{Ker}(\Delta^1)$ and as a positive operator on Im(δ) (its complement). The two sectors evolve independently. □

Remark. Theorem 3 is the central structural statement of the paper. It establishes that the σ-sector and persistent gauge sector are *complementary subspaces* of a common master configuration space $C^1(C_6)$, with the master dynamics splitting cleanly between them.

This is a different conceptual move from the draft's "quotient" framing (which posited the persistent sector as a quotient of the σ-sector). The correct framing is *direct-sum decomposition*: the two sectors are orthogonal components of the master configuration space, not in a quotient relationship. The persistent gauge sector emerges as the *complement* of the σ-sector inside Λ_master, not as a quotient of the σ-sector.

The σ-sector envelope dynamics is therefore not the "parent" theory from which the persistent gauge sector emerges by quotienting. Both are equally fundamental components of the master dynamics; they happen to be complementary, with the σ-sector accessing the exact-cochain subspace and the persistent gauge sector accessing the harmonic subspace.

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9. Theorem 4: Emergent Asymptotic Gauge Equivalence

Theorem 4 (Emergent asymptotic gauge equivalence on the persistent sector). *Let $\alpha, \alpha' \in C^1(C_6)$ with $\alpha' = \alpha + \delta\chi$ for some $\chi \in C^0(C_6)$ (i.e., α and α' differ by an exact 1-cochain — "differ by a gauge transformation" in conventional terminology). Then under the master Hodge heat flow, the distance between their evolved configurations decays exponentially:*

$$\|\alpha(\tilde{\tau}) - \alpha'(\tilde{\tau})\| = \|e^{(-\tilde{\tau}\Delta^1)}\, \delta\chi\| \le e^{(-\tilde{\tau})} \cdot \|\delta\chi\| \text{ for all } \tilde{\tau} \ge 0,$$

so that they converge to identical asymptotic configurations as $\tilde{\tau} \to \infty$. The harmonic components of α and α' are equal (since $\delta\chi$ has no harmonic component), and the exact components differ by an amount that decays at rate $e^{(-\tilde{\tau})}$.

*This is an **emergent asymptotic gauge equivalence**: two configurations differing by an exact 1-cochain become indistinguishable under the master dynamics in the asymptotic limit. The equivalence is **emergent** (it arises from the dissipative dynamics, not from a fundamental redundancy of the dynamics) and **asymptotic** (it holds in the limit $\tilde{\tau} \to \infty$, not at all $\tilde{\tau}$ as fundamental gauge symmetry would require).*

Proof. $\alpha(\tilde{\tau}) = e^{(-\tilde{\tau}\Delta^1)}\, \alpha(0)$, and similarly for $\alpha'(\tilde{\tau})$. Then

$$\alpha(\tilde{\tau}) - \alpha'(\tilde{\tau}) = e^{(-\tilde{\tau}\Delta^1)}\, (\alpha(0) - \alpha'(0)) = e^{(-\tilde{\tau}\Delta^1)}\, (-\,\delta\chi) = -\,\delta\, (e^{(-\tilde{\tau}\Delta^0)}\, \chi),$$

since δ commutes with the heat semigroup. The norm bound follows from $\|e^{(-\tilde{\tau}\Delta^0)}\| \le e^{(-\tilde{\tau})}$ on the orthogonal complement of $\mathrm{Ker}(\Delta^0)$, which contains the relevant components.

The harmonic component of $\alpha - \alpha' = -\delta\chi$ is zero (since $\delta\chi \in \mathrm{Im}(\delta)$, orthogonal to $\mathrm{Ker}(\delta^*)$), so the harmonic components of α and α' are equal at all times. □

Remark on emergent vs fundamental gauge equivalence. Theorem 4 is the rigorous version of "gauge redundancy emerges from local transport equivalence" from the original draft. But the structural relationship it establishes is more limited than that phrasing suggests, and the comparison to conventional gauge symmetry needs care.

- **Conventional gauge symmetry** (e.g. electromagnetism). Two configurations α and $\alpha + \delta\chi$ are the *same* physical state, described by different mathematical representatives. The identification is made at the level of the action: the action is invariant under $\alpha \mapsto \alpha + \delta\chi$, so the two representatives are physically indistinguishable at every moment by construction.
- **Theorem 4 phenomenon** (this framework). Two configurations α and $\alpha + \delta\chi$ are *physically distinct* (they carry different exact-1-cochain content) and the dissipative master flow erases the distinction over time. This is dissipative equilibration of an observationally-relevant degree of freedom, not gauge equivalence in the conventional sense.

