Lorentzian Completion of Transport-Generated Geometry in VERSF

Hyperbolic Causal Structure from Commitment–Coherence Compatibility, Cone-Compatible Metric Completion, and the Emergence of Effective Lorentzian Continuum Geometry

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General-Reader Summary

The previous paper in this programme showed that the VERSF substrate produces *directional* distortions of the geometry of space — a directional anisotropic correction sitting on top of an underlying uniform stretch. What it produced was, deliberately, only the geometry of *space*. It said nothing about *time*.

This paper completes that picture by showing how time enters — and, more importantly, why the geometry that combines them must be *Lorentzian*: the particular kind of geometry in which one direction (time) is treated fundamentally differently from the other directions (space), with a finite "speed limit" built into its structure.

The challenge is that we cannot simply *declare* the geometry Lorentzian. In ordinary physics, one writes down a minus sign in front of the time coordinate and moves on. In VERSF, time is not a primitive — it emerges from irreversible substrate events. So the minus sign has to come from somewhere: there must be a substrate reason why time looks different from space.

The reason developed in this paper is a compatibility condition that the substrate has to satisfy in order for *facts* to become stable. Roughly: for an irreversible commitment event in some region to actually become a stable fact, the region has to be able to coordinate with itself faster than it can break apart. If signals could move arbitrarily fast (no speed limit), this would be trivial; if signals could not move at all (rigid geometry), nothing would commit. The middle ground — *finite signal speed with consistent causal coordination* — is precisely what mathematicians call a hyperbolic propagation law, and hyperbolic propagation lives only on Lorentzian geometries. Riemannian (purely spatial) geometries cannot host it.

So the Lorentzian signature isn't put in by hand. It is *forced* by the requirement that facts can stably commit at all.

Once that signature is forced, the rest of the construction is essentially geometric bookkeeping. The directional spatial geometry built in the previous paper has to fit inside the Lorentzian geometry in exactly one way (up to a choice of time coordinate). The same anisotropic correction that bent paths through space now also tilts and squashes the cones of light — the invariant

signal cones that separate "can be reached" from "cannot be reached" at any point. A direction along which substrate distinguishability is more easily transported is also a direction along which the causal cone opens slightly wider. The geometry of cause-and-effect inherits the directional structure of the substrate.

What the paper does **not** do is derive Einstein's equations or say what the metric satisfies dynamically. That is the next step. What it *does* do is install the causal Lorentzian framework into which any future dynamical theory must fit, and it derives the wave operator that natively propagates substrate commitments through that framework. The open question is no longer "how can time enter *this layer* of VERSF geometry?" — that is settled here, for the layer this paper concerns — but rather "what dynamical equations govern the completed Lorentzian metric?" The broader question of time-emergence as a programme-wide matter is the subject of separate companion work and is not claimed to be closed here.

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Abstract

Previous papers in the programme established refinement-stable causal transport, tensorial transport curvature, conformal metric selection, and non-conformal anisotropic spatial metric corrections of the form

$$h_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij},$$

where $\hat{Q}_{ij}$ is the unique parity-even traceless quadratic contraction of the refinement-stable antisymmetric transport-curvature tensor $R^{(\infty)}_{ij}$. The present paper develops the **Lorentzian completion** of this spatial geometry.

The construction is *not* the routine addition of a time coordinate. We establish that, given the Causal–Coherence Compatibility (CCC) condition inherited from a companion programme, the continuum propagation law cannot be elliptic or parabolic and must be hyperbolic — and that the natural metric geometry supporting a hyperbolic second-order propagation operator is Lorentzian. Lorentzian signature therefore emerges as a *consequence* of substrate-level commitment dynamics, not an assumption.

The main results are:

1. **CCC selects strict hyperbolicity within the second-order metric-type class (Theorem 3.2).** Under finite admissible commitment-propagation speed and well-posed Cauchy evolution, the continuum propagation operator is strictly hyperbolic — *within* the second-order metric-type class; higher-order or non-metric-type completions are outside scope.

2. **Hyperbolic ⇒ Lorentzian signature (Theorem 3.4).** A second-order propagation operator of metric type is hyperbolic if and only if the underlying metric has Lorentzian signature.
3. **Cone-compatible completion (Theorem 4.2).** Given h_{ij} and the invariant admissible commitment-propagation speed $c_\star$, the leading Lorentzian metric is uniquely

   $ds^2 = - c_\star^2 \, d\tau^2 + h_{ij} \, dx^i \, dx^j$, up to a constant translation of the time coordinate. The speed $c_\star$ is inherited from the causal-cone sector, not inserted.

4. **CRE quotient covariance (Theorem 5.2).** Observable physics descends to equivalence classes of proto-histories under Commitment Reordering Equivalence, on which the proto-time ordering is operationally unobservable.
5. **Hyperbolic transport operator (Theorem 6.2).** The completed geometry carries a natural d'Alembertian $\Box_g$ of strictly hyperbolic type whose quasi-static limit reproduces the spatial transport geometry of the preceding paper.
6. **Determinant cancellation at O(λ) (Theorem 7.1).** The traceless correction $\hat{Q}_{ij}$ leaves the spatial volume element invariant to leading order — a direct algebraic consequence of $\mathrm{tr}(\hat{Q}) = 0$ — so the leading λ-correction to wave propagation is purely directional.
7. **Anisotropic cone deformation (Theorem 10.1).** The null cone of the completed metric is directionally deformed relative to the conformal cone whenever $\hat{Q}_{ij} \neq 0$; no scalar effective speed reproduces the deformation.
8. **Shift suppression at leading algebraic order (Proposition 11.1).** Within the parity-even algebraic sector, the leading shift vanishes; gradient-built O(λ) contributions are absorbed into a synchronous-foliation gauge, with substrate-derived suppression of the gradient sector remaining open.

The paper does **not** derive Einstein equations, a stress-energy tensor, the value of $c_\star$, or quantum corrections. Its contribution is structural: it completes the passage from refinement-stable spatial transport geometry to effective Lorentzian continuum geometry, and it identifies the causal-coordination origin of Lorentzian signature itself.

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1. Introduction

The preceding paper in the programme established that refinement-stable causal transport curvature generates non-conformal spatial metric corrections. The resulting spatial metric reads

$$h_{ij}(x) = \Omega^2(x) \, \delta_{ij} + \lambda \, \hat{Q}_{ij}(x),$$

where Ω^2 absorbs the trace-conformal content and $\lambda \, \hat{Q}_{ij}$ carries the genuinely anisotropic correction. The construction was deliberately restricted to the spatial sector: signature was Euclidean throughout, inherited from δ_{ij}, and the question of how the Lorentzian causal structure of physical geometry enters the picture was deferred.

The present paper closes that gap. It is not, however, the routine addition of a time coordinate.

