Operational Geometry in the VERSF Framework

Entropy-Induced Metrics, Admissible Measure, and the Geometry of Finite Distinguishability

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A General Reader's Summary

Why does nature stop counting?

There are three families of charged leptons — electron, muon, tau — and no fourth. There are six quarks, organised into three generations, and no seventh. The Standard Model of particle physics is finite. Yet there is no obvious reason it should be: in principle one could imagine indefinitely many further generations, each heavier than the last. Why doesn't nature produce a tower of them?

The recent *finite distinguishability packing* paper of this programme established the structural answer. There is a limit to how many distinct, stable, refinement-stable kinds of physical configuration the closure manifold of admissibility can support — a kind of cosmic sphere-packing limit. The number of stable particle generations is not a free parameter. It is bounded above by a factorised ceiling whose every factor is a structural invariant of the underlying admissibility architecture.

That paper *established* the bound. **The present paper explains why it is geometric.**

When the packing paper said "the admissible structures lie in a finite-dimensional space of operational distinguishability," it treated that space as a list of numbers — a counting device. This paper shows that the same space is in fact a *genuine geometry*: it has a metric (a notion of distance between admissible configurations), a dimension (a fixed number of independent directions), a volume (a finite measure of total operational room), a sense of continuity (a way to talk about smooth refinement), and a notion of compactness (no admissible structure can escape to infinity). All of these structures are derived from admissibility alone. They are not assumed; they are not borrowed from spatial intuition; they are not imported from the macroscopic geometry of relativistic physics.

The structural backbone is an inner-product theorem (Theorem 4.0): the same finite combinatorial closure architecture (the $K = 7$ $\mathbb{Z}_7$ structure) that gives rise to the Standard Model gauge group in VERSF *also* endows the admissible subspace with a natural Hilbert-space structure — the kind of geometry familiar from quantum mechanics, but here derived from admissibility rather than imposed by physical postulate. Once that Hilbert structure is in place, every geometric feature follows as a standard mathematical consequence. The packing theorem of the prior paper then becomes a one-step argument inside this Hilbert geometry, rather than a separately-postulated combinatorial bound.

A wider implication. If admissibility *already* generates geometry — metric, measure, dimension, transport, compactness — then the geometric structure of macroscopic physics (the emergent 4-manifold on which general relativity is formulated) need not be a primitive substrate on which physics is layered. It may itself be derived from the distinguishability geometry developed here. Whether that derivation can be carried out is left open (§13), but the structural ingredients are now in place.

In short: the previous paper established that nature's finite count is *bounded*. This paper establishes that nature's finite count is *geometric* — and suggests that the geometry of admissibility may be more fundamental than the geometry of physical space.

A note on scope. This paper does *not* derive the specific values of the structural invariants (the number of spectral channels N_spec, the per-channel rank C_max) from substrate primitives — those are established elsewhere in the VERSF corpus, through the substrate architecture papers, the $\mathbb{Z}_7$ closure programme, and the projected closure dynamics papers. What this paper shows is that *given* those invariants, the bound on stable particle generations is a *geometric* consequence — a sphere-packing inequality inside an admissibility-induced Hilbert space — rather than a combinatorial coincidence. A general reader curious about why N_spec and C_max take the values they do should consult the substrate-primitives programme separately; what is established here is that, whatever those values turn out to be, the structure of the bound is geometry, not arithmetic.

The technical body of the paper follows. Readers without a mathematical physics background may wish to skim §1 (Introduction) and §11 (Relation to Emergent Geometric Structure), which together convey the essential content in continuous prose, before consulting the theorems and proofs.

Abstract

Prior VERSF work established:

(i) discrete ledger realizability under bounded closure support, (ii) uniqueness *and* quantization of the admissible entropy partition, (iii) a unique closure scale ξ with k_max = 2π/ξ, (iv) finite admissible spectral channels N_spec from the $\mathbb{Z}_7$ closure architecture, (v) finite-rank projected closure dynamics under Ω_max, (vi) finite distinguishability packing with the factorised cardinality ceiling |Σ(M)| ≤ Vol_op(M) / Δ_op^d, d = N_spec · C_max.

The packing programme invoked several geometric objects — D_op, Vol_op(M), $\mathcal{A}$ = Im(Ω_max), d — operationally, but did not establish their status as genuine geometric structures.

The present paper develops the geometry. The structural backbone is an inner-product theorem (Theorem 4.0): the $\mathbb{Z}_7$ Fourier architecture (§2.4) endows the admissible subspace $\mathcal{A} = \mathrm{Im}(\Omega_{max})$ with a natural inner product $\langle\cdot,\cdot\rangle_{op}$ under which:

- $(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$ is a finite-dimensional Hilbert space of dimension $d_{op} = \dim \mathcal{A}$;
- the spectral-channel decomposition $\mathcal{A} = \bigoplus_\alpha V_\alpha$ is an orthogonal direct sum;
- Ω_{max} is the orthogonal projection from the ambient closure-state-space onto $\mathcal{A}$.

Every downstream geometric property then follows from standard Hilbert-space facts:

1. The operational distinguishability metric D_{op} is the norm-induced metric $\|\cdot\|_{op}$ (Theorem 4.1); metric axioms reduce to standard inner-product-norm properties.
2. Ω_{max} as orthogonal projection is non-expansive by Pythagoras (Theorem 5.1), acting as the identity on $\mathcal{A}$; refinement projection is continuous (Theorem 5.2).
3. The finite operational dimension $d_{op} = \sum_\alpha \dim V_\alpha \leq N_{spec} \cdot C_{max}$ is exhibited via the orthogonal channel decomposition (Theorem 6.1).
4. The operational Hausdorff measure $\mathscr{H}^{d_{op}}$*op exists on ($\mathcal{A}$, D_{op}), with Euclidean ball-volume identity* $vol_{op}(B(r)) = \omega\{d_{op}\}\ r^{d_{op}}$ (Theorem 7.2).
5. Bounded closure support of the *continuous* manifold M yields bounded continuous operational image $M_{op} := \Omega_{max}(M) \subseteq \mathcal{A}$ via non-expansiveness (Lemma 8.0), hence finite $\mathscr{H}^{d_{op}}_{op}(M_{op}) < \infty$ (Theorem 8.1). The continuous M_{op} carries the Hausdorff measure; the finite discrete $\Sigma(M) \subseteq M_{op}$ carries the packing count.
6. Heine–Borel in finite-dimensional Hilbert geometry gives total boundedness and precompactness, with covering scaling $N(\varepsilon) \lesssim Vol_{op}(M) / \varepsilon^{d_{op}}$ (Theorem 9.1, Corollary 9.2).

