Non-Conformal Metric Corrections from Refinement-Stable Causal Transport Curvature in VERSF

Symmetric Quadratic Contractions of Antisymmetric Transport Curvature, Trace–Traceless Decomposition, Algebraic Uniqueness, and Anisotropic Geodesic Structure Beyond the Leading-Conformal Geometry

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General-Reader Summary

Imagine the geometry of space as a stretchy rubber sheet. In an earlier paper in this programme, we showed that defects in the underlying VERSF substrate locally stretch this sheet — but the stretching there was always *uniform in every direction*. Pull on the sheet at one point and it stretched the same amount whether you measured left-right, up-down, or anything in between. Geometers call that kind of uniform local stretching *conformal*.

Real geometry — anything that hopes to one day look like gravity — has to do more than that. It has to bend things differently in different directions. A magnifying glass that just enlarged everything by the same factor everywhere would be a zoom, not a lens; an actual lens bends light differently along different paths. The geometry that physics needs has to be *directional*.

The question this paper asks is straightforward:

Can the VERSF substrate produce *directional* distortions of geometry, not just uniform stretching?

The answer is yes — but getting there means solving an awkward puzzle.

The mathematical object that carries directional information about how distinguishable facts get settled and propagated through the substrate — twisting and rotating as they travel — is called the **transport curvature**. The trouble is that it has the wrong symmetry to be a piece of geometry. Geometry needs quantities that look the same when you swap their two indices (symmetric); the transport curvature flips sign when you swap them (antisymmetric). You simply cannot plug it into a metric.

The resolution is to combine the transport curvature *with itself*, in a specific way that produces a symmetric directional quantity. Think of it as squaring a swirl: the swirl itself has no preferred orientation as a thing-that-points, but the *pattern of how strongly it swirls in each direction* does, and that pattern is symmetric. This gives a well-defined tensor that *can* sit inside the geometry.

Then comes a subtlety that earlier discussions glossed over. When you build a symmetric tensor this way, part of what you get is *still* uniform stretching — it just looks dressed up as something more elaborate. Only what remains *after* you subtract that uniform-stretching part is genuinely directional. The paper separates these two pieces cleanly. The uniform piece is absorbed into the overall stretch factor (where it belongs); the leftover piece is the genuinely new content.

What the paper establishes about this leftover, genuinely directional correction is that it:

- has the right symmetry to sit inside the geometry,
- is *unique* among the symmetric quantities you could build at this order — there is no other essentially different way to do it, only one number (a coupling strength) is left free,
- survives consistently as you zoom in further and further on the substrate (it does not wash out or blow up),
- is *concentrated* near substrate defects rather than smeared across all of space, and decays *twice as fast* with distance as the transport curvature does (because we built it by multiplying two copies of the curvature together, and two exponentials multiplied together fall off twice as fast),
- and produces real direction-dependent bending of paths — a particle moving north and a particle moving east through the same point now experience genuinely different deflection, in a way that no uniform stretching can mimic.

This is the first place in the VERSF programme where the substrate produces *directional* distortion of geometry, not just uniform local stretching.

What the paper does **not** do is derive Einstein's equations, or say how this geometry evolves in time, or fix how strong the directional correction is. Those are next. The causal side of the picture — how time enters this geometry, and which directions signals can travel in — is worked out separately in other parts of the programme; this paper is about the spatial side, where the directional stretching lives. What it *does* do is install the geometric structure that any future theory of gravity in VERSF will have to work with. The open question is no longer "can the substrate produce geometry?" — that was settled in earlier papers. The question now is: "what dynamics does this directional geometry obey, and what principle fixes the strength of the directional correction?"

Table of Contents

Abstract

Previous stages of the programme established flat continuum emergence (Stage V), localized coherence defects (Stage VIII), coupled global transport dynamics, tensorial transport curvature, refinement-stable continuum transport observables, and transport-action-selected conformal metric rescaling $g^{(0)}_{ij} = \delta_{ij} / \varepsilon_{gap}^2$.

The present paper develops the first non-conformal metric corrections arising from refinement-stable transport curvature. The refinement-stable transport-curvature tensor $R^{(\infty)}_{ij}$ is antisymmetric and therefore cannot directly equal the symmetric Ricci tensor of any metric geometry. We resolve this algebraic obstruction by constructing the leading symmetric quadratic contraction $Q_{ij} := R^{(\infty)}_{ik} R^{(\infty)}{}_j{}^k$ and isolating its genuinely non-conformal content via the trace–traceless decomposition.

We establish:

1. **Symmetric quadratic contractions (Theorem 3.2, Proposition 3.3).** Q is symmetric and positive-semidefinite, with kernel equal to $\ker R^{(\infty)}$.
2. **Trace–traceless decomposition (Theorem 3.5).** Q_{ij} splits uniquely into a trace part absorbable into the conformal factor and a traceless part $\hat{Q}_{ij}$ carrying genuine anisotropy.
3. **Algebraic uniqueness at quadratic order (Theorem 3.6).** Among all symmetric rank-2 tensors built algebraically from $R^{(\infty)}$ — no derivatives, bilinear, using only the flat-background invariant — the genuinely non-conformal content is uniquely $\lambda \hat{Q}_{ij}$ up to scalar normalization.
4. **Non-conformal metric corrections (Theorem 4.2).** The continuum metric acquires the form $g_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$; the second term cannot be removed by any conformal rescaling whenever $\hat{Q}_{ij} \neq 0$.
5. **Refinement stability (Theorem 4.5).** Under uniform $W^{1,p}_{loc}$ bounds on the refinement sequence with $p > D/2$, Rellich–Kondrachov upgrades weak convergence of $R^{(\infty)}$ to strong L^{2p}_{loc} convergence; Q_{ij} and $\hat{Q}_{ij}$ then converge in L^p_{loc}. Weak convergence of R alone is insufficient.
6. **Double-rate localization (Theorem 5.2).** Non-conformal corrections decay exponentially at Combes–Thomas rate $2\eta_\infty$ — double the rate for $R^{(\infty)}$ — as a direct consequence of bilinearity, with explicit constants tracked.
7. **Anisotropic geodesic structure (Theorem 6.3, Corollary 6.4).** The leading-order Christoffel correction induced by $\hat{Q}$ — given covariantly by the standard metric-perturbation linearization around the conformal background — drives a direction-dependent deflection of geodesics that no conformal correction can produce.

The paper does not derive Einstein equations, Lorentzian signature, stress-energy tensors, or gravitational dynamics. Its contribution is structural: refinement-stable transport curvature now generates genuine non-conformal continuum metric structure within VERSF, in a form fixed up to a single coupling λ.

1. Introduction

The previous paper in the programme established that the transport action determines a conformal rescaling of the flat Stage V continuum metric:

$g^{(0)}\_{ij}(x) = \delta\_{ij} / \varepsilon\_{gap}(x)^2$

That result was intentionally restricted to leading conformal order. The present paper develops the first genuinely anisotropic metric corrections beyond this leading-conformal structure.

