

Microscopic Origin of Refinement Transport in VERSF

Simplicial Commitment Foam, Discrete Holonomy Flow, Closure-Compatible Refinement Dynamics, and the Substrate Construction of the Continuum Refinement Transport Operator

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General-Reader Summary

The recent geometric papers of the VERSF programme established gravity as the continuum-limit shadow of irreversible commitment transport. The Lorentzian Completion paper produced an emergent Lorentzian continuum. The transport-geometry papers produced refinement-stable parallel transport on that continuum. The Einstein–Hilbert Emergence paper showed that the continuum connection, the Riemann curvature tensor, the invariant volume form, and the Einstein–Hilbert action itself all emerge as continuum-limit images of refinement-stable transport on the substrate.

However, one structural object was still being inherited rather than constructed: the refinement transport operator T_γ itself. Earlier papers established its continuum-limit behaviour — Cauchy-type convergence under refinement, Levi-Civita parallel transport in the limit — but did not construct T_γ from the underlying discrete commitment dynamics at finite refinement order. The Einstein–Hilbert Emergence paper labelled this gap OP2 (the existence question — what is T_γ as an explicit construction from void combinatorics) and OP8 (the stability question — why does it converge under refinement).

That is the gap this paper closes.

The central claim is that the refinement transport operator T_γ arises as discrete holonomy on the $K = 7$ simplicial commitment foam — the substrate architecture established in the foundational papers of the programme. Each oriented commitment edge carries a transport operator U_{ij} ; minimal simplicial loops impose closure constraints; and refinement of the foam produces a Cauchy-type sequence of finite-order transport operators whose limit, where it exists, is the continuum T_γ used by the geometric programme.

The paper establishes that this construction works: T_γ exists as a continuum limit, is path-reparametrisation invariant, and reproduces the Levi-Civita parallel transport of the inherited Lorentzian Completion geometry. The first-order generator of T_γ in the continuum limit coincides exactly with the $\Gamma^{\alpha}_{\beta\mu}$ of the Einstein–Hilbert Emergence paper (its Definition 3.4 and Theorem 3.5), closing OP2 of that paper. Closure compatibility of triangle loops forces

torsion-freeness directly at the discrete level. Refinement stability follows from BCB / TPB closure architecture and finite distinguishability, closing OP8.

The discrete construction also produces, almost as a byproduct, two further results. First, the substrate-level causal-coherence structure on the commitment foam carries the Lorentzian signature of the continuum-limit metric through coarse-graining; this is compatibility with — not re-derivation of — the Lorentzian Completion paper's continuum geometry. Second, irreversibility of commitment events means that earlier closure structure cannot be fully erased by later transport, generating a small but non-zero nonlocal memory kernel whose continuum-limit form is the bilocal kernel $K(x, x')$ of the memory sector of the variational paper. The matter sector, the closure-normalisation factor C_λ , and the full quantum completion remain open and are recorded separately.

The result therefore unifies two previously semi-independent strands of the VERSF programme: the simplicial-foam / BCB / TPB strand on one side, and the continuum-transport-geometry strand on the other. The microscopic substrate construction of T_γ on the $K = 7$ foam now sits below the continuum-limit emergence cascade Theorem 3.5 \rightarrow 4.1 \rightarrow 5.2 \rightarrow (6.2, 6.3) \rightarrow 7.1 \rightarrow 8.2 of the Einstein–Hilbert Emergence paper, with each layer derived rather than postulated.

The paper does not close: the closure-normalisation factor C_λ , the matter-coupling sector, the quantum completion of refinement transport, the substrate derivation of the κ / memory-amplitude / anisotropic Wilson coefficients, or subleading higher-curvature corrections. These are explicitly recorded as residual open problems and structurally inherit from the geometric programme.

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Abstract

The Einstein–Hilbert Emergence paper of the VERSF programme established the continuum connection $\Gamma^{\alpha}_{\beta\mu}$, the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$, the Ricci scalar R , the invariant volume form $\sqrt{|g|} d^{(D+1)}x$, the Einstein–Hilbert action \mathcal{S}_{EH} , and substrate-level GR recovery as continuum-limit consequences of refinement-stable transport on the emergent Lorentzian continuum. However, the refinement transport operator T_γ underlying that cascade was inherited from the transport-geometry programme rather than constructed from substrate combinatorics. The Einstein–Hilbert Emergence paper labelled this OP2 (the existence question) and OP8 (the stability question).

The present paper closes both.

We construct T_γ explicitly as discrete holonomy on the $K = 7$ simplicial commitment foam — the closure architecture established in the foundational VERSF papers. Edge transport operators U_{ij} , oriented by commitment ordering and constrained by the $K = 7$ closure structure, generate path holonomies that admit a refinement-Cauchy structure under closure-compatible refinement of the foam. The continuum limit T_γ exists, is path-reparametrisation invariant, and coincides with the Levi-Civita parallel transport of the inherited Lorentzian Completion geometry.

The paper establishes six principal results.

(i) Substrate Construction of T_γ (§§3–5). The refinement transport operator is constructed explicitly as the continuum-limit holonomy on the $K = 7$ commitment foam, with refinement stability following from BCB / TPB closure architecture and finite distinguishability.

(ii) Levi-Civita Emergence (§6, Theorem 6.2). The first-order generator of T_γ in the continuum limit coincides exactly with the $\Gamma^\alpha_{\beta\mu}$ of Einstein–Hilbert Emergence paper Definition 3.4 / Theorem 3.5, with torsion-freeness following directly from triangle-closure compatibility at the discrete level (Lemma 6.3).

(iii) Riemann Emergence at Substrate Level (§7, Theorem 7.2). The discrete plaquette curvature, defined as the leading non-trivial part of loop holonomy on minimal simplicial loops — second-order in the edge length, since the first-order term vanishes by triangle closure (Lemma 6.3) — converges in the continuum limit to the Riemann curvature tensor $R^\alpha_{\beta\mu\nu}$ of the inherited Levi-Civita geometry, supplying the substrate underpinning of Einstein–Hilbert Emergence paper Theorem 4.1.

(iv) Lorentzian Compatibility (§8, Theorem 8.2). The commitment-order causal structure on the $K = 7$ foam carries the Lorentzian signature of the inherited continuum-limit metric through coarse-graining. The present paper does not re-derive Lorentzian completion; it establishes the substrate-level compatibility statement that the Lorentzian Completion paper's continuum metric is consistent with the discrete commitment-order structure.

(v) Memory Kernel Emergence (§10, Theorem 10.2). Irreversibility of commitment events generates a small but non-zero residual closure-history operator at the substrate level, whose coarse-grained continuum-limit form is the bilocal memory kernel $K(x, x')$ of the variational paper's memory sector. The substrate construction therefore supplies the microscopic origin of memory geometry.

(vi) Microscopic GR Recovery (§11, Theorem 11.1). In the weak-memory, weak-anisotropy limit inherited from the variational paper, the substrate-derived $\mathcal{S}_{\text{VERSF}}$ reduces continuously to standard general relativity, with the geometric operator basis sourced from the $K = 7$ commitment foam and the field equations recovered through the Einstein–Hilbert Emergence cascade.

The paper therefore closes OP2 and OP8 of the Einstein–Hilbert Emergence paper and unifies the simplicial-foam / BCB / TPB strand of the VERSF programme with the continuum-transport-geometry strand.

The paper does **not** derive:

- the closure-normalisation factor C_λ ,
- the explicit values of the memory-amplitude / anisotropic Wilson coefficients,
- Standard-Model matter coupling,
- quantum fluctuations of T_γ or of the commitment foam itself,
- subleading higher-curvature corrections beyond leading admissible order,
- the substrate-derived form of the transport group \mathcal{G} (the identification $\mathcal{G} = \text{SO}(1, D)$ is inherited from the Lorentzian Completion paper).

Each is recorded as an explicit open problem.

The contribution is structural: it supplies the first microscopic substrate construction of refinement transport within the VERSF programme, closing the residual existence-and-stability gap below the continuum-limit geometric cascade.

1. Introduction

The VERSF gravity programme has established, in successive layers:

- emergent Lorentzian continuum geometry (Lorentzian Completion paper),
- refinement-stable parallel transport on the emergent continuum (transport-geometry papers),
- effective stress–energy structure (source-structure paper),
- Einstein-compatible field equations (dynamical-geometry paper),
- a unified covariant action $\mathcal{S}_{\text{VERSF}}$ (variational paper),
- substrate emergence of the Einstein–Hilbert action (Einstein–Hilbert Emergence paper).

The Einstein–Hilbert Emergence paper closed the continuum-limit emergence cascade for the geometric sector: the continuum connection $\Gamma^\alpha_{\beta\mu}$ (its Definition 3.4 and Theorem 3.5), the Riemann curvature tensor $R^\alpha_{\beta\mu\nu}$ (Theorem 4.1), the Ricci scalar R (Theorem 5.2), the invariant volume form $\sqrt{|g|} d^{(D+1)}x$ (Theorem 6.2 and Proposition 6.3), the Einstein–Hilbert action \mathcal{S}_{EH} (Theorem 7.1), and substrate-level GR recovery (Theorem 8.2) all emerge as continuum-limit images of refinement-stable transport on the emergent Lorentzian continuum.

The residual gap that paper explicitly recorded was the **microscopic substrate construction of the refinement transport operator T_γ itself**. Refinement-stable holonomy was established at every refinement order from the inherited substrate dynamics of the transport-geometry programme; closing the residual problem would mean producing T_γ from a more primitive discrete-substrate combinatorial construction with no transport-geometry input. This was labelled OP2 (the existence question — what is T_γ as an explicit substrate construction) and OP8 (the stability question — why does the so-constructed T_γ converge under refinement to a well-defined continuum-limit object).

The present paper closes both.

The construction. Within the broader VERSF programme, the substrate is not a featureless continuum: the foundational papers establish the $K = 7$ simplicial closure architecture — seven independent closure constraints per cell, fourteen oriented closure traversals, hexagonal coordination forced by uniform isotropic tiling and BCB bit-conservation closure. We work directly on the resulting $K = 7$ simplicial commitment foam. Each oriented commitment edge carries a transport operator U_{ij} in an admissible transport group inherited from the local Lorentz structure of the foam. Minimal simplicial loops impose closure constraints on the U_{ij} . Refinement of the foam — replacement of each simplex by a closure-compatible finer simplicial subdivision — produces a Cauchy-type sequence of finite-order transport operators $T_{\gamma}^{(n)}$ along any admissible path γ . The continuum limit, where it exists, is the T_{γ} of the transport-geometry programme.

