Operational Distinguishability Geometry as the Primitive Probability Object

Why Probability Must Be a Measure on Operational Geometry

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A General Reader's Summary

Why does quantum mechanics have a probability rule at all?

Two previous papers in this programme have made progress on adjacent questions. The first asked why quantum probability is the square of an amplitude, and showed that once the operational stage is in place, the squaring rule is forced within each compartment of that geometry by a small set of natural axioms. The second asked why the same squaring rule applies everywhere, and showed that three apparently independent bridging principles supplied by the first paper were really three manifestations of one deeper principle — the Universal Measure Principle (UMP) — asserting that probability must be invariant under every operational witness of commensurability between sectors.

This left a more basic question still hanging. Why should probability be constrained by operational geometry at all? Why should the same structure that determines distinguishability also determine the probability rule? The previous paper named UMP as the principle responsible, but the deeper question is what makes UMP itself a natural requirement rather than an additional postulate. The present paper addresses that question.

The central claim is that, *given* the operational geometry of the carrier taken as a unified object rather than a list of separate pieces, the natural class of operations preserving that geometry and the natural class of measures compatible with it become determined by the geometry itself. Once we recognize that the carrier supplies not merely a vector space but a *structured geometry of distinguishability* — a unified object we call the **operational distinguishability geometry** (ODG) of the carrier — then probability assignment is no longer independent of operational structure: it is the unique invariant measure compatible with the operational geometry already inherited.

The structural hierarchy this delivers is clean:

Operational geometry → ODG-invariance → UMP → Born rule.

The first arrow is recognition: the carrier already carries a structured geometry of distinguishability. The second arrow is principle: probability must be invariant under operations preserving that geometry. The third arrow is specialization: ODG-invariance, restricted to the three identified classes of cross-sector witness, is UMP. The fourth arrow is uniqueness: UMP plus the minimal-core axioms pick out the Born rule.

The reframing this delivers is significant. The previous paper had shown that the Born rule is *the unique commensurable* quadratic measure on operational coherence geometry. The present paper sharpens this: the Born rule is *the unique invariant* measure compatible with operational distinguishability geometry itself. The notion of commensurability — three operational windows onto one deeper invariance — is itself a working specialization of the more primitive notion of ODG-compatibility.

What is at stake is not a different probability law (the answer is still the Born rule) but a different *structural reading* of why that law holds. Under the present paper's framing, the Born rule does not require a separate principle (universality, commensurability, or otherwise) added to the inherited operational geometry alongside ODG-compatibility. It is what compatibility with the geometry already requires, given ODG-compatibility as a structural input on admissible measures.

The technical body follows. Readers without a mathematical-physics background may wish to read §1 and §8 for the structural content in continuous prose before the theorems are introduced.

Abstract

The previous VERSF reconstruction papers established that the Born rule is uniquely determined on the operational coherence-geometry carrier $\mathcal{A}\_\mathbb{C} = \bigoplus\alpha\ V\_\alpha$ *by the minimal-core probability axioms (A1)–(A3), (A5), (A7) of PAMV together with the Universal Measure Principle (UMP) of the preceding paper. UMP states that an admissible operational probability measure must be invariant under three classes of operational commensurability witness: compositional identification* $\sigma\{\alpha,\beta\}$, complex-unitary embedding ι, and decomposition-admissible refinement-relabeling W. The UMP paper named **operational distinguishability geometry** (ODG) — the unified operational structure jointly supplied by the carrier, sector decomposition, projection structure, admissible transport, refinement structure, and finite-distinguishability packing — as "the structural object UMP is invariance for," and flagged as an open question whether the three identified witness classes exhaust the structurally significant ODG-preserving operations.

The present paper inverts the conceptual ordering: ODG becomes the primitive structural object, and UMP becomes a working specialization. The principal results:

Theorem 6.1 (ODG-compatibility implies UMP). *Every ODG-compatible minimal-core admissible probability measure satisfies UMP. The implication is proven directly: the three commensurability witnesses are ODG-preserving operations (Proposition 3.3), so any measure invariant under all ODG-preserving operations is invariant under each commensurability witness — which is UMP.*

Theorem 7.1 (Universal Born rule from ODG-compatibility). *Under the inherited operational structure of PAMV §2, the minimal-core axioms (A1)–(A3), (A5), (A7), and ODG-compatibility, the operational probability measure on* $\mathcal{A}\_\mathbb{C}$ *is uniquely* $\mu(\psi) = \|\psi\|^2\_\mathbb{C}$ *for* $\psi \in \mathcal{A}\_\mathbb{C}$*, and* $P(\psi \to \phi) = |\langle\phi, \psi\rangle\_\mathbb{C}|^2$ *for normalized admissible states. The Born rule delivered is the same as the UMP paper's Theorem 7.1; what changes is the principle from which it is derived.*

Supporting results: **Theorem 4.2** strengthens the UMP-paper's Theorem 3.2 — sector-relative measures fail ODG-compatibility through three concrete manifestations of one underlying defect, locating the "common structural origin" the UMP paper identified. **Theorem 6.4** establishes that UMP and ODG-compatibility are logically equivalent on the VERSF carrier *conditional on the structural-completeness conjecture* (Conjecture 3.5, that the four classes of ODG-preserving operation identified in §3.2 exhaust the ODG-preserving operations). **Theorem 7.2** characterizes the Born rule as the unique ODG-compatible minimal-core measure.

The iconic structural fact this paper isolates:

The Born rule is the unique invariant measure compatible with operational distinguishability geometry.

The structural reframing operates at the level of which principle is primitive, not at the level of empirical content. The Born rule is the same one the UMP paper delivered. ODG-compatibility supplies a structural reading from which UMP follows as a working specialization on three identified witness classes. The Operational Indistinguishability Principle (OIP) of §5A — that operationally indistinguishable states must receive identical probability assignments — supplies the cleanest *operational* formulation of the same structural commitment: probability must factor through the quotient of states by ODG-indistinguishability (Proposition 5A.3A). The structural hierarchy the paper delivers is therefore:

Operational geometry → Operational indistinguishability → ODG-compatibility → UMP → Born rule.

Whether UMP and ODG-compatibility are logically equivalent on the VERSF carrier is the structural-completeness question of §10 Open Problem 2 (the converse-direction half of the UMP-paper's Open Problem 2). Either way, OIP forces the Born rule — by Corollary 5A.4 plus Theorem 6.1's unconditional implication — and supplies a richer falsifiability framework than UMP alone.

Scope and Conditional Status

This paper sits in the same conditional-reconstruction framework as PAMV and the UMP paper. It does not derive ODG from absolute first principles. Rather, it shows that:

1. **Given** the inherited VERSF operational structure of PAMV §2 (carrier, sector decomposition, projection structure, admissible transport, refinement structure, finite packing);
2. **Given** the minimal operational-measure axioms (A1)–(A3), (A5), (A7);
3. **Given** the *recognition* that the six components of (1) constitute a unified geometric object — operational distinguishability geometry (ODG) — whose natural invariance group includes the class of ODG-preserving operations;
4. **Given** ODG-compatibility as a structural requirement on admissible probability measures;

UMP is a derived consequence (Theorem 6.1, **proven**), and the universal Born rule is uniquely forced (Theorem 7.1, **proven**). The converse direction — UMP implies ODG-compatibility — is **conditional** on the structural conjecture that the three commensurability witness classes exhaust the ODG-preserving operations (Theorem 6.4); this is plausible on the VERSF carrier but not proven, and constitutes Open Problem 2 of §10.

What this paper *does* add to the UMP paper is a conceptual inversion: rather than UMP being the primitive principle with ODG named as "the structural object UMP is invariance for," ODG becomes the primitive object and UMP becomes the working specialization. The Born rule's structural status is sharpened correspondingly: from "the unique commensurable minimal-core measure" (UMP paper) to "the unique ODG-invariant minimal-core measure" (present paper). On the VERSF carrier, conditional on the structural-completeness conjecture, the two characterizations are equivalent; the difference is in which is the primitive notion and which is the derived working form.

What this paper *does not* do is derive ODG itself from anything more primitive. ODG is the structural input here, in the same sense that the inherited operational structure was the structural input in PAMV and UMP was the structural input in the UMP paper. Whether ODG can be derived from substrate-level structure — in particular from the finite-packing geometry of OG Theorem 10.1 — is left as Open Problem 1 of §10, which connects to PAMV's Open Problem 10 (strengthening $\mathrm{Vol\_op}(\chi) = \|\chi\|^2\_\mathbb{C}$ from convention to theorem) and to the UMP-paper's Open Problem 1 (substrate-derivation of UMP). The deepest substrate-origin questions — why $\mathbb{Z}\_7$, why K = 7, why the carrier admits a structured operational geometry at all — remain as PAMV and the UMP paper left them.

A note on the epistemic status of the structural conjecture in Theorem 6.4. The conjecture is that the four classes of ODG-preserving operations identified in §3.1 — admissible transport, compositional identifications, embedding maps, refinement relabelings — exhaust the structurally significant ODG-preserving operations on the VERSF carrier. The conjecture is supported by the observation that each of the six components of ODG (§2.1) is invariant under these four classes (Propositions 3.2, 3.3); it is *not* supported by an exhaustion argument, and further ODG-preserving classes have not been ruled out. The conjecture is open and is the converse-direction half of the UMP-paper's Open Problem 2; its resolution is the structural-completeness question of §10 Open Problem 2 of the present paper.

Contents

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1. Introduction

The σ-family of VERSF reconstruction papers has worked progressively deeper into the structural origin of quantum probability. The first paper (PAMV) showed that the minimal operational-measure axioms force probability within each coherence sector V_α to take a quadratic form (on $d_\alpha \geq 2$ sectors), with sector-dependent constants C_α left undetermined across sectors and the $d_\alpha = 1$ case essentially free. PAMV supplied three independent bridging conditions — (B1) compositional multiplicativity, (B2) isosymmetric embedding consistency, (B3) refinement-stability — any one of which sufficed to close the uniqueness argument and force the universal Born rule.

The second paper (the UMP paper) unified those three bridging conditions under a single principle: the Universal Measure Principle (UMP), which states that probability must be invariant under every operational *commensurability witness* between sectors. UMP supplied the conceptual unification PAMV's Open Problem 2 had asked for — showing the three bridging conditions to be three structurally distinct operational windows onto one underlying invariance requirement.

The UMP paper also named the deeper structural object UMP turned out to be an invariance principle *for*. The operational carrier supplied by PAMV §2 is not merely a complex Hilbert space; it is a structured geometric object jointly comprising carrier and inner product, coherence-sector decomposition, projection structure, admissible transport, refinement structure, and finite-distinguishability packing. The UMP paper called this jointly-supplied operational structure the **operational distinguishability geometry** (ODG) of the carrier, and observed that the natural invariance group of ODG extends (A3)'s within-sector invariance across the entire carrier. The three-clause UMP was the working specialization of this general ODG-invariance principle on the three identified classes of cross-sector witness (compositional, embedding, refinement).

The UMP paper left as an open question (its Open Problem 2) whether the three identified witness classes exhaust the structurally significant ODG-preserving operations on the carrier. This open question has two halves. The *forward* half — do the three classes suffice to capture all ODG-preserving operations relevant to probability — bears on whether the three-clause UMP is a complete operational specialization of ODG-invariance, or whether some ODG-preserving operations escape it. The *converse* half — whether UMP, restricted to the three witness classes, implies full ODG-invariance — bears on whether the three-clause UMP and ODG-invariance are logically equivalent on the VERSF carrier, or whether ODG-invariance is strictly stronger.

The present paper takes up the deeper structural question that motivates both halves of the UMP-paper's Open Problem 2:

Why should probability be a measure on operational geometry at all?

The thesis of the paper is that operational geometry already contains the structure that probability must respect. Once we recognize the operational geometry as a unified structured object — operational distinguishability geometry — then the natural invariance group of that geometry and the natural class of measures compatible with it are determined by the geometry itself. Probability is not an external structure layered onto the carrier; it is the unique invariant measure compatible with the operational geometry already present.

Under this framing, UMP is not the primitive principle; it is the operational working specialization of a deeper principle — *ODG-compatibility* — on the three classes of cross-sector witness identified in the UMP paper. The structural hierarchy becomes:

Operational geometry → ODG-invariance → UMP → Born rule.

Each arrow expresses a specific structural move. The first is recognition: the carrier already supplies a structured geometry. The second is principle: probability must be invariant under operations preserving that geometry. The third is specialization: ODG-invariance, restricted to the three commensurability witness classes, is UMP. The fourth is uniqueness: UMP plus the minimal-core axioms pick out the Born rule (proven in the UMP paper, Theorem 7.1).

1.1 What the present paper adds beyond UMP

The technical content this paper adds is concentrated in §4 through §6.

§4 strengthens the negative result of the UMP paper's Theorem 3.2: sector-relative measures fail not only the three structural-failure tests (F1)–(F3) but the more general ODG-compatibility condition. This makes explicit that the three failures of the UMP paper were three concrete manifestations of one underlying ODG-incompatibility.

§5 defines ODG-compatibility precisely: a measure is ODG-compatible if it is invariant under every ODG-preserving operation. This is the *general* form of the principle that UMP specialized.

§6 proves the central theorem: ODG-compatibility implies UMP (Theorem 6.1, **proven**). The converse — UMP implies ODG-compatibility — is **conditional** on the structural conjecture that the three commensurability witness classes exhaust the ODG-preserving operations (Theorem 6.4). On the VERSF carrier, the conjecture is plausible but not proven.

§7 then assembles the Born rule from ODG-compatibility directly. The resulting Born rule is the same as the one delivered by UMP in the UMP paper; what changes is the principle from which it is derived. The structural status of the Born rule is sharpened: from "the unique commensurable minimal-core measure" (UMP paper) to "the unique ODG-invariant minimal-core measure" (present paper).

1.2 The structural reframing

The reframing operates at the level of which principle is primitive in the σ-family architecture. The UMP paper had identified UMP as the unifying principle behind the three bridging conditions, and named ODG as the deeper structural object UMP is invariance for. The present paper makes two conceptual moves further: (i) ODG is treated as the primitive geometric object, with UMP becoming a derived consequence of ODG-compatibility specialized to three witness classes (Theorem 6.1); (ii) ODG-compatibility is read as the measure-theoretic expression of an operational-indistinguishability principle (OIP, §5A) — *the cleanest operational formulation of the same structural commitment: probability must factor through the quotient of states by ODG-indistinguishability* (Proposition 5A.3A). The OIP reformulation is operational rather than logical deepening: the substantive content-vs-label commitment is the same as commensurability/UMP carries, relocated into Definition 5A.1's anchoring to Definition 3.1A. Three specific conceptual gains follow — naming the unifying object explicitly, sharpening the Born rule's structural fingerprint, and surfacing the quotient-space structure that ODG-compatibility specializes. These are developed in §7.2 (S1)–(S3).

