Probability as Admissible Measure on the VERSF Operational Hilbert Geometry

Constraint-Intersection Recovery of the Born Rule from Projection Geometry, Finite Packing, and Isosymmetric Operational Structure

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A General Reader's Summary

Why is quantum probability the square of an amplitude?

Quantum mechanics gives extraordinarily accurate predictions about the world, but it does so through one famously strange rule. To predict the chance of getting a particular outcome when measuring a quantum system, the theory tells us to take a particular number — called the amplitude — and square it. That squared value is the probability. Squaring is the rule for everything: light hitting a detector, an electron's spin, the decay of an atom. No part of standard quantum theory explains *why* squaring. The rule is simply assumed.

This is unsatisfying. Probabilities in everyday life don't come from squaring anything — they come from counting how many ways something can happen. Why is the quantum world different? Several attempts have been made to derive the squaring rule from deeper principles. Some appeal to mathematical structure (Gleason's theorem); some to decision-theoretic arguments about what rational agents should bet on; some to information theory or thermodynamics. Each captures a piece of the answer, but the pieces look independent — different roads ending at the same destination, with no obvious reason they all converge there.

The earlier papers of this programme built up, step by step, a picture of physical reality in which space, time, matter, and quantum behaviour all emerge from a deeper underlying structure. Three of those papers established the stage on which quantum mechanics plays out: they showed that the right mathematical space for describing physical states arises naturally from more primitive ingredients, and they explained how this space supports both reversible motion (the smooth evolution of quantum systems) and irreversible commitment (what happens when a measurement gives a definite result). What those papers did *not* do was explain where the squaring rule itself comes from.

The present paper closes that gap. Its central claim is:

Once the underlying physical stage is in place, the squaring rule is not a free choice. It is the *only* probability rule consistent with the structure of that stage. The rule is forced — not added.

The argument runs through a sequence of natural requirements that any probability rule on this stage must obey. Independent options should give independent probabilities. Reversible motion should not change probabilities. Probabilities should vary smoothly rather than jump discontinuously. Probabilities should not depend on arbitrary choices about how to label things. Each requirement is mild on its own; together, they leave exactly one possibility standing — the squaring rule. The paper also shows that the various earlier roads to this conclusion (decision theory, information theory, thermodynamic arguments, and others) are not independent derivations after all. They are different views of the same underlying picture, each capturing one face of it. The seeming convergence of independent arguments is in fact a single argument expressed in many ways.

The picture that emerges: probability in quantum mechanics is a kind of *geometric volume* — a measure of how much physical "room" an outcome occupies within the space of possibilities. Squaring an amplitude is not a mysterious recipe; it is what you get when you measure that volume properly.

The technical body follows. Readers without a mathematical-physics background may wish to read §1, §6, and §11 next, where the structural content is given in continuous prose before the theorems are introduced.

Abstract

Prior work in the VERSF programme established the operational Hilbert geometry of admissibility (OG), the conformal-deformation curvature it carries under distinguishability-density gradients (OC), and the canonical polar decomposition of admissible refinement transport into unitary and positive-contractive factors with substrate-derived complex structure from $\mathbb{Z}_7$-equivariance (URHG). These supply ingredients (1)–(6) of the quantum-reconstruction list — Hilbert carrier, inner product, complex structure, reversible dynamics, self-adjoint generator, projection structure — leaving the probability measure layer as the remaining open input.

The present paper closes that gap. The argument has three structural inputs:

(A) Inherited operational structure. The carrier $(\mathcal{A}_\mathbb{C}, \langle\cdot,\cdot\rangle_\mathbb{C})$ is the substrate-derived complex Hilbert space of URHG Theorem 3.5; orthogonal channel decomposition $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$ is the URHG-extended OG decomposition; orthogonal projection P ($P^2 = P$, $P^\dagger = P$) is the admissibility-projection structure of OG; unitary reversible transport $U(\tau) = \exp(-i\tau H)$ is the URHG reversible regime, restricted to $\mathbb{Z}_7$-equivariant transports; finite distinguishability packing $|\Sigma(M)| \leq \mathrm{Vol}_{op}(M) / \Delta_{op}^{d_{op}}$ is OG Theorem 10.1.

(B) Operational measure axioms. An admissible operational probability measure is a map $\mu : \mathcal{A}_\mathbb{C} \to [0, \infty)$ satisfying a **minimal core** of five axioms — positivity, orthogonal additivity, admissible unitary invariance, continuity (finite-packing stability), and normalization — together with a **cross-channel bridging condition** taking any one of three structurally distinct forms.

Two further axioms — projection consistency and full-unitary measure invariance — turn out to be automatic on the quadratic solution and serve as structural confirmations rather than independent constraints (Definition 3.1, Remark 3.1.1).

(C) The uniqueness argument. The proof proceeds channel by channel and then bridges across channels. Admissible ($\mathbb{Z}_7$-equivariant) unitary invariance acts transitively on each within-channel norm-sphere (Lemma 5.1), forcing μ to take the form $\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_{\mathbb{C}})$ for per-channel continuous functions F_α (Theorem 5.2). Orthogonal additivity within each channel, combined with continuity, reduces each F_α via change of variable $G_\alpha(t) := F_\alpha(\sqrt{t})$ to the additive Cauchy equation on $[0, \infty)$; Kuczma's theorem on continuous additive functions on $[0, \infty)$ forces $G_\alpha(t) = C_\alpha t$, hence $F_\alpha(r) = C_\alpha r^2$ (Theorem 6.1). A separate **cross-channel bridging argument** — using either tensor-product compositional multiplicativity (B1), sub-channel isosymmetric embedding (B2), or refinement-stability under channel-decomposition changes (B3) — forces $C_\alpha = C$ uniformly across α (Theorem 6.2). Normalization fixes $C = 1$ (Theorem 6.3).

The central result is the **Common Operational Measure Theorem** (Theorem 8.1):

Any admissible probability assignment on $(\mathcal{A}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ satisfying the minimal core of axioms plus the cross-channel bridging condition is uniquely the quadratic operational measure $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$. The transition probability between admissible states ψ and a normalized outcome ϕ is therefore $P(\psi \to \phi) = |\langle\phi, \psi\rangle_{\mathbb{C}}|^2$, recovering the Born rule.

The paper further shows that the previously partially-independent VERSF derivational routes (Double Square pairwise geometry, entropic unfolding, isosymmetric reconstruction, tensor-product bilinearity, Gleason-type frame analysis, TPB normalization, finite packing) are equivalent operational projections of the same underlying measure geometry.

A notable corollary: because the present derivation uses orthogonal additivity on *vectors* (rather than frame-additivity on *projections*), the uniqueness conclusion holds in dimension $d_{op}^{\mathbb{C}} \geq 2$, whereas Gleason's theorem requires $d \geq 3$. This is a genuine structural advantage of the operational route (Remark 7.1.2).

Scope and Conditional Status

This paper should be read as a **conditional reconstruction theorem**. It does not claim to derive the Born rule from no assumptions whatsoever. Rather, it proves that once the VERSF operational Hilbert geometry is already in place — namely the admissible carrier, substrate-derived complex structure, orthogonal projection structure, finite-packing continuity, and admissible reversible transport (all supplied by OG, OC, URHG) — the remaining probability layer is no longer free. Under the minimal operational-measure axioms (positivity, orthogonal additivity, admissible unitary invariance, continuity, single-reference normalization) plus *one* explicit cross-channel bridging principle (compositional multiplicativity with character matching, sub-channel isosymmetric embedding, or refinement-stability), the only admissible probability measure is the quadratic norm.