The central methodological point. In ordinary physics, Lorentzian signature is *assumed*. One writes a minus sign in front of $d\tau^2$, extracts a wave operator, and proceeds. In VERSF, this move is unavailable: time is not a primitive of the substrate; it emerges through irreversible commitment events. Lorentzian signature must therefore come from somewhere — there has to be a substrate-level reason that the temporal direction is treated differently from the spatial directions, and a substrate-level reason that the difference takes the specific form of an indefinite metric with one negative eigenvalue.

That reason, developed below, is **Causal–Coherence Compatibility** (CCC), inherited from a companion programme. CCC states that stable irreversible facts can be produced only when coherent causal coordination across a region completes within the region's own causal crossing time — a condition that has both an upper bound (signals cannot be arbitrarily fast) and a lower bound (signals cannot be arbitrarily slow either, since the region would decohere before committing). The unique class of second-order continuum propagation laws compatible with both bounds, and admitting a well-posed Cauchy problem, is the hyperbolic class. The unique class of second-order metric geometries supporting a hyperbolic wave operator is the class of Lorentzian metrics.

This twin observation — *CCC forces hyperbolicity, and hyperbolicity forces Lorentzian signature* — is the spine of the paper. Everything else (the explicit form of the completion, the wave operator, the cone deformation by $\hat{Q}_{ij}$, the determinant cancellation, the geodesic compatibility) is bookkeeping once the spine is in place.

Bundle independence carries over. The parallel-emergence picture of the preceding paper is preserved here: the transport bundle whose holonomy generates $R^{(\infty)}_{ij}$ is not identified with the tangent bundle of the completed Lorentzian continuum. The two bundles couple only through the symmetric quadratic invariant $\hat{Q}_{ij}$ and through the cone-compatibility condition imposed in §4.

Inheritance, not reinvention. The Causal–Coherence Compatibility condition, the refinement-stable cone field, the invariant commitment-propagation speed $c_\star$, and the Commitment Reordering Equivalence quotient are all established in companion papers cited in §2; the present paper takes them as inputs and shows that their compatibility with the spatial metric h_{ij} is what produces a Lorentzian geometry.

Epistemic discipline. Results are labelled:

- **Proven** — established here under stated hypotheses, or in cited prior papers.
- **Conditional** — follows from proven results given an explicitly stated additional input (typically the value of $c_\star$, the zero-shift normalization, or the constancy of $c_\star$).
- **Conjectural** — plausibility argument only, flagged as such.

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2. Inherited Structures

We inherit, without re-derivation, the following structures from earlier papers in the programme.

2.1 Spatial transport-generated metric

From the non-conformal metric paper:

$$h_{ij}(x) = \Omega^2(x)\, \delta_{ij} + \lambda\, \hat{Q}_{ij}(x),$$

with

$$\Omega^2(x) := 1/\varepsilon_{gap}(x)^2 + (\lambda/D)\, \|R^{(\infty)}(x)\|_F^2,$$

$$\hat{Q}_{ij}(x) := Q_{ij}(x) - (1/D)\, \|R^{(\infty)}(x)\|_F^2 \cdot \delta_{ij},$$

$$Q_{ij}(x) := R^{(\infty)}{}_{ik}(x)\, R^{(\infty)}{}_j{}^k(x).$$

Here $R^{(\infty)}{}_{ij}$ is the refinement-stable antisymmetric causal transport-curvature tensor, and $\hat{Q}_{ij}$ is the unique parity-even non-conformal symmetric rank-2 correction at quadratic order in $R^{(\infty)}$ (Theorem 3.6 of the preceding paper).

2.2 Causal–Coherence Compatibility (CCC)

From the CCC programme: a continuum region $U \subset M$ of characteristic scale L and committed-record density ρ supports irreversible fact production only when the coherent-coordination scale

$$\chi(L) := \rho\, L^4 / (\hbar\, c_\star)$$

satisfies $\chi(L) \gtrsim 1$ — equivalently, when the region's causal crossing time

$$\tau_{cross}(U) := \mathrm{diam}_h(U) / c_\star$$

upper-bounds the commitment completion time

$$\tau_{commit}(U) \leq \tau_{cross}(U).$$

Here $c_\star$ is the invariant admissible commitment-propagation speed inherited from the causal-cone sector (the same $c_\star$ used throughout the present paper, expressed in the units of the CCC programme); its substrate derivation is open (§14, OP1) and outside the scope of this paper.

2.3 Refinement-stable cone field

From the continuum-limit programme: under stated regularity assumptions on TPB refinement sequences, the admissible commitment-propagation directions at each point $x \in M$ converge to a continuum cone field $C(x) \subset T_x M$. This is the refinement-stable invariant cone of admissible transport.

2.4 Commitment Reordering Equivalence (CRE)

From the observable-covariance programme: two proto-histories H_1 and H_2 are commitment-reordering equivalent, written $H_1 \sim_{CRE} H_2$, if they generate the same set of committed records, the same causal relation among committed records, the same invariant cone structure C(x), and the same transport-generated continuum geometry. Observable physics descends to the quotient

$$H_{obs} := H_{proto} / \sim_{CRE}.$$

2.5 Convention and notation

We work in $D \geq 2$ spatial dimensions throughout, with Lorentzian completion in D + 1 dimensions. Indices i, j, k, … denote spatial components (1 to D); indices μ, ν, … denote Lorentzian-frame components (0 to D), with 0 = τ. The convention ε_{gap} dimensionless and coordinates carrying length is inherited from §4.6 of the preceding paper. The dimensional triviality at D = 2 ($\hat{Q} \equiv 0$ by orbit counting; Proposition 3.5.1 of the preceding paper) carries over: the Lorentzian completion of the D = 2 spatial geometry produces a (2+1)-dimensional Lorentzian geometry with purely conformal spatial sector, no algebraic obstruction.

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3. The Causal–Coherence Origin of Lorentzian Signature

This section establishes the spine of the paper: CCC forces a hyperbolic propagation operator, and a metric of hyperbolic type is necessarily Lorentzian. Lorentzian signature is therefore not a postulate but a structural consequence of finite-speed commitment dynamics.

3.1 Motivation — why Riemannian geometry is insufficient [physical motivation]

The following structural-physical observation motivates the formal chain (Def 3.1 → Thm 3.2 → Thm 3.4) developed below; the *force* of the argument that signature must be Lorentzian comes from Theorems 3.2 and 3.4, not from this paragraph alone.

A purely Riemannian metric describes distances. It does not distinguish *can-influence* from *cannot-influence*: there is no preferred subset of directions in $T_x M$ along which information may propagate. But CCC requires precisely such a distinction — commitment events are stable only when they propagate within a causal cone. The continuum metric carrying observable physics must therefore distinguish *timelike/causally-connected* from *spacelike/causally-disconnected* directions. That is the role played by an indefinite metric of Lorentzian signature.

This motivates indefinite signature; the specialization to Lorentzian (as opposed to ultra-hyperbolic) is established in Theorem 3.4 below.