These are structurally different phenomena. The natural comparison is not "Theorem 4 is emergent gauge equivalence" but rather "Theorem 4 establishes that exact 1-cochain differences decay under the master flow; this is not the same structural relationship as fundamental gauge symmetry in conventional gauge theory." Whether the two phenomena are observationally equivalent in some limit, or whether the dissipative equilibration of Theorem 4 plays the *role* of gauge identification in the VERSF framework even though it has a different formal structure, is an open structural question and not settled by the present paper.

The conceptual significance of Theorem 4 is therefore best framed in conservative terms: it identifies a specific mathematical mechanism (dissipative decay of the exact subspace) that *might* give rise, in the long-time limit, to phenomena observationally similar to gauge identification; but the formal relation between dissipative equilibration and fundamental gauge identification is not established here.

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10. Which Obstacles Are Addressed

We summarise which of the four structural obstacles from the continuum-limit paper are addressed by the framework of this paper, and which remain.

Obstacle 4 (Kernel of Laplacian vs cohomology class): FULLY RESOLVED.

The Hodge theorem identifies $\mathrm{Ker}(\Delta^1)$ with $H^1(C_6)$ as the same one-dimensional subspace of $C^1(C_6)$ under two equivalent descriptions. The σ-sector-style description (Laplacian kernel) and the persistent-gauge-sector-style description (cohomology class) refer to literally the same object. The obstacle dissolves: there was never a genuine difference, only two languages describing the same harmonic 1-cochain subspace.

Obstacle 2 (Dissipative vs conservative): PARTIALLY ADDRESSED.

The persistent direction Λ_gauge is *invariant* under the master dissipative flow (Theorem 2: $\partial_{\tilde{\tau}} \alpha_{harm} = 0$). Invariance is a weak form of conservation: the harmonic component is conserved exactly, the dissipative dynamics does not act on it.

But invariance under a dissipative flow is not the same as having conservative Hamiltonian dynamics. The conventional persistent gauge sector supports propagating solutions, conserves a kinetic energy quantity, and has Hamiltonian structure. In the framework of this paper, the persistent direction has *no dynamics at all* under the master flow — it is static. To upgrade "invariant under dissipative flow" to "conservative under Hamiltonian flow" requires additional structure: a second-order temporal derivative term, a coupling to a Hamiltonian sector, or a separate variational principle on Λ_gauge.

The obstacle is therefore partially addressed: the persistent direction is *non-dissipative*, but it is not yet *dynamically conservative*. The full resolution requires identifying or postulating the Hamiltonian structure on Λ_gauge.

Obstacle 1 (Parabolic vs hyperbolic): PARTIALLY ADDRESSED, BY THE SAME OBSERVATION AS OBSTACLE 2.

The master Hodge heat flow on Λ_master is parabolic. The persistent gauge sector's invariance under this flow does not give it hyperbolic (wave-equation-like) dynamics on its own; it gives it *trivial* dynamics (constant in time). The conventional persistent gauge sector has wave-propagating dynamics; the framework of this paper gives static dynamics on the persistent subspace.

Upgrading static to hyperbolic requires the same additional structure as upgrading invariant to conservative: a second-order temporal derivative, a Hamiltonian coupling, or a separate dynamical principle on Λ_gauge.

Obstacle 3 (One-dimensional vs higher-dimensional): NOT ADDRESSED.

The rim cycle C_6 is one-dimensional. $H^1(C_6) \cong \mathbb{R}$ is one-dimensional. The persistent gauge sector identified in this framework is therefore one-dimensional: a single Wilson loop direction, not a higher-dimensional gauge field with curvature, anisotropy, or multiple cohomology classes.

To access higher-dimensional gauge structure, the framework would need to be extended from C_6 to higher-dimensional complexes (where H^1 has higher rank, H^2 becomes non-trivial, and higher-dimensional gauge fields can carry non-Abelian or topological content). The $K = 7$ architecture itself does not provide higher-dimensional spatial structure; the σ-sector continuum-limit paper made this point explicitly. The dimensional obstacle is structural to the σ-sector and is not resolved by the cochain-complex framework alone.