The result is therefore conditional but structurally strong: the freedom to choose an alternative probability rule is eliminated once the inherited VERSF geometry and a cross-channel uniformity principle are accepted. The two structural inputs the paper does *not* derive from absolute first principles are:

1. **The inherited operational Hilbert structure of §2** — supplied by OG, OC, URHG.
2. **A cross-channel bridging principle** — supplied by any one of (B1), (B2), (B3), each of which expresses an operational principle natural to the VERSF substrate but not derivable from admissible unitary invariance alone (Lemma 6.2A).

The paper makes both of these structural inputs explicit and shows why they are natural in the VERSF setting. What it forces is everything else: given these two inputs, the quadratic measure and the Born rule follow uniquely.

A note on substrate-origin questions. The choice of $\mathbb{Z}_7$ as the substrate symmetry group is also not derived in this paper. $\mathbb{Z}_7$ is inherited from the prior VERSF corpus (URHG §2.5, the K = 7 closure architecture established in earlier programme papers), where its substrate-level origin is treated as a separate structural question. The present paper accepts $\mathbb{Z}_7$ as part of the inherited operational structure and shows that *given* the $\mathbb{Z}_7$-substrate, the probability layer is forced. The deeper "why $\mathbb{Z}_7$ rather than some other finite cyclic structure" question is a substrate-origin question, outside the scope of a measure-reconstruction paper. It is listed as Open Problem 11 of §13, and is the eventual long-term target for the programme: making $\mathbb{Z}_7$ itself appear inevitable rather than selected.

Contents

1. Introduction

The Born rule

$P(\psi \to \phi) = |\langle\phi, \psi\rangle_\mathbb{C}|^2$

is one of the most successful and least foundationally settled features of quantum theory. Conventional formulations assume it as a postulate; reconstruction programmes derive parts of it from operational, informational, or geometric axioms (Gleason; Hardy; Masanes–Müller; Chiribella; Deutsch–Wallace), with each route capturing the quadratic measure under a different set of structural inputs.

The VERSF programme has developed multiple partially independent routes to the Born rule within its own framework:

- **Double Square pairwise geometry** — the Born rule as the unique pairwise-correlation measure on admissible path pairs;
- **Entropic unfolding** — the Born rule as the iso-entropic limit of a thermodynamic projection weighting;
- **Physical-necessity / admissibility elimination** — the Born rule as the unique measure surviving admissibility constraints under operational scrutiny;
- **Isosymmetric reconstruction** — the Born rule from operational task-class equivalence;
- **Tensor-product bilinearity** — the Born rule from compositional structure of admissible assemblies;
- **TPB normalization** — the Born rule from norm preservation under reversible transport.

These routes have been developed semi-independently. The structural question this paper addresses is whether they are genuinely separate derivations of one rule, or whether they are equivalent operational projections of one underlying measure geometry.

The thesis of the present paper is the latter:

The Born rule is the unique admissible operational measure preserving finite distinguishability geometry under refinement, projection, and reversible transport. The previously-distinct VERSF routes are equivalent operational projections of this one structure.

The argument proceeds in four stages. §2–§3 collect inherited structure and define the admissible operational measure, with explicit identification of the minimal axiomatic core versus the structural-consequence axioms. §4 establishes orthogonal additivity from independent-sector finite-packing structure. §5 establishes channel-wise norm-only dependence from admissible ($\mathbb{Z}_7$-equivariant) unitary invariance plus the within-channel transitivity of $U(V_\alpha)$. §6 closes the chain in two steps: (i) the additive Cauchy equation argument forces each per-channel function to be quadratic with potentially distinct constants C_α (Theorem 6.1), and (ii) any one of three structurally distinct cross-channel bridging arguments forces all C_α equal (Theorem 6.2). §7 recovers the Born rule as a corollary. §8 states the central Common Operational Measure Theorem unifying the previously-independent routes. §9–§14 develop interpretation, falsifiability, open problems, and conclusion.

The structural picture: admissible unitary invariance (restricted to $\mathbb{Z}_7$-equivariant transports) is genuinely weaker than full-unitary invariance, so the measure-uniqueness argument must proceed channel-by-channel and then bridge across channels via a separate structural input. The bridging step is the key technical point of the paper, and three structurally distinct bridging arguments are presented — compositional multiplicativity with character matching, sub-channel isosymmetry, and refinement-stability — any one of which suffices to close the uniqueness argument. This redundancy is itself a structural feature: the framework is robust against challenges to any single bridging condition, while the conditions themselves have distinct scope (e.g. (B2) becomes vacuous in the minimal $d_\alpha = 1$ case unless supplemented; see Remark 6.2.1).

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2. Inherited Operational Structure

We summarize the structural framework inherited from prior VERSF work. The three prior papers are referenced as **OG** (Operational Geometry), **OC** (Operational Curvature), and **URHG** (Unified Refinement Hilbert Geometry).

2.1 Operational Hilbert Space (OG Theorem 4.0)

The admissible subspace $\mathcal{A} = \mathrm{Im}(\Omega_{max})$ of the closure-state-space carries a real finite-dimensional Hilbert inner product $\langle\cdot,\cdot\rangle_{op}$ of dimension $d_{op} := \dim \mathcal{A} < \infty$.

2.2 Substrate-Derived Complex Structure (URHG Theorem 3.5)

The substrate $\mathbb{Z}_7$-equivariance forces a canonical complex structure J on the non-trivial spectral channels $\mathcal{A}_{nt} \subseteq \mathcal{A}$, satisfying $J^2 = -I$ and $J^\dagger = -J$. Under the trivial-channel-suppression option of URHG Remark 3.5.1(a), the resulting complex Hilbert space is

$$(\mathcal{A}_\mathbb{C}, \langle\cdot,\cdot\rangle_\mathbb{C}), \quad \langle x, y\rangle_\mathbb{C} := \langle x, y\rangle_{op} + i \cdot \langle Jx, y\rangle_{op},$$

of complex dimension $d_{op}^\mathbb{C} = \dim_\mathbb{R} \mathcal{A}_{nt} / 2$. The complex norm coincides with the real norm: $\|x\|^2_\mathbb{C} = \|x\|^2_{op}$ (URHG Definition 3.6). To avoid ambiguity we use $\|\cdot\|_\mathbb{C}$ throughout this paper after this point, even when the equivalent $\|\cdot\|_{op}$ would also apply.

2.3 Orthogonal Channel Decomposition (OG Theorem 4.0)

The admissible space decomposes orthogonally:

$$\mathcal{A}_\mathbb{C} = \bigoplus_{\alpha=1}^{N_{spec}} V_\alpha,$$

with channels V_α carrying distinct $\mathbb{Z}_7$-equivalence classes of complex characters. Every $\psi \in \mathcal{A}_\mathbb{C}$ has a unique decomposition $\psi = \sum_\alpha \psi_\alpha$ with $\psi_\alpha \in V_\alpha$, and $\|\psi\|^2_\mathbb{C} = \sum_\alpha \|\psi_\alpha\|^2_\mathbb{C}$. We denote $d_\alpha := \dim_\mathbb{C} V_\alpha$, so $\sum_\alpha d_\alpha = d_{op}^\mathbb{C}$.

2.3A Operational Meaning of the $\mathbb{Z}_7$-Channels

The orthogonal decomposition $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$ is not merely an abstract representation-theoretic decomposition. Each channel V_α has a direct operational interpretation within the admissibility framework.

Operationally, a channel represents a **transport-invariant coherence sector** of admissible distinguishability configurations — a subspace of $\mathcal{A}_\mathbb{C}$ whose elements:

- transform coherently under admissible reversible transport,
- preserve a common substrate-phase response under the $\mathbb{Z}_7$-action,
- and remain closed under admissible refinement evolution.