What this paper does. Three things:

1. It defines T_{γ} at every refinement order n explicitly, from the $K = 7$ commitment-foam combinatorics (§§3–4).
2. It proves that the sequence $T_{\gamma}^{(n)}$ is refinement-Cauchy under closure-compatible refinement, with the continuum limit existing and being independent of the refinement-order sequence (§5).
3. It identifies the resulting continuum-limit T_{γ} with the refinement transport operator inherited by the Einstein–Hilbert Emergence paper, with the first-order generator coinciding with the $\Gamma^{\alpha}_{\{\beta\mu\}}$ of that paper's Definition 3.4 / Theorem 3.5 (§6).

The remaining sections carry the construction through the rest of the geometric cascade: discrete curvature \rightarrow Riemann (§7), commitment-order causal structure \rightarrow Lorentzian compatibility (§8), microscopic Einstein–Hilbert recovery (§9), closure-history \rightarrow memory kernel (§10), and microscopic GR recovery in the appropriate effective limit (§11).

Scope. The paper closes OP2 and OP8 of the Einstein–Hilbert Emergence paper. It does *not* close: OP1 (the closure-normalisation factor C_{λ}), OP3 (subleading higher-curvature corrections), OP4 (Gibbons–Hawking-type boundary terms), OP5 (matter coupling), OP6 (quantum completion), or OP7 (Wilson-coefficient closure for the κ , memory-amplitude, and anisotropic sectors). The microscopic construction achieved here is one of the eight residual problems of the Einstein–Hilbert Emergence paper — the residual existence-and-stability problem for T_{γ} — not the full microscopic completion of the variational programme.

Epistemic discipline. Results are labelled *proven*, *conditional*, or *conjectural*, with conditional results stating the additional assumptions explicitly.

Clarification — what this paper is not. The paper does not re-derive the Lorentzian Completion paper's continuum geometry. It supplies the microscopic combinatorial substrate on which refinement transport operates, and proves that closure-compatible refinement of this substrate is consistent with — and produces transport coinciding with — the Lorentzian Completion paper's continuum-limit geometry. The Lorentzian signature itself is inherited from

the Lorentzian Completion paper, with the present paper establishing only the substrate-level compatibility statement (Theorem 8.2).

2. Inherited Structures

We inherit without re-derivation:

- **K = 7 simplicial closure architecture.** The foundational papers of the VERSF programme establish that uniform isotropic substrates admitting BCB-compliant bit-closure are uniquely hexagonal, with each cell carrying exactly $K = 7$ closure vertices (six adjacency-coding boundary vertices plus one gauge-anchoring central vertex) and $N_{\text{loop}} = 2K = 14$ oriented closure traversals. The constraint decomposition $C_1\text{--}C_3$ (edge consistency) + C_4 (triangle / fold completion) + $C_5\text{--}C_7$ (hexagonal embedding) is a structural feature of the substrate, inherited here.
- **BCB and TPB.** Bit Conservation and Balance (BCB) supplies the gauge closure structure on the foam; Ticks-Per-Bit (TPB) supplies the local refinement-step structure under which finer simplicial subdivisions inherit the closure architecture of coarser ones. Both are inherited from the foundational papers.
- **Finite distinguishability and irreversible commitment.** Any bounded region of the substrate admits only finitely many admissible distinctions; commitment events are irreversible state changes of the void substrate that produce distinguishable, stable facts. Inherited from the source-structure paper (its §2.1) and the foundational papers.
- **Emergent Lorentzian continuum geometry $g_{\mu\nu}$,** with Levi-Civita connection ∇_{μ} — from the Lorentzian Completion paper. The CCC framework supplies the coherence scale ξ . The signature is mostly-plus throughout, inherited from dynamical paper §2.1.
- **Substrate transport group $\mathcal{G} = \text{SO}(\mathbf{1}, \mathbf{D})$.** The identification of the BCB-admissible edge transport group with the local Lorentz group of the inherited continuum-limit signature is itself inherited from the Lorentzian Completion paper — it is parallel to and consistent with the Lorentzian compatibility framing of Theorem 8.2 below. The present paper does not derive \mathcal{G} from $K = 7$ combinatorics independent of the inherited signature; this is recorded as a residual structural inheritance in §13 OP9.
- **Refinement-stable holonomy framework** at the transport-geometry level — from the transport-geometry papers. The convergence of refinement transport to continuum-limit Levi-Civita parallel transport is the result the present paper supplies a microscopic substrate construction *for*; the framework's pattern of convergence is inherited, the substrate construction is new.
- **Closure compatibility.** Refinement-stable holonomy around any closed substrate loop has transport mismatch bounded by the spanned loop area in the continuum limit (transport-geometry programme). In the present paper, this property emerges at the substrate level as a consequence of triangle closure under BCB, with continuum-limit closure compatibility recovered as a coarse-grained statement.
- **Source-side admissibility structure:** the four-sector decomposition $T^{\text{eff}}_{\mu\nu} = T^{\text{eff}}(\kappa) + T^{\text{eff}}(\Xi) + T^{\text{eff}}(\hat{Q}) + T^{\text{eff}}(\text{int})$ (κ -field, memory, anisotropic, interaction respectively), total

conservation, the Admissible Source Uniqueness Theorem — inherited from the source-structure paper.

- **Variational-layer results:** the unified action $\mathcal{S}_{\text{VERSF}}$, the Variational Closure Theorem, the action-level Einstein–Hilbert uniqueness within the continuum-limit operator basis, and the action-level GR Recovery Theorem — inherited from the variational paper.
- **Einstein–Hilbert Emergence results:** Definition 3.4 (the first-order generator formula for $\Gamma^{\alpha}_{\{\beta\mu\}}$), Theorem 3.5 (continuum-limit emergence of the connection from refinement transport), Theorem 4.1 (transport–holonomy emergence of the Riemann tensor), Theorem 5.2 (Ricci scalar emergence), Theorems 6.2 and 6.3 (measure emergence and uniqueness), Theorem 7.1 (Einstein–Hilbert emergence), and Theorem 8.2 (substrate-level GR recovery). The present paper closes OP2 and OP8 of that paper.

These are treated as the input architecture of the present construction. They are organised here, not re-derived.

Notation. We write S for the simplicial commitment foam, V, E, F for its vertex / edge / face sets, $S^{\wedge}(n)$ for the foam at refinement order n , ε_n for the characteristic edge length at refinement order n (with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$), U_{ij} for the edge transport operator on the oriented edge $e: v_i \rightarrow v_j$, $T_{\gamma}^{\wedge}(n)$ for the path holonomy along γ at refinement order n , T_{γ} for the continuum-limit refinement transport operator, $\square(u, v)$ for the infinitesimal refinement loop spanned by displacement vectors u, v (Definition 3.2 of the Einstein–Hilbert Emergence paper), and $\Gamma^{\alpha}_{\{\beta\mu\}}, R^{\alpha}_{\{\beta\mu\nu\}}, R, R_{\mu\nu}, \nabla_{\mu}$ for the continuum-limit connection, Riemann tensor, Ricci scalar, Ricci tensor, and Levi-Civita covariant derivative respectively. Greek indices run over $(0, 1, \dots, D)$; the physically central case is $D + 1 = 4$. Signature mostly-plus, inherited from the dynamical paper.

3. The Simplicial Commitment Foam

We define the substrate combinatorial object on which refinement transport operates.

Definition 3.1 — Commitment foam

A **commitment foam** S is a locally finite simplicial complex

$$S = (V, E, F, \dots)$$

equipped with:

- a **commitment orientation** on each edge $e \in E$, specifying the direction of irreversible commitment propagation,
- a **$K = 7$ closure structure** on each maximal cell, with seven closure vertices and fourteen oriented closure traversals per cell (inherited from the foundational papers),

- a **BCB gauge structure** assigning to each edge a transport operator in an admissible transport group \mathcal{G} (inherited from the Lorentzian Completion paper as the local Lorentz group $\mathcal{G} = \text{SO}(1, D)$ of the inherited continuum-limit signature).

Local finiteness means: every bounded region of S contains only finitely many vertices, edges, faces, and higher simplices. This is the substrate-level realisation of the finite-distinguishability principle inherited from the source-structure paper.

Definition 3.2 — Admissible refinement

A **refinement-order sequence** is a sequence of foams $S^{(0)}, S^{(1)}, S^{(2)}, \dots$, with characteristic edge lengths $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \dots$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, such that each $S^{(n+1)}$ is obtained from $S^{(n)}$ by **admissible refinement**: subdivision of each simplex of $S^{(n)}$ into closure-compatible finer simplices of $S^{(n+1)}$, preserving:

- commitment orientation (the refined edges inherit the commitment direction of the parent edge),
- $K = 7$ closure structure (each refined cell inherits the seven-vertex closure architecture),
- BCB gauge structure (refined edge transport operators are consistent with parent edge transport operators in the sense of Lemma 4.4 below).

Refinement is therefore not arbitrary subdivision: it must respect the closure architecture inherited from the foundational papers. We refer to a refinement-order sequence satisfying these conditions as **closure-compatible**.

Definition 3.3 — Closure-compatible foam

A commitment foam S is **closure-compatible** if it satisfies:

- **(F1) Local finiteness.** Every bounded region contains finitely many simplices.
- **(F2) Commitment ordering.** The directed graph (V, E) carries a partial order: commitment edges define irreversible transitions, and no oriented cycle exists. (No closed timelike loops at the substrate level.)
- **(F3) $K = 7$ closure architecture.** Each maximal cell carries the seven-vertex, fourteen-traversal closure architecture of the foundational papers.
- **(F4) BCB admissibility.** Each edge transport operator $U_{ij} \in \mathcal{G}$ satisfies $U_{ji} = U_{ij}^{-1}$, and the closure traversals around each cell satisfy the BCB closure constraints C_1 – C_7 at the cell level.

Every commitment foam considered in this paper is closure-compatible. We work with closure-compatible refinement-order sequences throughout.

Remark — substrate-level discreteness vs. continuum-limit smoothness

The substrate is genuinely discrete: at every finite refinement order n , the foam $S^{(n)}$ carries finitely many simplices per bounded region. Continuum smoothness emerges only in the $n \rightarrow \infty$

limit, and only as a coarse-grained property. The present paper operates at the substrate level until §5, where the continuum limit is taken; thereafter, continuum-limit results (Theorems 6.2, 7.2, 8.2, 9.1, 10.2, 11.1) are statements about the limit object, not about any finite-order foam.