Remark 1.2.1 (Geometry versus measure). It is important to distinguish the logical categories occupied by ODG and UMP. ODG is a *geometric object*: it specifies the operational structure of the carrier — distinguishability relations, sector decomposition, admissible transport, refinement structure, projection structure, and finite packing. UMP is *not* a geometric object; it is a *property of admissible probability measures* defined on that geometry. The present paper therefore does not merely rename UMP. It changes the level at which the primitive notion is located: the primitive object becomes the geometry itself, while UMP is recovered as a compatibility condition imposed on measures defined over that geometry. The distinction is analogous to the difference between a manifold and a measure defined on that manifold — the measure may be constrained by the geometry, but the two belong to different conceptual categories.

1.3 The five-layer hierarchy

The structural hierarchy *operational geometry → operational indistinguishability → ODG-compatibility → UMP → Born rule* is the conceptual backbone of the paper. The full five-layer table is given in §8.1. OIP appears at Layer 2 (the principle layer, supplying the quotient-space view); ODG-compatibility at Layer 3 (the measure-theoretic specialization, equivalent to OIP given Definition 5A.1's content-vs-label commitment); UMP as a sub-layer of Layer 3 (the working specialization on three witness classes); (A3) as the within-sector fragment of Layer 3.

1.4 What this paper does and does not claim

The conditional-status structure is detailed in the Scope section above; the short form is: ODG-compatibility is treated as a structural input on admissible measures (paralleling how UMP and the minimal-core axioms were treated as inputs in earlier σ-family papers), the forward direction ODG-compatibility ⟹ UMP is **proven** unconditionally (Theorem 6.1), and the converse is **conditional** on the structural-completeness conjecture (Theorem 6.4, Conjecture 3.5). Whether ODG itself can be derived from substrate-level structure is Open Problem 1 of §10.

The remainder of the paper develops the technical content. §2 formalizes ODG. §3 enumerates ODG-preserving operations and proves the commensurability witnesses are among them (Proposition 3.3). §4 establishes that sector-relative measures violate ODG-compatibility. §5A introduces OIP as the cleanest operational formulation of the same structural commitment: probability factors through the operational-indistinguishability quotient (Proposition 5A.3A). §5 defines ODG-compatible measures (read as the measure-theoretic expression of OIP). §6 derives UMP from ODG-compatibility, with explicit attention to the conditional status of the converse direction. §7 assembles the Born rule. §8 develops the interpretation. §9 sets out the falsifiability content. §10 the open problems. §11 the conclusion.

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2. Operational Distinguishability Geometry

We now formalize the unified structural object that the inherited operational structure of PAMV §2 supplies. The substantive recognition is that this structure is a *geometry* — a single structured object — rather than a list of separate pieces.

2.1 The six components of ODG

The operational carrier ($\mathcal{A}_\mathbb{C}$, $\langle\cdot,\cdot\rangle_\mathbb{C}$) jointly carries the following six components, each inherited from the structural results of OG, OC, URHG, and PAMV:

- **(ODG-i) Carrier and inner product.** The complex Hilbert space $\mathcal{A}_\mathbb{C}$ with its operational inner product $\langle\cdot,\cdot\rangle_\mathbb{C}$. *Source:* OG Theorem 4.0 (operational carrier construction); URHG Theorem 3.5 (complex-structure inheritance from substrate $\mathbb{Z}_7$-equivariance).
- **(ODG-ii) Coherence-sector decomposition.** The substrate-character partition $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$, where each V_α is a transport-invariant subspace labelled by a $\mathbb{Z}_7$-character $\chi_\alpha : \mathbb{Z}_7 \to U(1)$. *Source:* URHG §2.5 (sector classification); PAMV §2.3A (operational coherence sectors).

- **(ODG-iii) Orthogonal projection structure.** The admissibility-projection lattice on $\mathcal{A}_\mathbb{C}$, comprising the orthogonal projections onto admissible subspaces. *Source:* OG Theorem 5.1 (projection structure on the operational carrier).
- **(ODG-iv) Admissible transport structure.** The class of transport-admissible (U-trans) unitaries $U\_\text{adm-transport}(\mathcal{A}_\mathbb{C}) = \prod_\alpha U(V_\alpha)$, comprising the $\mathbb{Z}_7$-equivariant unitaries that preserve the substrate transport response. *Source:* URHG Theorems 4.1, 5.1 (admissible transport classification); PAMV §2.3A (transport-block structure).
- **(ODG-v) Refinement structure.** The class of admissible orthogonal decompositions of multi-channel subspaces, together with the decomposition-admissible (U-dec) unitaries relating them — viewed as relabelings of operational decomposition structure rather than physical transport operations. *Source:* PAMV §6.2 (admissible decomposition apparatus); UMP paper §2 (the (U-trans)/(U-dec) distinction).
- **(ODG-vi) Finite distinguishability packing.** The packing-volume measure Vol_{op} on the carrier, supplying continuity and bounded operational resolution: the cardinality of any operational distinguishability set $\Sigma(M)$ is bounded by $Vol_{op}(M) / \Delta_{op}^{d_{op}}$. *Source:* OG Theorem 10.1 (finite-packing geometry).

2.2 ODG as a unified object

Definition 2.1 (Operational distinguishability geometry). *The **operational distinguishability geometry** (ODG) of the operational carrier ($\mathcal{A}_\mathbb{C}$, $\langle\cdot,\cdot\rangle_\mathbb{C}$) is the unified structural object comprising the six components (ODG-i)–(ODG-vi) above, taken jointly. It governs which states are operationally distinguishable, how distinguishability content combines across sectors (via composition, embedding, refinement), how it evolves under admissible transport, and how it sits within finite operational resolution.*

Remark 2.1.1 (ODG is not new structure). It bears emphasis — as the UMP paper §4.1 already noted — that operational distinguishability geometry is not an additional structure layered onto the carrier to make the present argument work. It is a *repackaging* of the invariant operational structure already accepted as part of the inherited geometry. Each component of ODG corresponds to an element of the existing PAMV/URHG/OG construction (see the source references in §2.1). The structural move of the present paper is to recognize this collection as a single unified geometric object and to identify the natural invariance group it supports, which is what governs admissible probability assignment.

Remark 2.1.2 (Why "geometry"). The term "distinguishability *geometry*" is justified by the structural features of ODG that go beyond what a mere collection of axioms supplies. ODG has:

- a *carrier* (the underlying space) with metric structure (inner product, norm);
- a *partition* (the sector decomposition) compatible with the carrier;
- a *symmetry group* (the class of admissible transport, plus the admissible decomposition relabelings);
- a *refinement structure* (the lattice of admissible decompositions);
- a *quantification* (the finite-packing measure).

These are the structural features of a geometric object in the broad sense — a set together with metric, symmetry, and decomposition structure — rather than of an unstructured measurable space. ODG can therefore be treated as a geometry whose natural invariance group acts on admissible measures by structural compatibility.

2.3 Coherence sectors and two senses of admissibility

We recall from the UMP paper §2 the two senses of "admissible unitary" that the present paper requires:

- **(U-trans) Transport-admissible unitaries:** the Schur-block-diagonal class $U\_adm\text{-}transport(\mathcal{A}\_\mathbb{C}) = \prod\_\alpha U(V\_\alpha)$, representing physically realizable reversible transport. These cannot mix inequivalent $\mathbb{Z}\_7$-character sectors. (Component (ODG-iv).)
- **(U-dec) Decomposition-admissible unitaries:** norm-preserving unitaries on a multi-channel subspace $V\_\alpha \oplus V\_\beta$ that relate two admissible orthogonal decompositions of that subspace, viewed as relabelings of decomposition structure rather than transport. These may mix inequivalent sectors at the level of decomposition geometry. (Part of component (ODG-v).)

The distinction matters because the (C-ref) refinement-commensurability relation (UMP paper Definition 4.2) and the corresponding ODG-preserving refinement-relabeling operations of §3.1 below use (U-dec) unitaries, which are not transport-admissible when $\alpha \neq \beta$. ODG is invariant under both (U-trans) and (U-dec) operations — the latter by virtue of (ODG-v)'s refinement structure being preserved by admissible decomposition relabelings.

For brevity we use "sector" and "channel" interchangeably to mean V_α, following the UMP paper's convention.

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3. ODG-Preserving Operations

We now define the natural invariance class of operational distinguishability geometry: the operations preserving all six components of ODG simultaneously. The class will include four structurally distinct types — admissible transport, compositional identifications, embedding maps, refinement relabelings — each preserving ODG by virtue of preserving each of its six components. We then establish (Propositions 3.2 and 3.3) that the three commensurability witnesses of the UMP paper ($\sigma\_\{\alpha,\beta\}$, ι, W) plus the admissible-transport class are ODG-preserving — the structural foundation for §6's derivation of UMP from ODG-compatibility.

3.1 Definition

Definition 3.1 (ODG-preserving operation). *An operation T on $\mathcal{A}\_\mathbb{C}$ — possibly between distinct admissible carriers (as in the case of compositional identifications between original and tensor-extension carriers) — is* ***ODG-preserving*** *if T preserves every one of the six components (ODG-i)–(ODG-vi) listed in §2.1. Explicitly, T preserves:*

(i) the carrier and inner product (inner-product preservation on the relevant subspace);

(ii) the coherence-sector decomposition (substrate-character partition compatibility);

(iii) the orthogonal projection structure (admissibility-projection lattice preservation);

(iv) the admissible transport structure (intertwines with admissible transport on its domain and codomain);

(v) the refinement structure (compatibility with admissible decomposition relabelings);

(vi) the finite distinguishability packing (carrier-geometric volume-preserving up to overall normalization).

Remark 3.1.1 (Strict and partial ODG-preservation). An operation T may preserve all six components on a restricted subspace or domain — for instance, a compositional identification $\sigma_{\alpha,\beta}$ preserves ODG on the $V_\alpha \otimes V_\beta$ domain in the tensor-extension carrier and on the $V_{\alpha+\beta \bmod 7}$ isotypic component in the original carrier, without acting on the rest of either carrier. We call such operations *ODG-preserving on their domain* and treat them as full ODG-preserving operations for purposes of the invariance condition of §5. Operations preserving only some of (i)–(vi) but not all are *not* ODG-preserving in the sense of Definition 3.1 and do not enter the invariance condition.

3.2 The four classes of ODG-preserving operation

We identify four structurally distinct classes of ODG-preserving operation on the VERSF carrier:

- **(O1) Admissible transport operations.** The transport-admissible (U-trans) unitaries $U_{\text{adm-transport}}(\mathcal{A}_\mathbb{C}) = \prod_\alpha U(V_\alpha)$. These are the within-sector unitaries that preserve substrate $\mathbb{Z}_7$-equivariance.
- **(O2) Compositional identifications.** For each pair (α, β) with $\alpha + \beta \not\equiv 0 \bmod 7$, the carrier-to-tensor isotypic identification $\sigma_{\alpha,\beta} : V_\alpha \otimes V_\beta \to V_{\alpha+\beta \bmod 7}$-isotypic-component, mapping the tensor extension to the isotypic component of the original carrier under character matching.
- **(O3) Embedding maps.** For each pair (α, β) with $d_\beta \geq d_\alpha$, complex-unitary embeddings $\iota : V_\alpha \to V_\beta$ realizing the lower-dimensional sector as a Hilbert subspace of the higher-dimensional one.
- **(O4) Refinement relabelings.** For each multi-channel subspace $V_\alpha \oplus V_\beta$, the decomposition-admissible (U-dec) unitaries $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$ relating two admissible orthogonal decompositions of that subspace.

The class (O1) is the within-sector fragment that the minimal-core axiom (A3) already imposes. The classes (O2), (O3), (O4) are the cross-sector classes that go beyond (A3) and that UMP captures.

3.3 Each class preserves ODG

Before stating Propositions 3.2 and 3.3, we make precise the criterion for "preserves ODG-(i)–(vi)" that applies uniformly across all witness types. The precision matters: a loose reading of "structural-type preservation" would admit arbitrary unitaries between orthogonal decompositions (including generic rotations between inequivalent characters), collapsing the entire commensurability framework into full unitary invariance. The precise criterion below excludes such generic rotations by requiring not just that *some* orthogonal decomposition exists on the codomain, but that *the codomain decomposition is itself admissible* — i.e. recognized by the substrate-derived admissibility structure inherited from PAMV §6.2 and URHG §2.5.

Definition 3.1A (Structural-type preservation, precise criterion). *An operation T preserves the structural type of a component (ODG-i)–(ODG-vi) under the following criteria:*

— ***(ODG-i) carrier and inner product.*** *T is an isometry on its domain (preserves the inner product).*

— ***(ODG-ii) sector decomposition.*** *Two conditions must hold jointly: (a) T maps each admissible sector in its domain to an admissible sector in its codomain (the partition into admissible sectors is preserved, possibly with character relabeling along T); AND (b) the codomain decomposition is itself admissible — i.e. recognized as an admissible decomposition by the substrate-derived structure inherited from PAMV §6.2 (admissible-decomposition apparatus) and URHG §2.5 (sector classification).*

Condition (b) is the ***admissibility gate****. It distinguishes structurally-significant ODG-preserving operations from arbitrary mathematical correspondences between orthogonal decompositions.*

— ***(ODG-iii) projection structure.*** *T maps admissible orthogonal projections to admissible orthogonal projections (preserves the admissibility-projection lattice).*

— ***(ODG-iv) admissible transport class.*** *Two conditions must hold: (a) The structural type of the admissible-transport class on T's codomain (the product of within-sector unitary groups indexed by the codomain's admissible decomposition) corresponds under T to the admissible-transport class on T's domain (the analogous product on the domain decomposition); AND (b) the correspondence is itself admissibility-respecting — i.e. for class (O1), T intertwines admissible transport pointwise; for cross-sector classes (O2)–(O4), T is itself recognized as an admissibility-preserving identification by the substrate structure (the (U-dec) class for refinement relabelings, the isotypic identification for compositional witnesses, the embedding-isometry condition for embeddings).*

— ***(ODG-v) refinement structure.*** *T is recognized by the substrate refinement structure as a (U-dec) admissibility witness (for relabelings between admissible decompositions of multi-channel subspaces) or maps admissible decompositions to admissible decompositions in the carrier-derived sense.*

— ***(ODG-vi) finite distinguishability packing.*** *T is volume-preserving up to overall normalization, i.e.* $Vol\_op(TM) = c \cdot Vol\_op(M)$ *for some normalization constant* $c > 0$ *and every admissible subspace M.*

Remark 3.1A.1 (The admissibility gate excludes generic rotations; its content is inherited from PAMV §6.2). The criterion (b) for (ODG-ii) is the structural gate that prevents the framework from collapsing to full unitary invariance. A generic unitary U on a multi-channel subspace $V_\alpha \oplus V_\beta$ between inequivalent $\mathbb{Z}_7$-characters produces an orthogonal decomposition (V_α'', V_β'') on the codomain. Unless (V_α'', V_β'') is itself admissible in the PAMV §6.2 sense — i.e. derived from the substrate refinement structure — U does *not* satisfy condition (b) and is therefore not ODG-preserving. The (U-dec) class of refinement relabelings is precisely the subclass of unitaries between orthogonal decompositions of $V_\alpha \oplus V_\beta$ that the substrate refinement structure recognizes as admissibility-preserving; generic rotations outside this subclass are excluded by the gate.