Equivalently: **a channel is an operational coherence sector of admissible distinguishability transport.**

The $\mathbb{Z}_7$-character label χ_α attached to a channel is therefore not interpreted as an arbitrary mathematical decoration. It labels the substrate-response class governing how distinguishability content transforms under admissible refinement transport. Distinct channels correspond to distinguishability sectors that can superpose geometrically within the full carrier $\mathcal{A}_\mathbb{C}$ but cannot be mixed by admissible reversible transport, because admissible transport is $\mathbb{Z}_7$-equivariant (URHG Definition 3.2).

Thus the channel structure has direct operational content:

Mathematical object	Operational meaning
V_α	admissible coherence sector
χ_α	substrate transport-response class
Block-diagonal admissible transport	preservation of operational coherence class
Cross-channel mixing	non-admissible transport operation

The decomposition therefore plays a role *analogous* to superselection structure in conventional quantum theory, but with a different origin. Conventional superselection sectors are typically imposed phenomenologically (charge superselection, mass superselection) or by appeal to external symmetry. Here the sectors arise from *admissibility constraints inherited from the substrate-level $\mathbb{Z}_7$-equivariant structure* — they are derived from the substrate, not imposed phenomenologically. The channels are therefore best thought of as **substrate-derived operational superselection sectors**: structurally similar to conventional superselection but with an admissibility-substrate origin.

Caveat on the superselection analogy. *The analogy is structural rather than literal.* Conventional superselection rules prohibit physical superposition across sectors (e.g. charge-superposed states are excluded from the physical Hilbert space). Admissible channels in the present framework are *not* superselection-restricted in this strong sense: the full carrier $\mathcal{A}_\mathbb{C}$ supports geometric superposition across channels — the decomposition $\psi = \sum_\alpha \psi_\alpha$ with non-trivial components in distinct V_α is an admissible state, and orthogonal additivity (A2) is *defined* over such superpositions. What the channel structure prohibits is not superposition but *admissible reversible transport between channels*: $\mathbb{Z}_7$-equivariant unitaries are block-diagonal and cannot mix inequivalent channels (URHG §2.5). The analogy with conventional superselection is therefore at the level of "transport blocks that cannot be mixed by allowed operations," not at the level of "states that cannot be superposed." A reader familiar with charge superselection should treat the analogy as a structural guide rather than as a literal identification.

Remark 2.3A.1 (Why channels matter operationally). The channels are not directly identified with observable particle species or externally measurable labels. Rather, they classify the *internal* operational transport structure of admissible distinguishability configurations. Their significance appears through:

- transport restrictions (admissible transport is block-diagonal across channels — URHG §2.5);
- compositional structure (tensor products of channels carry character-multiplied labels — Theorem 6.2 (B1));
- refinement stability (admissible refinement preserves channel content — OC Theorem 5.0);
- probability uniformity across admissible coherence sectors (the cross-channel bridging condition of §6.2).

The cross-channel bridging conditions of §6.2 are therefore *physically meaningful*: they state that probability assignment is universal across all admissible coherence sectors despite their differing substrate-response classes.

Why the bridging principle exists — operational summary.

Without a cross-channel bridging principle, the framework would permit each operational coherence sector to carry its own intrinsic probability weighting — a substrate-level parochialism in which distinct coherence sectors would not be probabilistically commensurable. Operationally indistinguishable preparations differing only in their substrate-response class would receive distinct probability weights. The bridging conditions of §6.2 restore probabilistic universality across coherence sectors: they assert that probability is a property of the abstract operational distinguishability content, not of the substrate-channel identification.

Lemma 6.2A makes this gap mathematically explicit; Remark 6.2A.2 makes the physical ugliness of non-uniform C_α concrete; the bridging principle resolves both. This is the single most *important conceptual content of the cross-channel structure, and it is what the universality of the Born rule across the operational carrier ultimately rests on.*

2.4 Orthogonal Projection Structure (OG Theorem 5.1, URHG §7)

Orthogonal projections $P : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ satisfy $P^2 = P$, $P^\dagger = P$. They are non-expansive and strictly remove distinguishability content orthogonal to Im(P) (URHG Theorem 7.1):

$$\|P\psi\|^2_{\mathbb{C}} = \|\psi\|^2_{\mathbb{C}} - \|(I - P)\psi\|^2_{\mathbb{C}}.$$

The universal admissibility projector Ω_{max} is the orthogonal projection onto $\mathcal{A}$.

2.5 Unitary Reversible Transport (URHG Theorems 4.1, 5.1, Corollary 6.1)

Reversible admissible transport on $\mathcal{A}_{\mathbb{C}}$ is a strongly continuous one-parameter unitary group $U(\tau) = \exp(-i\tau H)$ with self-adjoint generator H. By URHG Definition 3.2 condition (3), admissible transports are **$\mathbb{Z}_7$-equivariant**: they commute with the substrate $\mathbb{Z}_7$-action and hence (by Schur's lemma) are block-diagonal across inequivalent $\mathbb{Z}_7$-irreps. The admissible unitary group is

$$U_{adm}(\mathcal{A}_{\mathbb{C}}) = \prod_\alpha U(V_\alpha)$$

(if all V_α carry inequivalent $\mathbb{Z}_7$-irreps; more generally, the product of unitary groups of the isotypic components). It is *strictly smaller* than the full unitary group $U(d_{op}^{\mathbb{C}})$; admissible unitaries do not connect distinct $\mathbb{Z}_7$-channels. This restriction is the structural feature that necessitates the channel-by-channel argument of §5–§6.

2.6 Finite Distinguishability Packing (OG Theorem 10.1)

The admissible sector packing satisfies

$$|\Sigma(M)| \leq \mathrm{Vol}_{op}(M) / \Delta_{op}^{d_{op}},$$

with $\Delta_{op} > 0$ the universal distinguishability quantum. Finite packing induces finite operational resolution, bounded admissible sector density, and a smooth packing density at scales $\geq \Delta_{op}$ (OG §2.3) — the last of which supplies the continuity of admissible measures (axiom A5 below).

2.7 Dimensional Conventions

For the present paper, we require $d_{op}^{\mathbb{C}} \geq 2$ throughout. This suffices for the orthogonal-additivity arguments of §6 and for at least two channels (or at least a 2-dim channel) to support the cross-channel bridging. The $\mathbb{Z}_7$-architecture supplies $d_{op}^{\mathbb{C}} \geq 3$ (three non-trivial conjugate-character pairs $k \in \{1, 2, 3\}$ of URHG Theorem 3.5(i)), but the *measure-uniqueness* conclusion of this paper holds whenever $d_{op}^{\mathbb{C}} \geq 2$, which is a genuine structural advantage over Gleason's theorem (which requires $d \geq 3$). We make this comparison explicit in Remark 7.1.2.

These results are taken as inputs throughout.

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3. The Admissible Operational Measure

We now define the admissible probability measure on $\mathcal{A}_{\mathbb{C}}$. To match the structural role of each axiom precisely, we separate the **minimal core** axioms (those used directly in the uniqueness proof) from the **cross-channel bridging condition** (presented in three equivalent forms) and from the **structural-consequence** axioms (those automatic on the quadratic solution).

Reader orientation: numbering of axioms. The labels (A1)–(A8) below are not strictly sequential by use in the proof. The minimal-core axioms (A1), (A2), (A3), (A5), (A7) appear in the order in which they enter the uniqueness argument of §§4–6. The gaps in the sequence — (A4), (A6), (A8) — are not omissions: they correspond to derived consequences of the minimal core (projection consistency, cross-channel equality, full-unitary measure invariance) that emerge later in §6 rather than being assumed at the outset. Their numbering reflects historical axiom-system organization in earlier formulations of this paper; we retain it for continuity with the broader VERSF corpus and to signal that these are *structural-consequence* axioms rather than minimal-core or bridging conditions. See Remark 3.1.1 and Remark 3.1.2 for the structural relations among the eight labels.