The central obstacle is algebraic. The refinement-stable transport-curvature tensor $R^{(\infty)}\_{ij}$ is antisymmetric, whereas metric corrections must be symmetric. Direct identification with a Ricci-type tensor is impossible at first order in $R^{(\infty)}$. The resolution is to pass to the *quadratic* level: the leading symmetric tensor available from $R^{(\infty)}$ is the contraction

$Q\_{ij} := R^{(\infty)}\_{ik} R^{(\infty)}\_{j}{}^{k}$.

A subtlety, treated explicitly here, is that Q_{ij} carries a residual trace contribution that is *not* new geometric content: it is merely an additional conformal rescaling. The genuinely non-conformal information is contained in the traceless symmetric part $\hat{Q}\_{ij}$. The trace–traceless decomposition is the technical core of the paper, and the algebraic uniqueness of $\hat{Q}$ at this order (Theorem 3.6) ensures that the construction is not arbitrary.

Within the broader BCB/admissibility framework developed in companion papers of the programme, the refinement-stable transport-curvature tensor $R^{(\infty)}\_{ij}$ is to be read not merely as an abstract transport observable but as a coarse-grained measure of anisotropic *causal fact-commitment transport* on the admissible interface substrate. The substrate-level dynamics being coarse-grained — finite-capacity propagation of distinguishability, local fact-commitment flow, and anisotropic commitment transport near coherence defects — are the physical content of the "transport" described abstractly here. The present paper studies how the directional structure of that transport propagates into continuum metric anisotropy at quadratic order.

The construction also adopts the *parallel-emergence* interpretation developed in the interface-transport framework: transport structure and continuum metric structure emerge from common substrate-level admissibility dynamics, but the transport bundle (carrying its discrete substrate structure) is *not* identified with the continuum's spatial tangent bundle. The two are independent emergent layers, coupled only through symmetric invariants of $R^{(\infty)}$. This is the structural reason an antisymmetric $R^{(\infty)}$ cannot itself serve as a Ricci-type tensor: it lives on a different bundle. The metric anisotropy $\lambda \hat{Q}\_{ij}$ is the coarse-grained image of transport-bundle structure projected into the tangent-bundle sector, mediated by the bilinear invariant — not by a bundle isomorphism.

Three further structural results round out the analysis. First, the refinement-stability of Q is not automatic: weak convergence of $R^{(\infty)}$ does not imply weak convergence of bilinears in $R^{(\infty)}$. We therefore upgrade to strong convergence via Rellich–Kondrachov, leveraging the $W^{1,p}\_{loc}$ bound established in the Refinement-Stable Holonomy paper. Second, the exponential localization rate of Q is *double* that of $R^{(\infty)}$ by direct bilinearity — a sharp consequence rather than a heuristic. Third, the leading-order Christoffel correction induced by $\hat{Q}$ is computed in both covariant and coordinate form, and shown to produce direction-dependent geodesic deflection that no conformal correction can mimic.

Throughout, results are labelled by epistemic status:

- **Proven** — established here or in cited prior papers under the stated assumptions.
- **Conditional** — follows from proven results given an explicitly stated additional input (typically: the value of the coupling λ, or the strong-convergence hypothesis on $R^{(\infty)}$).
- **Conjectural** — plausibility argument only; flagged as such.

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2. Structural Setup and Inheritance

We inherit, without re-derivation, the following structures from prior papers in the programme:

Structure	Source
Flat continuum (Stage V)	Stage V emergence papers
Localized coherence defects (Stage VIII)	Stage VIII defect papers
Coupled global transport dynamics	Global Transport paper
Continuum transport-curvature tensor $R^{(\infty)}_{ij}$	Tensorial Transport Geometry paper
Refinement stability of $R^{(\infty)}$ in $W^{1,p}_{loc}$	Refinement-Stable Holonomy paper
Conformal metric $g^{(0)}_{ij} = \delta_{ij} / \varepsilon_{gap}^2$	Transport-Action Metric Selection paper
Combes–Thomas rate η_∞ for $R^{(\infty)}$ localization	Refinement-Stable Holonomy paper
BCB/admissibility framework; causal fact-commitment substrate dynamics	Interface-transport paper
Substrate-level toy models for finite-capacity propagation and anisotropy emergence	Interface-transport paper, Apps. G–H
Continuum-limit convergence of TPB refinement sequences and refinement-stable cone-field emergence	Continuum-Limit and Explicit Refinement Dynamics papers
Causal–coherence compatibility (CCC) and observable covariance under Commitment Reordering Equivalence (CRE)	CCC and CRE / observable-covariance programmes

The transport-curvature tensor satisfies the antisymmetry

$$R^{(\infty)}_{ij}(x) = - R^{(\infty)}_{ji}(x), \text{ for all } x \in M,$$

where M denotes the emergent D-dimensional continuum ($D \geq 2$). All index manipulations in this paper are performed with respect to the flat background δ_{ij}; corrections to the metric appear additively and are tracked to leading order in the (small) coupling λ.

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3. Symmetric Quadratic Contractions of Transport Curvature

Definition 3.1 — Leading Symmetric Contraction

Define the symmetric quadratic transport-curvature tensor

$Q_{ij}(x) := R^{(\infty)}{}_{ik}(x)\, R^{(\infty)}{}_{j}{}^{k}(x) = \Sigma_k\, R^{(\infty)}{}_{ik}(x)\, R^{(\infty)}{}_{jk}(x),$

where the second equality uses the flat background δ to raise k.

Theorem 3.2 — Symmetry [proven]

For all $x \in M$, $Q_{ij}(x) = Q_{ji}(x)$.

Proof. Pointwise, the components $R^{(\infty)}{}_{ik}(x)$ are real numbers and multiplication of reals is commutative. Hence

$Q_{ij} = \Sigma_k\, R^{(\infty)}{}_{ik}\, R^{(\infty)}{}_{jk} = \Sigma_k\, R^{(\infty)}{}_{jk}\, R^{(\infty)}{}_{ik} = Q_{ji}.$

The antisymmetry of $R^{(\infty)}$ is not used; only commutativity of scalar multiplication.

Proposition 3.3 — Positive Semidefiniteness [proven]

For any tangent vector $\xi \in T_x M$,

$Q_{ij}(x)\, \xi^i \xi^j = \Sigma_k \left(R^{(\infty)}{}_{ik}(x)\, \xi^i \right)^2 \geq 0,$

with equality if and only if $R^{(\infty)}{}_{ik}(x)\, \xi^i = 0$ for every k — equivalently, $\xi \in \ker R^{(\infty)}(x)$, where $R^{(\infty)}(x)$ is viewed as a real D×D antisymmetric matrix. (Antisymmetry gives $\ker R^{(\infty)} = \ker R^{(\infty)T}$, so the joint-kernel condition collapses to a single one.)

Proof. The middle equality is the sum-of-squares identity for the symmetric bilinear form $\Sigma_k (R^{(\infty)}{}_{ik}\, \xi^i)^2$. Nonnegativity is immediate. The kernel characterization is the equality clause of the same identity.