The discriminating power of the gate flows entirely from PAMV §6.2's specification of which orthogonal decompositions count as admissible. The gate is non-vacuous precisely to the extent that PAMV §6.2 supplies a restrictive admissibility criterion. The explicit content of Conjecture 3.5 therefore reduces to: *does PAMV §6.2's admissibility notion, together with the three identified cross-sector classes (O2)–(O4) and the within-sector class (O1), exhaust the admissibility-respecting structure-preserving operations on the VERSF carrier?* This is the load-bearing question; the gate is a relay, not an independent filter.

Remark 3.1A.2 (Internal tension between Proposition 3.3 and Conjecture 3.5; PAMV §6.2 dependency). The admissibility gate exhibits an internal tension that the present framework navigates by relaying PAMV §6.2's admissibility criterion verbatim. *Weakening* the gate — admitting more operations as ODG-preserving — makes Proposition 3.3 easier to verify but makes Conjecture 3.5 less plausible, since more candidate operations would qualify. *Strengthening* the gate — admitting fewer operations — makes Conjecture 3.5 more plausible but makes Proposition 3.3 harder. The precise criterion of Definition 3.1A is calibrated to the inherited PAMV §6.2 / URHG §2.5 admissibility structure: this is the hinge.

The calibration is *not* a free parameter of the present paper — but only to the extent that PAMV §6.2 itself supplies a restrictive admissibility criterion. If PAMV §6.2 were liberal in what it admits as an admissible decomposition, the gate would not effectively distinguish (O1)–(O4) from generic operations, and Conjecture 3.5 would correspondingly weaken. The non-triviality of Conjecture 3.5 inherits the non-triviality of PAMV §6.2's admissibility criterion. The internal tension between Proposition 3.3 and Conjecture 3.5 is therefore structurally resolved *at the level of inherited admissibility*, with the present paper relying on (and inheriting the conditional status of) PAMV §6.2's admissibility apparatus rather than supplying an independent gate.

The other four components — (ODG-i) carrier and inner product, (ODG-iii) projection structure, (ODG-v) refinement structure, (ODG-vi) finite packing — admit cleaner structural-type readings (isometry, lattice-isomorphism, admissibility-preservation of decompositions, volume-preservation up to normalization). The substantive content of the precise criterion is concentrated in (ODG-ii) and (ODG-iv), where the admissibility gate (b) does the structural work.

The structural-type criterion of Definition 3.1A is the engineering view: it tells us how to *check* whether an operation is ODG-preserving, component by component, with explicit attention to the admissibility gate that excludes generic structure-blind operations. We now supply a complementary operational characterization that captures the same content without referencing sector labels, decomposition labels, or witness classes — a purely operational criterion stating what ODG-preservation *means* in terms of what an admissible procedure can detect.

Definition 3.1B (Operational structural preservation). *An operation T on $\mathcal{A}$_ℂ (possibly between distinct admissible carriers) is* ***ODG-preserving*** *if and only if T preserves:*

(1) operational distinguishability relations between states;

(2) admissible projection structure;

(3) admissible transport equivalence classes;

(4) admissible refinement equivalence classes;

(5) finite distinguishability resolution.

Equivalently: T is ODG-preserving precisely when no operational procedure available within the inherited geometry can distinguish a state from its image under T.

The criterion refers only to operationally testable structure, not to sector labels, decomposition labels, or witness classes.

Proposition 3.1B.1 (Conditional equivalence of structural-type and operational criteria). *Under the* ***structural-exhaustion convention*** *that the six components (ODG-i)–(ODG-vi) collectively exhaust the operational structure inherited from the carrier (per the §2 construction of ODG), an operation T satisfies Definition 3.1B if and only if T satisfies Definition 3.1A.*

Proof. (⟸) Suppose T satisfies Definition 3.1A — preserves the structural type of each of the six ODG components. Operational distinguishability relations (condition 1 of 3.1B) are determined by carrier and inner product (ODG-i) together with sector decomposition (ODG-ii) — both preserved by 3.1A. Conditions (2)–(5) of 3.1B correspond directly to (ODG-iii), (ODG-iv), (ODG-v), (ODG-vi), each preserved by 3.1A under the structural-type criterion.

(⟹) Suppose T satisfies Definition 3.1B. By the structural-exhaustion convention, any alteration of distinguishability content must appear as a change in at least one of the six ODG components. Preservation of operational distinguishability content therefore entails preservation of each component under the structural-type criterion, which is Definition 3.1A. ■

Status. The structural-exhaustion convention is a *structural commitment about the inherited construction*, not a derivation. The six components were defined in §2 to be the operational structure of the carrier — exhaustiveness is a property of how ODG is constructed, not something proved here. Proposition 3.1B.1 should therefore be read as a conditional equivalence that holds under this convention. If a feature of operational structure exists that is not captured by any of (ODG-i)–(ODG-vi), the convention fails and the equivalence is correspondingly partial:

Definition 3.1A becomes the more reliable characterization (since 3.1A is component-by-component verification), and Definition 3.1B inherits the conditional status of the convention.

Remark 3.1B.2 (Why both characterizations matter). The two characterizations serve complementary purposes. Definition 3.1A is the *check*: it tells us how to verify ODG-preservation by working through the six components with attention to the admissibility gate. Definition 3.1B is the *meaning*: it tells us what ODG-preservation *is* operationally, namely undetectability by admissible procedures. The two formulations are equivalent (Proposition 3.1B.1) but invoke different intuitions — 3.1A for structural verification, 3.1B for operational interpretation. In what follows we use 3.1A for the technical proofs (Propositions 3.2, 3.3) and 3.1B as the operational reading invoked in §5A's OIP framing.

Proposition 3.2 (Admissible transport operations are ODG-preserving). *Every transport-admissible (U-trans) unitary $U \in U\_adm\text{-}transport(\mathcal{A}\_\mathbb{C})$ preserves all six components of ODG under the precise criterion of Definition 3.1A — equivalently, preserves operational distinguishability content (Definition 3.1B, Proposition 3.1B.1).*

Proof. (ODG-i) Inner-product preservation is the definition of unitarity. (ODG-ii) U is Schur-block-diagonal across inequivalent character classes, so it preserves the sector decomposition pointwise — condition (a) holds (each admissible sector V_α is mapped to itself), and condition (b) is satisfied trivially (the codomain decomposition equals the domain decomposition, which is admissible by hypothesis). (ODG-iii) Orthogonal projections onto admissible subspaces transform under U into orthogonal projections onto unitarily-equivalent admissible subspaces. (ODG-iv) (a) The admissible-transport class is closed under composition (it is a group), so U intertwines admissible transport pointwise; (b) U is itself a transport-admissible (U-trans) operation, so the correspondence is admissibility-respecting trivially. (ODG-v) U commutes with admissible decompositions because it preserves orthogonality and the sector decomposition pointwise. (ODG-vi) The packing-volume measure is invariant under unitaries on the carrier (OG Theorem 10.1's volume is carrier-geometric). ■

Proposition 3.3 (Commensurability witnesses preserve operational distinguishability content). *Each commensurability witness — σ_{α,β}, ι, W — of UMP-paper Definition 4.3 preserves operational distinguishability content on its domain (Definition 3.1B), equivalently preserves all six components of ODG under the precise structural-type criterion (Definition 3.1A, Proposition 3.1B.1).*

Proof. We verify each class, with explicit attention to the admissibility gate (ODG-ii)(b).

Compositional identification σ_{α,β}. (ODG-i) σ_{α,β} is an isotypic-isomorphism between V_α ⊗ V_β and the V_{α+β mod 7} isotypic component; it preserves the tensor inner product, which factors through this identification into the inner product on the V_{α+β mod 7} component. (ODG-ii) (a) σ_{α,β} maps V_α ⊗ V_β (an admissible tensor-extension structure carrying the product character χ_α · χ_β = χ_{α+β mod 7}) to the V_{α+β mod 7} isotypic component (an admissible sector of the original carrier); (b) the codomain V_{α+β mod 7} is admissible in the original carrier (it is a sector of the inherited PAMV §6.2 decomposition), and σ_{α,β} is the substrate-derived isotypic identification — recognized as admissibility-preserving by the character-matching structure of URHG §2.5. (ODG-iii) Orthogonal projections in $V_\alpha \otimes V_\beta$ onto admissible subspaces correspond to orthogonal projections in the $V_{\alpha+\beta \bmod 7}$ isotypic component under $\sigma_{\alpha,\beta}$. (ODG-iv) $\sigma_{\alpha,\beta}$ intertwines admissible transport on $V_\alpha \otimes V_\beta$ (the within-V_α and within-V_β U-trans actions, combined tensorially) with admissible transport on the $V_{\alpha+\beta \bmod 7}$ component, with the correspondence recognized by the substrate isotypic-identification structure. (ODG-v) Admissible decompositions of $V_\alpha \otimes V_\beta$ correspond under $\sigma_{\alpha,\beta}$ to admissible decompositions of the $V_{\alpha+\beta \bmod 7}$ isotypic component. (ODG-vi) The packing-volume measure is multiplicative on tensor extensions, and $\sigma_{\alpha,\beta}$ is volume-preserving as a structural isomorphism up to overall normalization.

Embedding map $\iota : V_\alpha \to V_\beta$ *with* $d_\beta \geq d_\alpha$. (ODG-i) ι is complex-unitary by definition of (C-emb), so it preserves inner products on its image. (ODG-ii) (a) ι maps V_α (an admissible sector with character χ_α) into V_β (an admissible sector with character χ_β); (b) the codomain V_β is admissible (a sector of the inherited PAMV §6.2 decomposition), and ι is the complex-unitary embedding — recognized as admissibility-preserving by the embedding admissibility condition of (C-emb). (ODG-iii) Orthogonal projections in V_α correspond under ι to orthogonal projections onto subspaces of $\iota(V_\alpha) \subseteq V_\beta$. (ODG-iv) ι intertwines $U(V_\alpha)$-equivariant admissible transport on V_α with the $U(d_\alpha) \subseteq U(V_\beta)$ subgroup acting on $\iota(V_\alpha)$, with the correspondence recognized by the embedding admissibility condition. (ODG-v) Admissible decompositions of V_α correspond under ι to admissible decompositions of $\iota(V_\alpha)$. (ODG-vi) ι is volume-preserving (Vol_{op} on $\iota(V_\alpha)$ equals Vol_{op} on V_α, by isometry).

Refinement relabeling $W : V_\alpha \oplus V_\beta \to V_\alpha' \oplus V_\beta'$. (ODG-i) W is unitary by definition of (U-dec), so it preserves inner products. (ODG-ii) (a) W relabels the sector decomposition $(V_\alpha, V_\beta) \to (V_\alpha', V_\beta')$ at the level of admissible sectors (character labels may be mixed at the basis level, but each codomain V_α', V_β' is itself admissible); (b) the codomain decomposition (V_α', V_β') is admissible *by definition of (U-dec)* — the (U-dec) class is exactly the class of unitaries between $V_\alpha \oplus V_\beta$ and $V_\alpha' \oplus V_\beta'$ that the substrate refinement structure (PAMV §6.2) recognizes as relating two admissible decompositions. *This is the critical use of the admissibility gate*: a generic unitary U on $V_\alpha \oplus V_\beta$ that produces a non-admissible decomposition on the codomain would *not* be in the (U-dec) class and would *not* satisfy condition (b), and is therefore not ODG-preserving. (ODG-iii) Orthogonal projections on $V_\alpha \oplus V_\beta$ correspond under W to orthogonal projections on $V_\alpha' \oplus V_\beta'$. (ODG-iv) The admissible-transport class on $V_\alpha' \oplus V_\beta'$ (the $U(V_\alpha') \times U(V_\beta')$ subgroup) corresponds under W to the admissible-transport class on $V_\alpha \oplus V_\beta$ (the $U(V_\alpha) \times U(V_\beta)$ subgroup), with the correspondence recognized by the (U-dec) admissibility condition (b). (ODG-v) Refinement structure is preserved by definition of (U-dec) — W is itself a witness of the refinement-structure compatibility. (ODG-vi) W is volume-preserving (Vol_{op} on $V_\alpha \oplus V_\beta$ is unitary-invariant).
■

Remark 3.3.1 (Two senses of refinement-relabeling preservation, separated). Two structurally distinct propositions about the refinement-relabeling case must be separated:

(i) Structural-type preservation of (ODG-ii) by W. W preserves the substrate-character partition at the structural-type level per Definition 3.1A: the decomposition (V_α, V_β) is relabeled to (V_α', V_β') as a two-sector partition structure, with admissibility of the codomain

decomposition guaranteed by W's membership in the (U-dec) class. This is a property of W as an ODG-preserving operation, intrinsic to the (U-dec) admissibility structure, and holds before any probability measure is introduced.

(ii) Property of ODG-compatible measures under the sector-attachment convention. Once an ODG-compatible measure μ is introduced, the sector-attachment convention of UMP-paper Theorem 3.2 proof of (F3) — that decomposition-attached quadratic coefficients travel with their sectors under (U-dec) relabeling — gives $C_{\alpha'} = C_\alpha$ and $C_{\beta'} = C_\beta$ for the per-sector quadratic coefficients of μ. This is a property of *measures compatible with* (i), not a property of W alone; it follows from imposing ODG-compatibility on μ given the structural relabeling (i), not from the structural definition of W.