Definition 3.1 (Admissible operational probability measure). An **admissible operational probability measure** is a function

$\mu : \mathcal{A}_{\mathbb{C}} \to [0, \infty)$

satisfying the following axioms.

Minimal core (used directly in the uniqueness proof):

(A1) **Positivity:** $\mu(\psi) \geq 0$ for all $\psi \in \mathcal{A}_{\mathbb{C}}$, with $\mu(0) = 0$.

(A2) **Orthogonal additivity:** If $\psi_1, \psi_2 \in \mathcal{A}_{\mathbb{C}}$ are complex-orthogonal ($\langle\psi_1, \psi_2\rangle_{\mathbb{C}} = 0$, equivalent to both $\mathrm{Re}\langle\psi_1, \psi_2\rangle_{op} = 0$ and $\langle J\psi_1, \psi_2\rangle_{op} = 0$), then

$\mu(\psi_1 + \psi_2) = \mu(\psi_1) + \mu(\psi_2)$.

(A3) **Admissible unitary invariance:** For every admissible ($\mathbb{Z}_7$-equivariant) unitary transport $U \in U_{adm}(\mathcal{A}_{\mathbb{C}}) = \prod_\alpha U(V_\alpha)$ and every $\psi \in \mathcal{A}_{\mathbb{C}}$,

$\mu(U\psi) = \mu(\psi)$.

(A5) **Continuity (finite-packing stability):** μ is continuous on $\mathcal{A}_\mathbb{C}$.

(A7) **Normalization (single-reference form).** There exists at least one state $\psi_0 \in \mathcal{A}_\mathbb{C}$ with $\|\psi_0\|_\mathbb{C} = 1$ such that $\mu(\psi_0) = 1$. This is genuine conventional scale-fixing: a single reference state has measure 1. The stronger statement "$\mu(\psi) = 1$ for all ψ with $\|\psi\|_\mathbb{C} = 1$" is itself a cross-channel uniformity condition; it is *derived* below as a consequence of the bridging argument of §6.2 combined with this single-reference normalization, not assumed.

Cross-channel bridging condition. Additionally, μ must satisfy *any one* of the following three structurally equivalent conditions (each suffices independently; their equivalence under the minimal core is shown in Theorem 6.2):

(B1) **Compositional multiplicativity with character matching.** For any decomposable admissible assembly $\psi \otimes \phi$ in a tensor-product extension of the carrier with $\psi \in V_\alpha$ and $\phi \in V_\beta$, the measure of the composite satisfies $\mu(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi)$. Additionally, the operational measure on $V_\alpha \otimes V_\beta$ — an isotypic component for the product character $\chi_\alpha \cdot \chi_\beta = \chi_{\alpha+\beta \bmod 7}$ in the tensor carrier — is consistent under character-matching with the operational measure on the $V_{\alpha+\beta \bmod 7}$ channel of the original carrier when $\alpha + \beta \not\equiv 0 \bmod 7$. The character-matching consistency expresses the operational principle that probability measures on $\mathbb{Z}_7$-isotypic components are determined by the character itself, not by the carrier of origin.

Clarification of (B1). The character-matching clause in (B1) is not derived from the minimal core alone. It is an additional operational identification principle: it states that the probability coefficient attached to an isotypic component is determined by its substrate character rather than by accidental carrier-specific multiplicity data. Without this identification, the minimal core permits channel-dependent quadratic constants (see Lemma 6.2A). Thus (B1) should be understood as a genuine cross-channel uniformity principle, not as a consequence of admissible unitary invariance. The substrate-level justification for character-matching is given in Step 1 of the (B1) proof in Theorem 6.2: the $\mathbb{Z}_7$-character is the only substrate-level label that survives inter-carrier identification, so dependence on anything else would constitute hidden carrier-specific data.

(B2) **Sub-channel isosymmetric embedding.** For any channel V_α and any complex-unitary embedding $\iota : V_\alpha \to V_\beta$ into another channel V_β with $d_\beta \geq d_\alpha$, the measure restricted to $\iota(V_\alpha)$ agrees with the measure on V_α: $\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha)$.

(B3) **Refinement-stability across channel decompositions.** The total measure $\mu(\psi)$ is independent of which orthogonal channel decomposition is used to compute it — i.e. for any unitary isomorphism W between admissible decompositions of $\mathcal{A}_\mathbb{C}$ into channel families $\{V_\alpha\}$ and $\{V_\alpha'\}$ that mixes V_α and V_β within a common dimension $d_\alpha + d_\beta$, the measure satisfies $\mu(W\psi) = \mu(\psi)$.

We write **(A6)** for the resulting derived axiom: "the per-channel quadratic coefficients C_α arising from the minimal core are equal across α." (A6) is *derived*, not assumed independently — it follows from the minimal core plus any one of (B1)–(B3) (Theorem 6.2).

Structural-consequence axioms (automatic on the quadratic solution; included for clarity):

(A4) **Projection consistency:** For every orthogonal projection $P : \mathcal{A}\_\mathbb{C} \to \mathcal{A}\_\mathbb{C}$ and every $\psi \in \mathcal{A}\_\mathbb{C}$,

$$\mu(P\psi) \leq \mu(\psi),$$

with equality iff $P\psi = \psi$. As Remark 6.1.2 shows, this is automatically satisfied by the quadratic solution forced by the minimal core plus (A6).

(A8) **Full-unitary measure invariance:** $\mu(U\psi) = \mu(\psi)$ for every $U \in U(\mathcal{A}\_\mathbb{C})$, not merely the admissible subgroup U_adm. Like (A4), this is automatic on the quadratic solution: $\|U\psi\|^2\_\mathbb{C} = \|\psi\|^2\_\mathbb{C}$ for any unitary U.

Remark 3.1.1 (Structure of the axiom system). The axiom system separates cleanly into:

- five minimal-core axioms (A1, A2, A3, A5, A7) plus one cross-channel bridging condition presentable as (B1), (B2), or (B3) — together these force the unique quadratic measure;
- two structural-consequence axioms (A4, A8) automatically satisfied by the quadratic solution.

The previous version of this paper (and most prior treatments) presented all eight as axioms on equal footing, which obscured the proof structure and quietly upgraded (A6) from an inter-system consistency condition to an intra-system invariance condition mid-proof. The present separation makes clear which axioms do the structural work, which is the bridging step, and which are confirmations of the result.

Remark 3.1.2 (Status of admissible vs full unitary invariance). Axiom (A3) is *admissible* unitary invariance, restricting to $\mathbb{Z}_7$-equivariant transports. This is the structurally honest condition: by URHG Definition 3.2, admissible transports are $\mathbb{Z}_7$-equivariant, so the operational principle "the measure is invariant under admissible transport" gives (A3), not the stronger (A8). The stronger (A8) — "the measure is invariant under all unitary relabelings of the carrier including those that mix $\mathbb{Z}_7$-channels" — is not directly motivated within the substrate, since non-equivariant unitaries are not admissible transports. However, on the quadratic solution forced by the minimal core plus a bridging condition, (A8) is automatically true. So (A8) is *recovered* as a structural consequence of the minimal core, not assumed as an independent axiom. This resolves the tension between admissibility-restricted transport and full-unitary measure invariance that was an unresolved structural issue in earlier formulations.

Remark 3.1.3 (What each axiom contributes). Axiom (A2) is the structural backbone: independent admissible distinguishability sectors carry independent measure content. Axiom (A3) is the substrate-derived symmetry: admissible reversible transport preserves operational distinguishability *within* each channel. Axiom (A5) supplies the regularity needed to exclude pathological Hamel-basis solutions of the Cauchy equation. The bridging condition supplies cross-channel uniformity through a structurally independent operational principle (composition, isosymmetric embedding, or refinement-stability). Axiom (A7) is the conventional probability normalization. Together they form the minimal sufficient axiom set; (A4) and (A8) are automatic.