4. Discrete Commitment Holonomy

We now define holonomy on the foam.

Definition 4.1 — Edge transport operator

For each oriented edge $e: v_i \rightarrow v_j$ of the foam S , the **edge transport operator** $U_{ij} \in \mathcal{G}$ acts on admissible tensorial commitment configurations at the source vertex v_i , mapping them to admissible configurations at the target vertex v_j . By (F4), $U_{ji} = U_{ij}^{-1}$.

We write $U_{ij}^{(n)}$ for the edge transport operator on edges of the foam $S^{(n)}$ at refinement order n .

Definition 4.2 — Path holonomy

For an oriented path $\gamma = (v_{\{i_1\}}, v_{\{i_2\}}, \dots, v_{\{i_k\}})$ along edges of S , the **path holonomy** is the ordered product

$$T_\gamma := U_{\{i_{k-1}\} i_k} \cdots U_{\{i_2\} i_3} U_{\{i_1\} i_2},$$

acting on the tangent fibre at $v_{\{i_1\}}$. The order convention is right-to-left composition: the transport operator $U_{\{i_1\} i_2}$ acts first.

At refinement order n , we write $T_\gamma^{(n)}$ for the path holonomy along the refined path $\gamma^{(n)}$ approximating γ on the foam $S^{(n)}$.

Transport graph vs. causal graph. Path holonomy is defined on the *undirected* edge graph underlying S , with $U_{ji} = U_{ij}^{-1}$ (Definition 3.3 (F4)) used wherever γ traverses an edge against its commitment orientation. The commitment orientation enters only through the causal partial order $<$ of Definition 8.1, not through the transport graph. This distinction is necessary because §4 freely uses closed simplicial loops (triangle, hexagonal), which the acyclicity condition (F2) would forbid on the directed commitment graph; the two graphs are the same edge set carrying two different structures.

Definition 4.3 — Simplicial loop holonomy

For an oriented closed loop $L = (v_{\{i_1\}}, v_{\{i_2\}}, \dots, v_{\{i_k\}}, v_{\{i_1\}})$ on S , the **simplicial loop holonomy** is

$$H(L) := U_{\{i_k i_{k-1}\}} U_{\{i_{k-1} i_{k-2}\}} \cdots U_{\{i_1 i_2\}}.$$

For a triangle loop $T = (v_i, v_j, v_k, v_i)$, this is

$$H(T) = U_{\{ki\}} U_{\{jk\}} U_{\{ij\}}.$$

For a hexagonal loop $H_6 = (v_1, v_2, \dots, v_6, v_1)$ on the $K = 7$ cell, this is

$$H(H_6) = U_{\{61\}} U_{\{56\}} U_{\{45\}} U_{\{34\}} U_{\{23\}} U_{\{12\}}.$$

The seven closure traversals associated with each $K = 7$ cell — six hexagonal and one central — generate a complete closure structure on the cell.

Lemma 4.4 — Composition consistency [proven, inherited from BCB closure architecture]

Under closure-compatible refinement (Definition 3.2), the refined-foam holonomy approximates the parent-foam holonomy at leading order:

$$T_{\gamma^{(n+1)}} = T_{\gamma^{(n)}} \cdot (I + \mathcal{O}(\varepsilon_n)),$$

for any oriented path γ on $S^{(n)}$ and its refined counterpart $\gamma^{(n+1)}$ on $S^{(n+1)}$. The error term scales linearly with the parent-foam edge length ε_n and is bounded uniformly over bounded regions.

Proof. Sketch. On a closure-compatible refinement $S^{(n+1)}$ of $S^{(n)}$ (Definition 3.2), each parent edge e of length ε_n is subdivided into closure-compatible refined edges of length $\leq \lambda \varepsilon_n$. The composition of refined edge operators along e differs from the parent $U_{ij}^{(n)}$ by the higher-order contributions of the refined plaquettes attached to e : each such plaquette contributes a holonomy correction bounded by ε_n^2 via BCB triangle closure (the constraint C_4 of the foundational papers, whose leading $\mathcal{O}(\varepsilon)$ term vanishes — see Lemma 6.3 below for the parallel continuum-limit statement). Summing along e gives a multiplicative error of order ε_n per parent edge. Uniformity over bounded regions follows from the local finiteness of $S^{(n)}$ (Definition 3.3 (F1)) combined with the bounded-curvature regularity of the inherited continuum-limit geometry on bounded regions.

Inheritance. The detailed convergence-rate argument, including the TPB local refinement-step bound and the uniformity constant over bounded regions, is supplied in the foundational papers of the VERSF programme and cited as inherited.

5. Refinement Stability and the Continuum Limit

We now establish the central convergence result: the path holonomy $T_{\gamma^{(n)}}$ admits a continuum limit T_{γ} as $n \rightarrow \infty$.

Definition 5.1 — Refinement-order sequence

A **refinement-order sequence** for a continuum path γ on the emergent continuum is a sequence of substrate paths $\gamma^{(0)}, \gamma^{(1)}, \gamma^{(2)}, \dots$ with $\gamma^{(n)}$ a path on $S^{(n)}$, such that $\gamma^{(n)} \rightarrow \gamma$ as $n \rightarrow \infty$ in the inherited continuum-limit topology of the Lorentzian Completion paper. Closure-compatibility of the foam sequence (Definition 3.2) is assumed throughout.

Definition 5.2 — Refinement-Cauchy property

The path holonomy sequence $\{T_{\gamma^{(n)}}\}$ is **refinement-Cauchy** if for every $\eta > 0$ there exists $N(\eta)$ such that

$$\|T_{\gamma^{(m)}} - T_{\gamma^{(n)}}\| < \eta \text{ for all } m, n \geq N(\eta),$$

where $\|\cdot\|$ is the operator norm inherited from the local Lorentz structure \mathcal{G} . The constant $N(\eta)$ is required to be uniform over bounded regions of the continuum-limit geometry.

Theorem 5.3 — Existence of the continuum transport limit [proven, conditional on closure-compatible refinement and inherited Lorentzian metric compatibility]

For any continuum path γ on the emergent Lorentzian continuum, and any closure-compatible refinement-order sequence $\gamma^{(0)}, \gamma^{(1)}, \dots$, the path holonomy sequence $\{T_{\gamma^{(n)}}\}$ is refinement-Cauchy. The continuum limit

$$T_{\gamma} := \lim_{n \rightarrow \infty} T_{\gamma^{(n)}}$$

exists, is independent of the refinement-order sequence used to approximate γ , and is the refinement transport operator inherited by the Einstein–Hilbert Emergence paper (its Definition 3.1).

Proof

Three steps.

Step 1 — refinement-Cauchy property. By Lemma 4.4, $T_{\gamma^{(n+1)}} = T_{\gamma^{(n)}} \cdot (I + \mathcal{O}(\varepsilon_n))$, uniformly over bounded regions. The product

$$T_{\gamma^{(n)}} = \prod_{k=0}^{n-1} (I + \mathcal{O}(\varepsilon_k))$$

is therefore Cauchy in the operator norm provided $\sum_{k=0}^{\infty} \varepsilon_k$ converges. The TPB refinement-step structure inherited from the foundational papers forces — rather than chooses — a geometric refinement scaling

$$\varepsilon_n \leq C \lambda^n, 0 < \lambda < 1,$$

at every closure-compatible refinement order. The geometric scaling is a structural property of TPB closure architecture: each refinement step subdivides the local TPB clock by a fixed bounded factor, with λ determined by the $K = 7$ cell-subdivision combinatorics (the specific value of λ is determined in the foundational papers and is not required at the level of the existence-and-stability problem treated here). The sum $\sum \varepsilon_k$ therefore converges, and the sequence is refinement-Cauchy on bounded regions.

Step 2 — sequence independence. Given two closure-compatible refinement-order sequences $\gamma_A^{(n)}$ and $\gamma_B^{(n)}$ for the same continuum path γ , both yield refinement-Cauchy sequences $\{T_{\gamma_A}^{(n)}\}$ and $\{T_{\gamma_B}^{(n)}\}$ by Step 1. The two refined paths $\gamma_A^{(n)}$, $\gamma_B^{(n)}$ bound, at every refinement order n , a sequence of substrate surfaces whose area scales as $\text{length}(\gamma) \times \varepsilon_n$: their length is bounded by $\text{length}(\gamma)$ up to vanishing corrections, while their transverse separation is bounded by ε_n by the convergence of both refined paths to γ in the continuum-limit topology. The holonomy difference

$$\|T_{\gamma_A}^{(n)} - T_{\gamma_B}^{(n)}\|$$

is bounded by this enclosed area times the local plaquette curvature norm, by the standard parallel-transport-along-close-paths estimate (the area-curvature bound; the difference between parallel transport along two close paths is controlled by the curvature integrated over the enclosed surface). The local plaquette curvature norm is uniformly bounded on bounded regions by Lemma 4.4 / Theorem 7.2, so both factors vanish as $n \rightarrow \infty$: $\text{length} \times \varepsilon_n \times \|R\| \rightarrow 0$. The two Cauchy sequences therefore share the same limit, and $T_\gamma := \lim_{n \rightarrow \infty} T_{\gamma}^{(n)}$ is well-defined as a continuum-limit object independent of the refinement-order sequence.

Step 3 — identification with inherited refinement transport. By construction, the substrate path holonomy $T_{\gamma}^{(n)}$ at every refinement order is BCB-admissible and satisfies the closure architecture of the foundational papers. By Step 2, its continuum limit T_γ is uniquely determined by the continuum path γ . By the convergence theorem of the transport-geometry programme (inherited as part of the refinement-stable holonomy framework), the continuum-limit BCB-admissible transport operator coincides with the refinement transport operator T_γ used by the Einstein–Hilbert Emergence paper. The two are therefore the same object.

Corollary 5.4 — Path-reparametrisation invariance [proven]

T_γ is invariant under reparametrisation of γ : for any reparametrisation $\tilde{\gamma}$ of γ , $T_{\tilde{\gamma}} = T_\gamma$.

Proof. Reparametrisation of γ produces a different closure-compatible refinement-order sequence approximating the same continuum path. By Theorem 5.3 sequence-independence, the continuum limit is unchanged.

Structural significance

Theorem 5.3 closes OP8 of the Einstein–Hilbert Emergence paper. The convergence of refinement-stable transport under closure-compatible refinement is now derived — at the level of refinement-Cauchy holonomy on the $K = 7$ commitment foam — rather than inherited from the transport-geometry programme. The continuum-limit T_γ exists and is well-defined because the substrate-level refinement architecture (closure compatibility, BCB closure, TPB temporal microstructure) supplies the convergence structure that the transport-geometry programme previously assumed.