Together these equip the closure manifold with a finite operational geometry. The prior finite distinguishability packing theorem becomes a direct geometric consequence (Theorem 10.1), reproducing

$$P(M) \leq Vol_{op}(M) / \Delta_{op}^{d_{op}}$$

with the same normalisation convention as the packing paper.

The result upgrades the VERSF programme from **finite admissibility** to **operational distinguishability geometry**: the closure manifold possesses, intrinsically, an admissibility-induced Hilbert structure, finite operational dimension, finite operational measure, non-expansive refinement transport, and finite distinguishability capacity — supplying the geometric substrate underlying the packing theorem and a structural bridge toward the emergent 4-manifold geometry of macroscopic physics.

Contents

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1. Introduction

The recent finite distinguishability packing programme established the factorised cardinality ceiling

$$|\Sigma(M)| \leq \mathrm{Vol}_{op}(M) / \Delta_{op}^d, \quad d = N_{spec} \cdot C_{max},$$

with every factor an admissibility invariant of M. The packing argument invoked the operational distinguishability function D_{op}, the operational d-volume $\mathrm{Vol}_{op}(M)$, the admissible subspace $\mathcal{A} = \mathrm{Im}(\Omega_{max})$, and the operational dimension d. It used these objects operationally to state and prove a cardinality bound, but did not establish their status as *geometric structures*. In particular:

- D_{op} was used as a distance function without proof that it is a metric;
- Vol_{op} was used as a measure without construction as a Hausdorff measure on a metric space;
- $\mathcal{A}$ was used as a finite-dimensional space without exhibiting the metric/topological/inner-product structure that supports Euclidean ball volumes and Heine–Borel;
- the precompactness of the admissible geometry, tacitly relied on by the sphere-packing argument, was never formally established.

The present paper develops the missing geometry. The structural backbone is an *inner-product theorem* (Theorem 4.0): the $\mathbb{Z}_7$ closure architecture (§2.4) endows the admissible subspace $\mathcal{A} = \mathrm{Im}(\Omega_{max})$ with a natural inner product $\langle\cdot,\cdot\rangle_{op}$, under which:

- $(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$ is a finite-dimensional Hilbert space of dimension d_{op};
- the spectral-channel decomposition $\mathcal{A} = \bigoplus_\alpha V_\alpha$ is an orthogonal direct sum;

- Ω_{max} is the orthogonal projection from the ambient closure-state-space onto $\mathcal{A}$.

Once this Hilbert structure is established, every downstream geometric property — operational metric, refinement non-expansiveness, finite dimension, Hausdorff measure, finite volume, total boundedness, precompactness, and finite covering numbers — follows as a standard consequence of finite-dimensional Hilbert geometry. The previous attempt to derive D_{op} via a path-length functional $\ell(\gamma)$ on undefined "refinement-admissible paths" is no longer necessary: D_{op} is simply the norm-induced metric $\|\cdot\|_{op}$, and triangle inequality, symmetry, and positivity reduce to standard inner-product-norm facts.

The prior finite distinguishability packing theorem then emerges as a direct geometric corollary (Theorem 10.1), reproducing

$$P(M) \leq Vol_{op}(M) / \Delta_{op}^{d_{op}}$$

with the same Δ_{op}-normalised unit convention adopted in the packing paper.

The paper closes by sketching the structural bridge from operational distinguishability geometry toward the emergent 4-manifold geometry of macroscopic physics (§11). The closure manifold already carries the basic ingredients of geometric structure — metric, measure, dimension, continuity, compactness, transport — derived from admissibility alone, prior to any emergent macroscopic continuum. This suggests that distinguishability geometry may be the more fundamental geometric layer.

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2. Prior Structural Results

The argument depends on six previously established VERSF results, summarised here for self-containment.

2.1 Discrete Ledger Realizability

The closure ontology admits only ledger-realizable states. The commitment ledger is discrete; bounded closure support admits only finitely many ledger-realizable states.

2.2 Unique Closure Scale

The closure-scale theorem establishes the unique scale

$$k_{max} = 2\pi / \xi,$$

with ξ the unique admissible closure length. Modes above k_{max} are phase-redundant.

2.3 Entropy Admissibility (Uniqueness and Quantization)

The admissible entropy programme established that

operational equivalence ≡ closure equivalence ≡ entropy equivalence

coincide uniquely, and that the resulting partition $\Pi_S(M)$ is operationally quantized: cells admit a strictly positive minimum operational separation, supplying the universal gap

$$\Delta_{op} > 0.$$

2.4 Finite $\mathbb{Z}_7$ Closure Architecture

The admissible closure geometry possesses finite $\mathbb{Z}_7$ Fourier structure, finite projected closure algebra, and a finite number $N_{spec} < \infty$ of admissible spectral channels.

Real-form convention. Admissible closure configurations are real-valued (operational entropy classes, ledger states), so the relevant geometric object is the real subspace of $\mathbb{Z}_7$-equivariant configurations — the real-linear span of the $\mathbb{Z}_7$-character decomposition, with the trivial character contributing one real dimension and each non-trivial character pair $\{\chi_\alpha, \chi_{-\alpha}\}$ contributing two real dimensions (the real and imaginary parts, equivalently cos and sin). On this real subspace, the standard finite-abelian-group form

$$\langle f, g\rangle = (1/|\mathbb{Z}_7|) \sum_{x \in \mathbb{Z}_7} f(x)\, g(x)$$

is positive-definite real-bilinear, and the real-form Schur orthogonality relations give pairwise orthogonality of distinct $\mathbb{Z}_7$-character components (real and imaginary parts of distinct characters remain orthogonal under the real inner product). This real-form inner product is the structural source of the Hilbert geometry of $\mathcal{A}$ developed in §4. All operational dimensions d_{op}, $\dim V_\alpha$, etc., refer to real dimension throughout.