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4. Orthogonal Additivity from Independent Distinguishability Sectors

We first establish the operational origin of the orthogonal-additivity axiom (A2).

Theorem 4.1 (Orthogonal additivity from independent sectors). *Under the operational interpretation of probability as normalized distinguishability volume, orthogonal additivity (A2) is forced by the structural independence of distinct admissible distinguishability sectors.*

Proof. Distinct complex-orthogonal sectors correspond to operationally independent admissible distinguishability channels: by URHG Theorem 7.1, an orthogonal decomposition of $\mathcal{A}_\mathbb{C}$ into $\mathrm{Im}(P) \oplus \ker(P)$ splits the operational distinguishability content additively in the squared-norm sense. By the orthogonal channel decomposition of OG (§2.3 above), every $\psi \in \mathcal{A}_\mathbb{C}$ decomposes uniquely as $\psi = \sum_\alpha \psi_\alpha$ across the $\mathbb{Z}_7$-spectral channels, with $\|\psi\|^2_\mathbb{C} = \sum_\alpha \|\psi_\alpha\|^2_\mathbb{C}$. Distinct V_α are orthogonal and carry inequivalent $\mathbb{Z}_7$-characters, so they support no admissible mixing under refinement transport (URHG Remark 3.2.1: $\mathbb{Z}_7$-equivariant operators are Schur-block-diagonal across inequivalent irreps).

Consequently, the operational distinguishability content of a vector $\psi_1 + \psi_2$ with $\psi_1 \perp_\mathbb{C} \psi_2$ is the sum of the contents of ψ_1 and ψ_2: they live in operationally non-interacting subspaces. Any admissible measure on this content must therefore be additive. The contrapositive is sharper: a measure violating orthogonal additivity would either (i) double-count distinguishability content shared between the two sectors (impossible since the sectors are orthogonal), or (ii) introduce a cross-sector contribution beyond the sector measures, which would violate the Schur-block-diagonal structure of admissible transport.

Remark on complex orthogonality. Throughout this paper, "orthogonality" without further qualification means *complex* orthogonality $\langle\psi_1, \psi_2\rangle_\mathbb{C} = 0$, which requires both $\mathrm{Re}\langle\psi_1, \psi_2\rangle_{op} = 0$ and $\langle J\psi_1, \psi_2\rangle_{op} = 0$ (from the definition $\langle x, y\rangle_\mathbb{C} = \langle x, y\rangle_{op} + i\langle Jx, y\rangle_{op}$ of URHG Definition 3.6). Complex orthogonality is strictly stronger than real orthogonality; under complex orthogonality, the Pythagorean identity $\|\psi_1 + \psi_2\|^2_\mathbb{C} = \|\psi_1\|^2_\mathbb{C} + \|\psi_2\|^2_\mathbb{C}$ holds with no cross-term.

Remark 4.1.1. Theorem 4.1 supplies the structural justification for axiom (A2). It is the operational analogue of the standard probability-theoretic statement that probabilities of mutually exclusive events add: in the operational reading, the "events" are admissibility assignments to distinct orthogonal sectors, and their operational independence is enforced by the Schur-block-diagonal structure of $\mathbb{Z}_7$-equivariant transport.

5. Channel-Wise Norm-Only Dependence from Admissible Unitary Invariance

We now establish that any measure satisfying the minimal core depends on each channel only through its norm.

Lemma 5.1 (Admissible unitary action on within-channel norm-spheres is transitive). *For each $\mathbb{Z}_7$-channel V_α with $d_\alpha := \dim_{\mathbb{C}} V_\alpha \geq 1$ and for each $r > 0$, the within-channel unitary group $U(V_\alpha)$ acts transitively on the within-channel norm-sphere*

$$S_\alpha(r) := \{\psi_\alpha \in V_\alpha : \|\psi_\alpha\|_{\mathbb{C}} = r\}.$$

Furthermore, every $U \in U(V_\alpha)$, extended by the identity on the orthogonal complement $\bigoplus_{\beta \neq \alpha} V_\beta$, lies in the admissible unitary group $U_{adm}(\mathcal{A}_{\mathbb{C}})$ and is therefore covered by axiom (A3).

Proof. Within V_α, the channel-restricted unitary group $U(V_\alpha)$ is the full unitary group of that complex Hilbert subspace. For $\psi_\alpha, \chi_\alpha \in S_\alpha(r)$: complete ψ_α/r and χ_α/r to orthonormal bases of V_α via Gram–Schmidt, and let $V \in U(V_\alpha)$ be the unitary mapping the first basis to the second. Then $V\psi_\alpha = \chi_\alpha$.

The extension $\tilde{V} := V \oplus I_{\bigoplus_{\beta \neq \alpha} V_\beta}$ is unitary on $\mathcal{A}_{\mathbb{C}}$ and acts as the identity on every V_β for $\beta \neq \alpha$. The $\mathbb{Z}_7$-action restricted to V_α is by the fixed character $\chi_\alpha : \mathbb{Z}_7 \to U(1)$, i.e. by scalar multiplication. Every complex-linear map $V_\alpha \to V_\alpha$ commutes with scalar multiplication, so $V \in U(V_\alpha)$ commutes with the $\mathbb{Z}_7$-action on V_α — hence $\tilde{V}$ is $\mathbb{Z}_7$-equivariant on V_α and (trivially) on every V_β for $\beta \neq \alpha$. This is the converse of Schur's lemma applied to the isotypic component V_α. Hence $\tilde{V} \in U_{adm}(\mathcal{A}_{\mathbb{C}})$.

Theorem 5.2 (Channel-wise norm-only dependence). *Let $\mu : \mathcal{A}_{\mathbb{C}} \to [0, \infty)$ satisfy the minimal-core axioms (A1), (A2), (A3), (A5) of Definition 3.1. Then for each channel index α there exists a continuous function $F_\alpha : [0, \infty) \to [0, \infty)$ with $F_\alpha(0) = 0$ such that, for every $\psi \in \mathcal{A}_{\mathbb{C}}$ with channel decomposition $\psi = \sum_\alpha \psi_\alpha$,*

$$\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_{\mathbb{C}}).$$

Proof. Fix α. By Lemma 5.1, $U(V_\alpha)$ (extended by identity, hence in $U_{adm}(\mathcal{A}_{\mathbb{C}})$) acts transitively on each within-channel sphere $S_\alpha(r)$ for $r > 0$. By axiom (A3), $\mu(U\psi_\alpha) = \mu(\psi_\alpha)$ for every $U \in U_{adm}(\mathcal{A}_{\mathbb{C}})$. For $\psi_\alpha, \chi_\alpha \in S_\alpha(r)$, Lemma 5.1 supplies $U \in U_{adm}$ with $U\psi_\alpha = \chi_\alpha$ (where ψ_α and χ_α here denote the full $\mathcal{A}_{\mathbb{C}}$-vectors with the V_α-component as specified and zero on V_β for $\beta \neq \alpha$). Hence $\mu(\chi_\alpha) = \mu(U\psi_\alpha) = \mu(\psi_\alpha)$. So μ is constant on each within-channel sphere $S_\alpha(r)$, regarded as a subset of $\mathcal{A}_{\mathbb{C}}$ via the inclusion $V_\alpha \hookrightarrow \mathcal{A}_{\mathbb{C}}$.

Define $F_\alpha(r) := \mu(\psi_\alpha)$ for any $\psi_\alpha \in V_\alpha$ with $\|\psi_\alpha\|_\mathbb{C} = r$ and zero V_β-components for $\beta \neq \alpha$, with $F_\alpha(0) := \mu(0) = 0$ by (A1). Continuity of F_α follows from continuity of μ (A5) and continuity of $\|\cdot\|_\mathbb{C}$ on V_α.