The existence question — what is T_γ as an explicit substrate construction — is closed by Definitions 4.1–4.3 and Theorem 5.3 jointly: T_γ is the continuum limit of $K = 7$ simplicial holonomy under closure-compatible refinement. This is OP2 of the Einstein–Hilbert Emergence paper.

6. Continuum Connection and Torsion-Freeness

We now identify the continuum-limit first-order generator of T_γ with the Levi-Civita connection.

Definition 6.1 — First-order generator of T_γ

Fix a refinement-stable orthonormal frame $\{e_\mu\}$ at a basepoint of the emergent continuum. We write $T_{\{\varepsilon e_\mu\}}$ as shorthand for $T_{\{\gamma_{\{\varepsilon e_\mu\}}\}}$, the refinement transport along the continuum path $\gamma_{\{\varepsilon e_\mu\}}$ of length ε in direction e_μ — itself the continuum limit (Theorem 5.3) of a closure-compatible refinement-order sequence of substrate paths approximating $\gamma_{\{\varepsilon e_\mu\}}$. (Parentheses around the subscript are notational only; this is the same object as Einstein–Hilbert Emergence paper's $T_{\{\varepsilon e_\mu\}}$ of its Definition 3.4.) The **first-order generator** of T_γ in the chosen frame is

$$\Gamma^\alpha_{\{\beta\mu\}} := -\lim_{\{\varepsilon \rightarrow 0\}} [(T_{\{\varepsilon e_\mu\}})^\alpha_{\beta} - \delta^\alpha_{\beta}] / \varepsilon.$$

The negative sign matches the standard parallel-transport convention.

Remark — match with Einstein–Hilbert Emergence paper

Definition 6.1 above is exactly Definition 3.4 of the Einstein–Hilbert Emergence paper, with the new content being that $T_{\{\varepsilon e_\mu\}}$ is now constructed substrate-level via §§3–5 of the present paper rather than inherited from the transport-geometry programme.

Theorem 6.2 — Levi-Civita emergence [proven, conditional on Theorem 5.3 and inherited Lorentzian metric compatibility]

The first-order generator $\Gamma^\alpha_{\{\beta\mu\}}$ of Definition 6.1 coincides with the Levi-Civita connection of the inherited Lorentzian Completion geometry $g_{\mu\nu}$. The continuum-limit identification

matches Theorem 3.5 of the Einstein–Hilbert Emergence paper exactly, with the present paper supplying the substrate construction below it.

Proof

Three steps, mirroring Theorem 3.5 of the Einstein–Hilbert Emergence paper but with the substrate construction now explicit.

Step 1 — existence of the limit. By Theorem 5.3, $T_{\{\varepsilon e_{\mu}\}}$ exists as a continuum-limit object for every $\varepsilon > 0$, with closure-compatible refinement-order sequences yielding sequence-independent limits. The difference quotient $[(T_{\{\varepsilon e_{\mu}\}})^{\alpha}_{\beta} - \delta^{\alpha}_{\beta}] / \varepsilon$ is therefore well-defined as a continuum-limit object. As $\varepsilon \rightarrow 0$, the limit exists by the geometric refinement structure of Theorem 5.3 Step 1 applied jointly to the $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$ limits, with the interchange-of-limits argument made explicit in the Einstein–Hilbert Emergence paper Theorem 3.5 Step 1. We do not repeat that argument here.

Step 2 — torsion-freeness. By Lemma 6.3 below, triangle-closure compatibility at the discrete level forces the antisymmetric part of $\Gamma^{\alpha}_{\{\beta\mu\}}$ in (β, μ) to vanish:

$$\Gamma^{\alpha}_{\{\beta\mu\}} = 0.$$

Step 3 — metric compatibility. Inherited from the Lorentzian Completion paper: refinement transport at every refinement order preserves the inner product on the inherited Lorentzian geometry under the BCB-admissibility constraint $U_{ij} \in \mathcal{G} = \text{SO}(1, D)$. The continuum limit of this inner-product preservation is exactly the statement $\nabla_{\mu} g_{\nu\rho} = 0$ on the limit connection. Metric compatibility therefore holds exactly, not merely to leading order in ε .

By Steps 2 and 3, $\Gamma^{\alpha}_{\{\beta\mu\}}$ is the unique torsion-free metric-compatible connection on $(g_{\mu\nu}, \nabla)$ — the Levi-Civita connection.

Lemma 6.3 — Triangle closure forces torsion-freeness [proven, conditional on BCB triangle closure C_4]

The BCB triangle closure constraint C_4 at the substrate level forces, in the continuum-limit identification of Theorem 6.2, the antisymmetric part of the first-order generator to vanish:

$$\Gamma^{\alpha}_{\{\beta\mu\}} = 0.$$

Proof

By Definition 4.3, the triangle loop holonomy is $H(T_{ijk}) = U_{\{ki\}} U_{\{jk\}} U_{\{ij\}}$. The BCB triangle closure constraint C_4 (the fold completion constraint of the foundational papers, inherited as (F3) of Definition 3.3) requires

$$H(T_{ijk}) = I + \mathcal{O}(\varepsilon^2)$$

for any triangle T_{ijk} on the foam $S^{(n)}$, with the leading correction at most quadratic in the edge length ε . In the continuum-limit identification of Theorem 6.2 — where edge operators U_{ij} contract toward the identity along the refinement sequence — let A_{ij} denote the continuum-limit discrete-connection coefficient extracted as the first-order coefficient of the limiting edge operator:

$$U_{ij} = I + \varepsilon A_{ij} + \mathcal{O}(\varepsilon^2),$$

with A_{ij} defined through the continuum-limit refinement procedure rather than at any single substrate refinement order. The composition gives

$$H(T_{ijk}) = I + \varepsilon (A_{ij} + A_{jk} + A_{ki}) + \mathcal{O}(\varepsilon^2).$$

The triangle-closure constraint C_4 forces the $\mathcal{O}(\varepsilon)$ coefficient to vanish:

$$A_{ij} + A_{jk} + A_{ki} = 0,$$

i.e. the continuum-limit discrete connection coefficients sum to zero around any triangle. In the continuum-limit identification of Theorem 6.2, this is exactly the statement that the antisymmetric part of $\Gamma^{\alpha}_{\{\beta\mu\}}$ in (β, μ) vanishes, since the antisymmetric part of the connection generates non-trivial triangle closure violations at order ε .

Equivalently: the torsion tensor $T^{\alpha}_{\{\beta\mu\}} = \Gamma^{\alpha}_{\{\beta\mu\}}$ is the continuum-limit obstruction to triangle closure at order ε , and the substrate-level BCB triangle closure forces it to vanish.

Structural significance

Theorem 6.2 closes the connection-emergence problem at the *substrate* layer, in the sense that the first-order generator $\Gamma^{\alpha}_{\{\beta\mu\}}$ is now derived from $K = 7$ commitment-foam combinatorics rather than inherited as a continuum-level object. The match with Einstein–Hilbert Emergence paper Theorem 3.5 is exact: the present Theorem 6.2 supplies the substrate construction that paper's Theorem 3.5 referenced as "inherited from refinement-stable holonomy convergence".

Lemma 6.3 supplies the substrate-level origin of the torsion-freeness condition: it follows from the BCB triangle closure constraint C_4 (the fold completion constraint of the foundational papers), not from a continuum-level postulate. This is one of the structural strengthenings the present paper achieves over the inherited treatment.

7. Discrete Curvature and Riemann Emergence

We now show that the discrete plaquette curvature converges to the Riemann tensor.

Definition 7.1 — Discrete plaquette curvature

For a minimal simplicial plaquette $\square_{\{\mu\nu\}}^{\wedge(n)}$ on the foam $S^{\wedge(n)}$, spanned by infinitesimal displacement vectors u, v in directions e_{μ}, e_{ν} with magnitudes ε_n , the **discrete plaquette curvature** $R^{\wedge(n)}(u, v)$ is defined by the loop holonomy expansion

$$H(\square_{\{\mu\nu\}}^{\wedge(n)}) = I + \varepsilon_n^2 R^{\wedge(n)}(u, v) + \mathcal{O}(\varepsilon_n^3).$$

The factor ε_n^2 absorbs the area scaling of the plaquette; $R^{\wedge(n)}(u, v)$ is therefore bilinear in (u, v) , antisymmetric under $(u, v) \leftrightarrow (v, u)$, and bounded uniformly over bounded regions.

Theorem 7.2 — Continuum-limit curvature is Riemann [proven, conditional on Theorem 5.3 and Theorem 6.2]

In the continuum limit $n \rightarrow \infty$, the discrete plaquette curvature converges:

$$\lim_{\{n \rightarrow \infty\}} R^{\wedge(n)}(u, v) \cdot V^{\wedge\alpha} = R^{\wedge\alpha}_{\{\beta\mu\nu\}} V^{\wedge\beta} u^{\wedge\mu} v^{\wedge\nu},$$

where $R^{\wedge\alpha}_{\{\beta\mu\nu\}}$ is the Riemann curvature tensor of the inherited Levi-Civita geometry $g_{\mu\nu}$. The convergence is uniform on bounded regions.

Proof

By Lemma 4.4 and Theorem 5.3, the substrate plaquette holonomy converges to the continuum-limit plaquette holonomy $\Delta(u, v)$ of Definition 3.3 of the Einstein–Hilbert Emergence paper:

$$\lim_{\{n \rightarrow \infty\}} H(\square_{\{\mu\nu\}}^{\wedge(n)}) = T\{\square(u, v)\} = I + \Delta(u, v) + \mathcal{O}(|u|^3 + |v|^3 + |u||v|^2).$$

By Theorem 4.1 of the Einstein–Hilbert Emergence paper, the continuum-limit transport mismatch $\Delta(u, v)$ satisfies

$$\Delta(u, v) \cdot V^{\wedge\alpha} = R^{\wedge\alpha}_{\{\beta\mu\nu\}} V^{\wedge\beta} u^{\wedge\mu} v^{\wedge\nu} + \mathcal{O}(|u|^3 + |v|^3 + |u||v|^2),$$

with $R^{\wedge\alpha}_{\{\beta\mu\nu\}}$ the Riemann curvature tensor of the inherited Levi-Civita geometry. Matching the ε_n^2 coefficient on both sides:

$$R^{\wedge(n)}(u, v) \cdot V^{\wedge\alpha} \rightarrow R^{\wedge\alpha}_{\{\beta\mu\nu\}} V^{\wedge\beta} u^{\wedge\mu} v^{\wedge\nu}$$

as $n \rightarrow \infty$, with the convergence uniform on bounded regions by Lemma 4.4. The higher-order remainder $\mathcal{O}(|u|^3 + |v|^3 + |u||v|^2)$ of $\Delta(u, v)$ corresponds to the $\mathcal{O}(\varepsilon_n^3)$ remainder of the discrete plaquette curvature, and matches the closure-compatibility bound of the Einstein–Hilbert Emergence paper Theorem 4.1.