2.5 Ω_{max}-Projected Dynamics

The projected closure operator Ω_{max} possesses finite admissible support, finite spectral rank $R_{max} = \mathrm{rank}(\Omega_{max}) < \infty$, stable refinement projection, and is *self-adjoint and idempotent with respect to the real-form $\mathbb{Z}_7$ inner product of §2.4* on the ambient closure-state-space (the standard projector property of refinement-stable closure operators). Ω_{max} is furthermore **$\mathbb{Z}_7$-equivariant** — i.e. it commutes with the $\mathbb{Z}_7$ action on the closure-state-space — so it preserves the spectral-channel decomposition: distinct V_α are mapped to themselves, and Ω_{max} splits as a direct sum of operators on each channel. Finite rank is established in the projected closure dynamics paper of the VERSF programme via the $\mathbb{Z}_7$ closure architecture, **without reference to admissible sector cardinality**, so its invocation here does not produce circularity.

2.6 Finite Distinguishability Packing

The packing theorem established

$$|\Sigma(M)| \leq P(M) \leq \mathrm{Vol}_{op}(M) / \Delta_{op}^d, \quad d = N_{spec} \cdot C_{max},$$

with every factor an admissibility invariant. The packing paper invoked D_{op}, Vol_{op}, $\mathcal{A}$, and d operationally. The present paper supplies their geometric construction.

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3. Definitions

We distinguish carefully between three objects that the previous version of the geometry programme conflated:

- the *closure manifold* M (continuous geometric substrate),
- the *operational quotient* $Q(M) = M / \sim_{op}$ (the entropy-equivalence quotient),
- the *admissible sector set* $\Sigma(M) \subseteq Q(M)$ (finite, discrete),
- the *admissible subspace* $\mathcal{A} = \mathrm{Im}(\Omega_{max})$ (finite-dimensional, continuous),
- the *operational image* $M_{op} := \Omega_{max}(M) \subseteq \mathcal{A}$ (bounded continuous region).

The Hausdorff measure of the packing programme lives on M_{op} (a *continuous* bounded region of $\mathcal{A}$); $\Sigma(M) \subseteq M_{op}$ is the discrete sector subset that gets counted by the packing bound. Confusing these objects — as the previous version did — collapses the geometry, because the Hausdorff measure of a finite set is zero.

Definition 3.1 (Closure manifold). The closure manifold M is the admissibility-complete continuous substrate of VERSF closure geometry, equipped with closure scale ξ, spectral architecture (§2.4), and projected closure operator Ω_{max} (§2.5). M is bounded under the ambient closure-state-space pre-metric — "bounded closure support" — and supports the discrete commitment ledger of §2.1.

Definition 3.2 (Operational equivalence and quotient). Let $\sim_{op}$ denote the entropy-equivalence relation on the ambient closure-state-space (uniqueness of which is established in §2.3). The **operational quotient** is

$$Q(M) := M / \sim_{op}.$$

Elements of Q(M) are entropy-equivalence classes of closure states.

Definition 3.3 (Admissible sector set). The **admissible sector set** $\Sigma(M) \subseteq Q(M)$ is the (finite, discrete) subset of equivalence classes corresponding to ledger-realizable, entropy-consistent, refinement-stable, spectrally admissible structures (Definition 3.2 of the packing paper). By §2.1,

$$|\Sigma(M)| \leq N_{ledger}(M) < \infty.$$

Definition 3.4 (Admissible subspace). The **admissible subspace** is

$$\mathcal{A} := \mathrm{Im}(\Omega_{max}),$$

equipped with the spectral-channel decomposition

$$\mathcal{A} = \bigoplus_{\alpha = 1}^{N_{spec}} V_\alpha,$$

where V_α is the Ω_{max}-image within $\mathbb{Z}_7$ spectral channel α (cf. §2.4–§2.5). The **operational dimension** is

$$d_{op} := \dim \mathcal{A} = \sum_{\alpha} \dim V_\alpha \leq N_{spec} \cdot C_{max},$$

with $C_{max} := \max_\alpha \dim V_\alpha$ as in the packing paper.

Definition 3.5 (Operational image). The **operational image** of M is

$$M_{op} := \Omega_{max}(M) \subseteq \mathcal{A},$$

i.e. the (continuous) projection of M onto the admissible subspace under refinement projection. $\Sigma(M) \subseteq M_{op}$ is the discrete admissible sector subset.

Definition 3.6 (Operational metric, forward reference). The operational distinguishability metric D_{op} on $\mathcal{A}$ is defined in §4 as the metric induced by the inner product $\langle\cdot,\cdot\rangle_{op}$ of Theorem 4.0:

$$D_{op}(x, y) := \|x - y\|_{op} = \sqrt{\langle x - y, x - y\rangle_{op}}.$$

For the purposes of definitions involving D_{op} below, this forward reference suffices.

Definition 3.7 (Distinguishability quantum).

$$\Delta_{op} := \min\{ D_{op}([S_i], [S_j]) : [S_i] \neq [S_j] \in \Sigma(M) \}.$$

Positivity $\Delta_{op} > 0$ was established in Lemma 4.1 of the packing paper via finiteness of $\Sigma(M)$.

Definition 3.8 (Operational topology). The **operational topology** τ_{op} on $\mathcal{A}$ is generated by the open D_{op}-balls $B(x, r) := \{ y \in \mathcal{A} : D_{op}(x, y) < r \}$.

Definition 3.9 (Operational Hausdorff measure).

$$\mathcal{H}^{d_{op}}op(E) := lim_{\delta \to 0^+} \inf\{ \textstyle\sum_i (\mathrm{diam}_{op}\, U_i)^{d_{op}} : E \subseteq \bigcup_i U_i,\ \mathrm{diam}_{op}\, U_i \leq \delta \},$$

the standard Carathéodory construction on the metric space $(\mathcal{A}, D_{op})$.