Definition 3.1 — Causal–Coherence Compatibility (operational form)

A continuum region $U \subset M$ satisfies causal–coherence compatibility if every stable commitment event supported in U can propagate its distinguishing record across U within the region's own causal crossing time:

$$\tau_{commit}(U) \leq \mathrm{diam}_h(U) / c_\star,$$

where h is the spatial transport-generated metric and $c_\star$ is the invariant admissible commitment-propagation speed.

Scope restriction — metric-type operators

We restrict to second-order linear differential operators of *metric type* — those whose principal symbol $\sigma(L)(x, k)$ is a quadratic form in k determined by an inverse metric — on the grounds that the spatial transport-Laplacian Δ_h of the preceding paper is the natural quasi-static limit of the completed propagation law (verified in Cor 6.3), and the Lorentzian completion must reduce to it. Higher-order operators (e.g. Dirac-like square roots, fourth-order completions) and non-metric-type operators are outside the scope of this paper and would require independent treatment.

Theorem 3.2 — CCC selects strict hyperbolicity within the second-order metric-type class [proven, conditional on CCC, well-posed Cauchy, and metric-type principal symbol]

Let L be a second-order linear differential operator of metric type governing the propagation of substrate commitment fields on the continuum. If

(H0) $\sigma(L)(x, k)$ is a quadratic form in k determined by an inverse metric (scope restriction above),

(H1) CCC holds in every sufficiently small region (finite admissible propagation speed $c_\star$),

(H2) the Cauchy problem for L is well-posed (commitment dynamics from given initial data is uniquely determined),

then L is strictly hyperbolic at every $x \in M$.

Proof. Classify second-order linear PDEs of metric type by the signature of their principal symbol $\sigma(L)(x, k)$ as a quadratic form on $T^*_x M$:

- *Elliptic* — σ is positive- or negative-definite. The Cauchy problem is ill-posed (Hadamard); equivalently, the domain of dependence is the entire manifold. Disturbances propagate with infinite effective speed. Excluded by (H1) and (H2).
- *Parabolic* — σ is degenerate, possessing a null direction. No finite-domain-of-dependence theorem of the strict-hyperbolic Friedrichs type applies; along the null direction the operator is characteristic and the Cauchy problem fails the standard well-

posedness criterion. Excluded by both (H1) (no finite signal speed in the strict sense) and (H2) (no well-posed Cauchy theory).

- *Ultra-hyperbolic* — σ is indefinite with $\min(p, q) \geq 2$ eigenvalues of each sign. The Cauchy problem is generically ill-posed: data on a non-characteristic hypersurface does not determine the solution uniquely. Excluded by (H2).
- *Strictly hyperbolic (Lorentzian type)* — σ is indefinite with signature (1, D) or (D, 1). The Cauchy problem is well-posed (Petrowski–Gårding); the domain of dependence is finite (Friedrichs); propagation respects the characteristic cone of σ.

Only the strictly hyperbolic class admits both finite propagation speed and a well-posed Cauchy problem. Hence (H0) ∧ (H1) ∧ (H2) ⇒ strict hyperbolicity.

Remark — what (H1) and (H2) encode

(H1) is the *physical* causal-coordination requirement: information cannot outrun the commitment-propagation speed. (H2) is the *consistency* requirement: given the substrate state at one proto-time slice, the subsequent commitment dynamics is determined. CCC supplies (H1) directly; (H2) is the standard expectation that substrate dynamics be deterministic at the level of admissible coarse-grainings. Neither is an arbitrary postulate within the programme.

Theorem 3.4 — Lorentzian signature [proven, conditional on metric origin of L]

Let L be a strictly hyperbolic second-order linear operator of metric type at $x \in M$, with principal symbol $\sigma(L)(x, k) = g^{\mu\nu}(x) k_\mu k_\nu$. Then $g^{\mu\nu}(x)$ has Lorentzian signature: exactly one eigenvalue of one sign and D eigenvalues of the opposite sign.

Proof. Strict hyperbolicity requires σ to be indefinite (excluding elliptic) and to have at most one eigenvalue of one sign (excluding ultra-hyperbolic, where minor characteristic-set components destroy uniqueness). The remaining possibilities are signatures (1, D) and (D, 1), which are physically equivalent under overall sign reversal of the metric. Both are Lorentzian.

Corollary 3.5 — Lorentzian completion is forced [proven]

Under CCC, well-posed Cauchy evolution, and the metric origin of the commitment-propagation operator, the continuum geometry on which observable physics is defined must be a Lorentzian metric whose spatial restriction (in any locally-defined synchronous frame) is the transport-generated metric h_{ij} of §2.1.

Structural interpretation

The argument inverts the conventional logic. In standard treatments, Lorentzian signature is *postulated*, and a wave equation is *derived*. Here, the physical requirement of *stable commitment dynamics* is postulated (via CCC), and Lorentzian signature is *derived* as the unique signature compatible with that requirement. The minus sign in ds^2 is not a convention; it is a fixed point of the substrate's causal-coordination structure.

4. Cone-Compatible Lorentzian Completion of the Spatial Metric

Given the spatial metric h_{ij} and the invariant commitment-propagation speed $c_\star$, we now identify the leading Lorentzian metric whose null cones coincide with the admissible causal cones inherited from the transport sector.

Definition 4.1 — Cone-compatible Lorentzian completion

A Lorentzian metric $g_{\mu\nu}$ on (D+1)-manifold $\tilde{M}$ is a *cone-compatible Lorentzian completion* of (M, h) at scale $c_\star$ if:

(C1) the spatial block agrees with the transport-generated metric: $g_{ij}(x, \tau) = h_{ij}(x)$ at leading order in λ, on each τ = const slice;

(C2) $g_{\mu\nu}$ has Lorentzian signature;

(C3) the null condition $g_{\mu\nu} v^\mu v^\nu = 0$ is equivalent to spatial propagation at invariant speed $c_\star$ with respect to h:

$$h_{ij} v^i v^j = c_\star^2 (v^0)^2.$$

Theorem 4.2 — Leading cone-compatible completion [proven, conditional on (S0) and (S1)]

Assume:

(S0) the ADM shift is taken to vanish at leading order in λ — either as a *synchronous-foliation gauge choice* (always available locally) absorbing any gradient-built $O(\lambda)$ shift contributions such as $\nabla^j \hat{Q}_{ij}$, or, on the substrate-derivation side, by Proposition 11.1 for the algebraic $O(\lambda)$ sector together with a separate gauge fixing of the gradient sector via the time-coordinate freedom of the CRE quotient;

(S1) the invariant speed $c_\star$ is spatially and temporally constant (the general spatially-varying case $c_\star = c_\star(x)$ is recorded in the remark following Corollary 4.3),

and choose τ to be the commitment-time coordinate inherited from the CRE quotient. Then the unique cone-compatible Lorentzian completion of h is

$$ds^2 = - c_\star^2 \, d\tau^2 + h_{ij} \, dx^i dx^j,$$

up to a constant translation $\tau \mapsto \tau$ + const of the time coordinate (and the trivial joint rescaling of τ and $c_\star$ that preserves the product $c_\star \, d\tau$).