For a general $\psi = \sum_\alpha \psi_\alpha$ with non-trivial decomposition: distinct V_α are complex-orthogonal (§2.3), so ψ_α and ψ_β are complex-orthogonal for $\alpha \neq \beta$. Applying (A2) iteratively across the orthogonal decomposition:

$$\mu(\psi) = \mu(\sum_\alpha \psi_\alpha) = \sum_\alpha \mu(\psi_\alpha) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_\mathbb{C}).$$

The full U_{adm} action is consistent with this formula: for $U = \prod_\alpha U_\alpha \in U_{adm}(\mathcal{A}_\mathbb{C})$ acting on $\psi = \sum_\alpha \psi_\alpha$ by $U\psi = \sum_\alpha U_\alpha \psi_\alpha$, with each U_α unitary on V_α and hence preserving the within-channel norm $\|U_\alpha \psi_\alpha\|_\mathbb{C} = \|\psi_\alpha\|_\mathbb{C}$, we have

$$\mu(U\psi) = \sum_\alpha F_\alpha(\|U_\alpha \psi_\alpha\|_\mathbb{C}) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_\mathbb{C}) = \mu(\psi),$$

confirming that axiom (A3) is fully respected — not just within-channel actions one at a time, but their simultaneous combination across all channels.

Remark 5.2.1 (What Theorem 5.2 does and does not give). Theorem 5.2 establishes channel-wise norm-only dependence: μ on $\mathcal{A}_\mathbb{C}$ is a sum of per-channel functions, each depending only on the within-channel norm. It does *not* establish full norm-only dependence $\mu(\psi) = F(\|\psi\|_\mathbb{C})$. The per-channel functions F_α may, at this stage of the argument, be structurally different functions. Forcing them all to be equal — i.e. forcing the C_α constants of Theorem 6.1 below to be uniform — requires a separate cross-channel argument (§6.2).

Remark 5.2.2 (Comparison with the previous version of this argument). Earlier versions of this paper (and informal treatments in the broader VERSF corpus) asserted full norm-only dependence $\mu(\psi) = F(\|\psi\|_\mathbb{C})$ directly from unitary invariance, by invoking transitivity of the full unitary group $U(\mathcal{A}_\mathbb{C})$ on the full norm-sphere S_1. This argument has a structural gap: $U(\mathcal{A}_\mathbb{C})$ is strictly larger than $U_{adm}(\mathcal{A}_\mathbb{C}) = \prod_\alpha U(V_\alpha)$, and (A3) only gives invariance under U_{adm}. Cross-channel unitaries (those mixing inequivalent $\mathbb{Z}_7$-irreps) are not admissible and so not directly available to bridge norm-spheres in distinct channels. The corrected argument given here proceeds channel-by-channel and bridges across channels through a separate structural argument (§6.2), making the role of cross-channel invariance explicit rather than implicit.

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6. The Quadratic Form Within Each Channel, and Cross-Channel Bridging

We close the chain in two steps. §6.1 establishes the quadratic form within each channel via the additive Cauchy equation. §6.2 supplies three structurally distinct arguments forcing the cross-channel quadratic coefficients to be equal, any one of which closes the uniqueness argument.

6.1 Per-Channel Quadratic Measure

Theorem 6.1 (Per-channel quadratic operational measure). *Let $\mu : \mathcal{A}_\mathbb{C} \to [0, \infty)$ satisfy the minimal-core axioms (A1), (A2), (A3), (A5). Then:*

(i) For each channel α with $d_\alpha \geq 2$, there exists a constant $C_\alpha \geq 0$ such that the per-channel function F_α of Theorem 5.2 satisfies $F_\alpha(r) = C_\alpha r^2$ for all $r \geq 0$.

(ii) For channels with $d_\alpha = 1$, no complex-orthogonal pair exists within V_α, so the within-channel Cauchy-equation argument of part (i) is vacuous. The minimal core leaves F_α undetermined beyond U(1)-phase invariance (A3), continuity (A5), positivity (A1), and $F_\alpha(0) = 0$; F_α may at this stage be any continuous non-negative function on $[0, \infty)$ vanishing at the origin. The quadratic form $F_\alpha(r) = C_\alpha r^2$ for $d_\alpha = 1$ channels is supplied entirely by the cross-channel bridging argument of §6.2.

Proof of (i). Fix a channel α with $d_\alpha \geq 2$. Take any $x, y \geq 0$ and choose complex-orthogonal vectors $\psi_{\alpha,1}, \psi_{\alpha,2} \in V_\alpha$ with $\|\psi_{\alpha,1}\|_\mathbb{C} = x$ and $\|\psi_{\alpha,2}\|_\mathbb{C} = y$. Such vectors exist because $d_\alpha \geq 2$ (pick orthonormal vectors $e_1, e_2 \in V_\alpha$ and set $\psi_{\alpha,1} := x \cdot e_1$, $\psi_{\alpha,2} := y \cdot e_2$). By complex Pythagoras (Remark in §4),

$$\|\psi_{\alpha,1} + \psi_{\alpha,2}\|^2_\mathbb{C} = \langle \psi_{\alpha,1} + \psi_{\alpha,2}, \psi_{\alpha,1} + \psi_{\alpha,2}\rangle_\mathbb{C} = \langle \psi_{\alpha,1}, \psi_{\alpha,1}\rangle_\mathbb{C} + \langle \psi_{\alpha,2}, \psi_{\alpha,2}\rangle_\mathbb{C} = x^2 + y^2,$$

with cross-terms vanishing by complex orthogonality.

By axiom (A2) applied to the complex-orthogonal sum $\psi_{\alpha,1} + \psi_{\alpha,2} \in V_\alpha$, and using Theorem 5.2's F_α (which on inputs supported only in V_α takes the form $F_\alpha(\|\psi_\alpha\|_\mathbb{C})$),

$$F_\alpha(\sqrt{x^2 + y^2}) = F_\alpha(x) + F_\alpha(y) \text{ for all } x, y \geq 0. \quad (*)$$

Define $G_\alpha : [0, \infty) \to [0, \infty)$ by

$$G_\alpha(t) := F_\alpha(\sqrt{t}).$$

Substituting $x = \sqrt{s}$, $y = \sqrt{t}$ into $(*)$ gives, for all $s, t \geq 0$,

$$G_\alpha(s + t) = F_\alpha(\sqrt{s + t}) = F_\alpha(\sqrt{(\sqrt{s})^2 + (\sqrt{t})^2}) = F_\alpha(\sqrt{s}) + F_\alpha(\sqrt{t}) = G_\alpha(s) + G_\alpha(t).$$

Hence G_α satisfies the additive Cauchy equation on $[0, \infty)$.

G_α is continuous: F_α is continuous (Theorem 5.2) and $\sqrt{\cdot}$ is continuous on $[0, \infty)$. By Kuczma's theorem on continuous solutions of Cauchy's functional equation on $[0, \infty)$ (Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Chapter XIII), every continuous additive $G_\alpha : [0, \infty) \to \mathbb{R}$ has the form $G_\alpha(t) = C_\alpha t$ for some $C_\alpha \in \mathbb{R}$. Positivity (A1) gives $F_\alpha \geq 0$, hence $G_\alpha \geq 0$, hence $C_\alpha \geq 0$. Therefore $F_\alpha(r) = G_\alpha(r^2) = C_\alpha r^2$.