Definition 3.10 (Operational volume). The **operational volume** of M is

$$\mathrm{Vol}_{op}(M) := (2^{d_{op}} / \omega_{d_{op}}) \cdot \mathcal{H}^{d_{op}}_{op}(M_{op}),$$

with ω_{d_op} the unit-ball volume in d_op-dimensional Euclidean space. The prefactor matches the normalisation of the packing paper (Definition 3.6 of that paper), so that Vol_op(M) / Δ_op^{d_op} is the cell count of a maximally tight Δ_op-pairwise-separated packing.

Proposition 3.11 (Admissible portion of Q(M) injects into 𝒜). *The admissible portion of Q(M) — the subset of Q(M) consisting of equivalence classes of admissible (refinement-stable) states — injects naturally into the admissible subspace 𝒜 ⊆ closure-state-space under the map [S] ↦ Ω_max(S). The image is the* **admissible image of 𝒜**, *a (possibly proper) subset of 𝒜.*

Proof. By §2.5, refinement-stable admissibility gives Ω_max(S) ~_op S for admissible S, so the map [S] ↦ Ω_max(S) is well-defined on admissible classes: if S ~_op S', then Ω_max(S) ~_op S ~_op S' ~_op Ω_max(S'), so the image depends only on [S], not on the representative.

Injectivity follows from refinement-stability and transitivity of ~_op. Suppose for contradiction that [S] ≠ [S'] are distinct admissible classes with Ω_max(S) = Ω_max(S'). Then Ω_max(S) ~_op Ω_max(S') trivially (equal elements are equivalent). Combined with refinement-stability — Ω_max(S) ~_op S and Ω_max(S') ~_op S' — transitivity of ~_op gives

S ~_op Ω_max(S) = Ω_max(S') ~_op S',

hence S ~_op S', i.e. [S] = [S'] in Q(M), contradicting the assumption. Therefore Ω_max(S) ≠ Ω_max(S') whenever [S] ≠ [S']. Injectivity is thus a direct consequence of refinement-stability (§2.5) and the transitivity of operational equivalence (§2.3), with no appeal to Δ_op (which is a downstream quantitative invariant, not a structural input here).

Image. The image of the admissible portion of Q(M) is, by definition, the admissible image of 𝒜 — the subset of 𝒜 attained by projecting admissible states. This subset may or may not equal all of 𝒜, depending on whether admissibility is non-trivial on every spectral channel; the question is structural and is addressed in the substrate-architecture papers of the VERSF programme. The downstream geometric arguments of §4–§10 do not require equality: they require only that the admissible image lies within the finite-dimensional Hilbert space 𝒜, with the operational dimension d_op = dim 𝒜 serving as an upper bound on the dimension of the admissible image.
■

Remark 3.11.1. *Scope of the identification.* The full quotient Q(M) may contain non-admissible equivalence classes — perturbations failing refinement-stability or spectral admissibility. The geometric structures developed in §4–§10 concern the admissible portion of Q(M), injected into 𝒜 via Proposition 3.11. Non-admissible classes lie outside the scope of the operational geometry developed here. Throughout the remainder of the paper, "the admissible operational geometry" refers to (𝒜, D_op), with the admissible portion of Q(M) sitting inside it as an injected subset.

4. Inner-Product Structure and the Operational Metric

This section establishes the structural backbone of the paper: 𝒜 carries a natural Hilbert-space structure inherited from the $\mathbb{Z}_7$ Fourier architecture. All subsequent geometric properties follow as standard finite-dimensional Hilbert facts.

Theorem 4.0 (Inner-product structure on the admissible subspace). *The admissible subspace 𝒜 = Im(Ω_max) carries a natural inner product*

⟨·, ·⟩_op : 𝒜 × 𝒜 → ℝ,

induced by the $\mathbb{Z}_7$ Fourier structure of §2.4, under which:

(i) *(𝒜, ⟨·,·⟩_op) is a finite-dimensional Hilbert space of dimension d_op;* (ii) *the spectral-channel decomposition 𝒜 = ⊕_α V_α is an orthogonal direct sum:*

⟨v_α, v_β⟩_op = 0 for v_α ∈ V_α, v_β ∈ V_β, α ≠ β;

(iii) *Ω_max is the orthogonal projection from the ambient closure-state-space onto 𝒜.*

Proof.

(i) Hilbert-space structure. By §2.4, the $\mathbb{Z}_7$ closure architecture carries the standard finite-abelian-group inner product on $\mathbb{Z}_7$-valued functions:

⟨f, g⟩*{$\mathbb{Z}_7$} = (1/|$\mathbb{Z}_7$|)* ∑{x ∈ $\mathbb{Z}_7$} f(x) g(x),

a positive-definite symmetric bilinear form. By §2.5, Ω_max acts on the ambient closure-state-space (which carries the $\mathbb{Z}_7$ Fourier structure) and produces a finite-dimensional image 𝒜 = Im(Ω_max). The ambient inner product restricts to 𝒜:

⟨v, w⟩*op := ⟨v, w⟩*{$\mathbb{Z}_7$} for v, w ∈ 𝒜.

Positive-definiteness, symmetry, and bilinearity are inherited from the ambient form. By §2.5, dim 𝒜 = d_op < ∞, so (𝒜, ⟨·,·⟩_op) is a finite-dimensional real Hilbert space.

(ii) Orthogonal direct sum. The spectral channels V_α correspond to $\mathbb{Z}_7$-character eigenspaces under the closure-architecture decomposition (§2.4), in the real-form convention of §2.4. Distinct $\mathbb{Z}_7$-characters are orthogonal under the real-form inner product — by real-form Schur orthogonality for finite abelian groups, real and imaginary parts of distinct characters χ_α, χ_β (α ≠ β) are pairwise orthogonal under the real-bilinear form of §2.4. By §2.5, Ω_max is $\mathbb{Z}_7$-equivariant (it commutes with the $\mathbb{Z}_7$ action), so it preserves the spectral-channel decomposition: V_α := Ω_max(χ_α-eigenspace) lies in the χ_α-eigenspace. Since V_α ⊆ χ_α-eigenspace and V_β ⊆ χ_β-eigenspace with the χ_α- and χ_β-eigenspaces pairwise orthogonal for α ≠ β (real-form Schur), the subspaces V_α and V_β are themselves pairwise orthogonal:

v_α ∈ V_α, v_β ∈ V_β, α ≠ β ⟹ ⟨v_α, v_β⟩_op = 0.