Proof. Under (S0), the most general leading-order ADM metric reduces to

$$ds^2 = -N(x, \tau)^2 d\tau^2 + h_{ij} dx^i dx^j$$

with lapse $N(x, \tau) > 0$ to be determined. The null condition

$$g_{\mu\nu} v^\mu v^\nu = -N^2 (v^0)^2 + h_{ij} v^i v^j = 0$$

gives $h_{ij} v^i v^j = N^2 (v^0)^2$. Cone compatibility (C3) requires this to equal $c_\star^2 (v^0)^2$ for every null v with $v^0 \neq 0$, forcing $N(x, \tau) \equiv c_\star$ pointwise. Different lapse functions would shift the null cone away from the invariant cone of admissible commitment propagation. The remaining freedom is the reparametrization $\tau \mapsto \tau' = f(\tau)$. Such a reparametrization preserves the null cone (which is intrinsic) but changes the coordinate metric to $-(c_\star/f'(\tau))^2 d\tau'^2 + h_{ij} dx^i dx^j$; preservation of the metric form with $c_\star$ held fixed by the causal sector requires $f'(\tau) = 1$, i.e., a constant translation $\tau \mapsto \tau + \text{const}$. Allowing a joint rescaling of $c_\star$ to absorb $f'(\tau)$ recovers affine f and gives the residual freedom stated in the theorem.

Corollary 4.3 — Completed VERSF metric [proven, conditional on (S0)]

Substituting the spatial transport metric,

$$ds^2 = -c_\star^2 d\tau^2 + (\Omega^2(x) \delta_{ij} + \lambda \hat{Q}_{ij}(x)) dx^i dx^j.$$

This is the leading-order Lorentzian completion of the non-conformal transport-generated geometry.

Remark — spatially varying $c_\star$

If (S1) is relaxed and $c_\star = c_\star(x)$ carries spatial dependence inherited from the causal sector, the cone-compatibility argument still forces $N(x) = c_\star(x)$ pointwise, giving

$$ds^2 = -c_\star(x)^2 d\tau^2 + h_{ij} dx^i dx^j.$$

This is an acoustic-metric-like geometry whose null structure varies from point to point even at leading conformal order. The wave-operator analysis of §6 then carries non-trivial spatial derivatives of $c_\star$ that drop out under (S1). Spatially-varying $c_\star$ is not pursued further here; it would couple the causal-sector dynamics back into the metric construction at a depth outside the present scope.

Remark — what $c_\star$ is and is not

The speed $c_\star$ enters here as the invariant admissible commitment-propagation speed *inherited from the causal-cone sector* (§2.3). It is not a free parameter of the present construction, and it is not introduced as an external background constant. Its substrate derivation — in terms of the TPB update rate, coherence length, and finite-distinguishability-flow capacity — is the subject of an open programme item (§14, OP1). For purposes of this paper, $c_\star$ enters as a single positive scalar fixed by the causal sector, and the cone-compatibility condition then locks the lapse to that value.

Remark — signature-agnosticity of $\hat{Q}_{ij}$ carries over

The previous paper established that the contraction $Q_{ij} = R^{(\infty)}_{ik} R^{(\infty)}_{jl} \cdot (\text{raising metric})^{kl}$ is signature-agnostic at the algebraic level: replacing δ^{kl} by η^{kl} in the definition produces a symmetric tensor by the same Theorem-3.2-style argument, and the trace–traceless decomposition operates identically with η^{ij} in place of δ^{ij}. The Lorentzian completion above therefore embeds $\hat{Q}_{ij}$ without algebraic distortion: the spatial $\hat{Q}_{ij}$ sits in the spatial block of $g_{\mu\nu}$ exactly as it sat in h_{ij}.

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5. Observable Time and the CRE Quotient

This section makes precise the sense in which the coordinate τ of the completed metric is the *observable* commitment time — not the underlying proto-time of the substrate.

5.1 Proto-time versus observable time

VERSF distinguishes two notions:

- **Proto-time** — the ordering parameter of reversible pre-commitment substrate evolution. Internal to the substrate; carries information that may not survive the commitment quotient.
- **Observable (committed) time** — the parameter ordering irreversible commitment events as encoded in the committed-record sector. This is the time appearing in any operational measurement.

The two need not coincide. Proto-histories with distinct proto-time orderings may produce identical commitment records and identical causal relations; CRE then identifies them as observationally equivalent.

Definition 5.1 — Commitment Reordering Equivalence

Two proto-histories H_1, H_2 are commitment-reordering equivalent, $H_1 \sim_{CRE} H_2$, if they generate:

(R1) the same set of committed records, (R2) the same causal relation among committed records, (R3) the same invariant admissible cone structure $C(x)$, (R4) the same transport-generated continuum geometry (h_{ij} and $R^{(\infty)}_{ij}$).

Theorem 5.2 — Observable covariance under CRE [proven, conditional on inherited CRE structure]

Any quantity defined as a function of committed records and their causal relations descends to the quotient space $H_{obs} := H_{proto} / \sim_{CRE}$. The proto-time ordering does not so descend; it is therefore operationally unobservable.

Proof. By (R1)–(R2), a function f of committed records and their causal order takes the same value on any two histories in the same $\sim_{CRE}$ equivalence class. Hence f factors through the quotient. Conversely, the proto-time ordering is not invariant under arbitrary reorderings preserving (R1)–(R4) — that is the content of CRE — so no operational quantity built from committed records distinguishes proto-time-equivalent histories.

Corollary 5.3 — The completed metric is CRE-invariant [proven]

The Lorentzian metric $g_{\mu\nu}$ of Corollary 4.3 is *CRE-invariant*: it is a tensor field on the continuum manifold (with completed dimension D + 1) that depends on a proto-history only through its $\sim_{CRE}$ equivalence class. The coordinate τ is the observable commitment-time parameter — the image under CRE of any proto-time parameter compatible with the same committed records.

Concretely, $g_{\mu\nu}$ depends on the proto-history only through $R^{(\infty)}_{ij}$*, ε_gap, and c⋆*, each of which is preserved by $\sim_{CRE}$ under (R3)–(R4); hence $g_{\mu\nu}$ descends to H_{obs} by Theorem 5.2.

Remark — why this matters for the completion

Without the CRE quotient, the choice of τ in §4 would be ambiguous (which proto-time?) and the cone-compatibility condition would inherit that ambiguity. The CRE quotient collapses the proto-time freedom into a single observable τ — well-defined up to the time-coordinate redefinitions allowed by Theorem 4.2 — and the completion is therefore unique on the observable continuum, not merely on a particular proto-history representative.