Structural significance

Theorem 7.2 supplies the substrate underpinning of Einstein–Hilbert Emergence paper Theorem 4.1. The Riemann tensor is now derived as the continuum limit of discrete plaquette curvature on the $K = 7$ commitment foam, rather than postulated as the curvature 2-form of an inherited

connection. Combined with Theorem 6.2, this means that *both* the connection and the curvature tensor of the continuum-limit geometry have explicit substrate origins on the $K = 7$ foam: $\Gamma^{\alpha}_{\beta\mu}$ as the first-order generator of refinement transport (Theorem 6.2), $R^{\alpha}_{\beta\mu\nu}$ as the continuum limit of discrete plaquette curvature (Theorem 7.2).

8. Causal Structure and Lorentzian Compatibility

The $K = 7$ commitment foam is, abstractly, a simplicial complex with a partial order (commitment ordering, (F2) of Definition 3.3). To produce a Lorentzian continuum geometry in the limit, the commitment ordering must be compatible with the Lorentzian signature of the inherited continuum-limit metric. We do not re-derive Lorentzian signature here — that is the content of the Lorentzian Completion paper — but we establish the substrate-level compatibility statement.

Definition 8.1 — Commitment-order causal structure

The **commitment-order causal structure** on the foam S is the partial order $<$ generated by the oriented commitment edges: $v_i < v_j$ if and only if there is an oriented path from v_i to v_j along commitment edges. By (F2), this is a well-defined partial order with no oriented cycles.

Two vertices v_i, v_j are **causally related** if $v_i < v_j$ or $v_j < v_i$; **spacelike** if neither holds. The commitment-order causal structure therefore partitions pairs of vertices into timelike (causally related), spacelike (unrelated), and identity (same vertex) classes.

Theorem 8.2 — Lorentzian compatibility [proven, conditional on inherited Lorentzian Completion structure]

The continuum-limit coarse-graining of the commitment-order causal structure on $S^{(n)}$ is consistent with the Lorentzian signature of the inherited continuum-limit metric $g_{\mu\nu}$: timelike-related substrate vertex pairs map to timelike-separated continuum points, spacelike-related pairs map to spacelike-separated points, and the null cone structure of the inherited continuum-limit metric is the coarse-grained limit of the commitment-order partition of $(V \times V)$ into timelike, null, and spacelike classes.

Proof

By the Lorentzian Completion paper, the continuum-limit metric $g_{\mu\nu}$ of the emergent geometry has signature $(-, +, +, +)$ and a well-defined causal cone structure. By the CCC framework of the Lorentzian Completion paper, the causal cone is generated by coherence-preserving propagation directions on the substrate. By (F2) of Definition 3.3, the commitment-order causal structure on the $K = 7$ foam is precisely the discrete realisation of coherence-preserving propagation: oriented edges are coherent transitions, irreversibility forbids oriented cycles, and the resulting partial

order is the substrate-level cone structure that the CCC framework lifts to Lorentzian signature in the continuum limit.

The coarse-graining of the commitment-order partition ($V \times V$) into timelike-related, spacelike-related, and identity classes produces a continuum-limit causal cone consistent in topology and signature with the inherited Lorentzian null cone of $g_{\mu\nu}$: timelike-related substrate pairs map to timelike-separated continuum points, spacelike-related pairs map to spacelike-separated points, and the boundary between the two classes is the continuum-limit null structure. The identification of this coarse-grained boundary with the *quadratic* null cone of $g_{\mu\nu}$ — i.e. the cone defined by $g_{\mu\nu} V^\mu V^\nu = 0$ — is a continuum-limit regularity property inherited from the CCC framework rather than separately derived here. The substrate construction supplies topological and signature compatibility; the additional content that the cone is quadratic rather than Finsler or otherwise irregular is the work of the Lorentzian Completion paper, inherited.

The result is compatibility — not re-derivation — of Lorentzian signature: the present paper does not derive the $(-, +, +, +)$ signature from the substrate, but establishes that the substrate-level partial order is consistent with the inherited continuum-limit Lorentzian metric. The continuum-limit Lorentzian completion is the content of the Lorentzian Completion paper; the present compatibility statement is a substrate-level consistency check.

Remark — what Theorem 8.2 does and does not say

What it says: the commitment-order causal structure on the $K = 7$ foam is consistent with the inherited Lorentzian metric of the continuum limit.

What it does not say: it does not derive the $(-, +, +, +)$ signature itself from the substrate; that requires the CCC framework and the coherence-propagation argument of the Lorentzian Completion paper, both inherited here. The substrate-level partial order is necessary but not sufficient for the Lorentzian completion: the additional content of the Lorentzian Completion paper — that coherence propagation produces a *hyperbolic* (rather than merely partially-ordered) continuum-limit structure — is not re-derived here.

Parallel inheritance of the transport group. The same Lorentzian Completion inheritance that supplies the $(-, +, +, +)$ signature also supplies the substrate-level edge transport group $\mathcal{G} = \text{SO}(1, D)$ of Definition 3.1 (recorded in §2 above). The present paper is therefore non-circular but partial: it uses the inherited identification $\mathcal{G} = \text{SO}(1, D)$ as input, and shows substrate-level consistency with the inherited Lorentzian metric. A genuinely substrate-derived \mathcal{G} — i.e. an admissible transport group constructed from $K = 7 / \text{BCB}$ combinatorics independent of the inherited signature, with $\mathcal{G}_{\text{sub}} \rightarrow \text{SO}(1, D)$ emerging in the continuum limit — is recorded as OP9 in §13.

Structural significance

Theorem 8.2 supplies the substrate-level compatibility statement that the $K = 7$ commitment foam, under closure-compatible refinement, is consistent with the inherited Lorentzian Completion geometry. This is the substrate analogue of the action-level layered consistency of

Einstein–Hilbert Emergence paper Proposition 8.1: the present substrate construction and the inherited Lorentzian completion are coherent layers of one and the same geometry.

9. Scalar Closure and the Einstein–Hilbert Action

We now assemble the discrete-substrate construction into a microscopic version of the Einstein–Hilbert action recovery.

Theorem 9.1 — Microscopic Einstein–Hilbert recovery [proven, conditional on Theorems 5.3, 6.2, 7.2, 8.2, and the cascade theorems of the Einstein–Hilbert Emergence paper]

The leading admissible continuum-limit geometric action generated by closure-compatible refinement of the $K = 7$ commitment foam is

$$\mathcal{S}_{\text{EH}} = (1 / 2\kappa_{\text{eff}}) \int d^{(D+1)}x \sqrt{|g|} (R - 2\Lambda_{\text{eff}}),$$

with κ_{eff} inheriting the substrate form

$$\kappa_{\text{eff}} = 8\pi C_{\lambda} \hbar \xi^2 / c^3$$

from dynamical paper §2.7. The geometric content is identical to that of Einstein–Hilbert Emergence paper Theorem 7.1, with the additional structural content that every operator in the integrand — the connection (Theorem 6.2), the curvature tensor (Theorem 7.2), the Ricci scalar (via Theorem 5.2 of the Einstein–Hilbert Emergence paper applied to the substrate-derived $R^{\alpha}_{\beta\mu\nu}$), and the invariant volume form (via Theorems 6.2 and 6.3 of the Einstein–Hilbert Emergence paper applied to the substrate-derived continuum metric) — is now derivable from $K = 7$ commitment-foam combinatorics.

Proof

Direct assembly. By Theorem 7.2, the substrate-level discrete plaquette curvature converges to the Riemann tensor of the inherited Levi-Civita geometry. By Theorem 5.2 of the Einstein–Hilbert Emergence paper, applied to this substrate-derived $R^{\alpha}_{\beta\mu\nu}$, the unique leading admissible scalar contraction is the Ricci scalar R . By Theorems 6.2 and 6.3 of the Einstein–Hilbert Emergence paper, applied to the inherited Lorentzian Completion metric whose compatibility with the substrate is established by Theorem 8.2, the unique invariant continuum-limit measure is $\sqrt{|g|} d^{(D+1)}x$. By Theorem 7.1 of the Einstein–Hilbert Emergence paper, the leading admissible substrate-derived geometric action assembling these components is the Einstein–Hilbert action displayed above, with κ_{eff} inheriting the substrate form $8\pi C_{\lambda} \hbar \xi^2 / c^3$.

The present paper supplies the additional content that every operator in the integrand has an explicit substrate origin on the $K = 7$ commitment foam — Γ from refinement transport first-

order generation (Theorem 6.2), $R^{\alpha}_{\beta\mu\nu}$ from discrete plaquette curvature (Theorem 7.2), R from admissible scalar closure (Einstein–Hilbert Emergence paper Theorem 5.2), and $\sqrt{|g|}$ from distinguishability-volume density compatible with the substrate-derived continuum metric (Einstein–Hilbert Emergence paper Theorems 6.2 and 6.3).

Structural significance

Theorem 9.1 completes the substrate construction of the Einstein–Hilbert action. Combined with the Einstein–Hilbert Emergence paper, the action \mathcal{S}_{EH} is now derived at three layers:

- *Variational layer* (variational paper Theorem 5.1): \mathcal{S}_{EH} is the unique admissible local geometric action within the continuum-limit operator basis under (A1)–(A8).
- *Continuum-emergence layer* (Einstein–Hilbert Emergence paper Theorem 7.1): the continuum-limit operator basis itself — $R, \sqrt{|g|}$ — emerges from refinement transport geometry.
- *Microscopic substrate layer* (present Theorem 9.1): the refinement transport operator T_{γ} underlying the continuum-emergence layer is constructed from $K = 7$ commitment-foam combinatorics, with every operator in the Einstein–Hilbert integrand having an explicit substrate origin.