Hence $\mathcal{A} = \bigoplus_\alpha V_\alpha$ is an orthogonal direct sum.

(iii) Ω_{max} as orthogonal projection. By §2.5, Ω_{max} is self-adjoint and idempotent on the ambient closure-state-space:

$$\Omega_{max}^2 = \Omega_{max}, \quad \Omega_{max}^\dagger = \Omega_{max}.$$

A self-adjoint idempotent on a Hilbert space is precisely an orthogonal projection — onto its image, which is $\mathrm{Im}(\Omega_{max}) = \mathcal{A}$. This is the standard spectral characterisation of orthogonal projections. ■

Remark 4.0.1. This theorem promotes what was previously a buried assumption — "the entropy quotient endows $\mathcal{A}$ with an inner product" — to an explicit derivation from the $\mathbb{Z}_7$ Fourier structure of §2.4 and the self-adjoint-idempotent structure of Ω_{max} from §2.5. Both inputs are independently established prior VERSF results. The Hilbert structure of $\mathcal{A}$ is therefore a *theorem of admissibility*, not a structural assumption.

Theorem 4.1 (Operational metric). *The function*

$$D_{op}(x, y) := \|x - y\|_{op} = \sqrt{\langle x - y, x - y\rangle_{op}}$$

is a metric on $\mathcal{A}$.

Proof. Standard inner-product-norm properties:

- **Positivity and identity of indiscernibles.** $\|\cdot\|_{op} \geq 0$ with equality iff $x = y$, by positive-definiteness of $\langle\cdot,\cdot\rangle_{op}$ (Theorem 4.0(i)).
- **Symmetry.** $\|x - y\|_{op} = \|y - x\|_{op}$ by bilinearity.
- **Triangle inequality.** $\|x - z\|_{op} \leq \|x - y\|_{op} + \|y - z\|_{op}$ by the standard Cauchy–Schwarz / Minkowski argument in any inner-product space. ■

Remark 4.1.1. The previous version of this theorem derived the triangle inequality via a path-length functional $\ell(\gamma)$ on undefined "refinement-admissible paths" — an undefined primitive. With the Hilbert structure of Theorem 4.0 now established, D_{op} is the norm-induced metric, and the metric axioms reduce to standard inner-product-norm facts. No path-length primitive is required; refinement-admissible paths, if introduced, are derived notions (e.g. straight-line interpolations in $\mathcal{A}$), not foundational definitions.

Corollary 4.2 (Hilbert metric structure). *($\mathcal{A}$, D_{op}) is a finite-dimensional Hilbert space of (real) dimension d_{op}, with metric D_{op} induced by the inner product $\langle\cdot,\cdot\rangle_{op}$. The standard Euclidean ball-volume identity*

$$\mathrm{vol}_{op}(B(x, r)) = \omega_{d_{op}} \cdot r^{d_{op}}$$

holds, where $\omega_{d_{op}}$ is the unit-ball volume in d_{op}-dimensional Euclidean space. ■

5. Refinement Non-Expansiveness and Continuity

The Hilbert structure of Theorem 4.0 makes refinement non-expansiveness a one-line corollary.

Theorem 5.1 (Ω_{max} is non-expansive). *Ω_{max} is a non-expansive map on the ambient closure-state-space (under the metric induced by $\langle\cdot,\cdot\rangle_{\mathbb{Z}_7}$):*

$$\|\Omega_{max}(x) - \Omega_{max}(y)\|_{op} \leq \|x - y\|,$$

with equality on $\mathcal{A}$, where Ω_{max} acts as the identity.

Proof. By Theorem 4.0(iii), Ω_{max} is the orthogonal projection onto $\mathcal{A}$. For orthogonal projections in Hilbert spaces, the Pythagorean identity gives

$$\|x\|^2 = \|\Omega_{max}(x)\|^2 + \|(I - \Omega_{max})(x)\|^2 \geq \|\Omega_{max}(x)\|^2.$$

Applied to $x - y$, this yields $\|\Omega_{max}(x - y)\|_{op} \leq \|x - y\|$. Linearity of Ω_{max} gives $\Omega_{max}(x - y) = \Omega_{max}(x) - \Omega_{max}(y)$, establishing non-expansiveness.

On $\mathcal{A}$ itself, Ω_{max} acts as the identity (Theorem 4.0(iii) — orthogonal projection onto its own image), so $\Omega_{max}(x) = x$ and equality holds. ■

Remark 5.1.1. The previous version asserted non-expansiveness of Ω_{max} on $Q(M)$ by appeal to a general "projections do not increase distance" claim, which is false in general metric spaces. With the Hilbert structure of Theorem 4.0 established, non-expansiveness is the standard Pythagorean fact for orthogonal projections in Hilbert spaces — and is genuine, not assumed.

Theorem 5.2 (Refinement continuity). *Ω_{max} acts continuously on the ambient closure-state-space, and in particular on $(\mathcal{A}, \tau_{op})$.*

Proof. 1-Lipschitz maps are continuous in the metric topology; Theorem 5.1 gives 1-Lipschitzness. For any open ball $B(\Omega_{max}(x), r)$,

$$\Omega_{max}^{-1}(B(\Omega_{max}(x), r)) \supseteq B(x, r),$$

so inverse images of open sets are open. ■

6. Finite Operational Dimension from Ω_{max}

Theorem 6.1 (Finite operational dimension). *The admissible operational geometry possesses finite operational dimension*

$$d_{op} = \dim \mathcal{A} = \sum_{\alpha=1}^{N_{spec}} \dim V_\alpha \le N_{spec} \cdot C_{max} < \infty,$$

with the orthogonal spectral-channel decomposition $\mathcal{A} = \bigoplus_\alpha V_\alpha$ *exhibited explicitly (Theorem 4.0(ii)).*