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6. Hyperbolic Transport Operator on the Completed Geometry

The Lorentzian completion carries a natural second-order propagation operator. Its quasi-static limit reproduces the spatial transport geometry of the preceding paper, providing a consistency check.

Setup

For the diagonal-lapse completion of Corollary 4.3:

$g_{00} = - c_\star^2$, $g_{0i} = 0$, $g_{ij} = h_{ij}$,

$g^{00} = - c_\star^{-2}$, $g^{0i} = 0$, $g^{ij} = h^{ij}$,

$|g| = c_\star^2 |h|$.

Definition 6.1 — Lorentzian transport wave operator

For a scalar commitment-transport field φ on the completed continuum, define

$$\Box_g \varphi := |g|^{-1/2} \partial_\mu (|g|^{1/2} g^{\mu\nu} \partial_\nu \varphi).$$

Substituting the block form,

$$\Box_g \varphi = - |g|^{-1/2} \partial_\tau (|g|^{1/2} c_\star^{-2} \partial_\tau \varphi) + |g|^{-1/2} \partial_i (|g|^{1/2} h^{ij} \partial_j \varphi).$$

In particular, if $c_\star$ is spatially and temporally constant,

$$\Box_g \varphi = - c_\star^{-2} \partial_\tau^2 \varphi - c_\star^{-2} (\partial_\tau \ln \sqrt{|h|}) \partial_\tau \varphi + \Delta_h \varphi,$$

where

$$\Delta_h \varphi := |h|^{-1/2} \partial_i (|h|^{1/2} h^{ij} \partial_j \varphi)$$

is the Laplace–Beltrami operator of the transport-generated spatial metric.

Theorem 6.2 — Strict hyperbolicity [proven]

The principal symbol of $\Box_g$ at any (x, τ) is

$$\sigma(\Box_g)(k) = g^{\mu\nu} k_\mu k_\nu = - c_\star^{-2} k_0^2 + h^{ij} k_i k_j.$$

Under positive-definiteness of h^{ij} (Theorem 4.3 of the preceding paper, for the parameter range stated there) and $c_\star^2 > 0$, σ has signature (1, D). Hence $\Box_g$ is strictly hyperbolic.

Proof. Positive-definiteness of h^{ij} gives D positive eigenvalues to the symbol; the prefactor $-c_\star^{-2} < 0$ contributes one negative eigenvalue. Signature is therefore (1, D), which is the strict-hyperbolicity condition.

Corollary 6.3 — Quasi-static reduction [proven, conditional on slow-τ regime]

In the quasi-static regime $|\partial_\tau \varphi| \ll |\partial_i \varphi| \cdot c_\star$, the wave operator reduces to

$$\Box_g \varphi \approx \Delta_h \varphi.$$

The previous paper's spatial transport-Laplacian dynamics is therefore the static limit of the Lorentzian theory, providing a non-trivial consistency check between the spatial-geometry programme and its present Lorentzian completion.

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7. Leading-Order Expansion in λ

We track how the non-conformal correction $\lambda \hat{Q}_{ij}$ enters the wave operator and the volume element. A clean cancellation at first order makes the directional content of $\hat{Q}$ stand out cleanly.

Convention on the λ-expansion [structural remark]

"Leading order in λ" in this section means *leading order in the non-conformal correction*, with the trace-conformal background $h^{(0)}_{ij} = \Omega^2 \delta_{ij}$ treated as the resummed background. Strictly, Ω^2 itself contains a λ-dependent trace contribution $\Omega^2 = 1/\varepsilon_{gap}^2 + (\lambda/D) \|R^{(\infty)}\|_F^2$, inherited from §4 of the preceding paper. The resummation is consistent because the trace piece is conformally absorbable (Theorem 3.5 of the preceding paper) and the natural expansion of physical interest separates the conformal (resummed) sector from the anisotropic (non-resummed) sector. A reader attempting to verify the determinant cancellation by a uniform expansion in λ — treating Ω^2 as expanded — will pick up extra cross-terms that cancel against the corresponding pieces of the trace; the resummed convention is cleaner and matches the convention of the preceding paper.

Inverse spatial metric

By Lemma 6.1 of the preceding paper,

$$h^{ij} = \Omega^{-2} \delta^{ij} - \lambda \Omega^{-4} \hat{Q}^{ij} + O(\lambda^2).$$

Determinant expansion

For $h_{ij} = h^{(0)}_{ij} + \lambda k_{ij}$ with $h^{(0)}_{ij} = \Omega^2 \delta_{ij}$ and $k_{ij} = \hat{Q}_{ij}$, the standard expansion gives

$$\sqrt{|h|} = \sqrt{|h^{(0)}|} \cdot (1 + (\lambda/2) h^{(0)ij} k_{ij}) + O(\lambda^2) = \Omega^D \cdot (1 + (\lambda/2) \Omega^{-2} \delta^{ij} \hat{Q}_{ij}) + O(\lambda^2).$$

Since $\delta^{ij} \hat{Q}_{ij} = 0$ by tracelessness of $\hat{Q}$:

$$\sqrt{|h|} = \Omega^D + O(\lambda^2).$$

Theorem 7.1 — First-order determinant cancellation [proven]

The traceless anisotropic correction does not alter the spatial volume element at first order in λ:

$$\sqrt{|h|} - \sqrt{|h^{(0)}|} = O(\lambda^2).$$

Proof. Direct from the expansion above using $\delta^{ij} \hat{Q}_{ij} = 0$.

Structural significance

The cancellation is not a coincidence; it is a tautological consequence of the trace–traceless decomposition that defines $\hat{Q}$. The volume element is sensitive only to the *trace* of the perturbation, which $\hat{Q}$ is constructed to lack. Physically: the anisotropic correction redistributes propagation directions without changing local volume — exactly the operational meaning of *pure anisotropy without bulk dilation*.

Corollary 7.2 — First-order spatial wave-operator correction [proven]

To first order in λ,

$$\Delta_h \varphi = \Omega^{-D} \partial_i (\Omega^{(D-2)} \delta^{ij} \partial_j \varphi) - \lambda \Omega^{-D} \partial_i (\Omega^{(D-4)} \hat{Q}^{ij} \partial_j \varphi) + O(\lambda^2).$$

The first term is the conformal Laplacian on $(M, \Omega^2 \delta)$; the second is a directional second-order perturbation sourced by $\hat{Q}$. Anisotropic transport curvature therefore modifies spatial propagation through a *directional* second-order operator — the wave-operator image of the geodesic-deflection result of the preceding paper.

---

8. Null Geodesics and Causal Cone Compatibility

Null geodesics of the completed metric should propagate at the inherited admissible speed $c_\star$ and along the inherited cone field $C(x)$. Both checks are immediate at this order.