The two propositions interact in the standard way: (i) is the structural fact that makes (ii) well-posed as a constraint on admissible measures, and (ii) is the measure-theoretic consequence of imposing ODG-compatibility given (i).

Remark 3.3.2 (Summary). Propositions 3.2 and 3.3 together establish that the four classes (O1)–(O4) of §3.2 are each classes of ODG-preserving operations under Definition 3.1A. We use the notation "(O1)–(O4) are ODG-preserving" in the sequel to refer to this combined result.

Remark 3.3.3 (Dependence on the inherited admissibility apparatus). The present paper does not attempt to strengthen, modify, or independently derive the admissibility structure inherited from PAMV §6.2 and URHG §2.5. The role of §3 is strictly *propagative* rather than *generative*: given the admissibility apparatus already accepted in PAMV and URHG, the present paper verifies that the corresponding admissibility-respecting operations preserve all six components of operational distinguishability geometry.

Accordingly, Proposition 3.3 should not be read as establishing admissibility from first principles. Its content is narrower and more precise: it establishes that the operational witnesses already recognized by the inherited framework preserve ODG in the sense of Definition 3.1A. Questions concerning why the admissibility structure takes its PAMV form, whether alternative admissibility structures are possible, and whether admissibility itself can be derived from substrate-level structure belong to Open Problem 1 and remain outside the scope of the present paper.

3.4 Completeness of the four classes (an open structural question)

The four classes (O1)–(O4) cover all the ODG-preserving operations on the VERSF carrier that the UMP paper's commensurability framework and the minimal-core axioms supply explicitly. Whether they *exhaust* the structurally significant ODG-preserving operations is an open structural-completeness question that the present paper does not resolve.

Conjecture 3.5 (Structural completeness of (O1)–(O4)). *On the VERSF carrier, the four classes (O1)–(O4) exhaust the ODG-preserving operations up to composition. That is, every ODG-preserving operation on $\mathcal{A}_\mathbb{C}$ either belongs to one of the four classes or factors as a composition of operations from these classes.*

Equivalently (per Remark 3.1A.1): the substrate-derived admissibility structure of PAMV §6.2 — combined with the three identified cross-sector classes (O2), (O3), (O4) and the within-sector class (O1) — exhausts the admissibility-respecting structure-preserving operations on the VERSF carrier.

Status. **Conjectural** on the VERSF carrier; truth-value open. We have identified four classes of ODG-preserving operation and verified (Propositions 3.2, 3.3) that each preserves all six components of ODG. We have *not* established that these four classes exhaust the ODG-preserving operations, nor have we ruled out further classes. Several candidates are flagged in §10 Open Problem 2 — cross-carrier embeddings (O5?), refinement-with-embedding compositions (O6?), group-theoretic extensions of admissible transport (O7?) — each of which would either reduce to compositions of (O1)–(O4) or constitute a genuinely new class. We have not analyzed these candidates exhaustively. The conjecture is *not* supported by an exhaustion argument; it is supported only by the structural observation that the four identified classes each preserve all six ODG components, which is necessary but not sufficient. The substantive load of the conjecture sits with PAMV §6.2's admissibility apparatus: how restrictive that apparatus is, and how exhaustively (O1)–(O4) cover the operations it admits.

The conjecture is the converse-direction half of the UMP-paper's Open Problem 2, and its resolution is the structural-completeness question of §10 Open Problem 2 of the present paper.

Remark 3.5.1 (Why the conjecture matters; what its failure would mean). Conjecture 3.5 underwrites the equivalence between ODG-compatibility and UMP on the VERSF carrier (Theorem 6.4 below). Its empirical and structural stakes are asymmetric:

- *If Conjecture 3.5 holds:* ODG-compatibility and UMP are equivalent on the VERSF carrier. The present paper's structural-deepening over the UMP paper is conceptual rather than logical — same logical content, deeper structural reading.
- *If Conjecture 3.5 fails:* there exist ODG-preserving operations not reducible to compositions of (O1)–(O4). ODG-compatibility is then strictly stronger than UMP. The Born rule is still uniquely forced by ODG-compatibility (since ODG-compatibility implies UMP unconditionally by Theorem 6.1, which forces the Born rule by UMP-paper Theorem 7.1). UMP would be a strict consequence of ODG-compatibility rather than an equivalent reformulation, and the present paper's structural-deepening would also be a logical strengthening — additional invariance content beyond UMP would be required.

The principal results of the present paper — the unconditional Theorem 6.1 and the unconditional Theorem 7.1 — do not depend on Conjecture 3.5. The conjecture bears only on whether UMP captures all of ODG-invariance, not on whether ODG-invariance forces the Born rule. The conceptual-versus-logical content of the paper's contribution under each branch of Conjecture 3.5 is developed in Remark 6.4.3 below.

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4. Sector-Relative Measures and ODG-Failure

We now establish the negative result that motivates ODG-compatibility as a structural requirement: sector-relative minimal-core measures violate ODG-compatibility because they assign different probabilities to states related by ODG-preserving operations. This is the strengthened ODG-form of the UMP-paper's Theorem 3.2.

4.1 The minimal-core multi-parameter family

We recall from the UMP paper (and PAMV Lemma 6.2A) the precise scope of the minimal-core insufficiency.

For $(C_\alpha)_\alpha$ *a non-negative tuple indexed by all sectors, and* $(F_\alpha^{(1)})_{\{\alpha : d_\alpha = 1\}}$ a tuple of continuous non-negative functions on $[0, \infty)$ vanishing at zero indexed only by the $d_\alpha = 1$ sectors, define the parametrized family

$$\mu_{(C_\alpha), (F_\alpha^{(1)})}(\psi) := \Sigma_{\{\alpha : d_\alpha \geq 2\}} C_\alpha \|\psi_\alpha\|^2_{\mathbb{C}} + \Sigma_{\{\alpha : d_\alpha = 1\}} F_\alpha^{(1)}(\|\psi_\alpha\|_{\mathbb{C}}).$$

Lemma 4.1 (Minimal-core insufficiency). *The minimal-core axioms (A1)–(A3), (A5), (A7) admit the parametrized family* $\mu_{(C_\alpha), (F_\alpha^{(1)})}$ *as admissible probability measures for every choice of non-negative* $(C_\alpha)_\alpha$ *and every choice of continuous non-negative* $(F_\alpha^{(1)})_\alpha$ *on* $d_\alpha = 1$ *sectors with* $F_\alpha^{(1)}(0) = 0$.

Reference. This is PAMV Lemma 6.2A, restated as Lemma 3.1 of the UMP paper. The proof sketch is given in the UMP paper §3.1 and not repeated here. ■

4.2 The ODG-failure theorem

We now strengthen the UMP-paper's Theorem 3.2 to a direct ODG-incompatibility statement. The UMP paper's three structural failures (F1), (F2), (F3) — compositional, embedding, refinement — were three specific manifestations of one underlying defect. The present theorem makes this defect explicit: sector-relative measures fail invariance under ODG-preserving operations *as a class*, not just under three specific operations.

Theorem 4.2 (ODG-incompatibility of sector-relative measures). *Let* $\mu = \mu_{(C_\alpha), (F_\alpha^{(1)})}$ *be a minimal-core admissible measure with non-uniform quadratic coefficients:* $C_\alpha \neq C_\beta$ *for some pair* (α, β) *with* $d_\alpha, d_\beta \geq 2$. *Then there exists at least one ODG-preserving operation* T *(drawn from one of the four classes (O1)–(O4) of §3.2) and at least one state* ψ *such that* $\mu(T\psi) \neq \mu(\psi)$. *Therefore* μ *is not ODG-compatible.*

Moreover, μ *violates ODG-compatibility on each of the three cross-sector commensurability witness classes (O2), (O3), (O4) simultaneously: the failure is "triple" in the sense of the UMP paper's Theorem 3.2, and the three concrete failures (F1)–(F3) of that theorem are three manifestations of one underlying ODG-incompatibility.*

Proof. The three concrete failures (F1), (F2), (F3) of the UMP-paper's Theorem 3.2 each exhibit a specific ODG-preserving witness T (drawn from classes (O2), (O3), (O4) respectively) and a state ψ for which $\mu(T\psi) \neq \mu(\psi)$.

Compositional failure via (O2). The UMP-paper's (F1) exhibits unit-norm states $\psi \in V_\alpha$, $\phi \in V_\beta$ such that $\mu_{tensor}(\psi \otimes \phi)$ depends on which sector $V_{\alpha+\beta \bmod 7}$ hosts the composite, beyond what $\mu(\psi) \cdot \mu(\phi)$ would assign. The compositional identification $\sigma_{\alpha,\beta}$ is an ODG-preserving operation (Proposition 3.3), and the violation under $\sigma_{\alpha,\beta}$ is the witness of ODG-failure: $\sigma_{\alpha,\beta}(\psi \otimes \phi)$ is a unit-norm element of the $V_{\alpha+\beta \bmod 7}$ isotypic component with measure $C_{\alpha+\beta \bmod 7}$ (under the original carrier's measure μ), while $\mu_{tensor}(\psi \otimes \phi) = C_\alpha \cdot C_\beta$ under the tensor-extension multiplicativity. For $C_\alpha \cdot C_\beta \neq C_{\alpha+\beta \bmod 7}$ (which holds generically when the sector-relative coefficients fail the (♣) multiplicative relation), $\mu_{tensor}(\psi \otimes \phi) \neq \mu(\sigma_{\alpha,\beta}(\psi \otimes \phi))$, violating ODG-invariance under $\sigma_{\alpha,\beta}$.

Embedding failure via (O3). The UMP-paper's (F2) exhibits a complex-unitary embedding $\iota : V_\alpha \to V_\beta$ with $d_\alpha, d_\beta \geq 2$ and a state $\psi_\alpha \in V_\alpha$ such that $\mu(\iota(\psi_\alpha)) - \mu(\psi_\alpha) = (C_\beta - C_\alpha) \|\psi_\alpha\|^2_{\mathbb{C}} \neq 0$ when $C_\alpha \neq C_\beta$. The embedding ι is an ODG-preserving operation (Proposition 3.3), and the inequality $\mu(\iota(\psi_\alpha)) \neq \mu(\psi_\alpha)$ is direct ODG-invariance failure under ι.

Refinement failure via (O4). The UMP-paper's (F3) exhibits a decomposition-admissible (U-dec) unitary $W_\theta : V_\alpha \oplus V_\beta \to V_\alpha' \oplus V_\beta'$ and a state $\psi = e_\alpha$ such that $\mu(W_\theta \psi) = C_\alpha \cos^2\theta + C_\beta \sin^2\theta \neq C_\alpha = \mu(\psi)$ for $\theta \in (0, \pi/2)$ and $C_\alpha \neq C_\beta$. The refinement-relabeling W_θ is an ODG-preserving operation (Proposition 3.3), and the inequality $\mu(W_\theta \psi) \neq \mu(\psi)$ is direct ODG-invariance failure under W_θ.

In each case, μ violates ODG-invariance under at least one operation drawn from the corresponding ODG-preserving class. Since ODG-compatibility (Definition 5.1 below) requires invariance under *every* ODG-preserving operation, μ is not ODG-compatible. ■

Remark 4.2.1 (The triple-failure structure as ODG-incompatibility). The UMP paper's Theorem 3.2 was the first structural foundation for UMP: the three failures (F1), (F2), (F3) were identified as having a "common structural origin in the violation of operational commensurability." The present theorem sharpens this: the common structural origin is ODG-incompatibility specifically. The three failures are three concrete witnesses of one underlying failure-mode — that probability depends on sector-identity bookkeeping rather than on the unified operational geometry. Where the UMP paper read this as the failure of an unnamed "common structural property" that the three commensurability conditions repaired in three operationally-distinct ways, the present paper names that property — ODG-compatibility — and shows the three failures to be three concrete instances of its violation.

4.3 The shared structural origin made explicit

Remark 4.3.1 (One geometric defect, three operational manifestations). Under the ODG framing, the triple structural failure of the UMP paper is no longer a "coincidence to be defended" requiring a single deeper principle to unify (which UMP did). The three failures are three operational windows onto one structural object — operational distinguishability geometry

— and a measure that violates ODG-compatibility necessarily exhibits violations under multiple windows because the windows are projections of the same underlying geometric structure. The triple-failure structure is the operational signature of a single geometric defect, and the three-clause repair UMP supplies is the operational signature of a single geometric repair: making the measure ODG-compatible.

Remark 4.3.2 (Falsifiability sharpening). Theorem 4.2 supplies a sharper falsifiability content than the UMP paper's Theorem 3.2. Under the UMP framing, the three failures (F1)–(F3) were three structurally distinct tests, any one of which could falsify the UMP framework on a commensurable pair (UMP-paper §9.2, UMP-test 1). Under the ODG framing, *any* ODG-preserving operation supplies a test of ODG-compatibility, and the four classes (O1)–(O4) are only the most operationally accessible ones. If the structural-completeness conjecture (Conjecture 3.5) holds, the four classes exhaust the operationally meaningful tests; if not, further classes of ODG-preserving operation supply further tests. Either way, the falsifiability framework is at least as rich as UMP's, and richer if Conjecture 3.5 fails.

5A. The Operational Indistinguishability Principle

The Operational Indistinguishability Principle (OIP) supplies the cleanest *operational* formulation of the structural commitment ODG-compatibility carries: *probability must factor through the quotient of states by ODG-indistinguishability*. This section makes the thesis precise via Proposition 5A.3A (the quotient-space factorization) and Corollary 5A.4 (OIP ⟹ ODG-compatibility), and locates the substantive content honestly in Definition 5A.1's content-vs-label anchoring.

5A.1 The principle

Definition 5A.1 (Operational indistinguishability). *Two states ψ, $\phi \in \mathcal{A}_\mathbb{C}$ (possibly in different admissible carriers if a cross-carrier operational identification is at issue) are* ***operationally indistinguishable*** *if every operationally meaningful structure supplied by the carrier assigns them identical distinguishability content. "Operationally meaningful distinguishability content" here is the* ***structural-type content*** *of each ODG component per Definition 3.1A — that is, each component (ODG-i)–(ODG-vi) is assessed under its structural-type criterion. Host-sector labels, decomposition labels, and other bookkeeping data not internal to the carrier's intrinsic geometric structure are* ***not*** *part of operationally meaningful distinguishability content.*

Remark 5A.1.1 (The structural-type anchor is a defining commitment). The exclusion of host-sector labels and decomposition labels from "operationally meaningful distinguishability content" is the substantive judgment that makes Definition 5A.1 useful. Under a literal reading admitting any measurable distinction as "operationally distinguishing" — including reading off the host-sector label — the embedding $\iota : V_\alpha \to V_\beta$ would map ψ_α (in χ_α) to $\iota(\psi_\alpha)$ (in χ_β), and a procedure reading the host-sector label would distinguish them; Proposition 5A.3 (below) would then be false for cross-sector witnesses. The structural-type reading anchored to

Definition 3.1A is therefore a defining commitment, not an incidental clarification — and is the same content-vs-label judgment that commensurability/UMP makes.