Theorem 3.4 — Trace Computation [proven]

The trace of Q with respect to the flat background is

$\mathrm{tr}(Q)(x) := \delta^{ij} Q_{ij}(x) = \Sigma_{i,k} (R^{(\infty)}_{ik}(x))^2 = \|R^{(\infty)}(x)\|_F^2$,

the squared Frobenius norm of $R^{(\infty)}(x)$. In particular $\mathrm{tr}(Q)(x) \geq 0$, with equality iff $R^{(\infty)}(x) = 0$. Because $R^{(\infty)}$ is antisymmetric, $R^{(\infty)}_{kk} = 0$ for all k, so only the $D(D-1)$ off-diagonal components contribute to the sum.

Theorem 3.5 — Trace–Traceless Decomposition [proven]

Q_{ij} decomposes uniquely as

$Q_{ij}(x) = \hat{Q}_{ij}(x) + (1/D) \|R^{(\infty)}(x)\|_F^2 \cdot \delta_{ij}$,

where the **traceless part** is

$\hat{Q}_{ij} := Q_{ij} - (1/D) \|R^{(\infty)}\|_F^2 \cdot \delta_{ij}, \delta^{ij} \hat{Q}_{ij} = 0$.

The trace part $(1/D) \|R^{(\infty)}\|_F^2 \cdot \delta_{ij}$ is proportional to δ_{ij} and is therefore conformally equivalent to a rescaling of the background metric. The traceless part $\hat{Q}_{ij}$ carries the genuinely non-conformal content.

Proof. The decomposition is the canonical split of a symmetric tensor into trace and traceless parts under the orthogonal-group action on $\mathbb{R}^D$. Uniqueness follows from the direct-sum decomposition $\mathrm{Sym}^2(\mathbb{R}^D) = \mathbb{R}\cdot\delta \oplus \mathrm{Sym}^2_0(\mathbb{R}^D)$ under the Frobenius inner product, where Sym^2_0 denotes the traceless symmetric matrices. That $\hat{Q}_{ij}$ is traceless by construction is immediate from Theorem 3.4.

Proposition 3.5.1 — Dimensional Triviality at D = 2 [proven]

In dimension $D = 2$, $\hat{Q} \equiv 0$ identically, and the entire non-conformal-correction construction collapses to a trace-only term absorbable into Ω^2.

Proof. In $D = 2$, antisymmetry leaves a single independent component $r := R^{(\infty)}_{12} = -R^{(\infty)}_{21}$, so

$Q_{11} = R_{12}^2 = r^2$, $Q_{22} = R_{21}^2 = r^2$, $Q_{12} = Q_{21} = R_{11} R_{21} + R_{12} R_{22} = 0$,

i.e., $Q = r^2 \cdot \delta$. Then $\|R^{(\infty)}\|_F^2 = 2r^2$ and $\hat{Q}_{ij} = Q_{ij} - (1/2)(2r^2) \cdot \delta_{ij} = r^2 \cdot \delta_{ij} - r^2 \cdot \delta_{ij} \equiv 0$.

Remark — operative range. The vanishing in $D = 2$ is not an algebraic accident: the symmetric traceless quadratic of a single antisymmetric tensor has no orbit content under O(2). The construction of this paper is therefore non-trivial only for $\mathbf{D \geq 3}$, which we tacitly assume in every result requiring $\hat{Q} \not\equiv 0$ — notably Theorem 4.2 (whose hypothesis $\hat{Q}_{ij}(x_0) \neq 0$ is vacuous in $D = 2$), Corollary 6.4, and the "first directional structure" claims of the General-Reader Summary and §8. For $D = 2$, the corrected metric reduces to the purely conformal $g_{ij} = (\Omega^2 + \lambda$

$r^2) \delta_{ij}$ (since the trace part of Q absorbs into Ω^2) and the programme's leading conformal results apply unchanged.

Theorem 3.6 — Algebraic Uniqueness at Quadratic Order [proven]

Let $S_{ij}(R)$ denote any symmetric rank-2 tensor on M built algebraically from $R^{(\infty)}$ alone (no derivatives), bilinear in $R^{(\infty)}$, and constructed using only the **parity-even** invariants of the flat background (i.e., δ_{ij} and δ^{ij}; parity-odd constructions using the Levi-Civita symbol ε are excluded by hypothesis and treated separately in the remark below). Then there exist real constants α, β such that, pointwise,

$$S_{ij}(R) = \alpha \cdot Q_{ij} + \beta \cdot \|R^{(\infty)}\|_F^2 \cdot \delta_{ij}.$$

Equivalently, after rearranging using Theorem 3.5,

$$S_{ij}(R) = \alpha \cdot \hat{Q}_{ij} + \gamma \cdot \|R^{(\infty)}\|_F^2 \cdot \delta_{ij},$$

for some $\gamma \in \mathbb{R}$. The non-conformal (traceless) content at this order is therefore uniquely $\alpha \cdot \hat{Q}_{ij}$ up to overall scalar normalization.

Proof. Any bilinear, δ-built, symmetric rank-2 tensor in an antisymmetric R can be written

$$S_{ij} = T_{ij}^{abcd} \cdot R_{ab} \cdot R_{cd},$$

where T is a tensor built solely from δ. Up to relabelling of contracted indices and use of $R_{ab} = -R_{ba}$, the inequivalent contraction patterns with two free indices (i, j) and four R-indices (a, b, c, d) — each pair (a, b) and (c, d) attached to a single R-factor — fall into the following exhaustive list:

(i) **One free index on each R-factor.** $S^{(1)}_{ij} = \delta^{kl} R_{ik} R_{jl} = R_{ik} R_j^{\ k} = Q_{ij}$. Any reordering of which free index sits on which R-factor is absorbed by Theorem 3.2.

(ii) **Both free indices on δ, R-factors fully contracted.** $S^{(2)}_{ij} = \delta_{ij} \cdot \delta^{km} \delta^{ln} \cdot R_{kl} R_{mn} = \delta_{ij} \cdot \|R\|_F^2$.

(iii) **Both free indices on one R-factor.** $S^{(3)}_{ij} = R_{ij} \cdot (\delta^{kl} R_{kl})$. But $\delta^{kl} R_{kl} = 0$ by antisymmetry, so $S^{(3)} \equiv 0$.

(iv) **Both free indices on the same R-factor with the other contracted with itself.** $S^{(4)}_{ij} = R_{ij} \cdot R_{kk} = 0$, again by antisymmetry.

(v) **Mixed patterns.** Patterns where the two free indices sit on different R-factors with internal contractions traversing both — for instance $R^a_{\ i} R_{aj}$, or its image under relabelling of the dummy index — produce symmetric output but reduce to $\pm Q_{ij}$ via the antisymmetry $R_{ab} = -R_{ba}$ together with relabelling of summation indices. The collapse is by R-antisymmetry, not by the (i, j)-symmetry hypothesis directly; a reader running the enumeration in coordinates will encounter $R^a{}_{i} R_{aj}$ before Q_{ij}, and the equivalence is what closes the case. Patterns producing antisymmetric output in (i, j) are separately excluded by the symmetry hypothesis. Either way, no new contraction class arises beyond (i) and (ii).

The general symmetric bilinear tensor is therefore $S_{ij} = \alpha S^{(1)}_{ij} + \beta S^{(2)}_{ij} = \alpha Q_{ij} + \beta \|R\|_F^2 \delta_{ij}$. The traceless-part statement follows from Theorem 3.5.