Definition 8.1 — Null transport geodesic

A curve $\gamma(s) = (\tau(s), x^i(s))$ is a *null transport geodesic* if

$$g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu = 0 \text{ and } \ddot{\gamma}^\mu + \Gamma^\mu_{\alpha\beta} \dot{\gamma}^\alpha \dot{\gamma}^\beta = 0.$$

For the completed metric, the null condition reads

$$- c_\star^2 \dot{\tau}^2 + h_{ij} \dot{x}^i \dot{x}^j = 0, \text{ i.e. } h_{ij} \dot{x}^i \dot{x}^j = c_\star^2 \dot{\tau}^2.$$

Theorem 8.2 — Null geodesics propagate at $c_\star$ [proven]

Null geodesics of the completed metric propagate at speed $c_\star$ with respect to the spatial metric h_{ij}.

Proof. From the null condition, assuming $\dot{\tau} \neq 0$,

$$h_{ij} (dx^i/d\tau)(dx^j/d\tau) = c_\star^2.$$

The left-hand side is the squared spatial speed in h-units; the right-hand side is $c_\star^2$.

Corollary 8.3 — Compatibility with the refinement-stable cone field [proven; consistency check, not an independent result]

If the continuum cone field $C(x) \subset T_x M$ is defined as the set of admissible commitment-propagation directions at speed $c_\star$ with respect to h, then the null cone of $g_{\mu\nu}$ coincides with $C(x)$ (extended trivially in the τ-direction) to leading order in λ. This consistency is built into the cone-compatibility condition (C3) of Definition 4.1; the corollary records the consistency explicitly rather than supplying an independent result.

Proof. Both cones are defined by the same speed-$c_\star$ condition on the same spatial metric h_{ij}.

Remark — what this guarantees

Cone compatibility (Corollary 8.3) is the *consistency* between the propagation cones defined by two independent constructions: the refinement-stable cone field inherited from the transport sector, and the null cone determined by the Lorentzian completion. The agreement is structural — it is built into the cone-compatibility condition (C3) of Definition 4.1 — but it is worth recording explicitly. A construction that *failed* this check would have placed the inherited cone field in geometric conflict with the metric carrying observable causality.

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9. Timelike Curves, Proper Commitment Time, and Time Dilation

Subluminal commitment trajectories inherit a proper-time structure and the associated time dilation — both as algebraic consequences of the Lorentzian completion, not as added postulates.

Definition 9.1 — Timelike admissible curve

A curve γ(s) is *timelike admissible* if

$g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu < 0$, i.e. $h_{ij} \dot{x}^i \dot{x}^j < c_\star^2 \dot{\tau}^2$.

This is the substrate condition that the committed record advances through the continuum at strictly subluminal admissible-transport speed.

Definition 9.2 — Proper commitment time

The *proper commitment time* along a timelike admissible curve is

$d\sigma^2 := d\tau^2 - c_\star^{-2} h_{ij} dx^i dx^j$,

$d\sigma = d\tau \cdot \sqrt{(1 - c_\star^{-2} h_{ij} v^i v^j)}$, where $v^i := dx^i/d\tau$.

Theorem 9.3 — Time dilation from admissible-cone confinement [proven]

For timelike admissible motion,

$0 < d\sigma \le d\tau$,

with equality if and only if $v^i = 0$.

Proof. Timelike admissibility gives $0 \le c_\star^{-2} h_{ij} v^i v^j < 1$, so the square root in Definition 9.2 lies in (0, 1], with the upper bound achieved iff $v^i = 0$.

Structural interpretation

Time dilation in this framework is not a postulated kinematic effect: it is the direct algebraic shadow of the requirement that committed records propagate inside the admissible causal cone. Equality $d\sigma = d\tau$ corresponds to spatial rest in the chosen τ-frame; strict inequality is forced by any spatial motion within the cone. The dilation factor $\sqrt{1 - c_\star^{-2} h_{ij} v^i v^j}$ is the *operational image* of the cone-compatibility condition (C3).

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10. Anisotropic Cone Deformation by $\hat{Q}_{ij}$

The non-conformal correction $\lambda \hat{Q}_{ij}$ does not merely bend geodesics in the spatial sector; it directionally deforms the null cone of the completed Lorentzian metric.

Setup

The null condition on the completed metric reads

$$c_\star^2 (v^0)^2 = h_{ij} v^i v^j = \Omega^2 \delta_{ij} v^i v^j + \lambda \hat{Q}_{ij} v^i v^j.$$

The first term is the conformal contribution (isotropic in v^i); the second is the directional perturbation.

Theorem 10.1 — Anisotropic cone deformation [proven, D ≥ 3]

Suppose $\hat{Q}_{ij}(x) \neq 0$ at some x and $\lambda \neq 0$. Then the null cone of $g_{\mu\nu}$ at x is anisotropically deformed relative to the conformal null cone (the cone of $g^{(0)}_{\mu\nu} = \mathrm{diag}(-c_\star^2, \Omega^2 \delta_{ij})$).

Proof. The conformal null cone at x is determined by $v \mapsto \Omega^2(x) \delta_{ij} v^i v^j$, depending only on the δ-norm $|v|_\delta^2$. With $\hat{Q} \neq 0$ at x, the corrected null condition is

$$\Omega^2(x) |v|_\delta^2 + \lambda \hat{Q}_{ij}(x) v^i v^j = c_\star^2 (v^0)^2.$$

Since $\hat{Q}(x)$ is a nonzero traceless symmetric tensor (which exists for $D \geq 3$ only — in $D = 2$, $\hat{Q} \equiv 0$ algebraically by Proposition 3.5.1 of the preceding paper), it has at least one positive and one negative eigenvalue. Choose eigenvectors u_0, w_0 for eigenvalues of opposite sign and rescale to common δ-norm $|u|_\delta = |w|_\delta$. Then

$$\hat{Q}(u, u) \neq \hat{Q}(w, w).$$

The corrected null condition therefore distinguishes u from w even though the conformal condition does not. The null cone is anisotropically deformed.

Corollary 10.2 — No scalar effective speed reproduces the deformation [proven, D ≥ 3]

There is no direction-independent function $c_{eff}(x)$ such that $h_{ij} v^i v^j = c_{eff}(x)^2 (v^0)^2$ on the corrected null cone.

Proof. If such c_{eff} existed, the corrected null condition would reduce to $\delta_{ij} v^i v^j \propto (v^0)^2$ with a direction-independent proportionality — i.e., an isotropic cone. This contradicts Theorem 10.1.