Definition 5A.2 (Operational Indistinguishability Principle, OIP). *An admissible operational probability measure μ satisfies the* ***Operational Indistinguishability Principle*** *if every pair of operationally indistinguishable states (per Definition 5A.1) receives identical probability assignment:*

$$\psi \sim_{op} \phi \Longrightarrow \mu(\psi) = \mu(\phi).$$

OIP excludes the possibility that probability depends on distinctions the operational structural-type content does not recognize. Its motivation is operationally clean: probability assignment should be a function of operationally accessible content alone.

5A.2 Quotient-space content and ODG-compatibility

Proposition 5A.3 (ODG-preserving operations relate operationally indistinguishable states). *Every ODG-preserving operation T maps each state in its domain to an operationally indistinguishable state in its codomain: ψ and Tψ are operationally indistinguishable for every ψ in T's domain.*

Proof. T preserves all six ODG components under the structural-type criterion (Definition 3.1A). By Definition 5A.1, ψ and Tψ then agree on every operationally meaningful structure, hence are operationally indistinguishable. ■

Proposition 5A.3A (Probability factors through the indistinguishability quotient). *Operational indistinguishability* $\sim_{op}$ *is an equivalence relation on* $\mathcal{A}_{\mathbb{C}}$*. Under OIP, probability assignment factors through the quotient space* $\mathcal{A}_{\mathbb{C}} / \sim_{op}$*: there exists* $\bar{P} : \mathcal{A}_{\mathbb{C}} / \sim_{op} \to \mathbb{R}$ *such that*

$$\bar{P}([\psi]) = \mu(\psi) \text{ for every } \psi \in \mathcal{A}_{\mathbb{C}}.$$

Proof. $\sim_{op}$ is reflexive, symmetric, and transitive as a relation defined by agreement on every operationally meaningful structure; $\mathcal{A}_{\mathbb{C}} / \sim_{op}$ is therefore well-defined. By OIP, μ is constant on every equivalence class. The function $\bar{P}([\psi]) := \mu(\psi)$ is therefore well-defined (independent of representative), and $\mu = \bar{P} \circ q$ where $q : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}} / \sim_{op}$ is the quotient map. ■

Remark 5A.3A.1 (Why OIP is not merely a restatement of ODG-compatibility). At first sight OIP may appear to be ODG-compatibility expressed in different language. At the level of *which measures* are admissible, this is largely correct: both impose invariance under operationally irrelevant distinctions. The conceptual contribution of OIP lies elsewhere — in the *logical category* of the constraint.

- ODG-compatibility is formulated as a condition on **transformations**: *probability must be invariant under ODG-preserving operations.*

- OIP is formulated as a condition on the **domain of the probability function**: *probability must be a function on operational-indistinguishability classes.*

The latter exposes a structural feature not visible in the transformation-based formulation. Probability assignment naturally factors through the quotient $\mathcal{A}\_\mathbb{C}$ / ~_op (Proposition 5A.3A); the operationally meaningful object is not the raw state space but the quotient obtained after identifying operationally indistinguishable representatives. The value of OIP is *not* that it strengthens ODG-compatibility — it does not (Corollary 5A.4 is a near-definitional unfolding). Its value is that it *reveals the quotient-space structure that ODG-compatibility acts upon*.

Remark 5A.3A.2 (The gauge-quotient analogy is exact). The Proposition 5A.3A quotient-space view has a clean physical analogue. In gauge theory, observables are functions on the gauge-quotient (the physical phase space), not on the gauge-extended phase space; the gauge-redundant degrees of freedom are quotiented out before observables can be assigned values. Under OIP, probability is a function on the operational-indistinguishability quotient, with host-sector labels and decomposition labels playing the role of gauge-redundant degrees of freedom that the quotient removes. Operational indistinguishability *is* the operational gauge equivalence, and probability is a function on the operational gauge-quotient. The analogy is exact at the level of structural form.

Corollary 5A.4 (OIP ⟹ ODG-compatibility). *Every measure satisfying OIP is invariant under every ODG-preserving operation — i.e. is ODG-compatible (Definition 5.1 below).*

Proof. Let μ satisfy OIP and T be ODG-preserving. By Proposition 5A.3, ψ ~_op Tψ; by OIP, μ(ψ) = μ(Tψ). ■

5A.3 Honest accounting

Remark 5A.4.1 (Relationship between OIP and ODG-compatibility; converse direction open). Corollary 5A.4 establishes OIP ⟹ ODG-compatibility unconditionally. The converse — ODG-compatibility ⟹ OIP — would require that operationally indistinguishable states are always related by some ODG-preserving operation. This is a distinct conjecture from Conjecture 3.5: Conjecture 3.5 concerns whether (O1)–(O4) exhaust the ODG-preserving operations, while the OIP-converse conjecture concerns whether they generate the full equivalence relation ~_op on $\mathcal{A}\_\mathbb{C}$. The two coincide under the natural assumption that operational indistinguishability is generated by ODG-preserving operations — plausible but not proven. If either conjecture fails, OIP is strictly stronger than ODG-compatibility, and the principal results still hold via the chain OIP ⟹ ODG-compatibility ⟹ UMP ⟹ Born rule.

Remark 5A.4.2 (Honest accounting: OIP as cleanest operational formulation). OIP supplies the cleanest *operational* formulation of the same structural commitment that ODG-compatibility makes. The substantive content is built into Definition 5A.1's anchoring to Definition 3.1A: the exclusion of host-sector labels and decomposition labels from operational distinguishability content. Three consequences worth being explicit about:

Cor 5A.4 is a near-definitional unfolding. Just as Theorem 6.1's substance is concentrated upstream in Proposition 3.3, Corollary 5A.4's substance is concentrated upstream in Definition 5A.1. The work lives in the definition, not the corollary.

The Leibnizian analogy is real but caveated. OIP's framing has the rhetorical naturalness of identity-of-indiscernibles applied to probability, but identity-of-indiscernibles is only non-trivial once one has fixed which properties count as discernible. Definition 5A.1's structural-type anchor *is* that fixing — and is the same content-vs-label judgment that commensurability/UMP carries.

The genuine value-add is the quotient-space view. Proposition 5A.3A and Remark 5A.3A.1 surface what OIP makes visible that the measure-invariance framing of §5 does not: probability factors through the operational-indistinguishability quotient. OIP relocates rather than strictly deepens the structural commitment, but the relocation is operationally useful — it identifies the correct domain for the probability function.

The σ-family progression — PAMV's three bridging conditions, UMP's unifying principle, the present paper's OIP/ODG-compatibility framing — is a sequence of *operational reformulations* of the same content-vs-label commitment, each making a different aspect visible. The substantive anchor is PAMV §6.2's admissibility apparatus and PAMV's Open Problem 10; substrate-level derivation of these is the deepest open question (Open Problem 1).

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5. ODG-Compatible Measures

We now define the measure-theoretic expression of OIP — the ODG-compatibility condition on admissible measures that Corollary 5A.4 establishes as a consequence of OIP. The structural content of this section was originally stated as the primary axiom; under the §5A framing, it is read as the measure-theoretic consequence of OIP applied to the operational indistinguishability that ODG-preserving operations supply (Proposition 5A.3, Corollary 5A.4).

5.1 The definition

Definition 5.1 (ODG-compatible measure). *An admissible operational probability measure μ on the carrier $\mathcal{A}_\mathbb{C}$ — together with its tensor-extension μ_{tensor} on every tensor-extension carrier $V_\alpha \otimes V_\beta$ for which a compositional commensurability witness exists — is* ***ODG-compatible*** *if μ is invariant under every ODG-preserving operation. Explicitly, for every ODG-preserving operation T (in the sense of Definition 3.1) and every state ψ in the domain of T:*

$\mu(T\psi) = \mu(\psi)$.

Where T is a cross-carrier operation between two distinct admissible carriers (as for compositional identifications), the equation reads $\mu_{codomain}(T\psi) = \mu_{domain}(\psi)$, with each side computed by the operational measure on the appropriate carrier.

By Corollary 5A.4, every measure satisfying OIP (Definition 5A.2) is ODG-compatible. ODG-compatibility is therefore the measure-theoretic specialization of OIP on the equivalence relation generated by ODG-preserving operations.

Remark 5.1.1 (Tensor-extension multiplicativity inherited). The tensor-extension multiplicativity $\mu\_tensor(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi)$ on separable composite states is, as in the UMP paper §5.1.2, a structural input from the operational-structure background — not a derived consequence of (A1)–(A7). Under ODG-compatibility this multiplicativity is required to interact consistently with the compositional ODG-preserving identification $\sigma\_\{\alpha,\beta\}$: the tensor-carrier measure and the original-carrier measure must agree under $\sigma\_\{\alpha,\beta\}$, with μ_tensor multiplicative on separable composites.

Remark 5.1.2 (Geometric form of the substrate-only principle, refined). ODG-compatibility was previously characterized (UMP paper §5.3.3) as the geometric form of the substrate-only principle. Under the §5A framing, this characterization is refined: ODG-compatibility is the *measure-theoretic specialization* of OIP — itself the precise statement of the substrate-only principle — on the equivalence relation generated by ODG-preserving operations. The substrate-only principle states that probability depends only on substrate-derived operational structure; OIP makes this precise at the level of operationally indistinguishable equivalence classes; ODG-compatibility implements the principle on the four identified classes of ODG-preserving operation. The three statements are connected via Corollary 5A.4 plus (on the VERSF carrier, conditional on Conjecture 3.5) Theorem 6.4.

5.2 ODG-compatibility extends (A3)

Proposition 5.2 (ODG-compatibility extends admissible-transport invariance). *Every ODG-compatible measure satisfies the admissible-transport invariance axiom (A3) of the minimal core. More precisely: the (A3) invariance is the restriction of ODG-compatibility to the class (O1) of admissible transport operations.*

Proof. (A3) asserts $\mu(U\psi) = \mu(\psi)$ for every $U \in U\_adm\text{-}transport(\mathcal{A}\_\mathbb{C})$. By Proposition 3.2, every U_adm-transport unitary is ODG-preserving (it belongs to class (O1) of ODG-preserving operations). The ODG-compatibility condition therefore implies (A3) on its restriction to class (O1). Conversely, (A3) is only the within-sector fragment of ODG-compatibility; it does not constrain invariance under classes (O2), (O3), (O4). ■

Remark 5.2.1 (ODG-compatibility as the natural extension of (A3)). Under the framing of Proposition 5.2, ODG-compatibility is the natural cross-sector extension of (A3)'s within-sector invariance principle to the full class of ODG-preserving operations. The minimal core's (A3) covers only class (O1); ODG-compatibility covers all four classes (O1)–(O4) (and any further classes that may exist beyond Conjecture 3.5). The structural shape of ODG-compatibility — invariance under a class of structure-preserving operations — is the same as (A3)'s; what changes is the class. This is the precise sense in which probability becomes "a measure on operational geometry" rather than "a measure on a carrier with an admissible-transport invariance constraint."

5.3 ODG-compatibility's content beyond the minimal core

Proposition 5.3 (ODG-compatibility's content beyond the minimal core). *On the inherited VERSF carrier of full commensurability (UMP-paper Remark 4.6.2), ODG-compatibility supplies, beyond the minimal-core axioms:*

(a) ***Quadraticity on $d_\alpha = 1$ sectors:*** *the per-sector function F_α on every $d_\alpha = 1$ sector is forced to be quadratic, $F_\alpha(r) = C r^2$, with the coefficient C inherited from higher-dimensional commensurable sectors via the appropriate ODG-preserving witness.*

(b) ***Cross-sector uniformity:*** *the quadratic coefficients C_α agree across all sectors, $C_\alpha = C$ uniformly.*

Together with single-reference normalization (A7), this yields C = 1 and hence $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$.

Proof. By Theorem 6.1 below (ODG-compatibility implies UMP) plus Theorem 5.3 of the UMP paper (UMP forces uniform quadratic operational measure). The combined chain ODG-compatibility ⟹ UMP ⟹ uniform Born rule establishes the claim. ■

Remark 5.3.1. Proposition 5.3 says that ODG-compatibility supplies (at least) everything UMP supplied — quadraticity on $d_\alpha = 1$ sectors and cross-sector uniformity — both of which were the substantive structural content of UMP beyond the minimal core. The proposition does not say ODG-compatibility supplies *more* than UMP; whether it does is the converse-direction question of §6.

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6. Derivation of the Universal Measure Principle

We now establish the central conceptual result of the paper: ODG-compatibility entails UMP unconditionally (Theorem 6.1, **proven**), and UMP entails ODG-compatibility conditionally on the structural-completeness conjecture (Theorem 6.4, **conditional**). The asymmetry of these two directions is the key structural fact about the relationship between ODG-compatibility (the geometric primitive of the present framing) and UMP (the operational specialization).

Honest calibration on where the work lives. The substantive structural work has already been done in §3, specifically in Proposition 3.3's verification that the three commensurability witnesses preserve all six components of ODG under Definition 3.1A. Theorem 6.1 is a one-line consequence of Proposition 3.3 plus the definition of ODG-compatibility (Definition 5.1): once the commensurability witnesses are established as ODG-preserving operations, any measure invariant under all ODG-preserving operations is automatically invariant under each commensurability witness, which is UMP. The conceptual content of the unification is concentrated in Proposition 3.3; Theorem 6.1 is the definitional unfolding that gives the unification its name.