Remark — parity-odd contractions in D = 4. In dimension D = 4, the Hodge dual $*R^{(\infty)}_{ij} := (1/2)\, \varepsilon_{ijkl} R^{(\infty)kl}$ is also antisymmetric, and the parity-odd contraction

$$S^{odd}_{ij} := R^{(\infty)}_{ik} \cdot *R^{(\infty)}{}_j{}^{k}$$

is symmetric and structurally independent of Q_{ij}. This contraction is excluded by the parity-even hypothesis above and would require independent treatment in any parity-non-conserving extension of the framework. The uniqueness statement of Theorem 3.6 is therefore correctly read as *uniqueness within the parity-even sector*; opening the parity-odd sector adds at most one further independent symmetric invariant per dimension in which the Hodge dual produces a tensor of the right rank.

Remark — significance. Algebraic uniqueness within the parity-even sector is structurally important: any future principle that selects the *form* of the non-conformal correction at quadratic order in $R^{(\infty)}$ and respects parity is forced to land on $\hat{Q}_{ij}$ up to normalization. Only the coupling constant λ is free; the geometric channel is fixed.

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4. Non-Conformal Metric Corrections

Definition 4.1 — Corrected Metric

Let $\lambda \in \mathbb{R}$ be a coupling discussed in §4.6. Define the corrected continuum metric

$$g_{ij}(x) := \delta_{ij} / \varepsilon_{gap}(x)^2 + \lambda Q_{ij}(x).$$

Using Theorem 3.5, this is equivalent to

$$g_{ij}(x) = \Omega^2(x)\, \delta_{ij} + \lambda \hat{Q}_{ij}(x),$$

with **effective conformal factor**

$$\Omega^2(x) := 1/\varepsilon_{gap}(x)^2 + (\lambda/D) \cdot \|R^{(\infty)}(x)\|_F^2,$$

and $\lambda \hat{Q}_{ij}$ the genuinely non-conformal correction.

Interpretive Remark — What $\hat{Q}_{ij}$ Is Not [effective-theory framing]

$\hat{Q}_{ij}$ is *not* a Ricci tensor, an Einstein tensor, or a fundamental continuum curvature primitive. It is an *effective* anisotropic geometric response induced by refinement-stable causal transport structure — the leading refinement-stable irrelevant geometric operator surviving coarse-graining of the admissible transport substrate (§8, conditional on the substrate-level RG flow). The suppression and localization of $\lambda \hat{Q}_{ij}$ established below are therefore effective-theory consequences of coarse-grained causal transport dynamics, not properties of a primitive geometric object. This framing matters when interpreting Theorems 4.2, 5.2, and 6.3: the non-conformality, the double-rate localization, and the direction-dependent geodesic deflection are all *effective* signatures rather than fundamental ones.

Theorem 4.2 — Non-Conformality of the Correction [proven]

Suppose $\hat{Q}_{ij}(x_0) \neq 0$ at some $x_0 \in M$ and $\lambda \neq 0$. Then there exists no smooth positive function $\omega(x)$ on any neighbourhood U of x_0 such that $g_{ij}(x) = \omega(x)\, \delta_{ij}$ on U.

Proof. Suppose, for contradiction, that $g_{ij} = \omega\, \delta_{ij}$ on a neighbourhood U of x_0. Taking the traceless part of both sides at x_0:

$$g_{ij}(x_0) - (1/D)\, g^k{}_k(x_0) \cdot \delta_{ij} = \omega(x_0)\, \delta_{ij} - (1/D)(D\, \omega(x_0))\, \delta_{ij} = 0.$$

But by Definition 4.1 and Theorem 3.5, the left-hand side equals $\lambda \hat{Q}_{ij}(x_0)$. Hence $\hat{Q}_{ij}(x_0) = 0$, contradicting the hypothesis.

Theorem 4.3 — Positive-Definiteness of the Corrected Metric [conditional]

Assume $\varepsilon_{gap}(x) > 0$ throughout M.

(i) If $\lambda \geq 0$, then $g_{ij}(x)$ is positive-definite at every $x \in M$.

(ii) If $\lambda < 0$, positive-definiteness holds provided

$$|\lambda| \cdot \|Q(x)\|_{op} < 1/\varepsilon_{gap}(x)^2 \text{ for all } x,$$

where $\|\cdot\|_{op}$ denotes the operator norm of Q viewed as a symmetric bilinear form.

Proof. Q is positive-semidefinite by Proposition 3.3. For $\lambda \geq 0$, g_{ij} is the sum of a positive-definite tensor $(1/\varepsilon_{gap}^2)\, \delta_{ij}$ and a positive-semidefinite tensor λQ_{ij}; the sum is positive-definite. For $\lambda < 0$, the smallest eigenvalue of $g_{ij}(x)$ is bounded below by $1/\varepsilon_{gap}(x)^2 - |\lambda| \cdot \|Q(x)\|_{op}$, which is positive under the stated bound.

Theorem 4.4 — Vacuum Recovery [conditional on prior vacuum normalization]

In the canonical vacuum where $R^{(\infty)}{}_{ij}(x) \equiv 0$, we have $Q_{ij} \equiv 0$ and $\hat{Q}_{ij} \equiv 0$, so

$$g_{ij}(x) = \delta_{ij} / \varepsilon_{gap\_vac}^2,$$

where ε_{gap_vac} is the vacuum value of ε_{gap} fixed in the conformal-metric paper. The vacuum geometry is conformally flat, in agreement with the leading-order result.

Theorem 4.5 — Refinement Stability of Q and Q̂ [proven under stated regularity]

Let $(R^{(n)}_{ij})_{\{n \geq 1\}}$ be the sequence of refinement-level transport-curvature tensors with limit $R^{(\infty)}_{ij}$. Assume $R^{(n)}$ is uniformly bounded in $W^{1,p}_{loc}(M; \mathbb{R}^{D\times D})$ for some $p > D/2$ (with $p \geq 1$). Then:

(i) by the Rellich–Kondrachov compact embedding $W^{1,p}_{loc} \hookrightarrow\hookrightarrow L^{2p}_{loc}$ — which holds throughout $p > D/2$ (with the embedding factoring through Hölder spaces when $p \geq D$) — a subsequence, which we continue to denote $(R^{(n)})$, converges strongly in L^{2p}_{loc} to $R^{(\infty)}$;

(ii) along this subsequence, the bilinear contractions converge strongly in L^p_{loc}:

$Q^{(n)}_{ij} := R^{(n)}_{ik} R^{(n)}{}_j{}^k \longrightarrow Q^{(\infty)}_{ij} = R^{(\infty)}_{ik} R^{(\infty)}{}_j{}^k$ in $L^p_{loc}(M)$;

(iii) consequently $\hat{Q}^{(n)}_{ij} \longrightarrow \hat{Q}^{(\infty)}_{ij}$ in L^p_{loc}.

Proof. Statement (i) is Rellich–Kondrachov applied to the uniform $W^{1,p}_{loc}$ bound. For (ii), write the bilinear difference as

$Q^{(n)}_{ij} - Q^{(\infty)}_{ij} = (R^{(n)}_{ik} - R^{(\infty)}_{ik}) R^{(n)}{}_j{}^k + R^{(\infty)}_{ik} (R^{(n)}{}_j{}^k - R^{(\infty)}{}_j{}^k)$.