Proof of (ii). For $d_\alpha = 1$, $V_\alpha \cong \mathbb{C}$ as a complex Hilbert space. There are no non-zero complex-orthogonal pairs within V_α (any non-zero $\psi_{\alpha,1}, \psi_{\alpha,2} \in V_\alpha \cong \mathbb{C}$ are complex scalar multiples of each other, hence $\langle \psi_{\alpha,1}, \psi_{\alpha,2} \rangle_\mathbb{C} \neq 0$). Within V_α alone, axiom (A2) imposes no constraint on F_α — there are no within-channel orthogonal sums to which (A2) applies. Axiom (A3) (U(1) acting by phase) makes F_α a function of $|z|$ only; axiom (A5) supplies continuity; positivity (A1) gives $F_\alpha \geq 0$; $F_\alpha(0) = 0$ by (A1) at the origin. These leave F_α as an arbitrary continuous non-negative function on $[0, \infty)$ vanishing at zero.

The quadratic form $F_\alpha(r) = C_\alpha r^2$ is supplied entirely by the cross-channel bridging step of §6.2 — directly under (B1) (which forces the quadratic form on each channel through the $\mathbb{Z}_7$-character-multiplication argument, operating on character labels rather than within-channel dimensions) or directly under (B3) (which elevates $U(V_\alpha) \times U(V_\beta)$ to the full $U(V_\alpha \oplus V_\beta)$ on multi-channel subspaces, making the Cauchy-equation argument applicable across the $(d_\alpha + d_\beta)$-dimensional subspace $V_\alpha \oplus V_\beta$ with the quadratic constant equal across the union). (B2)'s direct form is vacuous for $d_\alpha = 1$ unless supplemented by another route, since it requires a $d_\beta \geq 2$ channel to embed into; see Remark 6.2.1 for details.

Remark 6.1.1 (Where continuity is essential). Cauchy's functional equation has non-continuous solutions on $[0, \infty)$ (uncountably many pathological "Hamel basis" solutions assuming the axiom of choice). The continuous solutions are exactly $G_\alpha(t) = C_\alpha t$. Without continuity (axiom A5), the orthogonal-additivity argument would only force G_α to be additive — admitting pathological non-linear additive G_α, hence pathological F_α. The finite-packing structure of OG (Theorem 10.1), which supplies axiom (A5), is therefore structurally essential to the uniqueness argument: it excludes the substrate-discontinuous pathological measures that the bare additivity-plus-norm-dependence argument cannot exclude. Standard references for the Cauchy equation typically state the result on $\mathbb{R}$ (e.g. Aczél); the $[0, \infty)$ version with continuity is treated explicitly in Kuczma, op. cit.

Remark 6.1.2 (Projection consistency is automatic). With per-channel quadratic measures $F_\alpha(r) = C_\alpha r^2$ and the channel decomposition $\mu(\psi) = \sum_\alpha C_\alpha \|\psi_\alpha\|^2_\mathbb{C}$, projection consistency (A4) follows automatically. Let P be an orthogonal projection. For each channel α, $\|(P\psi)_\alpha\|_\mathbb{C} \leq \|\psi_\alpha\|_\mathbb{C}$ (orthogonal projection is non-expansive, and the decomposition into channels is itself orthogonal). Hence $\mu(P\psi) = \sum_\alpha C_\alpha \|(P\psi)_\alpha\|^2_\mathbb{C} \leq \sum_\alpha C_\alpha \|\psi_\alpha\|^2_\mathbb{C} = \mu(\psi)$, with equality iff $(P\psi)_\alpha = \psi_\alpha$ for every α, equivalently $P\psi = \psi$. So (A4) is satisfied without separate imposition — it is a *structural-consequence* axiom, not part of the minimal core. The same observation applies to (A8) (full-unitary measure invariance), which follows from $\|U\psi\|^2_\mathbb{C} = \|\psi\|^2_\mathbb{C}$ once the cross-channel bridging step forces $\mu(\psi) = C\|\psi\|^2_\mathbb{C}$.

6.2 Cross-Channel Bridging: Three Structurally Distinct Arguments

We now establish that the per-channel quadratic coefficients C_α are equal across α. Before presenting the three bridging routes that force this uniformity, we first establish *why* a bridging principle is needed at all.

Lemma 6.2A (Minimal-core insufficiency). *The minimal-core axioms (A1), (A2), (A3), (A5), (A7) alone do not force a unique Born measure.*

Proof. By Theorem 5.2, the minimal core implies

$\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_\mathbb{C})$.

For all channels with $d_\alpha \geq 2$, Theorem 6.1(i) gives $F_\alpha(r) = C_\alpha r^2$. For channels with $d_\alpha = 1$, Theorem 6.1(ii) leaves F_α as an arbitrary continuous non-negative function vanishing at zero. However, admissible unitary invariance (A3) acts only inside each $\mathbb{Z}_7$-channel (via the within-channel subgroup $U(V_\alpha)$) and does not mix inequivalent channels (URHG §2.5, the Schur-block-diagonal structure of admissible transport). Therefore no axiom in the minimal core forces $C_\alpha = C_\beta$ for $\alpha \neq \beta$.

Hence the family

$\mu_{(C_\alpha)}(\psi) := \sum_\alpha C_\alpha \|\psi_\alpha\|^2_\mathbb{C}$, $C_\alpha \geq 0$ arbitrary,

satisfies positivity (A1), orthogonal additivity (A2), admissible unitary invariance (A3), and continuity (A5) for *every* assignment of non-negative constants $(C_\alpha)_\alpha$ to the channels. (For $d_\alpha = 1$ channels, the within-channel function $F_\alpha(r) = C_\alpha r^2$ is consistent with the minimal core; the minimal core does not exclude it but also does not force it. Allowing more general F_α further widens the family of admissible measures consistent with the minimal core.) Single-reference normalization (A7) — there exists ψ_0 with $\mu(\psi_0) = 1$, $\|\psi_0\|_\mathbb{C} = 1$ — fixes only the channel constant in the channel containing ψ_0 (or one global scaling when the constants are already uniform); it does not by itself force uniformity across channels.

Therefore an additional cross-channel bridging condition is *necessary* to force $C_\alpha = C$ for all α and to recover the unique quadratic measure $\mu(\psi) = \|\psi\|^2_\mathbb{C}$.

Remark 6.2A.1 (Significance of Lemma 6.2A). The lemma makes explicit what the bridging conditions (B1), (B2), (B3) of §6.2 are doing: they fill a genuine structural gap left by the minimal core. The previous version of this paper (and informal treatments in the broader VERSF corpus) sometimes presented admissible unitary invariance as sufficient to force the Born rule on its own — Lemma 6.2A shows this is false. The cross-channel bridging is doing real structural work, not merely consistency-checking. This is why the paper carries three distinct bridging conditions: each fills the gap in a structurally distinct way, supplying robustness against challenges to any single bridging principle.

Remark 6.2A.2 (Physical intuition: why non-uniform C_α would be physically ugly). Lemma 6.2A proves mathematically that the minimal core permits non-uniform constants $(C_\alpha)_\alpha$. Physically, what would such a non-uniform measure look like? It would mean that distinct substrate channels carry *different intrinsic probability weightings* despite being governed by the same admissibility geometry. The same admissible Hilbert structure (with the same operational distinguishability content, the same finite packing, the same unitary transport) would attach different probability weights to states in different $\mathbb{Z}_7$-channels — a kind of substrate-level "channel parochialism" in the probability assignment.

Such a theory would preserve channel-wise quadraticity (Theorem 6.1) — within each channel, probabilities would still go as the squared norm — but it would break the *universality* of probability assignment across the operational carrier. Two operationally indistinguishable preparations differing only in which $\mathbb{Z}_7$-channel they occupy would receive different probability weights. Worse, the channel-specific constants C_α would be free parameters of the theory, not determined by anything in the operational structure — a hidden parameterization of probability by channel index that the admissibility geometry does not constrain.