6.1 ODG-compatibility implies UMP

Theorem 6.1 (ODG-compatibility ⟹ UMP). *Every ODG-compatible minimal-core admissible probability measure μ on the inherited VERSF carrier satisfies the Universal Measure Principle. Explicitly, μ satisfies each of the three clauses UMP-comp, UMP-emb, UMP-ref of UMP-paper Definition 5.1.*

Proof. By Proposition 3.3, each commensurability witness $\sigma_{\{\alpha,\beta\}}$, ι, W is an ODG-preserving operation. The ODG-compatibility condition (Definition 5.1) requires μ to be invariant under every ODG-preserving operation. Restricting this invariance to the three classes (O2), (O3), (O4) of commensurability witnesses recovers exactly the three clauses of UMP:

- For each (C-comp) pair (α, β) and each $\psi \otimes \phi \in V_\alpha \otimes V_\beta$, ODG-compatibility under $\sigma_{\{\alpha,\beta\}}$ gives $\mu_{tensor}(\psi \otimes \phi) = \mu_{original}(\sigma_{\{\alpha,\beta\}}(\psi \otimes \phi))$, which is UMP-comp.
- For each (C-emb) pair (α, β) with witness $\iota : V_\alpha \to V_\beta$ and each $\psi_\alpha \in V_\alpha$, ODG-compatibility under ι gives $\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha)$, which is UMP-emb.
- For each (C-ref) pair (α, β) with witness $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$ and each $\psi \in V_\alpha \oplus V_\beta$, ODG-compatibility under W gives $\mu(W\psi) = \mu(\psi)$, which is UMP-ref.

Therefore μ satisfies all three clauses of UMP. ■

Remark 6.1.1 (Where the work lives). The proof of Theorem 6.1 is essentially Proposition 3.3 in disguise: every step depends on the witnesses being ODG-preserving, which is Proposition 3.3's content. The honest framing is that Proposition 3.3 is the load-bearing structural result of the paper, and Theorem 6.1 is its one-line corollary. The conceptual unification UMP ⟹ ODG-derived therefore rests on the verification that each ODG-component (i)–(vi) is preserved by each of the three witness types, under the unified preservation reading of §3.3. Where the work lives is in the six-times-three table of preservation checks in Proposition 3.3's proof, not in Theorem 6.1's deductive step. Theorem 6.1's status is **proven** unconditionally given Proposition 3.3 and Definition 5.1, but the substance of what is proven is concentrated upstream.

6.2 Does UMP imply ODG-compatibility?

The converse direction — whether UMP implies ODG-compatibility — depends on whether the three classes of commensurability witness (plus admissible transport) exhaust the ODG-preserving operations. If they do, UMP and ODG-compatibility are logically equivalent; if not, ODG-compatibility is strictly stronger than UMP.

Remark 6.2.0 (Restricted converse, by definition). For each class (Oi) ∈ {(O2), (O3), (O4)} of cross-sector commensurability witnesses, UMP-restricted-to-(Oi) asserts μ-invariance under every witness in (Oi); ODG-compatibility-restricted-to-(Oi) asserts the same invariance. The two restricted conditions therefore have identical logical content tautologically, by definition of UMP's three clauses. The non-trivial converse — whether UMP across its three clauses (plus (A3)) implies ODG-compatibility across *all* ODG-preserving operations — is the content of Theorem 6.3 below, which is conditional on Conjecture 3.5.

Theorem 6.3 (Full converse, conditional). *Under Conjecture 3.5 (structural completeness of (O1)–(O4)), UMP — restricted to the three commensurability witness classes (O2), (O3), (O4)*

— plus the admissible-transport invariance axiom (A3) — restricted to class (O1) — together imply ODG-compatibility on the full class of ODG-preserving operations.

Proof (conditional on Conjecture 3.5). By Conjecture 3.5, every ODG-preserving operation T either belongs to one of the four classes (O1)–(O4) or factors as a composition of operations from these classes.

Case 1: T belongs to class (O1). Then T is a transport-admissible (U-trans) unitary, and (A3) gives $\mu(T\psi) = \mu(\psi)$.

Case 2: T belongs to class (O2), (O3), or (O4). Then T is a commensurability witness, and the corresponding clause of UMP gives $\mu(T\psi) = \mu(\psi)$.

Case 3: T factors as $T = T_n \circ \dots \circ T_1$ where each T_i belongs to one of the four classes. Then by induction, $\mu(T_i\psi) = \mu(\psi)$ for each i (applying Cases 1 or 2 at each step), so $\mu(T\psi) = \mu(T_n \circ \dots \circ T_1 \psi) = \mu(\psi)$.

Therefore $\mu(T\psi) = \mu(\psi)$ for every ODG-preserving operation T, i.e. μ is ODG-compatible. ■

Theorem 6.4 (Equivalence of UMP and ODG-compatibility on the VERSF carrier, conditional). *On the VERSF carrier of full commensurability (UMP-paper Remark 4.6.2), conditional on the structural-completeness conjecture (Conjecture 3.5), the following are equivalent for an admissible operational probability measure μ:*

(a) μ is ODG-compatible; (b) μ satisfies UMP (the three-clause Definition 5.1 of the UMP paper).

The unconditional implication (a) ⟹ (b) is Theorem 6.1. The conditional implication (b) ⟹ (a) is Theorem 6.3 (conditional on Conjecture 3.5).

Status. The forward direction (a) ⟹ (b) is **proven** unconditionally. The converse (b) ⟹ (a) is **conditional** on Conjecture 3.5, which is **conjectural** on the VERSF carrier. The full equivalence is **conditional** on Conjecture 3.5 holding.

6.3 What ODG-compatibility adds beyond UMP

Remark 6.4.1 (Three conceptual gains). Even when ODG-compatibility and UMP are equivalent (Theorem 6.4, conditional), the ODG framing supplies three conceptual gains over the UMP framing:

- **Geometric grounding.** ODG-compatibility is a measure-compatibility condition on a unified geometric object (operational distinguishability geometry). UMP is a list of three operational invariance clauses. The two have the same logical content (conditionally), but the former is geometrically motivated by structural compatibility with the unified operational object, while the latter is operationally motivated by the three named bridging conditions of PAMV.

- **Structural-completeness as the natural converse question.** Under the ODG framing, the converse direction "UMP ⟹ ODG-compatibility" has a precise structural content: whether the three commensurability witness classes exhaust the ODG-preserving operations. Under the UMP framing, the converse question was less sharply posed (since UMP did not have a natural concept of "additional invariance witness" beyond its three clauses).
- **Falsifiability extension.** ODG-compatibility supplies a richer falsifiability framework than UMP alone. Any failure of probability invariance under any ODG-preserving operation falsifies the framework; the three commensurability witnesses are merely the most operationally accessible classes. If further ODG-preserving operations exist (Conjecture 3.5 failing), they supply additional falsification routes that UMP alone would miss.

Remark 6.4.2 (Status: same Born rule, deeper principle). Theorems 6.1 and 6.4 together imply that, conditional on Conjecture 3.5, ODG-compatibility and UMP are equivalent on the VERSF carrier. The Born rule that follows is therefore the same as the one delivered by the UMP paper (Theorem 7.1 of the UMP paper). What changes is the principle from which it is derived: from UMP (a list of three operational invariance clauses) to ODG-compatibility (a measure-compatibility condition on a unified geometric object). The Born rule's structural fingerprint is sharpened from "the unique commensurable minimal-core measure" to "the unique ODG-invariant minimal-core measure."

If Conjecture 3.5 fails — i.e. if there exist ODG-preserving operations not covered by classes (O1)–(O4) — then ODG-compatibility is strictly stronger than UMP. In that case, ODG-compatibility still forces the Born rule (since it implies UMP, by Theorem 6.1, which forces the Born rule by UMP-paper Theorem 7.1). UMP would then be a strict consequence of ODG-compatibility rather than an equivalent reformulation; ODG-compatibility would be the "right" principle for the Born rule, with UMP as a useful operational specialization that captures most but not all of the invariance content.

In neither case does the Born rule's status change. What changes, conditional on Conjecture 3.5, is whether UMP and ODG-compatibility are equivalent reformulations or whether one is strictly stronger than the other.

Remark 6.4.3 (Conceptual and logical novelty). The significance of the present paper does not depend on the truth-value of Conjecture 3.5.

If Conjecture 3.5 holds, ODG-compatibility and UMP are logically equivalent on the VERSF carrier. In that case the advance is conceptual:

- the primitive object becomes ODG rather than UMP;
- operational indistinguishability (OIP) becomes the underlying principle;
- and the Born rule is identified as the unique measure compatible with operational geometry itself.

If Conjecture 3.5 fails, ODG-compatibility is strictly stronger than UMP. In that case the advance is both conceptual and logical: the same three conceptual gains hold, and ODG-compatibility additionally carries invariance content beyond UMP, supplying a richer falsifiability framework and a stricter measure-compatibility constraint.

The conceptual contribution is therefore *unconditional* — independent of the structural-completeness question. The logical-strengthening contribution is *conditional* on the outcome of Conjecture 3.5. The present paper's principal results (Theorem 6.1 unconditional, Theorem 7.1 unconditional) hold in both branches; what is settled by Conjecture 3.5 is only whether the framework's novelty over UMP is purely conceptual or also logical.

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7. The Born Rule as the Unique ODG-Compatible Measure

We now assemble the main result: the Born rule emerges as the unique minimal-core measure compatible with operational distinguishability geometry.

7.1 The universal Born rule under ODG-compatibility

Theorem 7.1 (Universal Born rule from ODG-compatibility). *Under the inherited operational structure of PAMV §2, the minimal-core axioms (A1)–(A3), (A5), (A7), and ODG-compatibility (Definition 5.1) on the inherited VERSF carrier of full commensurability, the operational probability measure on* $\mathcal{A}_\mathbb{C}$ *is uniquely*

$$\mu(\psi) = \|\psi\|^2_\mathbb{C} \text{ for every } \psi \in \mathcal{A}_\mathbb{C},$$

with the normalization convention of UMP-paper Theorem 7.1 (single-reference normalization (A7) fixes the overall scale, so $\mu(\psi_{ref}) = 1$ *for the chosen reference state, and* μ *assigns probability 1 to unit-norm states). The transition probability between admissible states is*

$$P(\psi \to \phi) = |\langle\phi, \psi\rangle_\mathbb{C}|^2$$

for every pair of normalized admissible states $\psi, \phi \in \mathcal{A}_\mathbb{C}$ *(with* $\|\psi\|_\mathbb{C} = \|\phi\|_\mathbb{C} = 1$*), following the same projection-based identification as PAMV Theorem 7.1 and UMP-paper Theorem 7.1.*

Proof. By Theorem 6.1, ODG-compatibility implies UMP. By UMP-paper Theorem 7.1, UMP plus the minimal-core axioms on a fully commensurable carrier forces $\mu(\psi) = \|\psi\|^2_\mathbb{C}$. The transition-probability statement follows by UMP-paper Theorem 7.1 (using PAMV Theorem 7.1's projection-based identification $P(\psi \to \phi) = \mu(P_\phi \psi) = \|P_\phi \psi\|^2_\mathbb{C} = |\langle\phi, \psi\rangle_\mathbb{C}|^2$ for normalized ϕ). ■

Theorem 7.2 (Born rule as the unique ODG-compatible minimal-core measure). *Among all minimal-core admissible probability measures parametrized by Lemma 4.1 on the inherited VERSF carrier of full commensurability, the Born rule is the unique one compatible with operational distinguishability geometry.*

Proof. Lemma 4.1 (inherited from the UMP paper) parametrizes the minimal-core admissible measures as the family $\mu_{(C_\alpha), (F_\alpha^{(1)})}$. Theorem 4.2 establishes that any non-uniform choice of (C_α) violates ODG-compatibility; admissible-transport invariance (class (O1)) forces $F_\alpha^{(1)} = 0$ within each sector. The unique remaining ODG-compatible member of the parametrized family is therefore $\mu(\psi) = \|\psi\|^2_\mathbb{C}$. ■

Caveat on scope. Uniqueness is within the Lemma-4.1-parametrized family. If a minimal-core measure exists outside Lemma 4.1's parametrization — i.e. a measure satisfying (A1)–(A7) but not of the form $\mu_{(C_\alpha), (F_\alpha^{(1)})}$ — the present argument does not cover it. Lemma 4.1's parametrization is the UMP paper's structural result that within-sector quadraticity plus the minimal-core axioms restrict the measure to this two-parameter family per sector; the exhaustiveness of Lemma 4.1 is inherited from the UMP paper and is not re-established here.

Remark 7.2.0 (Existence is automatic; the work lives in uniqueness). Every ODG-preserving operation T is an isometry by Definition 3.1A's (ODG-i) preservation condition. The Born rule $\mu(\psi) = \|\psi\|^2_\mathbb{C}$ is therefore *automatically* invariant under every ODG-preserving operation — $\mu(T\psi) = \|T\psi\|^2_\mathbb{C} = \|\psi\|^2_\mathbb{C} = \mu(\psi)$ — and so is automatically ODG-compatible, regardless of how many ODG-preserving classes exist or whether Conjecture 3.5 holds. The entire content of Theorem 7.2 therefore lives in *uniqueness* (the commensurability witnesses connecting distinct sectors force C_α uniform across all sectors), not in *existence*. This preempts a potential satisfiability worry in the Conjecture-3.5-fails branch: if 3.5 fails and additional ODG-preserving classes exist beyond (O1)–(O4), these additional classes still consist of isometries (by (ODG-i)), so they cannot make the Born rule inconsistent — they can at most provide additional invariance constraints that the Born rule already satisfies. The structural work in establishing the Born rule as the unique ODG-compatible measure is therefore concentrated in showing that the *uniqueness*-forcing classes (the commensurability witnesses (O2)–(O4)) suffice to pin uniformity across all sector pairs, given full commensurability of the VERSF carrier — not in checking compatibility of the Born rule with the invariance conditions, which is automatic by isometry.

7.2 What has been sharpened versus the UMP paper

The UMP paper established (its Theorem 7.2):

Born rule is the unique commensurable minimal-core measure.

The present paper sharpens this to:

Born rule is the unique ODG-invariant minimal-core measure.

The two formulations are logically equivalent (conditional on Conjecture 3.5; Theorem 6.4) but conceptually distinct in three ways.

(S1) The unifying object is named explicitly. In the UMP paper, "commensurable" referred to invariance under the three commensurability witness classes — three operational windows onto a deeper invariance principle. The present paper names that deeper principle (ODG-compatibility), identifies the unified object it is invariance for (operational distinguishability geometry), and treats commensurability as the operational working form on three identified witness classes.

(S2) The structural fingerprint is sharpened. The Born rule's fingerprint moves from "commensurability" (a list of three operational invariance conditions) to "ODG-invariance" (a measure-compatibility condition on a unified geometric object). Where the UMP paper named three operational windows onto an unspecified deeper structure, the present paper names the deeper structure itself.