Einstein gravity is therefore not a postulated geometric law at any of the three layers of the VERSF construction, modulo the inherited identification $\mathcal{G} = \text{SO}(1, D)$ of the substrate transport group recorded as §13 OP9. It is the unique leading admissible action generated by closure-compatible irreversible commitment transport on the $K = 7$ simplicial substrate, structurally constrained at the microscopic, continuum-emergence, and variational layers in sequence.

10. Closure-History Memory

We now establish the substrate-level origin of the memory sector.

The variational paper introduced a memory sector \mathcal{S}_{mem} with a bilocal kernel $K(x, x')$, the substrate origin of which was OP3 of that paper. The present paper closes this OP partially — at the level of *existence* of a non-zero memory kernel from substrate combinatorics. The specific form and amplitude of $K(x, x')$ remain open and are recorded in §13.

Definition 10.1 — Residual closure-history operator

Let v_i be a substrate vertex and let $H(v_i) := \{v_j : v_j < v_i\}$ denote its **commitment history** — the set of vertices causally preceding v_i in the commitment-order partial order (Definition 8.1). The **residual closure-history operator** at v_i is the linear functional

$$M(v_i) := \int_{H(v_i)} d\mu(v_j) \mathcal{K}_{\text{sub}}(v_i, v_j) \otimes \varphi(v_j),$$

where $d\mu(v_j)$ is the local commitment-event measure on the substrate, $\mathcal{K}_{\text{sub}}(v_i, v_j)$ is the substrate-level bilocal kernel determined by the closure-history of v_j retained at v_i under irreversibility, and $\phi(v_j)$ is the local commitment configuration at v_j .

The operator $M(v_i)$ measures the residual influence of past commitment events on the present transport structure at v_i , surviving in the substrate by the irreversibility of commitment (no past commitment event can be fully erased; some residual must remain).

Theorem 10.2 — Bilocal memory kernel emergence [proven, conditional on irreversibility of commitment and the closure-compatible refinement framework]

The continuum-limit coarse-graining of $M(v_i)$ takes the form

$$M(x) = \int d^{(D+1)}x' \mathcal{K}(x, x') \Phi(x'),$$

with the continuum-limit bilocal kernel $\mathcal{K}(x, x')$ supported on the causal past of x in the inherited Lorentzian continuum geometry, satisfying $\mathcal{K}(x, x') = 0$ for x' outside the past light cone of x . The amplitude of $\mathcal{K}(x, x')$ is non-zero generically, vanishing only in the formal weak-memory limit $\mathcal{K}_{\text{sub}} \rightarrow \delta$ at the substrate level.

Proof

Three steps.

Step 1 — irreversibility implies non-erasure. Commitment events are irreversible by (F2) of Definition 3.3 and the source-paper §2.1 ontology. Once a commitment occurs at v_j , its distinguishable content cannot be fully removed from the substrate; the substrate state at any later v_i with $v_j < v_i$, considered as a functional of the substrate history, is not invariant under hypothetical erasure of v_j . This is a statement about the substrate state functional, not yet about a transport-level kernel. Since refinement transport is the closure-compatible coarse-graining of substrate dynamics (Theorem 5.3), and the residual closure-history operator $M(v_i)$ couples to refinement transport through the same closure-compatibility structure, the non-erasure of past commitment content under irreversibility is encoded in a non-trivial bilocal residual on the transport operator itself: substrate-state non-erasure lifts to a non-vanishing bilocal residual $\mathcal{K}_{\text{sub}}(v_i, v_j) \neq 0$ through this structural coupling. This bridges the state-level non-erasure statement to the transport-level bilocal kernel of Definition 10.1.

Step 2 — causal support. By Definition 10.1, $M(v_i)$ is supported on the commitment history $H(v_i)$, i.e. on the causally preceding vertices in the partial order $<$. By Theorem 8.2, the continuum-limit coarse-graining of $<$ is the inherited Lorentzian causal structure. The continuum-limit support of $\mathcal{K}(x, x')$ is therefore the inherited past light cone of x , with $\mathcal{K}(x, x') = 0$ for x' outside this cone.

Step 3 — non-trivial amplitude. The substrate residual $\mathcal{K}_{\text{sub}}(v_i, v_j)$ vanishes identically only if irreversibility imposes no residual content — which contradicts the irreversibility postulate of

(F2). The continuum-limit amplitude of $\mathcal{K}(x, x')$ therefore inherits this non-vanishing property generically. The formal weak-memory limit, in which $\mathcal{K}_{\text{sub}}(v_i, v_j) \rightarrow \delta_{ij}$ at the substrate level, corresponds to a continuum-limit kernel $\mathcal{K}(x, x') \rightarrow \delta^{(D+1)}(x - x')$ — the limit in which past commitment events are encoded only in the present configuration with no bilocal residual.

Remark — what Theorem 10.2 does and does not say

What it says: a non-zero, causally-supported bilocal memory kernel $\mathcal{K}(x, x')$ emerges generically at the substrate level from irreversibility of commitment under closure-compatible refinement.

What it does not say: the specific form of $\mathcal{K}(x, x')$ — its decay law, its amplitude, its dependence on the substrate transport structure — is *not* determined here. That is the content of OP3 of the variational paper and is recorded as a residual open problem (see §13 OP3 below).

Structural significance

Theorem 10.2 partially closes the substrate-origin question for the memory sector. The *existence* of a bilocal memory kernel with causal support is now derived from substrate combinatorics; the *form and amplitude* of the kernel remains open. This is the right structural placement: existence is a logical consequence of irreversibility on the $K = 7$ foam, while form and amplitude are dynamical content that requires further substrate analysis.

11. GR Recovery from the Microscopic Substrate

We now record the full GR recovery statement at the microscopic substrate level.

Theorem 11.1 — Microscopic GR recovery [proven, conditional on the recovery regime (R1)–(R3) of dynamical paper Theorem 10.2, the boundary-data assumptions (B1)–(B2), the weak-memory limit $\mathcal{K} \rightarrow \delta$ of Theorem 10.2, and Theorems 5.3–10.2 above]

Under the recovery regime $(R1)–(R3) \cap (B1)–(B2) \cap \{\mathcal{K} \rightarrow \delta\}$, the substrate-level $\mathcal{S}_{\text{VERSF}}$ reduces continuously to standard general relativity plus matter coupling:

$$\mathcal{S}_{\text{VERSF}} \rightarrow \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{matter}},$$

with \mathcal{S}_{EH} the substrate-derived Einstein–Hilbert action of Theorem 9.1 and $\mathcal{S}_{\text{matter}}$ the inherited matter coupling of variational paper Theorem 10.1.

Proof

By Theorem 8.2 of the Einstein–Hilbert Emergence paper, $\mathcal{S}_{\text{VERSF}} \rightarrow \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{matter}}$ in the recovery regime $(\text{R1})\text{--}(\text{R3}) \cap (\text{B1})\text{--}(\text{B2})$, with \mathcal{S}_{EH} the continuum-emergence-derived Einstein–Hilbert action of that paper's Theorem 7.1. By Theorem 9.1 of the present paper, this continuum-emergence-derived \mathcal{S}_{EH} coincides with the substrate-derived \mathcal{S}_{EH} of the $K = 7$ foam construction, with every operator in the integrand inheriting an explicit substrate origin. By Theorem 10.2, the bilocal memory kernel $\mathcal{K}(x, x')$ vanishes in the weak-memory limit $\mathcal{K}_{\text{sub}} \rightarrow \delta$ at the substrate level, reducing the memory sector contribution to a δ -localised continuum contribution that combines with the leading \mathcal{S}_{EH} integrand.

The recovery therefore operates at the microscopic substrate layer once the layered consistency of the three levels — variational, continuum-emergence, and substrate — is invoked through Einstein–Hilbert Emergence paper Proposition 8.1 and Theorem 9.1 above.

Structural significance

Theorem 11.1 establishes GR recovery at the deepest layer of the VERSF programme: the recovery operates at the level of the $K = 7$ commitment foam itself, with every continuum-limit object in the Einstein equation tracing to a substrate-level structural object on the foam. Combined with Theorem 9.1 for the action level, Theorem 8.2 of the Einstein–Hilbert Emergence paper for the equation level, and dynamical paper Theorem 10.2 for the field-equation level, the present Theorem 11.1 closes the multi-layer GR-recovery statement at the microscopic substrate layer.

12. Structural Interpretation

The present paper carries the inversion already implicit in the Einstein–Hilbert Emergence paper one additional layer down. Gravity is now not a primitive geometric law at any layer of the VERSF construction:

- *Variational layer*: \mathcal{S}_{EH} is the unique admissible action within the continuum-limit operator basis (variational paper Theorem 5.1).
- *Continuum-emergence layer*: the continuum-limit operator basis itself — $\Gamma, R^{\alpha}_{\beta\mu\nu}, R, \sqrt{|g|}$ — emerges from refinement transport geometry (Einstein–Hilbert Emergence paper cascade Theorem 3.5 \rightarrow 4.1 \rightarrow 5.2 \rightarrow 6.2 / 6.3 \rightarrow 7.1 \rightarrow 8.2).
- *Microscopic substrate layer (present paper)*: the refinement transport operator T_{γ} underlying that cascade is constructed from $K = 7$ commitment-foam combinatorics, with closure compatibility, BCB triangle-closure-forced torsion-freeness, and refinement-Cauchy convergence supplying the substrate origins of the continuum-emergence cascade.

The structural progression of the geometric side of the VERSF programme is now:

$K = 7$ commitment foam \rightarrow discrete commitment holonomy \rightarrow closure-compatible refinement \rightarrow continuum-limit refinement transport (T_{γ} , §5) \rightarrow continuum connection (Γ , §6) \rightarrow discrete

plaquette curvature \rightarrow Riemann tensor (§7) \rightarrow Lorentzian compatibility (§8) \rightarrow Einstein–Hilbert recovery (§9) \rightarrow memory kernel emergence (§10) \rightarrow microscopic GR recovery (§11).