Summary table.

Obstacle	Status after this paper
4. Kernel of Laplacian vs cohomology class	Fully resolved (Hodge theorem identification)
2. Dissipative vs conservative	Partially addressed (invariance, not yet Hamiltonian)
1. Parabolic vs hyperbolic	Partially addressed (static persistent direction, not yet propagating)
3. 1D vs higher-D	Not addressed (structural to σ-sector)

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11. What This Establishes — and What It Does Not

What this paper establishes:

1. The σ-sector envelope dynamics is identical, after temporal rescaling, to the Hodge heat flow on 0-cochains of the rim cycle C_6 (§4).
2. The natural extension of this dynamics to 1-cochains is the Hodge heat flow on $C^1(C_6)$, generated by the Hodge Laplacian Δ^1 (§5.1).
3. The master configuration space $C^1(C_6)$ admits an orthogonal Hodge decomposition into exact and harmonic subspaces (§5.2).

4. The exact subspace Im(δ) carries the σ-sector dynamics and dissipates exponentially under the flow (Theorem 1).
5. The harmonic subspace Ker(δ*) ≅ $H^1(C_6)$ is exactly invariant under the flow and is one-dimensional, generated by the Wilson-loop 1-cochain (Theorem 2).
6. The σ-sector and persistent gauge sector are complementary orthogonal subspaces of the master configuration space, dynamically decoupled by the Hodge structure (Theorem 3).
7. Gauge equivalence in this framework is emergent and asymptotic: configurations differing by an exact 1-cochain become indistinguishable under the dissipative flow as $\tilde{\tau} \to \infty$, but are not exactly equivalent at finite $\tilde{\tau}$ (Theorem 4).
8. The fourth structural obstacle from the continuum-limit paper (Laplacian kernel vs cohomology class) is fully resolved by the Hodge theorem identification (§10).

What this paper does not establish:

- Conservative Hamiltonian dynamics on the persistent gauge sector. The framework gives static (invariant) dynamics on the harmonic subspace, not conservative propagating dynamics. The conventional persistent gauge sector's Hamiltonian structure requires additional input not provided here.
- Hyperbolic wave-equation dynamics. The master flow is parabolic; the persistent direction is invariant rather than propagating. To get hyperbolic dynamics on the persistent sector, a second-order temporal structure or Hamiltonian coupling is needed.
- Higher-dimensional gauge structure. The framework gives one-dimensional H^1 on the rim cycle, hence a single Wilson loop. Higher-rank gauge groups, non-Abelian structure, or higher-dimensional spatial domains are not accessed.
- Fundamental gauge symmetry. The gauge equivalence in this framework is emergent and asymptotic (Theorem 4); it is *not* the fundamental gauge symmetry of conventional gauge theory. Whether the two are observationally equivalent at large $\tilde{\tau}$ is an open question.
- Derivation of the master-action variational principle on $C^1(C_6)$. The extension of the σ-sector envelope dynamics to 1-cochains via the Hodge heat flow is *postulated* in §5.1; its derivation from the K = 7 constraint catalogue extended to 1-cochains is open.
- Identification with the broader VERSF persistent cohomological transport sector. The framework here gives a specific one-dimensional gauge sector on the rim cycle; whether this matches the persistent sector identified in other VERSF papers (with potentially different spatial domain, dynamics, and cohomological content) is an open question.

This is a framework paper. It establishes the cochain-complex setting in which the master-action unification programme can proceed rigorously, and addresses one of four structural obstacles fully and two more partially. **The achievement is *kinematic*: the gauge sector is identified as a one-dimensional invariant subspace of the master configuration space, with explicit cohomological structure. The achievement is *not dynamical*: the gauge sector has no propagating dynamics under the master flow, only static invariance. Gauge dynamics — propagating wave solutions, Hamiltonian structure, energy conservation — requires the upgrade flagged in P1 below.** The full unification — with conservative dynamics on the persistent sector, higher-dimensional structure, and connection to the conventional persistent gauge sector — is the subject of subsequent papers.

12. Relation to the Persistent Cohomological Transport Sector

The broader VERSF programme has, in separate work, identified a persistent cohomological transport sector with specific features: gauge redundancy from refinement equivalence, Maxwell-like transport on $H^1(G(\Lambda))$, conservative dynamics, hyperbolic propagation. The natural conjecture is that this sector and the persistent gauge sector identified in §§5–7 of this paper are different descriptions of the same physical object.