(S3) The status of universality is geometric, not operational. Under the UMP paper's framing, universality of the Born rule across sectors was supplied by UMP — a structural principle with three operational manifestations. Under the present paper's framing, universality is supplied by ODG-compatibility: by recognizing the operational geometry as a unified object whose invariance group probability must respect, universality is supplied as a natural geometric requirement rather than as a separate principle. UMP becomes the working form of that geometric requirement on three identified witness classes.

7.3 The structural reframing

Remark 7.2.1 (Three levels of structural reading). The σ-family papers' progressively sharpened readings of the Born rule's structural status are:

- **PAMV (the first paper).** The Born rule is the unique minimal-core measure forced by quadraticity within sectors plus one of three bridging conditions (B1), (B2), (B3) supplying cross-sector uniformity. Quadraticity within sectors was treated as the primary content, with cross-sector universality as a separate bridging input.
- **UMP paper (the second paper).** The Born rule is the unique *commensurable* minimal-core measure. Universality is the deepest structural content; quadraticity within $d_\alpha \geq 2$ sectors is a derived consequence of orthogonal additivity. The three bridging conditions of PAMV are three operational windows onto one unifying principle (UMP).
- **ODG paper (present paper).** The Born rule is the unique *ODG-invariant* minimal-core measure. UMP is the operational working specialization of ODG-compatibility on three identified witness classes. The OIP layer of §5A supplies the cleanest operational formulation: probability factors through the operational-indistinguishability quotient (Proposition 5A.3A).

Each successive reading identifies the structural object responsible for the Born rule at a different organizational level; the Born rule itself does not change. The σ-family progression has been a sequence of operational reformulations — PAMV's three bridging conditions, UMP's unifying principle, the present paper's geometric primitive plus quotient-space framing — with each layer surfacing a different aspect of the same content-vs-label commitment (see Remark 5A.4.2 for the honest accounting).

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8. Interpretation

Convention on the term "ODG-invariance" and "ODG-compatibility." Throughout the paper, **ODG-invariance** and **ODG-compatibility** are used interchangeably to refer to the condition that an admissible probability measure is invariant under every ODG-preserving operation. The terminology is parallel to the UMP paper's use of "universality" — both terms name the structural condition under which probability respects the operational geometry of the carrier rather than depending on sector-identity bookkeeping. ODG-compatibility makes the geometric object explicit; UMP names the operational specialization on three witness classes.

8.1 The five-layer structural hierarchy

The full structural hierarchy that the σ-family delivers, as developed by the present paper, can be stated in five layers:

Layer	Structural object	Role
Layer 1	Operational geometry (ODG)	The unified inherited structure: carrier, sector decomposition, projection structure, admissible transport, refinement structure, finite packing
Layer 2	Operational indistinguishability + OIP (§5A)	The principle layer: states with identical ODG-content are operationally indistinguishable; OIP demands they receive identical probability assignment
Layer 3	ODG-compatible measures (§5)	The measure-theoretic specialization of Layer 2: probability invariant under every ODG-preserving operation (the four classes (O1)–(O4) of §3.2). Implied by Layer 2 via Corollary 5A.4
Layer 4	UMP (working specialization)	The operational specialization of Layer 3 on three identified cross-sector witness classes (O2)–(O4); (A3) is the within-sector fragment (O1)
Layer 5	The Born rule	The unique Layer-3 measure satisfying the minimal core (Theorem 7.1)

The descent from Layer 2 to Layer 3 (OIP ⟹ ODG-compatibility) is unconditional (Corollary 5A.4). The descent from Layer 3 to Layer 4 (ODG-compatibility ⟹ UMP) is unconditional (Theorem 6.1). The descent from Layer 4 back to Layer 3 (UMP ⟹ ODG-compatibility on the VERSF carrier) is conditional on Conjecture 3.5 (Theorem 6.3/6.4). The descent to Layer 5 is by Theorem 7.1. The hierarchy is therefore strict from Layer 2 downward (each higher layer is at least as strong as the lower); whether the Layer 2 → Layer 3 implication is one-way or two-way depends on whether OIP is strictly stronger than ODG-compatibility, which is the OIP-converse question of Remark 5A.4.1.

8.2 Probability as derivative of operational geometry

The conceptual picture this delivers is the following.

In the conventional reading of quantum theory, the operational structure of a quantum system and its probability rule are two separate inputs: one specifies the kinematic structure (the Hilbert space, with its operators, observables, and dynamics), and one specifies the probability rule (Born). The connection between them — that the probability rule must be the squared norm of an inner-product structure on the Hilbert space — is an additional postulate, contingent on the choice of probability rule.

Under the σ-family reconstruction, this picture has been progressively dissolved:

- PAMV established that *within each coherence sector*, the operational structure forces the probability rule to be quadratic — i.e. that local probability structure is determined by local operational geometry. This eliminated the local independence of probability and operational structure.
- The UMP paper established that *across sectors*, the probability rule is also forced to be quadratic and uniform — by a single principle (UMP) that captures the cross-sector invariance content. This eliminated the global independence of probability and operational structure.
- The present paper establishes that the single principle UMP itself follows from recognizing the operational structure as a *unified geometric object* (ODG) plus imposing ODG-compatibility as a measure-compatibility condition. Given ODG-compatibility as a structural input on admissible measures, the cross-sector principle UMP is no longer separately needed: it is the working specialization of ODG-compatibility on three identified witness classes.

The end result, *given ODG-compatibility as a structural input*: probability assignment is not independent of operational structure. It is the measure-theoretic counterpart of the operational geometry, derived by the natural compatibility relation between geometry and measure given the structural-input status of ODG-compatibility. The Born rule is what ODG-compatibility with the operational geometry already requires.

8.3 Why this matters for foundations of quantum mechanics

The structural significance of this reframing for foundations of quantum mechanics can be stated in three points.

First, given ODG-compatibility as a structural input, the Born rule is not a separately-postulated axiom. Under the present framing, *given the structural-input layer*, the Born rule is the unique invariant measure compatible with the operational geometry of the carrier. Once the carrier is supplied with its operational structure (Layer 1) and ODG-compatibility is imposed as a measure-compatibility condition (Layer 3, equivalent to OIP at Layer 2 via Corollary 5A.4), the remaining structural content is determined by compatibility. There is no logical room for a different probability rule consistent with the same operational geometry under the same structural-input layer. ODG-compatibility plays the structural role that the Born rule itself played in conventional formulations; whether the structural-input layer can be derived from substrate-level structure is Open Problem 1 of §10.

Second, the unification of three bridging conditions is no longer mysterious. PAMV's three independent bridging conditions appeared as a structural coincidence in need of explanation. The

UMP paper supplied UMP as the single unifying principle. The present paper supplies the structural reading: the three conditions are three operational windows onto the natural invariance group of a unified geometric object. The convergence is not a coincidence; it is a structural reflection of the fact that ODG is itself unified. The OIP layer (§5A) restates the same structural content as a probability-on-equivalence-classes principle, which adds rhetorical and conceptual organization without strengthening the underlying commitment.

Third, the conditional-status pyramid is clarified. Each of the three σ-family papers has a conditional-status structure. PAMV is conditional on (i) the inherited operational geometry of §2 and (ii) the minimal-core axioms. The UMP paper adds (iii) UMP. The present paper supplies two further structural layers: (iii') the recognition of the inherited operational structure as a unified geometric object (ODG), and (iv') OIP as the cleanest operational formulation of ODG-compatibility (Proposition 5A.3A's quotient-space view). The two new layers reorganize the same content-vs-label commitment that commensurability/UMP carries (Remark 5A.4.2). The conditional pyramid bottoms out at PAMV §6.2's admissibility apparatus, PAMV's Open Problem 10 (whether the packing-volume identification is a derived theorem), and the present paper's Open Problem 1 (substrate-derivation of ODG and the content-vs-label judgment).

In each successive layer, less is asserted as a separately-postulated axiom and more is derived from structural compatibility given the conditional-status input of that layer. The reductionist destination is a complete substrate-level derivation of the Born rule, with the structural-input layer pushed down to substrate primitives. Open Problem 1 of §10 — substrate-level derivation of ODG and its admissibility content — is the natural next reduction.

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9. Predictions and Falsifiability

Following the §9 structure of the UMP paper, we distinguish inherited from novel predictions.

9.1 Predictions inherited from the UMP paper

The UMP paper's UMP-test 1 through UMP-test 4 (UMP-paper §9.2) are all inherited and unaffected. Under ODG-compatibility, they retain their roles as falsifiers of the framework:

- **(UMP-test 1, inherited):** Joint validity of compositional, embedding, and refinement consistency. Failure of any single (B_i) on a commensurable pair falsifies UMP (UMP-paper §9.2), and therefore — by Theorem 6.1 — falsifies ODG-compatibility.
- **(UMP-test 2, inherited):** Universality across newly-coupled sectors. Failure under ODG-preserving dynamic coupling falsifies the framework.
- **(UMP-test 3, inherited):** Sector-relative probability under operational isolation. ODG-compatibility constrains only ODG-related sectors; operationally isolated sectors are not constrained, in agreement with UMP.
- **(UMP-test 4, inherited):** Detection of triple structural failure under sector-relativity. The triple-failure structure of Theorem 4.2 generalizes the UMP-paper's Theorem 3.2.

The Sorkin third-order interference test (PAMV §12.1) is similarly inherited and unaffected. It tests orthogonal additivity (A2), which is part of the minimal core and is required by ODG-compatibility together with the other minimal-core axioms.

9.2 ODG-specific predictions

The ODG framing supplies one new class of prediction beyond the UMP paper's falsifiability content, conditional on Conjecture 3.5.

(ODG-test 1) ODG-preserving operations beyond the three commensurability classes. If Conjecture 3.5 holds, the three commensurability witness classes plus admissible transport exhaust the ODG-preserving operations on the VERSF carrier, and ODG-compatibility is equivalent to UMP. If Conjecture 3.5 fails — i.e. if there exist ODG-preserving operations not reducible to compositions of (O1)–(O4) — then ODG-compatibility is strictly stronger than UMP, and probability would have to satisfy invariance conditions beyond UMP's three clauses.

The test decomposes into two structurally distinct questions:

- *(structural question)* Does there exist an ODG-preserving operation on the VERSF carrier not reducible to compositions of (O1)–(O4)? This is resolved by structural analysis of the carrier's operational geometry — enumerating operationally-meaningful maps between sectors, checking which preserve all six ODG components, and checking which factor through (O1)–(O4) compositions. The question is not empirical; it is settled (or left open) by structural reasoning alone. Open Problem 2 of §10 develops the structural shape of this attack further.
- *(empirical question, conditional on the structural question being resolved positively)* Suppose such an additional ODG-preserving operation T is identified. Then the Born rule's invariance under T is a testable prediction: under ODG-compatibility, $\mu(T\psi) = \mu(\psi)$ for every state ψ in T's domain. Experimental failure of this invariance would falsify ODG-compatibility while potentially leaving UMP intact (depending on whether T's failure correlates with any of the three commensurability witness classes (O2)–(O4)).

The two questions are independent in the following sense: the structural question's resolution does not require experiment, while the empirical question's resolution presupposes the structural question's positive resolution (since one cannot test invariance under a class of operations not yet identified). Confusing them — treating the existence of additional ODG-preserving operations as an empirical question — would misframe the test's logical structure. The natural attack sequence is structural-first: enumerate candidate operations, check ODG-preservation, check reducibility to (O1)–(O4); only if a non-reducible candidate is identified does the empirical Born-rule-invariance test become well-posed.

(ODG-test 2) Geometric compatibility of probability and packing volume. Open Problem 1 of §10 (substrate-level derivation of ODG) sketches a candidate route from finite-packing geometry to ODG-compatibility. The candidate derivation predicts that probability and packing-volume should agree up to overall normalization (a sharpening of PAMV Proposition 9.1). Experimental detection of a substantive discrepancy between probability assignments and packing-volume assignments on operationally indistinguishable states — beyond what overall normalization can absorb — would falsify the candidate derivation and would suggest that ODG and packing-volume are structurally independent rather than co-determined by substrate-level structure.

9.3 General falsifiers

Each of the following would falsify the present paper's framework:

- demonstration that the Born rule fails to be invariant under any ODG-preserving operation (this would falsify ODG-compatibility regardless of which class the operation belongs to);
- demonstration that the three commensurability witness classes do not all preserve every component of ODG (this would falsify Proposition 3.3 and remove the structural connection between UMP and ODG-compatibility);
- demonstration of an admissible probability measure on the VERSF carrier that satisfies ODG-compatibility but is *not* the Born rule (this would falsify Theorem 7.1);
- demonstration of a sector-relative measure that is ODG-compatible (this would falsify Theorem 4.2);
- demonstration that ODG-compatibility cannot be derived from substrate-level structure (this would not falsify the present paper directly, but would leave Open Problem 1 in negative status, indicating that ODG is genuinely a structural input rather than a derivable consequence).

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10. Open Problems

The open problems are ordered by structural leverage. The first three are the highest-leverage directions for the next stage of the programme — substrate-level derivation of ODG, structural-completeness of the four classes, and the deeper origin of operational geometry — each developed to a level of specificity that supports independent attack. Problems 4–7 are further structural questions of varying difficulty.

1. Substrate-level derivation of ODG

Whether ODG can itself be derived from a more primitive substrate-level structure — in particular from the finite-packing geometry of OG Theorem 10.1, or from the substrate ℤ_7-equivariance directly. The present paper treats ODG as a structural input on the operational measure-theory; whether it descends from packing or character-theoretic structure at a deeper level is the natural next question. Resolving this open problem would convert ODG from a structural input to a derived consequence — the cleanest possible closure of the present paper's conditional status, and the natural successor to the UMP paper's Open Problem 1.

Candidate sketch. The candidate derivation runs parallel to the UMP paper's Open Problem 1, but with ODG as the target rather than UMP. The packing-volume measure of OG Theorem 10.1

is intrinsically carrier-geometric: it is a substrate-derived quantity that depends on Vol_op(M) and Δ_op, not on any choice of sector-decomposition or measure. If the packing-volume can be shown to be a *structurally determined function of the carrier* — i.e. if PAMV's Open Problem 10 can be resolved by upgrading Vol_op(χ) = ‖χ‖²_ℂ from definitional convention to derived theorem — then the operational geometry of the carrier (ODG) is itself a derived consequence of the substrate-level packing structure, since each of the six components (ODG-i)–(ODG-vi) is determined by carrier-geometric data once Vol_op is fixed structurally.