Applying Hölder's inequality with conjugate exponents (2p, 2p) on any compact $K \subset M$:

$\|Q^{(n)}_{ij} - Q^{(\infty)}_{ij}\|_{L^p(K)} \leq \|R^{(n)} - R^{(\infty)}\|_{L^{2p}(K)} \cdot (\|R^{(n)}\|_{L^{2p}(K)} + \|R^{(\infty)}\|_{L^{2p}(K)})$.

The right-hand side tends to zero by (i) and the uniform L^{2p} bound (the Sobolev embedding ensures the right-hand factor is finite and uniformly bounded). Statement (iii) follows from (ii) by linearity of the trace and continuity of the projection onto the traceless subspace.

Remark — necessity of the strong-convergence upgrade. Weak convergence of $R^{(n)}$ alone does *not* imply weak convergence of the bilinear contraction $Q^{(n)}$: weak limits do not commute with quadratic operations, a fact that underlies concentration-compactness phenomena. The strong-convergence upgrade obtained from the $W^{1,p}_{loc}$ bound is essential. The required bound is exactly what the Refinement-Stable Holonomy paper establishes; the present construction therefore relies on that earlier work in a way that the leading-conformal construction did not.

4.6 — The Coupling λ [conditional / partly conjectural]

The coupling λ is not derived in this paper. Three statements can be made:

1. **Dimensional analysis [proven, convention-dependent].** *Convention*: throughout the programme ε_gap is taken dimensionless and coordinates carry the standard length dimensions, so g_ij components are dimensionless (readers using a convention where g_ij carries units of length squared should rescale λ accordingly; the *ratio* $\lambda/\ell\_\star^4$ is dimensionless and convention-independent). Under this convention, if $R^{(\infty)}$*ij carries dimensions of inverse length squared, then Q_ij carries dimensions of inverse length fourth. For λ Q_ij to share dimensions with the conformal metric* $\delta\_ij / \varepsilon\_gap^2$*, λ must carry dimensions of length to the fourth. The natural substrate scale is the tick length ℓ⋆*, suggesting $\lambda \sim \ell\_\star^4$ up to a dimensionless coefficient of order unity.
2. **Sign constraint [proven, given desired positive-definiteness].** Theorem 4.3 restricts λ if positive-definiteness of g_ij is required everywhere. For $\lambda \geq 0$, positive-definiteness is automatic. For $\lambda < 0$, the bound $|\lambda| \cdot \|Q\|\_op < 1/\varepsilon\_gap^2$ must hold pointwise.
3. **Microscopic determination [conjectural].** A derivation of λ from substrate dynamics — analogous to the way the transport action selected ε_gap^{-2} in the previous paper — is left to future work. Two candidate routes: (a) variational selection from a second-order transport action containing the contraction $R^{(\infty)}\_ik\ R^{(\infty)}\_j\textasciicircum k$ structurally — under Theorem 3.6, any such action (quadratic in $R^{(\infty)}$, built from parity-even flat-background invariants, no derivatives) has, up to a separately conformal trace term, no choice but to produce $\lambda\ \hat{Q}\_ij$ as its leading non-conformal source, so the variational programme is not searching for the *form* of the correction (already fixed by §3) but only for its *coupling* λ — a much more constrained problem; (b) renormalization-flow fixed-point conditions on admissible coarse-grainings.

Throughout the remainder of the paper, λ is treated as a fixed real parameter and results are stated at leading order in $|\lambda|$.

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5. Localized Geometric Backreaction

Theorem 5.1 — Inherited Exponential Decay of $R^{(\infty)}$ [cited from prior work]

From the Refinement-Stable Holonomy paper: in the presence of a Stage VIII coherence defect localized near $x_0 \in M$, the transport-curvature tensor satisfies the Combes–Thomas bound

$|R^{(\infty)}ij(x)| \leq C\_R \cdot \exp(-\eta\infty \cdot d(x, x_0))$

for all x outside a defect-core region, with rate $\eta\_\infty > 0$ determined by the spectral gap of the transport operator.

Theorem 5.2 — Double-Rate Localization of the Correction [proven, with explicit constants]

Under the hypotheses of Theorem 5.1, the following bounds hold pointwise outside the defect core. Using $R^{(\infty)}\_kk = 0$ (from antisymmetry) the term counts tighten as follows:

$|Q_{ii}(x)| \leq (D - 1) \cdot C_R^2 \cdot \exp(-2\eta_\infty \cdot d(x, x_0))$ (diagonal entries; sum runs over $k \neq i$),

$|Q_{ij}(x)| \leq \max(D - 2, 0) \cdot C_R^2 \cdot \exp(-2\eta_\infty \cdot d(x, x_0))$ (off-diagonal $i \neq j$; sum runs over $k \notin \{i, j\}$),

$\mathrm{tr}(Q)(x) = \|R^{(\infty)}(x)\|_F^2 \leq D(D - 1) \cdot C_R^2 \cdot \exp(-2\eta_\infty \cdot d(x, x_0))$,

$|\hat{Q}_{ij}(x)| \leq 2(D - 1) \cdot C_R^2 \cdot \exp(-2\eta_\infty \cdot d(x, x_0))$ (uniform over all i, j).

The non-conformal metric correction $\lambda \hat{Q}_{ij}$ is therefore exponentially localized at *twice* the rate η_∞ of $R^{(\infty)}$ itself.

Proof. For Q_{ii}, antisymmetry gives $Q_{ii} = \Sigma_{k \neq i} (R^{(\infty)}_{ik})^2$, *a sum over* $D - 1$ *terms each bounded by* $C_R^2 \cdot \exp(-2\eta_\infty d)$. For Q_{ij} with $i \neq j$, the contributions from $k = i$ ($R^{(\infty)}_{ii} = 0$) and $k = j$ ($R^{(\infty)}_{jj} = 0$) vanish, leaving $D - 2$ nonzero terms (zero terms when $D = 2$). The trace bound follows from $\mathrm{tr}(Q) = \Sigma_i Q_{ii} \leq D \cdot (D - 1) C_R^2 \cdot e^{-2\eta_\infty d}$. For $\hat{Q}_{ij}$ with $i \neq j$, $|\hat{Q}_{ij}| = |Q_{ij}| \leq (D - 2) C_R^2 \cdot e^{-2\eta_\infty d}$; for $i = j$,

$|\hat{Q}_{ii}| \leq |Q_{ii}| + (1/D) \cdot |\mathrm{tr}(Q)| \leq (D - 1) C_R^2 \cdot e^{-2\eta_\infty d} + (D - 1) C_R^2 \cdot e^{-2\eta_\infty d} = 2(D - 1) C_R^2 \cdot e^{-2\eta_\infty d}$.

The uniform bound $2(D - 1) \cdot C_R^2 \cdot e^{-2\eta_\infty d}$ covers both cases.