Interpretive remark — causal anisotropy at the substrate level

In the BCB reading of the programme, the cone deformation is operationally meaningful. Fix a unit δ-direction $\hat{u}$ and write the null condition as $s_\delta^2 (\Omega^2 + \lambda \hat{Q}(\hat{u}, \hat{u})) = c_\star^2$, so the δ-coordinate null speed reads $s_\delta = c_\star / \sqrt{(\Omega^2 + \lambda \hat{Q}(\hat{u}, \hat{u}))}$. Directions with *negative* $\hat{Q}(\hat{u}, \hat{u})$ — those along which the h-length of a unit δ-step is *smaller* than the conformal prediction, equivalently those along which substrate distinguishability is more easily transported in the h-metric — carry a slightly *larger* δ-coordinate null speed, i.e. the causal cone *opens slightly wider* in those directions. Conversely, positive-$\hat{Q}$ directions carry h-distances *longer* than the conformal prediction (transport is harder) and the corresponding δ-cone *narrows*. The geometry of admissible causation inherits the directional structure of the underlying admissibility profile for fact-commitment flow, with the sign correlation fixed: *easier transport ↔ wider δ-cone*. This is the Lorentzian-completion image of the direction-dependent commitment-cost remark of the preceding paper. (The statement "wider/narrower" is meaningful only in coordinates that distinguish h from δ — in h-units the null speed is $c_\star$ in every direction tautologically.)

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11. ADM Generalization and Shift Suppression

The leading construction of §4 imposed zero shift. The most general ADM metric has

$$ds^2 = - N^2 d\tau^2 + h_{ij} (dx^i + N^i d\tau)(dx^j + N^j d\tau),$$

with lapse $N(x, \tau)$ and shift vector $N^i(x, \tau)$. We now justify (S0).

Proposition 11.1 — Leading-algebraic-order shift suppression [proven for algebraic O(λ); gradient contributions remain]

Suppose:

(P1) the observable continuum is defined by the CRE quotient (no preferred proto-time-dependent spatial drift frame at leading order), and

(P2) the substrate dynamics generating the metric correction at order λ is *parity-even* and built algebraically (no derivatives) from $\hat{Q}_{ij}$ at this order.

Then there is no nonzero shift vector N^i at algebraic $O(\lambda)$: $N^i = 0$ at this order.

Proof. A shift vector is a rank-1 tensor field. The available algebraic ingredients at order λ are: the spatial metric δ_{ij}, the scalar Ω, and the symmetric traceless tensor $\hat{Q}_{ij}$. From these, by parity-even algebraic combinations alone, the only rank-1 quantity that can be built is the zero vector: combining $\hat{Q}_{ij}$ with itself yields rank-2 objects ($\hat{Q}^2 = \hat{Q}_{ik} \hat{Q}_j{}^k$), with δ yields a scalar ($\mathrm{tr}(\hat{Q}) = 0$), and with vectors requires an additional input vector that is unavailable at this order. Hence $N^i = 0$ at algebraic $O(\lambda)$ under (P1)–(P2).

Remark — what is not suppressed

Proposition 11.1 is an *algebraic* statement at the leading order in λ. It does not rule out:

- *Gradient corrections*: a parity-even, rank-1, order-λ contribution to the shift built from derivatives of $\hat{Q}_{ij}$ — most directly, $N^i \propto \lambda \nabla^j \hat{Q}_{ij}$ — is *not* suppressed by the algebraic argument above. This piece is genuinely $O(\lambda)$ and survives in a generic foliation. It enters at sub-leading order in a gradient expansion but is *not* subleading in λ. The leading completion of §4 absorbs this contribution into a synchronous-foliation gauge choice (the time-coordinate freedom of the CRE quotient), making the *gauge condition* (S0) hold by construction rather than the *substrate-derived* result. The gauge is always available locally; whether a substrate-level derivation suppresses the gradient piece independently of gauge choice is open (§14, OP3).
- *Higher-order terms*: $N^i = O(\lambda^2)$ is consistent with the proposition; only the $O(\lambda)$ algebraic vanishing is claimed.
- *Parity-odd extensions*: introduction of an orientation, a Levi-Civita symbol, or a preferred axis in the substrate would supply additional rank-1 invariants and could populate the shift at $O(\lambda)$.

The proposition is therefore best read as: *within the parity-even algebraic sector to which the preceding paper's uniqueness theorem also applies, the leading algebraic shift vanishes*; the gradient $O(\lambda)$ sector is gauge-fixed by the synchronous-foliation choice underlying (S0) of Theorem 4.2.

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12. Relation to Future Gravitational Dynamics

The present paper develops kinematics and causal geometry only. It does *not* derive field equations for $g_{\mu\nu}$. What it does is install the kinematic framework in which such equations would be formulated.

The completed metric

$$g_{\mu\nu} = \mathrm{diag}(-c_\star^2, \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij})$$

is a deterministic functional of refinement-stable transport observables (ε_{gap}, $R^{(\infty)}_{ij}$). A future dynamical paper would need to identify a principle — variational, fixed-point, or otherwise — selecting an equation of the schematic form

$$G_{\mu\nu}[g] + \Lambda g_{\mu\nu} = 8\pi G \cdot T^{eff}_{\mu\nu},$$

where $T^{eff}_{\mu\nu}$ is sourced by some combination of committed-record density, causal-coherence flux, transport curvature, and matter coupling. The present paper makes no claim that $g_{\mu\nu}$ of the form above will solve such an equation, and no claim about which equation is the right one.

What can be said is structural: the *form* of the spatial-metric correction at this order is already fixed (Theorem 3.6 of the preceding paper), so the dynamical programme is not searching for the form of the spatial sector — only for the equation it satisfies and the value of the coupling λ. This is a substantially more constrained problem than it would be without the algebraic-uniqueness result already in hand.

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13. Epistemic Status

Result	Status
Spatial metric $h_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$	inherited from preceding paper
CCC finite-causal-coordination condition	inherited
Refinement-stable cone field C(x)	inherited
CRE quotient $\sim_{CRE}$	inherited
Necessity of indefinite signature (§3.1)	structural necessity argument
CCC ⇒ hyperbolic propagation (Thm 3.2)	proven, conditional on metric-type principal symbol and well-posed Cauchy

Result	Status
Hyperbolic ⇒ Lorentzian signature (Thm 3.4)	proven
Cone-compatible Lorentzian completion (Thm 4.2)	proven, conditional on (S0) [algebraic O(λ) by Prop 11.1; gradient O(λ) by synchronous-foliation gauge] and (S1) [constant $c_\star$]
CRE quotient covariance (Thm 5.2)	proven, conditional on inherited CRE
Hyperbolic wave operator $\Box_g$ (Thm 6.2)	proven
Quasi-static reduction to Δ_h (Cor 6.3)	proven, conditional on slow-τ regime
First-order determinant cancellation (Thm 7.1)	proven
Null geodesics propagate at $c_\star$ (Thm 8.2)	proven
Cone-field compatibility (Cor 8.3)	proven, conditional on inherited cone definition
Time dilation (Thm 9.3)	proven
Anisotropic cone deformation by $\hat{Q}$ (Thm 10.1)	proven for $D \geq 3$
Shift suppression at algebraic O(λ) (Prop 11.1)	proven for parity-even algebraic sector
Substrate derivation of $c_\star$	open (OP1)
Einstein-type field equations	open (OP4)
Effective stress-energy tensor	open (OP5)
Quantum fluctuations of $g_{\mu\nu}$	open (OP6)

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14. Limitations and Open Problems

The paper does not derive:

- Einstein equations or any field equations for $g_{\mu\nu}$,
- a stress-energy tensor or matter coupling,
- the value of $c_\star$ from substrate dynamics,
- the substrate origin of the lapse normalization beyond $N = c_\star$ at leading order,
- nonzero shift vectors (algebraic O(λ) suppressed by Proposition 11.1; gradient O(λ) present but absorbed into a synchronous-foliation gauge; possibly also present at $O(\lambda^2)$ or in the parity-odd sector),
- quantum corrections to $g_{\mu\nu}$,
- gradient-order corrections to the wave operator and metric.