Four-link structural chain. The candidate derivation decomposes into four structurally distinct links (one more link than the UMP-paper's chain, reflecting the deeper target):

- *Link (i).* PAMV Open Problem 10's strengthening of Vol_op(χ) = ‖χ‖²_ℂ from convention to theorem. (Same as UMP-paper Open Problem 1 Link (i).)
- *Link (ii).* Derivation of the six components (ODG-i)–(ODG-vi) of ODG from the strengthened Vol_op identification plus the substrate ℤ_7-equivariance, establishing that each component is structurally determined by substrate data rather than separately postulated. This is the genuinely new structural work the present paper's Open Problem 1 calls for, going beyond the UMP-paper's chain. *Link (ii) is itself a multi-step structural project, not a single derivation.* Each of the six components requires its own sub-derivation from Vol_op and substrate ℤ_7-equivariance, with prospects varying significantly by component:
    - *Sub-link (ii.a).* (ODG-i) carrier and inner product: derivable from OG Theorem 4.0's carrier construction with the strengthened Vol_op identification supplying the inner-product structure.
    - *Sub-link (ii.b).* (ODG-vi) finite distinguishability packing: direct from OG Theorem 10.1, requiring no further derivation.
    - *Sub-link (ii.c).* (ODG-ii) sector decomposition: derivable from substrate ℤ_7-equivariance via Schur's lemma applied to the admissible-transport representation.
    - *Sub-link (ii.d).* (ODG-iii) projection structure: derivable from (ODG-i) and (ODG-ii) via the admissibility-projection construction of OG Theorem 5.1.
    - *Sub-link (ii.e).* (ODG-iv) admissible transport: requires URHG-level input (Theorems 4.1, 5.1) plus the strengthened Vol_op. The prospects are moderate; the within-sector class structure may follow from ℤ_7-equivariance, but the global admissibility constraints may require additional structural input.
    - *Sub-link (ii.f).* (ODG-v) refinement structure: the most structurally delicate sub-link. The admissible-decomposition apparatus of PAMV §6.2 carries structural commitments (the (U-dec) class, the sector-attachment convention) that may not reduce cleanly to Vol_op plus ℤ_7-equivariance.

  Sub-links (ii.a), (ii.b), (ii.c), (ii.d) are tractable; (ii.e) is moderate; (ii.f) is the open structural risk. A successful resolution of Link (ii) requires all six sub-links to close; failure of any single sub-link would leave that ODG-component as a structurally independent input, requiring a partial-derivation framework rather than full substrate-derivation of ODG.

- *Link (iii).* Derivation of the natural invariance group of ODG from the structural-determination of its six components. Once each component is substrate-determined under Link (ii), the natural invariance group is the operations preserving all six simultaneously — which by Conjecture 3.5 includes the four classes (O1)–(O4).
- *Link (iv).* Derivation of ODG-compatibility from the structural-determination of admissible measures by their invariance under the ODG-preserving operations. This is essentially immediate once Links (i)–(iii) hold: it is the substrate-only principle (UMP-paper Remark 5.3.3) applied to the operational geometry of the carrier.

Coupling to PAMV Open Problem 10 and UMP-paper Open Problem 1. Resolving PAMV's Open Problem 10 supplies Link (i) directly, which is shared with the UMP-paper's chain. Link (ii) is the new step the present open problem calls for. Links (iii) and (iv) are essentially immediate consequences. Resolution of PAMV's Open Problem 10 is therefore necessary but not sufficient for closing the present Open Problem 1; the genuinely new structural work is Link (ii) — deriving the six components of ODG from the strengthened packing structure plus substrate $\mathbb{Z}\_7$-equivariance.

The relationship to the UMP-paper's Open Problem 1 is that resolving the present Open Problem 1 *supersedes* the UMP-paper's Open Problem 1: once ODG-compatibility is derived from substrate-level structure (Link (iv) of the present chain), UMP follows as a working specialization on three identified witness classes (by Theorem 6.1), and the substrate-level derivation of UMP is a corollary of the substrate-level derivation of ODG. The present open problem is therefore strictly stronger than the UMP-paper's Open Problem 1; closing it closes both.

2. Structural completeness of (O1)–(O4)

Whether the four classes of ODG-preserving operation identified in §3.2 — admissible transport (O1), compositional identifications (O2), embedding maps (O3), refinement relabelings (O4) — exhaust the structurally significant ODG-preserving operations on the VERSF carrier. This is the structural-completeness conjecture (Conjecture 3.5), and it is the converse-direction half of the UMP-paper's Open Problem 2.

Stakes. If the conjecture holds, ODG-compatibility and UMP are equivalent on the VERSF carrier (Theorem 6.4), and the present paper's structural-deepening over the UMP paper is conceptual rather than logical: the same Born rule with a deeper structural reading, but the same logical content for the principles involved. If the conjecture fails, ODG-compatibility is strictly stronger than UMP, and the present paper's structural deepening is also a logical strengthening: probability would need to satisfy invariance conditions beyond UMP's three clauses, and the Born rule would follow from a strictly stronger principle than UMP alone.

Structural attack. The natural attack on the conjecture is to enumerate the operationally-meaningful maps between sectors of the VERSF carrier, check whether each preserves all six components of ODG, and check whether each reduces to a composition of (O1)–(O4). Candidate further classes of ODG-preserving operation to consider:

- **(O5?) Cross-carrier embeddings.** Embeddings $\iota : V_\alpha \to V_\beta$ between sectors of *distinct* admissible carriers (not just within the original carrier). The UMP-paper's Open Problem 8 noted that cross-carrier embeddings would need definitional work before being usable in witness-composition; this open problem is the structural-completeness analog of that.
- **(O6?) Refinement composition with embedding.** Compositions of refinement-relabelings with embeddings, producing operations that change both the decomposition structure *and* the sector dimension. These compositions are not literally elements of (O3) or (O4) but may reduce to compositions of them under appropriate factorization.
- **(O7?) Group-theoretic extensions of admissible transport.** Subgroups of $U(\mathcal{A}_\mathbb{C})$ that preserve the substrate $\mathbb{Z}_7$-equivariance but are not block-diagonal — e.g. group extensions involving cross-character intertwining. These would be transport operations of a more general kind than the strictly Schur-block-diagonal class (O1).

Each candidate either reduces to (O1)–(O4) compositions (in which case Conjecture 3.5 holds for that candidate) or constitutes a genuinely new class of ODG-preserving operation (in which case Conjecture 3.5 fails for that candidate). The structural-completeness question reduces to checking each candidate against this dichotomy and ruling out exhaustively.

Connection to UMP-paper Open Problem 8. The UMP-paper's Open Problem 8 (strengthening Theorem 6.2's converse) is partially superseded by the present Open Problem 2: if Conjecture 3.5 holds, then any single (B_i) plus (A3) plus the structural-completeness of (O1)–(O4) implies ODG-compatibility (hence UMP via Theorem 6.1, hence all three (B_i)). Resolving Open Problem 2 of the present paper therefore also resolves Open Problem 8 of the UMP paper in the direction of "single-test confirmation possible." However, the UMP-paper's Open Problem 8 also asked about witness-composition mechanics across distinct commensurability classes, which is a separate structural question not directly settled by Open Problem 2 of the present paper.

3. The deeper origin of operational geometry itself

Why does the inherited VERSF carrier supply a structured operational geometry — ODG — at all? The six components of ODG are not independent: each is determined by the inherited substrate structure of URHG. The structurally cleanest framing recognizes three possibilities:

- **(3a) Forced by substrate $\mathbb{Z}_7$-equivariance alone.** If the $\mathbb{Z}_7$-equivariance of the substrate plus admissible-transport structure suffices to determine all six components of ODG, then ODG is essentially automatic on any $\mathbb{Z}_7$-equivariant admissible operational carrier. This would make ODG a structural consequence of the symmetry choice.
- **(3b) Forced by the $K = 7$ closure architecture specifically.** If the $K = 7$ minimal-fact architecture (rather than $\mathbb{Z}_7$ alone) is what supplies the unified structure of ODG — for instance if the refinement structure (ODG-v) or the finite-packing structure (ODG-vi) requires the $K = 7$ closure to be well-defined — then ODG is contingent on the $K = 7$ selection (PAMV Open Problem 11).
- **(3c) Contingent feature of the URHG carrier.** If neither (3a) nor (3b) holds, ODG is a non-trivial structural feature of the inherited URHG carrier that might not hold on other admissible operational carriers. The present paper's structural framework would then

preserve its content on the VERSF carrier specifically, with no general structural guarantee for other substrates.

Distinguishing (3a), (3b), (3c) is the substantive open question. The structural attack is to identify which features of the URHG construction are actually used in establishing each of the six components of ODG, and whether those features generalize to other substrates of the same symmetry type or closure architecture.

This open problem is the deepening of the UMP-paper's Open Problem 2 (the deeper origin of commensurability itself) — where the UMP paper asked about the origin of the three commensurability relations, the present open problem asks about the origin of the deeper operational geometry of which those three relations are surface manifestations.

4. ODG under partial commensurability

The UMP-paper Remark 5.4 addressed UMP under partial commensurability — the case where the commensurability graph splits into disconnected components. Under ODG-compatibility, the analogous question is more general: what does ODG-compatibility require when the operational geometry is *itself* partially disconnected, in the sense that ODG splits into structurally independent components with no ODG-preserving operations connecting them?

The structural framework is robust enough to handle this case: ODG-compatibility forces invariance within each component, with the resulting measures potentially differing across components. This generalizes the UMP-paper's "multi-component admissible measures" framework to the ODG setting. The full structural treatment, and the question of whether physically meaningful substrates can produce partially-disconnected ODG, is open.

5. ODG and operational entropy

Whether operational entropy on the admissible carrier interacts with ODG in a structurally non-trivial way. Candidate: under ODG-compatibility, operational entropy would be invariant under every ODG-preserving operation as a function of the universal Born rule, supplying a clean ODG-formulation of the iso-entropic limit. The entropic-unfolding route to the Born rule (PAMV §8 (v)) might admit an ODG-reformulation in which the iso-entropic limit operates on the entire ODG and not merely on commensurable sector-classes.

6. Infinite-dimensional ODG

Extension of ODG to infinite-dimensional carriers. The structural definitions of §2 (six components of ODG) generalize naturally; the technical content of §3 (four classes of ODG-preserving operation) requires care in infinite dimensions because of convergence subtleties in the channel-decomposition sum, the embedding-class structure (countably many sectors), and the refinement-class structure (admissible decompositions of infinite-dimensional subspaces). The structural-completeness conjecture (Conjecture 3.5) requires explicit treatment in infinite dimensions.

7. Lorentzian ODG

Whether ODG-compatibility survives the Riemannian-to-Lorentzian operational signature transition conjectured in OC and URHG. Lorentzian signature changes the inner-product structure on the carrier (one negative direction), and the resulting operational distinguishability geometry may have a modified structure: the projection lattice (ODG-iii) and the admissible transport class (ODG-iv) both depend on the signature, and the natural invariance class may include or exclude operations specific to the signature change. The natural attack: identify which components of ODG are signature-independent (the carrier and sector decomposition are signature-independent; the inner product, projection lattice, and transport class are not) and determine the resulting modification of the invariance group.

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11. Conclusion

The σ-family of VERSF reconstruction papers has worked progressively deeper into the structural origin of quantum probability. PAMV established quadraticity within sectors and supplied three independent bridging conditions for cross-sector uniformity. The UMP paper unified those three conditions under a single principle — the Universal Measure Principle — and named **operational distinguishability geometry** (ODG) as the deeper structural object UMP turned out to be an invariance principle for. The UMP paper left as Open Problem 2 the question of whether the three commensurability witness classes exhaust the structurally significant ODG-preserving operations.

The present paper reorganizes the σ-family architecture by two further conceptual moves. First (§§2–4), ODG is treated as the primitive geometric object, with UMP becoming the operational working specialization on three identified witness classes (Theorem 6.1). Second (§5A), ODG-compatibility is read as the measure-theoretic expression of an operational-indistinguishability principle (OIP) — the cleanest *operational* formulation of the same structural commitment: probability must factor through the quotient of states by ODG-indistinguishability (Proposition 5A.3A). The structural hierarchy this delivers is:

Operational geometry → Operational indistinguishability → ODG-compatibility → UMP → Born rule.

The central technical content:

- **ODG-incompatibility of sector-relative measures.** Theorem 4.2: sector-relative measures violate ODG-compatibility through three concrete manifestations of one underlying defect — probability depending on sector-identity rather than on operational geometry.
- **OIP implies ODG-compatibility (unconditionally).** Corollary 5A.4: every measure satisfying OIP is ODG-compatible. Proposition 5A.3A: under OIP, probability factors through the operational-indistinguishability quotient $\mathcal{A}_\mathbb{C} / \sim_{op}$.

- **ODG-compatibility implies UMP (unconditionally).** Theorem 6.1: every ODG-compatible minimal-core measure satisfies UMP, because the three commensurability witnesses are among the ODG-preserving operations.
- **UMP implies ODG-compatibility (conditionally).** Theorem 6.4: under Conjecture 3.5, UMP plus admissible transport invariance imply ODG-compatibility. Equivalent to the question of whether PAMV §6.2's admissibility apparatus plus (O1)–(O4) exhaust the admissibility-respecting operations.
- **The Born rule is the unique ODG-compatible minimal-core measure.** Theorems 7.1 and 7.2: under the inherited operational structure plus the minimal core plus ODG-compatibility, $\mu(\psi) = \|\psi\|^2_\mathbb{C}$ is the unique Lemma-4.1-parametrized measure compatible with operational distinguishability geometry.

The σ-family progression may be summarized by three successive questions.

PAMV asked: *Why must quantum probability be quadratic?*

The UMP paper asked: *Why must the same probability rule apply across all sectors?*

The present paper asks: *Why should probability respect operational geometry at all?*

The answer developed here is that probability naturally factors through operational indistinguishability (Proposition 5A.3A). Once probability is required to depend only on operationally meaningful distinguishability content, admissible measures become constrained by operational distinguishability geometry itself. ODG-compatibility then yields UMP as an operational specialization (Theorem 6.1), and UMP yields the Born rule (Theorem 7.1). The resulting picture is concise:

Operational geometry → Operational indistinguishability → ODG-compatibility → UMP → Born rule.

Under this reading, the Born rule is not merely the unique commensurable minimal-core measure (the UMP paper's reading). It is the unique invariant measure compatible with operational distinguishability geometry.

The Born rule is the unique invariant measure compatible with operational distinguishability geometry.