Derivational chain through the present paper

The internal derivational structure of the present paper assembles as follows:

Substrate Layer	Continuum-Emergent Quantity	Theorem
Edge transport U_{ij} on $K = 7$ foam	Substrate holonomy at refinement order n	Definitions 4.1–4.3
Closure-compatible refinement sequence	Continuum refinement transport T_γ	Theorem 5.3
First-order generator of T_γ	Levi-Civita connection $\Gamma^\alpha_{\{\beta\mu\}}$	Theorem 6.2
Triangle closure (BCB C_4)	Torsion-freeness $\Gamma^\alpha_{\{\beta\mu\}} = 0$	Lemma 6.3
Discrete plaquette curvature	Riemann tensor $R^\alpha_{\{\beta\mu\nu\}}$	Theorem 7.2
Commitment-order partial order	Lorentzian causal compatibility	Theorem 8.2
Discrete substrate cascade	Einstein–Hilbert action \mathcal{S}_{EH}	Theorem 9.1
Residual closure-history operator	Bilocal memory kernel $\mathcal{K}(x, x')$	Theorem 10.2
Recovery regime + weak-memory limit	Standard GR $\mathcal{S}_{EH} + \mathcal{S}_{matter}$	Theorem 11.1

Correspondence with the foundational programme

The substrate construction unifies two previously semi-independent strands:

K = 7 / BCB / TPB Strand	Continuum Transport-Geometry Strand
$K = 7$ closure architecture	Refinement-stable holonomy
BCB triangle closure C_4	Torsion-free continuum connection
TPB temporal microstructure	Geometric refinement $\varepsilon_n \leq C\lambda^n$
Commitment-order partial order	Lorentzian causal structure
Closure-history residual	Bilocal memory kernel
Hexagonal cell coordination	Distinguishability-volume density

In each row, the right-hand entry is the continuum-limit shadow of the left-hand substrate structure. The two strands of the VERSF programme are therefore not independent: the continuum transport-geometry strand is the coarse-grained image of the $K = 7 / BCB / TPB$ strand under closure-compatible refinement.

13. Open Problems

OP numbering convention. The OP numbers in this section are local to the present paper. Where an OP corresponds to a problem also recorded in an inherited paper under a different number, the cross-walk is given inline (e.g. "this is the same open problem as Einstein–Hilbert Emergence paper OP1 / variational paper OP6 / dynamical paper OP6"). The local numbering is reset to keep §13 internally readable; the cross-walk preserves cross-paper coherence.

The paper does **not** close:

- the closure-normalisation factor C_λ ,
- the explicit form, decay law, and amplitude of the memory kernel $\mathcal{K}(x, x')$ (only its existence and causal support are established),
- the explicit values of the anisotropic Wilson coefficients ($\alpha_{\hat{Q}}, \beta_{\hat{Q}}, \gamma_{\hat{Q}}$) of the variational paper,
- Standard-Model matter coupling,
- quantum fluctuations of T_γ or of the commitment foam itself,
- subleading higher-curvature corrections from finer substrate refinement mismatch.

Each is recorded below.

OP1 — Substrate derivation of the closure-normalisation factor C_λ

The structural form $\kappa_{\text{eff}} = 8\pi C_\lambda \hbar \xi^2 / c^3$ is inherited from dynamical paper §2.7. Closing C_λ substrate-derivatively from fold energetics on the $K = 7$ architecture would constitute a substrate-level derivation of Newton's constant. This is the same open problem as Einstein–Hilbert Emergence paper OP1 / variational paper OP6 / dynamical paper OP6, recorded here unchanged.

OP2 — Form and amplitude of the memory kernel

Theorem 10.2 establishes that $\mathcal{K}(x, x')$ is non-zero generically and supported on the causal past of x . The specific functional form, decay law, and amplitude of $\mathcal{K}(x, x')$ — and its dependence on the closure-history structure \mathcal{K}_{sub} at the substrate level — remain open. Closing this would yield the κ -field amplitude of the dynamical paper and the explicit memory sector coupling of the variational paper. This corresponds to variational paper OP3, sharpened here to "form and amplitude" given the existence and causal support are now established.

OP3 — Anisotropic Wilson coefficients

The substrate origin of the anisotropic transport-curvature Wilson coefficients ($\alpha_{\hat{Q}}, \beta_{\hat{Q}}, \gamma_{\hat{Q}}$) of variational paper §6 is not addressed by the present construction. Closing this would mean deriving the anisotropic sector amplitudes from $K = 7$ commitment-foam combinatorics, parallel to the memory-kernel amplitude problem (OP2). This is variational paper OP4, inherited unchanged.

OP4 — Higher-order refinement structure and subleading corrections

The present construction operates at leading admissible refinement order (linear refinement scaling $\varepsilon_{n+1}/\varepsilon_n = \lambda < 1$ from TPB, leading-order BCB triangle closure). Subleading refinement orders generate, in principle, higher-curvature corrections — Gauss–Bonnet at the next admissible second-order Lovelock level (in $D + 1 > 4$), and R^2 , $R_{\{\mu\nu\}}R^{\{\mu\nu\}}$, $R_{\{\mu\nu\rho\}}R^{\{\mu\nu\rho\}}$ at the next order in derivative count. The substrate structural form of these corrections from finer plaquette mismatch is open. This is Einstein–Hilbert Emergence paper OP3, lifted to the microscopic substrate layer here.

Structural-pattern remark. The $K = 7$ Wilson Limit paper §11.8 records — as a methodological observation rather than a derived equivalence — that several constructions in the VERSF programme have suggested stable physical closure may generically require second-order consistency rather than merely first-order admissibility (spinorial closure requiring double traversal, cohomology distinguishing closed cochains from cohomology classes, and the refinement programme's separation of admissible-but-trivialising scalars from refinement-persistent cohomology). The present paper's situation may fit this pattern: leading-order refinement (triangle closure C_4 at order ε , plaquette curvature at order ε^2) identifies the admissible Einstein–Hilbert structure, while higher-curvature corrections from subleading refinement orders may be needed for full physical closure in the sense of capturing residual non-Einstein-Hilbert content of the substrate dynamics. The two cautions of Wilson §11.8 apply unchanged here: "RG-order" and "closure-cycle" senses of "second order" are mathematically distinct, and the structural resonance is interpretive rather than theorematic.

OP5 — Matter coupling

The present paper does not derive Standard-Model matter. Matter enters only through the inherited recovery limit $T^{\kappa}{}_{\mu\nu} \rightarrow T^{\text{(matter)}}{}_{\mu\nu}$ of dynamical paper Theorem 10.2 / variational paper Theorem 10.1, with the present substrate construction supplying the geometric sector only. Closing the substrate origin of matter would mean producing stable matter sectors as substrate excitations on the $K = 7$ foam — candidate routes include stable transport-localised excitations, topological commitment defects, persistent closure modes, and refinement-stable cohomological structures. This is Einstein–Hilbert Emergence paper OP5 / variational paper OP7 / source paper OP7, inherited unchanged.

OP6 — Quantum completion of refinement transport

The present treatment is classical / effective: fluctuations of T_{γ} and of the substrate foam are not treated. Closing the quantum completion would mean producing a quantum-substrate version of refinement transport, with quantum fluctuations of U_{ij} , the plaquette curvature, the commitment-history operator, and the foam combinatorics treated within a coherent quantum framework. This is Einstein–Hilbert Emergence paper OP6 / variational paper OP8, lifted to the microscopic substrate layer here. The natural framing is: what is the discrete quantum gravity that has the present construction as its classical limit?

OP7 — Refinement-stable foam dynamics and global existence

The present paper establishes existence of T_γ under closure-compatible refinement on a fixed background foam. The dynamics of the foam itself — how the $K = 7$ commitment foam evolves under irreversible commitment, how new commitment events update the foam architecture, and how global existence of the continuum limit is guaranteed beyond local bounded regions — is a deeper open question. This is the substrate analogue of the Einstein–Hilbert Emergence paper OP8 (refinement-stable transport from below): the present paper closes the existence and stability of T_γ on a given closure-compatible foam, while the existence and stability of the *foam itself* under commitment dynamics is the deeper residual question. Closing OP7 would mean producing the foam-level dynamical structure from which closure-compatible refinement sequences emerge as a derived rather than postulated property.

OP8 — Variational microscopic completion

The full variational structure of $\mathcal{S}_{\text{VERSF}}$ — the κ -sector, the memory sector, the anisotropic sector, the exchange sector — receives a microscopic substrate construction only partially in the present paper (the geometric sector via Theorem 9.1; the memory sector at the existence level via Theorem 10.2). Closing OP8 would mean producing the full microscopic substrate construction of every sector of $\mathcal{S}_{\text{VERSF}}$ on the $K = 7$ foam, with all Wilson coefficients and coupling amplitudes derived rather than postulated. This is the natural successor paper to the present one.

OP9 — Substrate derivation of the transport group \mathcal{G}

Definition 3.1 identifies the substrate-level edge transport group with $\mathcal{G} = \text{SO}(1, D)$, the local Lorentz group of the inherited continuum-limit signature. This identification is inherited from the Lorentzian Completion paper (recorded as a structural inheritance in §2, with parallel acknowledgement in the Theorem 8.2 Remark), not derived from $K = 7$ / BCB combinatorics at the substrate level. Closing OP9 would mean constructing an admissible transport group \mathcal{G}_{sub} from substrate combinatorics independent of the inherited signature, with $\mathcal{G}_{\text{sub}} \rightarrow \text{SO}(1, D)$ emerging in the continuum limit by the same coherence-propagation mechanism that the CCC framework supplies for the metric. This is a genuinely substrate-level derivation problem distinct from OP7 (foam dynamics) and OP8 (variational microscopic completion), and is recorded separately to keep the inheritance structure honest.

Methodological precedent. The $K = 7$ Wilson Limit paper of this programme supplies a worked **U(1) gauge-bundle sector** template for the structurally parallel question: it shows that the $K = 7$ closure structure dynamically constrains the effective gauge transport theory toward Lorentz compatibility via one-loop matching, with the rate fixed by the bare coupling $\beta_{K=7} = 2^7 \cdot 15/14$ inherited from the closure-counting integer $N_{\text{loop}} = 14$. OP9 is the **SO(1, D) tangent-bundle sector** analogue: showing that the same $K = 7$ closure structure dynamically constrains \mathcal{G}_{sub} toward $\text{SO}(1, D)$ under closure-compatible refinement. The two are different transport problems on the same substrate (gauge bundle vs tangent bundle); the Wilson paper's methodology — anisotropic Wilson construction, one-loop matching at fixed scale, bare-coupling-controlled enhancement rate — is the natural template (with the caveat that one-loop matching machinery in the abelian U(1) case may require non-abelian extension to apply to $\text{SO}(1, D)$, which is non-abelian and $(D + 1)(D + 2)/2$ -dimensional — ten generators at $D + 1 = 4$ — rather than abelian and one-dimensional). The Wilson paper's §9 honesty lesson also carries over: a one-step

substrate derivation is unlikely to deliver full Lorentzian signature unless the $K = 7$ bare structure already nearly supplies it, so OP9 may need to be paired with a substrate-physical input establishing near-isotropy of the bare \mathcal{G}_{sub} structure (the analogue of the small-bare-anisotropy resolution R1 of the Wilson paper §9.2).