Specific open problems:

OP1 — Substrate derivation of $c_\star$. The invariant commitment-propagation speed is inherited from the causal sector. A complete derivation should express $c_\star$ in terms of the TPB update rate, the substrate coherence length, the finite-distinguishability-flow capacity, and the irreversibility threshold. Until OP1 is closed, $c_\star$ appears in the completion as a single free scalar (one fewer than the spatial-only theory but one more than the eventual full derivation will leave).

OP2 — Dynamical lapse. Theorem 4.2 fixes $N \equiv c_\star$ at leading order. In a fully dynamical completion, lapse may vary: $N = N(x, \tau)$. Identifying the substrate origin of lapse variation — whether it tracks commitment-density gradients, gravitational-potential analogues, or both — is the natural next problem.

OP3 — Shift sector. Proposition 11.1 suppresses N^i at algebraic $O(\lambda)$ within the parity-even sector. The gradient piece $N^i \propto \lambda \nabla^j \hat{Q}_{ij}$ is parity-even, rank-1, and order-λ; it is *not* suppressed by the algebraic argument and is presently absorbed by gauge choice (synchronous foliation, §11). Whether substrate dynamics suppress the gradient piece independently of gauge — or whether the parity-odd / orientation sectors populate the shift at $O(\lambda)$ — is open.

OP4 — Einstein-type field equations. What dynamical principle selects an equation governing $g_{\mu\nu}$? Candidate routes: a second-order action quadratic in transport curvature (in which case the form is heavily constrained by the algebraic-uniqueness theorem of the preceding paper); a renormalization-group fixed-point condition on admissible coarse-grainings; or a variational principle on the CRE quotient.

OP5 — Effective stress-energy tensor. What combination of committed-record density ρ, coherence flux, $R^{(\infty)}_{ij}$, and $\hat{Q}_{ij}$ sources the right-hand side of any such equation? The paper supplies only the kinematic stage; the source structure is open.

OP6 — Quantum fluctuations. The completed metric is a deterministic functional of refinement-stable observables. The quantum-substrate fluctuations of the underlying transport sector — and hence of $g_{\mu\nu}$ itself — are not characterized here.

OP7 — Gluing across regions failing CCC. CCC is a local condition; regions violating $\chi(L) \gtrsim 1$ cannot host stable commitment dynamics. The continuum geometry near such regions is undefined in the present construction. The gluing between CCC-compatible and CCC-incompatible regions is the geometric analogue of horizon physics and warrants separate treatment.

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15. Conclusion

This paper completes the transition from refinement-stable spatial transport geometry to effective Lorentzian continuum geometry in VERSF.

The preceding paper established the spatial metric

$$h_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}.$$

The present paper shows that, given the Causal–Coherence Compatibility condition and well-posed Cauchy evolution inherited from companion programmes, the continuum propagation law must be strictly hyperbolic *within the second-order metric-type class* (Theorem 3.2), and a strictly hyperbolic propagation law of metric type forces Lorentzian signature (Theorem 3.4). Lorentzian signature is therefore derived — within that scope — not assumed. The cone-compatibility condition (Definition 4.1) then locks the leading completion uniquely (Theorem 4.2) to

$$ds^2 = - c_\star^2 \, d\tau^2 + (\Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}) \, dx^i dx^j,$$

up to redefinition of the observable commitment-time coordinate and modulo shift contributions suppressed at algebraic $O(\lambda)$ by Proposition 11.1.

Eight structural results secure the construction:

- **Hyperbolic necessity (Thm 3.2)** — CCC plus well-posed Cauchy forces strict hyperbolicity.
- **Lorentzian signature (Thm 3.4)** — strict hyperbolicity of a metric-type operator forces signature (1, D).
- **Cone-compatible completion (Thm 4.2)** — the leading Lorentzian metric is unique up to constant translation of the time coordinate (and the trivial joint rescaling of τ and $c_\star$ that preserves $c_\star \, d\tau$).
- **CRE covariance (Thm 5.2)** — observable physics descends to the quotient; proto-time is unobservable.
- **Hyperbolic wave operator (Thm 6.2)** — the natural d'Alembertian is strictly hyperbolic; its quasi-static limit is the previous paper's spatial transport Laplacian.
- **Determinant cancellation (Thm 7.1)** — tracelessness of $\hat{Q}$ makes the leading volume-element correction vanish at $O(\lambda)$; the directional content is therefore *pure anisotropy without bulk dilation*.
- **Anisotropic cone deformation (Thm 10.1)** — $\hat{Q}_{ij}$ directionally tilts the null cone; no scalar effective speed reproduces the deformation (Cor 10.2).
- **Shift suppression (Prop 11.1)** — within the parity-even *algebraic* sector, the leading shift vanishes; gradient-built $O(\lambda)$ contributions are absorbed into a synchronous-foliation gauge, with substrate-derived suppression of the gradient sector remaining open.

The spine of the construction is the inversion of conventional logic: Lorentzian signature is *forced* by the requirement of stable substrate commitment dynamics, rather than postulated as a kinematic backdrop. The minus sign in ds^2 is a fixed point of the substrate's causal-coordination structure, not a convention.

Taken together with the algebraic-uniqueness theorem of the preceding paper, the present construction closes the *kinematic and causal* side of the geometry programme at this order. The form of $g_{\mu\nu}$ is fixed up to two scalars (λ from the spatial sector, $c_\star$ from the causal sector), and even these are heavily constrained — λ by dimensional analysis to $\sim \ell_\star^4$, and $c_\star$ by its substrate identification as the invariant admissible commitment-propagation speed.

The structural question at this layer of the programme is therefore no longer

"How can time enter this layer of VERSF geometry?"

— that is settled here, for the Lorentzian completion of the spatial transport metric — but

"What dynamical equations govern the completed Lorentzian metric $g_{\mu\nu}$, and what effective stress-energy tensor is sourced by irreversible commitment flow?"

That is the subject of the next paper. The broader question of time-emergence as a programme-wide matter — including the substrate origin of $c_\star$ (OP1), the proto-time / observable-time relationship beyond the CRE quotient used here, and the gluing across CCC-violating regions (OP7) — is the subject of separate companion work and is not claimed to be closed by the present construction.