Structural Interpretation

The double-rate decay is not an artefact of the construction; it is the *signature* of a genuinely bilinear backreaction. Single-power transport-curvature observables decay at the Combes–Thomas rate η_∞ inherited from the spectral gap of the transport operator. The leading metric backreaction, being bilinear in $R^{(\infty)}$, must decay at rate $2\eta_\infty$ by the product rule for exponentials. The localization rate of the non-conformal metric structure is therefore not an independent parameter — it is *fixed* by the substrate spectral gap, with the doubling factor enforced by the algebraic order of the construction. Conversely, an empirical determination of the metric-correction decay rate would directly probe η_∞ (with the factor of two known a priori).

Physical Interpretation [BCB framing]

Within the BCB framework, higher-order geometric structures correspond to increasingly coordinated causal fact-commitment processes on the substrate. The doubled localization rate of $\hat{Q}_{ij}$ relative to $R^{(\infty)}$ is therefore physically natural: bilinear anisotropic transport coherence requires two simultaneously coordinated commitment-transport channels, and decays more rapidly than the first-order transport structure from which it is built. The mathematical doubling factor $2\eta_\infty$ is the coarse-grained shadow of this compounded coherence requirement. The substrate-level toy models constructed in Apps. G–H of the interface-transport paper — exhibiting explicit finite-capacity propagation, localization, anisotropy emergence, and harmonic structure — serve as discrete witnesses that the picture being coarse-grained here is realizable at the substrate level, not merely assumed.

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6. Anisotropic Geodesic Structure

Lemma 6.1 — Leading-Order Inverse Metric [proven]

To leading order in λ, the inverse of $g_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$ is

$$g^{ij} = \Omega^{-2} \delta^{ij} - \lambda \Omega^{-4} \hat{Q}^{ij} + O(\lambda^2),$$

where $\hat{Q}^{ij} := \delta^{ik} \delta^{jl} \hat{Q}_{kl}$.

Proof. Standard Neumann-series expansion. Writing $g = \Omega^2(\delta + \lambda \Omega^{-2} \hat{Q})$ and using $(I + \lambda A)^{-1} = I - \lambda A + O(\lambda^2)$,

$$g^{-1} = \Omega^{-2} (\delta - \lambda \Omega^{-2} \hat{Q} + O(\lambda^2)).$$

The expansion converges in operator norm whenever $|\lambda| \cdot \Omega^{-2} \cdot \|\hat{Q}\|_{op} < 1$; by Theorem 5.2 this holds on any region whose distance from the defect core exceeds an explicit threshold.

Theorem 6.2 — Christoffel Correction [proven, at leading order in λ]

The Christoffel symbols of g_{ij} decompose as

$$\Gamma^k_{ij} = \Gamma^{(0)k}_{ij} + \lambda \cdot \Delta\Gamma^k_{ij} + O(\lambda^2),$$

where $\Gamma^{(0)k}_{ij}$ are the Christoffel symbols of the leading conformal metric $g^{(0)}_{ij} = \Omega^2 \delta_{ij}$, and the correction is given covariantly by the standard metric-perturbation linearization identity around the conformal background:

$$\Delta\Gamma^k_{ij} = (1/2)\, g^{(0)kl} (\nabla^{(0)}_i \hat{Q}_{jl} + \nabla^{(0)}_j \hat{Q}_{il} - \nabla^{(0)}_l \hat{Q}_{ij}),$$

where $\nabla^{(0)}$ denotes the Levi-Civita connection of $g^{(0)}$.

Proof. Apply $\Gamma^k_{ij} = (1/2)\, g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ to $g_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$ using $g^{kl} = \Omega^{-2} \delta^{kl} - \lambda \Omega^{-4} \hat{Q}^{kl} + O(\lambda^2)$ (Lemma 6.1), and collect terms by order in λ. The $O(\lambda^0)$ terms reconstruct $\Gamma^{(0)k}_{ij}$. At $O(\lambda^1)$, two channels contribute and *both* must be tracked: (α) the $O(\lambda^0)$ piece of g^{kl} contracted with the $O(\lambda)$ piece of ∂g (the $\partial\hat{Q}$ contribution), and (β) the $O(\lambda)$ piece of g^{kl} contracted with the $O(\lambda^0)$ piece of ∂g (which carries the variation of Ω). The sum of (α) and (β) is precisely the linearization $(1/2)\, g^{(0)kl} (\nabla^{(0)}_i \hat{Q}_{jl} + \nabla^{(0)}_j \hat{Q}_{il} - \nabla^{(0)}_l \hat{Q}_{ij})$ of $g^{(0)} \mapsto g^{(0)} + \lambda \hat{Q}$; channel (β) is exactly what packages the $\partial\Omega$ terms into the covariant derivatives, restoring tensorial form.

Corollary 6.2.1 — Explicit Coordinate Form [proven]

Expanding $\nabla^{(0)}$ in terms of Ω and its derivatives, the correction reads, with indices raised by δ:

$\Delta\Gamma^k_{ij} = (1/2)\,\Omega^{-2}\,(\partial_i \hat{Q}_j{}^{k} + \partial_j \hat{Q}_i{}^{k} - \partial^k \hat{Q}_{ij}) - \Omega^{-2}\,(\hat{Q}^k{}_j \cdot \partial_i \ln \Omega + \hat{Q}^k{}_i \cdot \partial_j \ln \Omega - \delta_{ij} \cdot \hat{Q}^{kl} \cdot \partial_l \ln \Omega)$.

In the constant-Ω limit $\partial_i \ln \Omega \to 0$, the second line vanishes and ΔΓ reduces to the bracketed first line alone, recovering the simpler partial-derivative form as a consistency check.

Remark — preservation of Corollary 6.4. Every term in ΔΓ contains either $\partial\hat{Q}$ or a factor $\hat{Q} \cdot \partial \ln \Omega$. All terms therefore vanish identically when $\hat{Q} \equiv 0$, so the conformal-mimicry obstruction of Corollary 6.4 is unaffected by the additional $\partial \ln \Omega$ contributions: the direction-dependent forcing in Theorem 6.3 still vanishes if and only if $\hat{Q} \equiv 0$.

Theorem 6.3 — Direction-Dependent Geodesic Deflection [proven, at leading order]

Let $\gamma^{(0)}(s)$ be a geodesic of the leading conformal metric $g^{(0)}_{ij}$, and let $\gamma(s) = \gamma^{(0)}(s) + \lambda\, \delta\gamma(s) + O(\lambda^2)$ be the geodesic of the full metric g_{ij} with the same initial position and velocity. Then $\delta\gamma^k$ satisfies the inhomogeneous Jacobi-type equation

$$D^2(\delta\gamma^k)/Ds^2 + R^{(0)k}{}_{lmn} \cdot u^{(0)l} \cdot u^{(0)m} \cdot (\delta\gamma)^n = -\,\Delta\Gamma^k_{ij}(\gamma^{(0)}) \cdot u^{(0)i} \cdot u^{(0)j},$$

where $u^{(0)} := d\gamma^{(0)}/ds$, D/Ds is the covariant derivative along $\gamma^{(0)}$ in $g^{(0)}$, and $R^{(0)}$ is the Riemann tensor of $g^{(0)}$.