The framework of this paper supports a partial identification:

- Both sectors are one-dimensional gauge structures (in this paper, $H^1(C_6) \cong \mathbb{R}$; in the broader programme, the persistent gauge sector on appropriate cohomology).
- Both have the Wilson-loop interpretation (in this paper, the constant 1-cochain; in the broader programme, the holonomy around closed cycles).
- Both are characterised cohomologically (in this paper, by $H^1(C_6)$; in the broader programme, by $H^1(G(\Lambda))$ for the appropriate refinement complex).

The identification is incomplete in the following respects:

- The conventional persistent gauge sector has conservative Hamiltonian dynamics; the gauge sector in this paper has only static (invariant) dynamics. The Hamiltonian upgrade is required for full identification.
- The conventional persistent gauge sector is hyperbolic and supports wave-propagating solutions; the gauge sector here is static. Hyperbolic dynamics requires additional temporal structure.
- The conventional persistent gauge sector may live on higher-dimensional refinement complexes; the gauge sector here is restricted to the one-dimensional rim cycle.
- The "gauge redundancy" in the conventional persistent gauge sector is presumably *fundamental* (built into the action); the gauge equivalence here is *emergent and asymptotic* (arising from the dissipative dynamics). The relation between these two notions of gauge structure is itself an open question.

A full identification of the two sectors therefore requires:

1. Upgrading the master dynamics on Λ_gauge from static to Hamiltonian (addressing obstacles 1 and 2 fully).
2. Extending the cochain-complex framework from C_6 to higher-dimensional complexes (addressing obstacle 3).
3. Either showing that the emergent asymptotic gauge equivalence here recovers the fundamental gauge symmetry of the broader sector, or showing that they are distinct structures with the same observational content.

These are non-trivial pieces of subsequent work. The present paper does not address them.

13. Open Problems

P1 (load-bearing). Hamiltonian dynamics on the persistent gauge sector. The framework's choice of master dynamics — Hodge heat flow Δ^1 everywhere on $C^1(C_6)$, including the harmonic subspace — gives static dynamics on Λ_gauge. A different choice — postulating a Hamiltonian operator on Ker(δ*) alongside the dissipative Hodge flow on Im(δ) — would give non-trivial gauge dynamics with conservative (potentially hyperbolic) structure. The framework *permits* the Hamiltonian upgrade; whether it is *forced* by an extended K = 7 master-action principle, or requires independent structural input, is the substantive open question. Possible mechanisms include: (i) extending the K = 7 master-action variation to 1-cochains and seeing whether the resulting constraint catalogue forces a Hamiltonian-plus-dissipative structure rather than purely dissipative; (ii) coupling the σ-sector to a separate Hamiltonian sector elsewhere in the VERSF programme; (iii) postulating a second-order temporal structure as an additional principle. The choice between these has consequences for how the gauge dynamics relates to the rest of the programme.

P2. Derivation of the extended master dynamics on $C^1(C_6)$. The Hodge heat flow on 1-cochains is postulated in §5.1, not derived. Does the K = 7 master-action variational principle, extended naturally from 0-cochains (spoke envelopes) to 1-cochains (rim gauge fields), force this specific dynamics? Or does it admit additional terms (e.g., higher-derivative operators) that modify Theorems 1–4 at sub-leading order?

P3. Higher-dimensional generalisation. The rim cycle C_6 is one-dimensional. Higher-dimensional gauge structures (non-Abelian groups, curvature, multiple cohomology classes) require higher-dimensional complexes. What is the natural higher-dimensional extension of the K = 7 architecture? Does the σ-sector / persistent-sector decomposition extend to higher dimensions, and if so, how do the four obstacles transform?