Proof. By §2.5, $\operatorname{rank}(\Omega_{max}) < \infty$. By Theorem 4.0(ii), $\operatorname{Im}(\Omega_{max})$ decomposes as an orthogonal direct sum

$$\mathcal{A} = \bigoplus_\alpha V_\alpha,$$

with V_α the Ω_{max}-image within spectral channel α. Each V_α has finite dimension $\dim V_\alpha \le C_{max}$. The total dimension is

$$d_{op} = \dim \mathcal{A} = \sum_\alpha \dim V_\alpha \le N_{spec} \cdot C_{max} < \infty. \blacksquare$$

Corollary 6.2. *$\Sigma(M) \subseteq M_{op} \subseteq \mathcal{A}$ — with $\Sigma(M)$ sitting in M_{op} as a finite discrete subset of the continuous bounded operational image — and $(\mathcal{A}, D_{op})$ is a finite-dimensional Hilbert space of dimension d_{op}.* ■

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7. Operational Hausdorff Measure

Theorem 7.1 (Euclidean structure of $\mathcal{A}$). *The admissible subspace $(\mathcal{A}, D_{op})$ is a finite-dimensional Euclidean space of dimension d_{op}, satisfying the standard ball-volume identity*

$$\operatorname{vol}_{op}(B(x, r)) = \omega_{d_{op}} \cdot r^{d_{op}}.$$

Proof. Immediate from Corollary 4.2 (Hilbert structure + finite dimension = Euclidean). ■

Theorem 7.2 (Existence of operational Hausdorff measure). *The operational Hausdorff measure $\mathcal{H}^{d_{op}}_{op}$ of Definition 3.9 exists on $(\mathcal{A}, D_{op})$.*

Proof. By Theorem 4.1, D_{op} is a metric. By Theorem 6.1, $d_{op} < \infty$. The Carathéodory construction applies on any metric space and produces a Borel-regular outer measure for every finite dimension parameter. The d_{op}-coverings of E:

$$\mathcal{H}^{d_{op}}_{op,\delta}(E) := \inf\Big\{ \sum_i (\operatorname{diam}_{op} U_i)^{d_{op}} : E \subseteq \bigcup_i U_i,\ \operatorname{diam}_{op} U_i \le \delta \Big\},$$

are monotone non-increasing in δ; the limit

$$\mathcal{H}^{d_{op}}_{op}(E) = \lim_{\delta \to 0^+} \mathcal{H}^{d_{op}}_{op,\delta}(E) \in [0, +\infty]$$

exists. ■

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8. Finite Operational Volume

We now establish finiteness of Vol_op(M) by routing through the *continuous* operational image M_op = Ω_max(M), avoiding the previous version's confusion between M_op and the finite discrete Σ(M).

Lemma 8.0 (Boundedness of the operational image). *Let M have bounded closure support — i.e. M is bounded in the ambient closure-state-space pre-metric (the metric induced by ⟨·,·⟩_{ℤ₇}). Then the operational image*

M_op := Ω_max(M) ⊆ 𝒜

is a bounded subset of (𝒜, D_op).

Proof. Bounded closure support means M ⊆ B_{ambient}(0, r_M) for some r_M < ∞ in the ambient closure metric (translating so that the bounding ball is centred at the origin). Ω_max is linear with Ω_max(0) = 0 (orthogonal projection through the origin) and non-expansive (Theorem 5.1), so for every x ∈ M,

‖Ω_max(x)‖_op = ‖Ω_max(x) − Ω_max(0)‖_op ≤ ‖x − 0‖_ambient = ‖x‖_ambient ≤ r_M.

Hence M_op = Ω_max(M) ⊆ B_op(0, r_M) ⊆ 𝒜, establishing D_op-boundedness of M_op. ■

Theorem 8.1 (Finite operational volume). *Vol_op(M) < ∞.*

Proof. By Lemma 8.0, M_op is a bounded subset of (𝒜, D_op). By Theorem 7.1, (𝒜, D_op) is d_op-dimensional Euclidean. Any bounded subset of finite-dimensional Euclidean space has finite d_op-dimensional Hausdorff measure:

ℋ^{d_op}_op(M_op) < ∞.

By Definition 3.10,

Vol_op(M) = (2^{d_op} / ω_{d_op}) · ℋ^{d_op}_op(M_op) < ∞. ■

Remark 8.1.1. *The continuous–discrete distinction made clean.* The operational Hausdorff measure is computed on M_op — the *continuous* operational image of M under Ω_max, a bounded continuous region of 𝒜 — not on Σ(M), the finite discrete sector set. The two are related but structurally distinct:

Σ(M) ⊆ M_op ⊆ 𝒜.

Σ(M) is the finite discrete subset of admissible sectors (cardinality ≤ N_ledger(M), zero d_op-dimensional Hausdorff measure for d_op ≥ 1). M_op is the continuous bounded region within which Σ(M) sits as a discrete subset; M_op carries the Hausdorff measure that supplies Vol_op. The packing bound counts Σ(M); the packing volume is supplied by M_op. The previous version's Lemma 8.0 routed boundedness through ledger finiteness, confusing M_op with Σ(M) and inadvertently making the Hausdorff measure zero. Routing through non-expansiveness of Ω_max on continuous M preserves the continuous structure of M_op and gives finite-positive Vol_op(M).

9. Total Boundedness, Precompactness, and Finite Covering Numbers

Theorem 9.1 (Total boundedness and precompactness of M_op). *The operational image M_op ⊆ 𝒜, equipped with D_op, is totally bounded and hence precompact.*

Proof. By Theorem 7.1, (𝒜, D_op) is a d_op-dimensional Euclidean Hilbert space. By Lemma 8.0, M_op is a bounded subset.

In any finite-dimensional Euclidean space, Heine–Borel applies: bounded ⟺ totally bounded ⟺ precompact (with the closure compact). Hence M_op is totally bounded under D_op, and its closure in 𝒜 is compact. ■

Corollary 9.2 (Finite covering numbers). *For every ε > 0, the admissible covering number*

N(ε) := min{ |𝒰| : 𝒰 a D_op-covering of M_op by balls of radius ε }

is finite, with explicit scaling

N(ε) ≲ Vol_op(M) / ε^{d_op}.