The forcing term $-\Delta\Gamma^k_{ij} \cdot u^{(0)i} \cdot u^{(0)j}$ depends bilinearly on $u^{(0)}$ through the *full* tensor structure of $\hat{Q}$ — not merely through its norm. Two geodesics through the same point with distinct tangent directions therefore experience *distinct* deflections, even after rescaling to common speed.

Proof. Substitute $\gamma = \gamma^{(0)} + \lambda\, \delta\gamma$ into the geodesic equation

$$d^2\gamma^k/ds^2 + \Gamma^k_{ij}(\gamma) \cdot (d\gamma/ds)^i \cdot (d\gamma/ds)^j = 0,$$

and expand to first order in λ using Theorem 6.2. The homogeneous part gives the standard Jacobi equation in $g^{(0)}$; the inhomogeneous part is the forcing shown.

Corollary 6.4 — Conformal Corrections Cannot Mimic the Anisotropy [proven]

Suppose, hypothetically, that the metric correction is purely a trace correction, so that $\hat{Q}_{ij} \equiv 0$ and only the trace part of Q contributes (absorbed into Ω^2). Then $\Delta\Gamma^k_{ij} \equiv 0$ at this order, and the forcing term in Theorem 6.3 vanishes identically. The direction-dependent deflection is therefore *strictly* a $\hat{Q}$-effect — not a generic geodesic-perturbation phenomenon.

Proof. By the remark following Corollary 6.2.1, every term in $\Delta\Gamma^k{}_{ij}$ — whether read off the covariant form of Theorem 6.2 or the explicit form of Corollary 6.2.1 — contains either $\partial\hat{Q}$ or a factor of $\hat{Q} \cdot \partial \ln \Omega$. Each such term carries $\hat{Q}$ as a factor, so if $\hat{Q} \equiv 0$ the expression vanishes identically, and the forcing term in Theorem 6.3 inherits this vanishing.

Combined with Theorem 4.2 (non-conformality of $\lambda \hat{Q}$) and Theorem 3.6 (algebraic uniqueness), Corollary 6.4 establishes that direction-dependent geodesic deflection at quadratic order in $R^{(\infty)}$ is *uniquely* sourced by the traceless contraction $\lambda \hat{Q}_{ij}$. No alternative geometric mechanism at this order can produce the same kinematic signature.

Interpretive Remark — Causal Fact-Commitment Geometry [BCB framing]

Read through the BCB lens of the interface-transport framework, $\lambda \hat{Q}_{ij}$ is not merely a tensorial deformation of an abstract metric: it is anisotropic *causal fact-commitment geometry*. Distinct directions through a point correspond to distinct admissibility profiles for committing distinguishable facts on the underlying substrate, with directionally varying causal-update efficiency. Theorem 6.3 and Corollary 6.4 then say something physically sharper than a kinematic deviation: paths through the substrate carry *direction-dependent commitment costs* that no purely conformal rescaling can encode. The continuum's directional structure is the coarse-grained image of the directional structure of admissible fact-commitment flow.

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7. Higher-Order Contractions and Power Counting [structural remarks]

The construction here is leading-order in two senses: leading in λ, and leading in powers of $R^{(\infty)}$. The next-order corrections in the latter expansion are quartic in $R^{(\infty)}$ and divide into three structural classes:

(I) **Iterated quadratic.** $Q_{ik} Q_j{}^k = R^{(\infty)}_{ia} R^{(\infty)}{}_k{}^a R^{(\infty)}_{jb} R^{(\infty)kb}$. This is bilinear in Q itself and contributes at $O(\lambda^2)$ to the metric through Q^2, not at a new independent order.

(II) **Cycle-pattern quartic invariant** [stated; quartic-order uniqueness not proven here]. $R^{(\infty)}_{ik} R^{(\infty)kl} R^{(\infty)}_{lm} R^{(\infty)}{}_j{}^m$. The contraction pattern threads the four R-factors in a single cycle rather than as two squared pairs, which we *assert* — without proof at quartic order; see Open Problem 5 of §9 — to be structurally independent of $Q_{ik} Q_j{}^k$. If independent, it would require its own coupling, call it λ'.

(III) **Gradient corrections.** Terms such as $\nabla_i R^{(\infty)}_{kl} \cdot \nabla_j R^{(\infty)kl}$ and $\nabla_i \nabla_j \|R^{(\infty)}\|_F^2$. These have different kinematic structure (involving derivatives of $R^{(\infty)}$) and would enter through a distinct effective-action sector with its own dimensional couplings.

By the trace-decomposition argument of §3 applied at each order, every higher symmetric contraction splits into a conformally absorbable trace part and a traceless anisotropic part. An analogue of Theorem 3.6 — a uniqueness statement at the next order — would determine the structurally allowed forms within each class. The leading non-conformal content is uniquely $\lambda \hat{Q}_{ij}$ at quadratic order in $R^{(\infty)}$; the next non-conformal content enters at quartic order through (I), (II), and (III), each with its own coupling and each separately decomposable into trace and traceless pieces.

This power-counting is conditional on the regularity of the substrate dynamics that generate the higher-order terms and on the absence of structural cancellations. A systematic effective-action treatment is the natural sequel.

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8. Structural Interpretation

The progression of the geometry programme now reads:

Stage	Structural content
Stage V	Flat continuum
Stage VIII	Scalar defect structure
Global Transport	Localization and transport dynamics
Tensorial Transport Geometry	Holonomy and antisymmetric transport curvature $R^{(\infty)}$
Refinement Stability	Continuum transport observables in $W^{1,p}_{loc}$
Transport-Action Metric Selection	Conformal metric $g^{(0)}_{ij} = \delta_{ij} / \varepsilon_{gap}^2$
Present paper	**Non-conformal anisotropic metric correction $\lambda \hat{Q}_{ij}$**

The transport geometry now feeds into continuum metric structure through two complementary channels: a scalar channel (the conformal factor Ω^2 built from ε_{gap} and the trace of Q), and a tensorial channel (the traceless anisotropic correction $\lambda \hat{Q}_{ij}$). The combined metric

$$g_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$$

is the first VERSF continuum metric carrying *directional* information sourced by transport curvature. The algebraic uniqueness of $\hat{Q}$ at this order (Theorem 3.6) ensures the tensorial channel is structurally fixed: only the coupling λ remains free, and even that is constrained by dimensional analysis to $\lambda \sim \ell_\star^4$ up to a dimensionless coefficient.

Parallel Emergence and Bundle Distinction

The construction presupposes — and is consistent with — the *parallel emergence* of transport structure and continuum metric structure from common substrate-level admissibility dynamics. The transport bundle whose holonomy generates $R^{(\infty)}$ is *not* identified with the continuum's spatial tangent bundle. The two are independent emergent layers, coupled only through the symmetric quadratic invariants of $R^{(\infty)}$ catalogued in Theorem 3.6. This is the structural answer to the question of why an antisymmetric $R^{(\infty)}$ cannot be a Ricci tensor: it lives on a different bundle. The metric anisotropy $\lambda \hat{Q}_{ij}$ is the coarse-grained image of transport-bundle structure projected into the tangent-bundle sector, mediated by a bilinear invariant — not by a bundle isomorphism.