P4. Structural relation between dissipative equilibration and fundamental gauge identification. Theorem 4 establishes that exact 1-cochain differences decay under the master dissipative flow. Conventional gauge theory identifies gauge-equivalent configurations at the level of the action (a different kind of identification: identification by definition, not by long-time dynamics). Are these two mechanisms — dissipative equilibration and fundamental identification — formally related? Does either reduce to the other in some structural limit? In particular: is there a sense in which the long-time limit of the master dissipative flow can be reformulated as a quotient by a fundamental gauge symmetry of a different action principle? This is the structural version of the question that the §9 remark left open. *No observational claim is made; $\tilde{\tau}$ in the present framework is a methodologically-chosen rescaling without an established physical interpretation, so deviations at "finite $\tilde{\tau}$" are not yet observational predictions.*

P5. Non-Abelian extension. The framework here is intrinsically Abelian ($H^1(C_6) \cong \mathbb{R}$ is one-dimensional and commutative). Non-Abelian gauge structure requires non-commutative cohomology and higher-dimensional gauge groups. The cohomological framework of this paper does not extend straightforwardly; a substantive structural extension is required.

P6. Coupling to matter sectors. The framework gives a gauge sector but not its coupling to charged matter. The conventional gauge sector couples via minimal coupling (covariant derivatives, gauge-covariant currents). What is the analogous coupling in VERSF, and is it forced by the $K = 7$ constraint catalogue extended to matter sub-sectors?

P7. Identification with the broader persistent cohomological transport sector. As discussed in §12, the gauge sector identified here is partially but not fully identified with the persistent cohomological transport sector of the broader VERSF programme. Completing this identification requires resolving P1, P3, and P4 above.

P8. Renormalisation-group structure. The master Hodge heat flow has a natural RG interpretation: long-time behaviour is dominated by the harmonic component, while short-time behaviour involves all modes. Does the σ-sector / gauge sector decomposition admit an RG interpretation, with the σ-sector as the "high-energy" component and the gauge sector as the "low-energy" component?

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14. Conclusion

The σ-sector and persistent cohomological transport sector of the VERSF programme appeared, in previous papers, to have opposite dynamical characters — the σ-sector dissipative and parabolic, the persistent sector conservative and hyperbolic. The σ-sector continuum-limit paper identified four structural obstacles to their identification.

This paper develops the cochain-complex framework in which the master-action unification can be precisely formulated. The σ-sector envelope dynamics is identified with the Hodge heat flow on 0-cochains of the rim cycle C_6; its natural extension to 1-cochains is the Hodge heat flow on $C^1(C_6)$. The Hodge decomposition splits the master configuration space orthogonally into exact 1-cochains (carrying the σ-sector dynamics, dissipating under the flow) and harmonic 1-cochains (isomorphic to $H^1(C_6)$, invariant under the flow).

The persistent direction is identified simultaneously as a Laplacian kernel and a cohomology class — the same one-dimensional subspace under two equivalent descriptions, by the Hodge theorem. The fourth structural obstacle dissolves: "kernel of Hodge Laplacian" and "first cohomology class" are not different objects, only different languages.

The first two structural obstacles (parabolic vs hyperbolic; dissipative vs conservative) are partially addressed: the persistent direction is invariant under the dissipative flow (hence non-dissipative), but it is not yet dynamically conservative or hyperbolic — it is static. Upgrading invariance to Hamiltonian dynamics requires additional structure beyond the dissipative master flow. The third obstacle (one-dimensional vs higher-dimensional) is structural to the σ-sector and is not addressed in this framework.

The framework also produces a structural observation: the dissipative master flow drives the exact subspace of configurations to zero in the asymptotic limit, leaving only the harmonic subspace observationally distinguishable at large $\tilde{\tau}$. Whether this dissipative equilibration phenomenon is formally related to fundamental gauge identification — which in conventional gauge theory is built into the action at every moment, not arrived at asymptotically — is a structural question (P4). No observational claims are made: $\tilde{\tau}$ in the present framework is a methodologically-chosen rescaling without an established physical interpretation.

The σ-sector and the persistent gauge sector are therefore not in opposition. They are complementary orthogonal subspaces of a common master configuration space, with dynamically decoupled behaviour: dissipative on the σ-sector subspace, invariant on the persistent gauge subspace. The master-action unification programme can now proceed within this framework, with the four obstacles individually addressable. Two of the obstacles remain genuinely open (the Hamiltonian upgrade and the dimensional lifting); they are the natural subjects of subsequent papers.

What is established: a rigorous framework for the master-action unification, with the cohomological structure of the persistent gauge sector made precise within the $K = 7$ architecture. What remains: the Hamiltonian upgrade of the persistent dynamics, the higher-dimensional extension, the non-Abelian generalisation, the coupling to matter, and the identification with the broader VERSF persistent transport sector. The unification is *opened* by this paper, not closed.