Proof. Finiteness follows from Theorem 9.1 (total boundedness). The scaling follows from the d_op-dimensional Euclidean volume identity (Theorem 7.1): a covering of a region of d_op-volume V by balls of radius ε has cardinality on the order of V / (ω_{d_op} · ε^{d_op}), up to a geometric constant of order unity. The Δ_op-normalised convention of Definition 3.10 absorbs the geometric constant into Vol_op. ■

10. Operational Geometry and Finite Packing

The prior finite distinguishability packing theorem (§2.6) emerges as a direct geometric corollary of the operational geometry developed above.

Theorem 10.1 (Packing as a corollary of operational geometry). *Under the hypotheses of §2.1–§2.5, the admissible packing number satisfies*

P(M) ≤ Vol_op(M) / Δ_op^{d_op},

reproducing the bound of the packing paper (§2.6).

Proof. Let F ⊆ Σ(M) be any pairwise-Δ_op-separated family:

D_op(x_i, x_j) ≥ Δ_op for all x_i ≠ x_j ∈ F.

By Corollary 6.2 and Theorem 7.1, F ⊆ M_op ⊆ 𝒜, a d_op-dimensional Euclidean Hilbert space. Around each x ∈ F place an open D_op-ball B(x, Δ_op / 2). By the triangle inequality (Theorem 4.1) and pairwise Δ_op-separation, these balls are pairwise disjoint.

By Theorem 7.1, each ball has d_op-volume ω_{d_op} · (Δ_op / 2)^{d_op}. Disjointness gives

|F| · ω_{d_op} · (Δ_op / 2)^{d_op} ≤ ℋ^{d_op}_op(M_op).

By Definition 3.10, ℋ^{d_op}*op(M_op) = (ω*{d_op} / 2^{d_op}) · Vol_op(M). Substituting,

|F| · ω_{d_op} · (Δ_op / 2)^{d_op} ≤ (ω_{d_op} / 2^{d_op}) · Vol_op(M),

so

|F| · Δ_op^{d_op} ≤ Vol_op(M),

hence

|F| ≤ Vol_op(M) / Δ_op^{d_op}.

Taking the supremum over admissible families F yields the bound on P(M). ■

Remark 10.1.1. The packing theorem is therefore a direct geometric consequence of:

- the Hilbert structure of 𝒜 (Theorem 4.0),
- the operational metric D_op (Theorem 4.1, Corollary 4.2),
- the finite Euclidean dimension d_op (Theorem 6.1, Theorem 7.1),
- the operational Hausdorff measure and finite operational volume (Theorem 7.2, Theorem 8.1),
- the total boundedness of M_op under D_op (Theorem 9.1).

The packing paper introduced these objects operationally; the present paper shows that, jointly, they constitute a complete operational geometry — with the packing bound following as a one-step Euclidean sphere-packing corollary.

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11. Relation to Emergent Geometric Structure

The closure manifold has now been shown to possess, intrinsically:

- a Hilbert-space inner product (⟨·,·⟩_op, Theorem 4.0);
- a metric (D_op, Theorem 4.1);
- a topology (τ_op);
- a finite dimension (d_op, Theorem 6.1);
- a measure (ℋ^{d_op}_op, Theorem 7.2);
- finite volume (Vol_op(M), Theorem 8.1);
- compactness (Theorem 9.1);
- finite covering numbers (Corollary 9.2);
- non-expansive transport (Ω_max, Theorem 5.1).

These are the basic ingredients of geometric structure. None requires the emergent 4-manifold geometry of macroscopic physics as input — each is derived from admissibility alone, through the discrete ledger (§2.1), the entropy partition (§2.3), the $\mathbb{Z}_7$ closure architecture (§2.4), and the finite-rank refinement projector (§2.5).

This raises a structural possibility:

The geometric structure of macroscopic physics — the emergent 4-manifold on which dynamical and gravitational phenomena are formulated — may itself emerge from operational distinguishability geometry, rather than being a primitive substrate on which admissibility is layered.

Within VERSF, the emergent 4-manifold geometry is constructed from substrate primitives (the fold, irreversible commitment events, the BCB and TPB layers). These primitives generate the K = 7 closure architecture and the associated $\mathbb{Z}_7$ Fourier structure (§2.4) — which is the *immediate* input to Theorem 4.0, the inner-product theorem underlying the Hilbert geometry of this paper. The full inheritance chain is therefore:

substrate primitives → K = 7 / $\mathbb{Z}_7$ closure architecture → operational Hilbert geometry (this paper) → emergent 4-manifold geometry.

The $\mathbb{Z}_7$ closure architecture is the structural pivot: it is generated by the substrate primitives below it, and it generates (via Theorem 4.0) the Hilbert structure on $\mathcal{A}$ that drives this paper's geometry. The operational Hilbert space ($\mathcal{A}$, ⟨·,·⟩_op) is therefore a candidate for the geometric layer between substrate and macroscopic continuum: a Hilbert geometry constructed directly from admissibility, on which the emergent 4-manifold of macroscopic physics is in turn constructed.

We do not attempt the full derivation in the present paper. We note only that the structural ingredients for such a derivation are now available, and that the resulting picture is consistent with the broader VERSF programme: time is emergent, the macroscopic geometric continuum is emergent, and operational distinguishability geometry is the substrate-side scaffold on which both arise.

Remark 11.1. This section deliberately avoids reference to "spacetime" — within VERSF, time is emergent, and the macroscopic geometric continuum is the emergent 4-manifold whose construction from substrate primitives is the subject of the geometric emergence programme.

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12. Predictions and Falsifiability

The operational geometry predicts:

- a finite operational dimension $d_{op} \leq N_{spec} \cdot C_{max}$ for every admissibility-complete closure manifold;
- a strictly positive universal distinguishability quantum $\Delta_{op} > 0$;
- an orthogonal spectral-channel decomposition $\mathcal{A} = \bigoplus_\alpha V_\alpha$;
- self-adjoint idempotent Ω_{max} as orthogonal projection onto $\mathcal{A}$;
- 1-Lipschitz refinement projection (non-expansive Ω_{max});
- finite operational volume $\mathrm{Vol}_{op}(M) < \infty$ on every closure manifold with bounded closure support;
- finite covering numbers $N(\varepsilon) \lesssim \mathrm{Vol}_{op}(M) / \varepsilon^{d_{op}}$;
- compactness of the admissible operational geometry under D_{op};
- the packing ceiling $P(M) \leq \mathrm{Vol}_{op}(M) / \Delta_{op}^{d_{op}}$ as a one-step Euclidean corollary.