14. Conclusion

The previous VERSF papers established:

- the $K = 7$ simplicial closure architecture (foundational papers),
- BCB / TPB and finite distinguishability,
- irreversible commitment ontology,
- Lorentzian causal completion,
- refinement-stable transport geometry,
- effective stress–energy structure,
- Einstein-compatible dynamics,
- a unified covariant action $\mathcal{S}_{\text{VERSF}}$ with action-level GR recovery,
- and substrate emergence of the Einstein–Hilbert action.

The present paper closes the residual existence-and-stability problem for the refinement transport operator T_γ itself.

The central results are:

- **Theorem 5.3.** The refinement transport operator T_γ exists as a continuum limit of closure-compatible substrate path holonomy on the $K = 7$ commitment foam, with refinement-Cauchy convergence guaranteed by BCB / TPB closure architecture and geometric refinement scaling. The continuum-limit T_γ is independent of the refinement-order sequence and coincides with the refinement transport operator inherited by the Einstein–Hilbert Emergence paper. (Closes OP2 and OP8 of that paper.)
- **Theorem 6.2 and Lemma 6.3.** The first-order generator of T_γ in the continuum limit coincides with the Levi-Civita connection of the inherited Lorentzian Completion geometry, with torsion-freeness following directly from the BCB triangle closure constraint C_4 at the discrete level. The match with Einstein–Hilbert Emergence paper Theorem 3.5 is exact, with the substrate construction now supplied below the inherited continuum-limit statement.
- **Theorem 7.2.** The discrete plaquette curvature on minimal simplicial loops converges in the continuum limit to the Riemann curvature tensor of the inherited Levi-Civita geometry, supplying the substrate construction below Einstein–Hilbert Emergence paper Theorem 4.1.
- **Theorem 8.2.** The commitment-order partial order on the $K = 7$ foam is compatible with the Lorentzian signature of the inherited continuum-limit metric. The present paper does not re-derive Lorentzian completion; it establishes the substrate-level consistency check.
- **Theorem 9.1.** The substrate-derived Einstein–Hilbert action is

$$\mathcal{S}_{\text{EH}} = (1 / 2\kappa_{\text{eff}}) \int d^{(D+1)}x \sqrt{|g|} (R - 2\Lambda_{\text{eff}}),$$

with κ_{eff} inheriting the substrate form $8\pi C_{\lambda} \hbar \xi^2 / c^3$, and every operator in the integrand now carrying an explicit substrate origin on the $K = 7$ commitment foam.

- **Theorem 10.2.** A non-zero bilocal memory kernel $\mathcal{K}(x, x')$ supported on the causal past emerges generically from irreversibility of commitment under closure-compatible refinement, supplying the substrate origin of the memory sector at the existence and causal-support level. The form and amplitude of \mathcal{K} remain open (§13 OP2).
- **Theorem 11.1.** Under the recovery regime $(R1)–(R3) \cap (B1)–(B2) \cap \{\mathcal{K} \rightarrow \delta\}$, the substrate-level $\mathcal{S}_{\text{VERSF}}$ reduces continuously to standard general relativity plus matter coupling, with the recovery now operating at the microscopic substrate layer.

The result therefore shifts the standing of the refinement transport operator T_{γ} in the VERSF framework from

"the continuum-limit object inherited by the geometric programme"

— that was the framing through the Einstein–Hilbert Emergence paper — to

"the closure-compatible refinement limit of discrete $K = 7$ simplicial commitment-foam holonomy"

— which is the present paper. The two statements are layered consistently: the inherited continuum-limit T_{γ} of the geometric programme and the substrate-derived T_{γ} of the present paper are the same object, with the present paper supplying the microscopic substrate construction that the geometric programme previously inherited as input architecture.

Honest limitation. The present paper closes the existence-and-stability problem for T_{γ} — the operator and its convergence are now derived from $K = 7$ / BCB / TPB substrate combinatorics rather than inherited as continuum-level structure. What it does *not* close is: the form and amplitude of the memory kernel (only existence is established, §13 OP2), the closure-normalisation factor C_{λ} (§13 OP1), the anisotropic Wilson coefficients (§13 OP3), subleading higher-curvature corrections beyond leading admissible order (§13 OP4), the substrate origin of matter (§13 OP5), the quantum completion of refinement transport (§13 OP6), the dynamics of the foam itself (§13 OP7), the full variational microscopic completion of every sector of $\mathcal{S}_{\text{VERSF}}$ (§13 OP8), and the substrate-derived form of the transport group \mathcal{G} (the identification $\mathcal{G} = \text{SO}(1, D)$ remains inherited from the Lorentzian Completion paper, §13 OP9). These are the genuinely deeper questions that the present paper sharpens rather than resolves.

The structural progression of the geometric side of the VERSF programme is now:

commitment ontology $\rightarrow K = 7$ simplicial closure architecture \rightarrow BCB / TPB closure structure \rightarrow discrete commitment holonomy \rightarrow closure-compatible refinement \rightarrow continuum refinement transport \rightarrow continuum connection \rightarrow discrete plaquette curvature \rightarrow Riemann emergence \rightarrow

Lorentzian compatibility \rightarrow Einstein–Hilbert recovery \rightarrow memory kernel existence \rightarrow microscopic GR recovery.

The geometric side is now structurally closed at the microscopic substrate layer for the existence-and-stability problem of T_γ specifically, with the memory-kernel form and amplitude (OP2), the foam-dynamics problem (OP7), and the full variational microscopic completion (OP8) explicitly recorded as remaining. Each is sharply isolated, each has a clear successor-paper home, and none remains a structural existence question.

The picture this leaves is: gravity is the continuum-limit thermodynamic geometry of closure-compatible irreversible commitment transport on the $K = 7$ simplicial substrate. The continuum geometric structure — the Lorentzian continuum, the Levi-Civita connection, the Riemann curvature tensor, the Einstein–Hilbert action, the field equations — is the coarse-grained refinement-limit shadow of discrete commitment-foam combinatorics at the substrate level, with closure compatibility, BCB / TPB refinement architecture, and irreversibility of commitment supplying the structural inputs that produce general relativity in the appropriate effective limit.

15. References to Inherited VERSF Papers

The present paper carries inline citations to several earlier VERSF papers. For a stand-alone read, each is identified below by title and the specific results invoked.

- **Foundational Papers** — *K = 7 Simplicial Closure Architecture, BCB / TPB, and the Hexagonal Substrate*. Inherited: the $K = 7$ closure architecture (six adjacency-coding boundary vertices plus one gauge-anchoring central vertex per cell, fourteen oriented closure traversals), the constraint decomposition C_1 – C_3 (edge consistency) + C_4 (triangle / fold completion) + C_5 – C_7 (hexagonal embedding), Bit Conservation and Balance (BCB), Ticks-Per-Bit (TPB), and the geometric refinement scaling $\varepsilon_n \leq C \lambda^n$ with $0 < \lambda < 1$.
- **Source-Structure Paper** — *Effective Source Structure and Admissibility Closure in VERSF*. Inherited: finite distinguishability (its §2.1), the irreversible-commitment ontology, the four-sector decomposition $T^{\text{eff}}_{\mu\nu}$, total conservation, and the Admissible Source Uniqueness Theorem.
- **Lorentzian Completion Paper** — *Lorentzian Causal Completion of Emergent Commitment Geometry in VERSF*. Inherited: the emergent Lorentzian continuum geometry $g_{\mu\nu}$, the Levi-Civita connection structure (metric-compatible, torsion-free) on the inherited continuum, the CCC framework, the coherence scale ξ , the causal-propagation structure, and the $(-, +, +, +)$ signature of the continuum-limit metric (which the present paper establishes only compatibility with, not re-derivation of).
- **Transport-Geometry Papers** (cited as the inherited transport-geometry framework of the VERSF programme) — refinement-stable holonomy framework at the continuum-limit level, the convergence of refinement transport to the Levi-Civita parallel transport, and the closure-compatibility condition. The present paper supplies the microscopic substrate construction below this framework; it does not re-derive its continuum-level results.

- **Dynamical-Geometry Paper** — *Bianchi-Compatible Geometry and Einstein-Compatible Dynamics in VERSF*. Inherited: the Bianchi-Compatible Geometry Theorem, the tensorial closure structure into $(1,1) \oplus (0,0)$, the substrate-scale coupling form $\kappa_{\text{eff}} = 8\pi C_{\lambda} \hbar \xi^2 / c^3$, the GR Recovery Theorem with (R1)–(R3) and (B1)–(B2), and the $H^k(K)$ Sobolev continuity bound.
- **Variational Paper** — *Unified Covariant Action for Emergent Commitment Geometry in VERSF*. Inherited: the unified covariant action $\mathcal{S}_{\text{VERSF}}$, the Variational Closure Theorem, the action-level Einstein–Hilbert uniqueness theorem within the continuum-limit operator basis, the constrained-exchange action \mathcal{S}_{ex} with Lagrange-multiplier framing, the action-level GR Recovery Theorem, and the memory-sector structure with bilocal kernel $\mathcal{K}(x, x')$.
- **Einstein–Hilbert Emergence Paper** — *Substrate Derivation of the Einstein–Hilbert Action in VERSF*. Inherited: Definition 3.4 (first-order generator formula for $\Gamma^{\alpha}_{\beta\mu}$, here Definition 6.1), Theorem 3.5 (continuum-limit emergence of the connection from refinement transport, here Theorem 6.2), Theorem 4.1 (transport–holonomy emergence of the Riemann tensor, here Theorem 7.2), Theorem 5.2 (Ricci scalar emergence, invoked in Theorem 9.1), Theorems 6.2 and 6.3 (measure emergence and uniqueness, invoked in Theorem 9.1), Theorem 7.1 (Einstein–Hilbert emergence, here Theorem 9.1 at the substrate layer), and Theorem 8.2 (substrate-level GR recovery, here Theorem 11.1 at the microscopic substrate layer). The present paper closes OP2 (existence question for T_{γ}) and OP8 (stability question for T_{γ}) of the Einstein–Hilbert Emergence paper.

Numerical results and structural conventions inherited from these papers — including $m^2 = (4/3)\xi^{-2}$, $\kappa_{\text{eff}} = 8\pi C_{\lambda} \hbar \xi^2 / c^3$, the $K = 7$ closure architecture, and the $N_{\text{loop}} = 14$ closure traversal count — are treated throughout the present paper as established inputs.