$\hat{Q}_{ij}$ as a Refinement-Stable Irrelevant Operator [interpretive]

Subject to a substrate-level renormalization-group flow whose existence is conjectured in companion papers but not constructed here, the following interpretive picture applies. Under the coarse-grained RG flow of the interface transport substrate, isotropy emerges asymptotically while symmetry-protected irrelevant operators survive at suppressed order. The traceless tensor $\hat{Q}_{ij}$ is naturally interpreted as the *leading refinement-stable irrelevant geometric operator* surviving this flow — the symmetric-bilinear analogue, in the geometric sector, of the harmonic structures (e.g. the surviving 6Ω mode discussed in the interface-propagation framework) that persist in the transport sector. Its survival rather than wash-out is RG-protected by the very symmetry structure that forbids it from appearing at first order: there is simply no symmetric rank-2 operator linear in an antisymmetric $R^{(\infty)}$, so the first non-trivial isotropy-breaking content arrives at quadratic order, where Theorem 3.6 makes it unique within the parity-even sector.

Compatibility with the Lorentzian and Observable-Covariance Programmes [pointer]

The present paper studies anisotropic spatial continuum geometry only. The Lorentzian causal structure and observable covariance that complete the full continuum picture are established independently within companion programmes and are *not* re-derived here.

Two compatibility conditions are inherited. *Causal–coherence compatibility* (CCC programme): stable irreversible facts can be produced only when coherent causal coordination across a region completes within the region's own causal crossing time. This selects hyperbolic over Euclidean causal organization for the observable continuum and identifies the invariant propagation cone as the maximal coordination structure compatible with stable commitment dynamics. *Commitment Reordering Equivalence* (CRE / observable-covariance programme): observable physics depends only on equivalence classes of proto-histories generating identical committed records and identical invariant signal-cone structure, so Lorentz invariance emerges as a quotient property rather than a postulate, while proto-time ordering — though physically real at the substrate level — becomes operationally unobservable in the committed-record sector.

The metric structure $g_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$ constructed here is intended to be compatible with — and the spatial restriction of — the Lorentzian completion supplied by those programmes. The contraction defining Q is signature-agnostic at the algebraic level: replacing δ^{kl} by a Lorentzian inverse metric η^{kl} in $Q_{ij} := R^{(\infty)}_{ik} R^{(\infty)}_{jl} \cdot (\text{raising metric})^{kl}$ yields, by exactly the symmetry argument of Theorem 3.2, a symmetric tensor in either signature, and the trace–traceless decomposition (Theorem 3.5) operates identically with η^{ij} replacing δ^{ij} in the trace. $\hat{Q}_{ij}$ therefore carries no algebraic obstruction to embedding in the Lorentzian completion and does not break the cone structure inherited from the transport sector; it deforms the spatial-geometric content within whatever signature the completion supplies. Full Lorentzian–spatial gluing remains an open structural problem (Open Problem 1 of §9).

9. Limitations and Open Problems

The paper does not derive:

- Einstein equations or any second-order field equations for g_{ij},
- Lorentzian signature (the construction is signature-Euclidean throughout, inherited from δ_{ij}),
- a stress-energy tensor or matter coupling,
- the coupling constant λ from microscopic substrate dynamics,
- the higher-order contractions explicitly, beyond identifying their structural classes.

Specific open problems:

1. **Explicit Lorentzian–spatial gluing.** The Lorentzian causal structure of the continuum is supplied independently by the CCC programme (§8). What remains open is the explicit gluing between that Lorentzian completion and the spatial-metric structure $g_{ij} = \Omega^2 \delta_{ij} + \lambda \hat{Q}_{ij}$ constructed here — in particular, the construction of a unified Lorentzian metric whose spatial restriction reproduces g_{ij} and whose causal cone matches the CCC-supplied invariant cone. The σ-duality and emergent-time structure of the sequential-interface-transport programme is the natural intermediary.
2. **Transport-action variational origin of λ.** A second-order transport action containing $R^{(\infty)}{}_{ik} R^{(\infty)}{}_{j}{}^{k}$ structurally would fix λ; constructing it is the immediate sequel to the conformal-selection paper.
3. **Field-equation principle.** Under what additional principle does the corrected metric g_{ij} satisfy dynamical equations of motion, rather than being a static functional of the underlying transport data?
4. **Bundle-level formulation.** The construction here is on the base manifold; a full bundle-theoretic treatment of how transport-bundle curvature is projected into tangent-bundle anisotropy — under the parallel-emergence picture of §8, in which the two bundles are *not* identified — is outstanding.
5. **Higher-order uniqueness.** An analogue of Theorem 3.6 at quartic order — establishing the structural completeness of classes (I)–(III) of §7 — would close the algebraic side of the effective-action expansion.
6. **Quantization.** The metric g_{ij} here is a deterministic functional of refinement-stable observables; the quantum fluctuations of the substrate that source g_{ij} have not been characterized.

10. Conclusion

The present paper establishes that refinement-stable transport curvature generates non-conformal continuum metric corrections in VERSF. The continuum metric

$$g_{ij}(x) = \Omega^2(x)\, \delta_{ij} + \lambda\, \hat{Q}_{ij}(x)$$

cleanly separates transport-selected conformal structure Ω^2 from anisotropic transport-curvature backreaction $\lambda\, \hat{Q}_{ij}$. Six structural results secure the construction:

- the trace–traceless decomposition (Theorem 3.5) isolates the genuinely non-conformal content;
- the algebraic uniqueness theorem (Theorem 3.6) fixes the form of the non-conformal correction at quadratic order in $R^{(\infty)}$, leaving only the coupling λ free;
- the strong-convergence-upgraded refinement-stability argument (Theorem 4.5) secures the construction at the continuum level;
- the double-rate Combes–Thomas localization (Theorem 5.2) ties the decay rate of the correction to the substrate spectral gap, with the doubling factor *forced* by the algebraic order of the construction rather than assumed — inverting the usual logic in which decay rates are independent inputs — and with explicit, tightened constants tracked;
- the anisotropic Christoffel correction (Theorem 6.3) is computed in closed form at leading order in λ;
- the conformal-mimicry corollary (Corollary 6.4) establishes that direction-dependent geodesic deflection at this order is *strictly* a $\hat{Q}$-effect.

This is the first stage of the programme in which transport curvature directly induces *directional* geometric structure rather than merely conformal rescaling. Read through the BCB lens of the interface-transport framework, the directional content is anisotropic causal fact-commitment geometry — the coarse-grained signature of directionally varying admissibility profiles for fact-commitment flow on the substrate, surviving RG suppression as the leading refinement-stable irrelevant geometric operator. Taken together with the independently established Lorentzian completion (CCC) and observable-covariance framework (CRE) cited in §8, the spatial-metric layer constructed here closes the *geometric* side of the picture at this order; what remains is its *dynamics*. The structural question is therefore no longer

"Can geometry emerge from transport?"

but

"What dynamical equations govern the transport-generated geometry $g_{ij} = \Omega^2\, \delta_{ij} + \lambda\, \hat{Q}_{ij}$ now in place, and what microscopic principle selects the coupling λ?"

These are the subjects of the next papers in the programme.