Potential falsifiers include:

- demonstration of an admissibility-complete closure manifold with infinite operational dimension;
- experimental observation of operational distinguishability below Δ_{op};
- failure of Ω_{max} to act as a self-adjoint idempotent (violation of §2.5 projector structure);
- failure of orthogonality between distinct spectral channels (violation of §2.4 $\mathbb{Z}_7$ Fourier structure);
- failure of refinement projection to act non-expansively;
- failure of the admissible operational geometry to be totally bounded;
- detection of admissible structure violating the covering scaling $N(\varepsilon) \lesssim \mathrm{Vol}_{op}(M) / \varepsilon^{d_{op}}$;
- variation of Δ_{op} or d_{op} across operational sub-manifolds in violation of admissibility invariance.

Each falsifier corresponds to a violation of one of §2.1–§2.5, each independently testable through its own VERSF programme.

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13. Open Problems

The operational geometry is established. Several geometric problems remain open:

1. **Explicit evaluation of Vol_op(M).** Closed-form computation of the operational d_op-volume for the physical closure manifold, in terms of substrate primitives.
2. **Operational curvature.** Whether (𝒜, ⟨·,·⟩_op) admits a natural curvature tensor under refinement-dynamical deformations of the inner product, and what role such curvature plays in admissible transport.
3. **Refinement geodesics.** Whether refinement dynamics endows 𝒜 with a notion of geodesic (beyond the trivial Hilbert straight lines) — e.g. minimum-commitment paths between admissible sectors — and how such geodesics relate to the operational metric.
4. **Entropy-flow asymmetry.** Whether entropy flow induces a directed structure on (𝒜, D_op) — capturing irreversible commitment within an otherwise symmetric Hilbert geometry.
5. **Construction of emergent 4-manifold geometry from operational geometry.** Explicit construction of the macroscopic geometric continuum as a limiting structure derived from (𝒜, ⟨·,·⟩_op) and refinement dynamics.
6. **Emergent Lorentzian structure.** Whether the asymmetric refinement dynamics on (𝒜, D_op) give rise to the Lorentzian signature of the macroscopic continuum.
7. **Operational field equations.** Derivation of the field equations governing operational transport on (𝒜, D_op), and their reduction to the emergent macroscopic field equations.
8. **Quantum operational geometry.** Whether the Hilbert structure (𝒜, ⟨·,·⟩_op) is the natural carrier of the VERSF quantum reconstruction programme, and whether the Born rule emerges from operational measure on this Hilbert space — a natural candidate given that quantum mechanics is itself a finite-dimensional Hilbert-space theory at the level of each closed system.

Each open problem is a geometric problem on the operational Hilbert geometry of M.

14. Conclusion

The previous finite distinguishability packing theorem established the factorised cardinality ceiling

|Σ(M)| ≤ Vol_op(M) / Δ_op^{d_op}

with every factor an admissibility invariant. The packing paper invoked Vol_op, Δ_op, 𝒜, and d_op operationally, leaving their geometric status implicit.

The present paper supplies the geometry. The structural backbone is the inner-product theorem (Theorem 4.0): the $\mathbb{Z}_7$ Fourier architecture (§2.4) endows the admissible subspace 𝒜 = Im(Ω_max) with a natural inner product ⟨·,·⟩_op, making (𝒜, ⟨·,·⟩_op) a finite-dimensional Hilbert space, with orthogonal spectral-channel decomposition 𝒜 = ⨁_α V_α, and Ω_max acting as the orthogonal projection from the ambient closure-state-space onto 𝒜.

Every downstream geometric property follows as a standard Hilbert-space fact:

- D_op is the norm-induced metric (Theorem 4.1; metric axioms reduce to standard inner-product-norm properties);
- Ω_max is non-expansive by Pythagoras (Theorem 5.1) and continuous (Theorem 5.2);
- the operational dimension is d_op = ∑_α dim V_α ≤ N_spec · C_max (Theorem 6.1);
- the operational Hausdorff measure ℋ^{d_op}_op exists with Euclidean ball volumes (Theorem 7.2, Theorem 7.1);
- bounded closure support of *continuous* M yields bounded continuous operational image M_op via non-expansiveness (Lemma 8.0), hence finite Vol_op(M) (Theorem 8.1);
- M_op is totally bounded and precompact under Heine–Borel (Theorem 9.1);
- covering numbers scale as N(ε) ≲ Vol_op(M) / ε^{d_op} (Corollary 9.2);
- the prior packing theorem P(M) ≤ Vol_op(M) / Δ_op^{d_op} emerges as a one-step Euclidean sphere-packing corollary (Theorem 10.1).

The closure manifold therefore possesses, intrinsically, a **finite operational Hilbert geometry**, derived from admissibility alone. The continuous–discrete distinction is preserved cleanly: the Hausdorff measure lives on the continuous operational image M_op = Ω_max(M), while the discrete sector set Σ(M) ⊆ M_op is what gets counted by the packing bound.

The result upgrades the VERSF programme from **finite admissibility** to **operational distinguishability geometry**: not merely the claim that admissible sectors are finite — established already in §2.1 — but the identification of the Hilbert-geometric structures (inner product, metric, dimension, measure, volume, non-expansive transport, compactness) that admissibility induces on the closure manifold, and through which finiteness, packing, and the emergent 4-manifold geometry of macroscopic physics may be understood.

Admissibility does not merely count. It geometrises — and geometrises as Hilbert structure. The closure manifold carries an operational Hilbert geometry whose structural invariants (⟨·,·⟩_op, Δ_op, Vol_op(M), d_op, N_spec, C_max) control both the cardinality of admissible sectors and the geometric substrate from which the emergent 4-manifold of macroscopic physics is built.