The bridging principles of §6.2 exclude precisely this possibility. They assert, in three operationally distinct formulations, that probability assignment cannot depend on substrate-specific channel-identification data: it must depend only on the abstract Hilbert-geometric content of the state. This is what makes the Born rule *universal* in the VERSF setting — not merely channel-wise quadratic but cross-channel uniform — and it is what the cross-channel bridging principle delivers that the minimal core does not.

We now state Theorem 6.2 with three structural arguments that close this gap.

Theorem 6.2 (Cross-channel equality of quadratic coefficients). *Under the minimal-core axioms (A1)–(A3) + (A5) plus any one of the three structural conditions (B1), (B2), (B3) of Definition 3.1, the per-channel quadratic coefficients C_α of Theorem 6.1 satisfy $C_\alpha = C$ for a single constant $C \geq 0$.*

Proof via (B1) (compositional multiplicativity with character matching). Take $\psi \in V_\alpha$ and $\phi \in V_\beta$. Without loss of generality, take both factors of unit norm — the relation $C_{\alpha+\beta \bmod 7} = C_\alpha \cdot C_\beta$ derived below extends to arbitrary norms by the quadratic form of Theorem 6.1. With $\|\psi\|_\mathbb{C} = \|\phi\|_\mathbb{C} = 1$, *the tensor-product extension carrier $\mathcal{A}_\mathbb{C} \otimes \mathcal{A}_\mathbb{C}$ inherits an admissible Hilbert structure with $\|\psi \otimes \phi\|^2_\mathbb{C} = 1$. The diagonal $\mathbb{Z}_7$-action on $V_\alpha \otimes V_\beta$ is by the product character $\chi_\alpha \cdot \chi_\beta = \chi_{\alpha+\beta \bmod 7}$, so $V_\alpha \otimes V_\beta$ is an isotypic component for character* $\chi_{\alpha+\beta \bmod 7}$.

Step 1 (Character-matching consistency). The operational measure on the tensor extension is compatible with the operational measure on the original carrier under character matching: when $\alpha + \beta \not\equiv 0 \bmod 7$, the quadratic constant on $V_\alpha \otimes V_\beta$ in the tensor carrier equals $C_{\alpha+\beta \bmod 7}$ of the original carrier. This expresses the structural principle that probability measures on $\mathbb{Z}_7$-isotypic components are determined by their character, not by which carrier they originate in — an operational requirement built into (B1)'s formulation.

Substrate-level justification of character matching. The $\mathbb{Z}_7$-character of an isotypic component is the *only* substrate-level label that survives the operational identification of isotypic components across carriers (URHG §2.5 and Definition 3.2's $\mathbb{Z}_7$-equivariance). The dimension d_α of V_α is a multiplicity datum specific to a particular carrier; the character χ_α is intrinsic to the substrate $\mathbb{Z}_7$-action. Any non-character-dependent constant on isotypic components would constitute substrate-level data not reducible to the character lattice — equivalent to introducing a hidden carrier-specific parameter into the operational measure, which violates the substrate-only principle that admissibility is defined by the character structure. Character matching therefore follows from the substrate-only principle: operational measure depends only on substrate-derived structure, and the character is the only such structure surviving inter-carrier identification.

Under character matching, for $\alpha + \beta \not\equiv 0 \bmod 7$:

$\mu(\psi \otimes \phi) = C_{\alpha+\beta \bmod 7} \cdot \|\psi \otimes \phi\|^2_{\mathbb{C}} = C_{\alpha+\beta \bmod 7}$.

By (B1) plus the per-channel quadratic form on unit-norm states from Theorem 6.1:

$\mu(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi) = C_\alpha \cdot C_\beta$.

Combining:

$C_{\alpha+\beta \bmod 7} = C_\alpha \cdot C_\beta$ for all $\alpha, \beta \in \{1, ..., 6\}$ with $\alpha + \beta \not\equiv 0 \bmod 7$. (♣)

Step 2 (Multiplicative-homomorphism argument). Relation (♣) makes the assignment $\alpha \mapsto C_\alpha$ a partial multiplicative homomorphism on the non-trivial elements of the cyclic group ($\mathbb{Z}_7$, +) into ($\mathbb{R}_{>0}$, ×). Extending naturally by $C_0 := 1$ (consistent with multiplicativity when $\alpha + \beta = 0$ is permitted), we get a full group homomorphism $C : \mathbb{Z}_7 \to \mathbb{R}_{>0}$, equivalently $\log C : \mathbb{Z}_7 \to \mathbb{R}$ a homomorphism. Since $\mathbb{Z}_7$ is finite cyclic of order 7 and $\mathbb{R}$ is torsion-free, the only homomorphism is the zero map — hence $C_\alpha = 1$ for all α.

Step 3 (Explicit verification). From (♣) with $\alpha = \beta = 1$: $C_2 = C_1^2$. Iterating with $\beta = 1$, each intermediate partial sum $k \in \{2, 3, 4, 5, 6\}$ is non-zero mod 7, so (♣) applies at every step and yields $C_k = C_1^k$ for $k = 1, ..., 6$. (The chain stops before $k = 7 \equiv 0$, where (♣) would no longer apply.)

The seventh-power closure follows from any pair $(\alpha, \beta) \in \{1, ..., 6\}^2$ with $\alpha + \beta \equiv 1 \bmod 7$. Concretely, taking $(\alpha, \beta) = (4, 4)$ in (♣): since $(4 + 4) \bmod 7 = 1$, the relation gives $C_1 = C_4 \cdot C_4 = C_4^2$. But from the chain above, $C_4 = C_1^4$, so $C_4^2 = C_1^8$. Hence $C_1 = C_1^8$, i.e. $C_1^7 = 1$. The unique positive real seventh root of 1 is 1, so $C_1 = 1$; by the chain, $C_\alpha = C_1^\alpha = 1$ for all non-trivial α.

Step 4 (Scale fixing via weakened (A7)). With all $C_\alpha = 1$, the measure takes the form $\mu(\psi) = \sum_\alpha \|\psi_\alpha\|^2_{\mathbb{C}} = \|\psi\|^2_{\mathbb{C}}$. The weakened normalization (A7) — there exists ψ_0 with $\|\psi_0\|_{\mathbb{C}} = 1$ and $\mu(\psi_0) = 1$ — is automatically consistent: $\|\psi_0\|^2_{\mathbb{C}} = 1 = \mu(\psi_0)$. The (B1) bridging therefore supplies cross-channel uniformity through the $\mathbb{Z}_7$-character-multiplication structure, independently of (A7)'s role as scale-fixer.

Proof via (B2) (sub-channel isosymmetric embedding). Consider channels V_α and V_β with $d_\beta \geq d_\alpha$ (relabel if necessary, with both ≥ 2; the $d = 1$ case is handled below). Let $\psi_\alpha \in V_\alpha$ with $\|\psi_\alpha\|_{\mathbb{C}} = r$. By (B2) there exists a complex-unitary embedding $\iota : V_\alpha \to V_\beta$. The image $\iota(\psi_\alpha)$ lies in a d_α-dimensional complex Hilbert subspace of V_β with $\|\iota(\psi_\alpha)\|_{\mathbb{C}} = r$. By (B2), $\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha) = C_\alpha r^2$. But by Theorem 6.1 applied within V_β (using only orthogonal pairs lying in the image $\iota(V_\alpha) \subseteq V_\beta$, which exist because $d_\alpha \geq 2$ and complex-orthogonal pairs within $\iota(V_\alpha)$ are also complex-orthogonal as elements of V_β), $\mu(\iota(\psi_\alpha)) = C_\beta r^2$. Hence $C_\alpha = C_\beta$.

For $d_\alpha = 1$: embed V_α into any $d_\beta \geq 2$ channel V_β by (B2), and apply Theorem 6.1 to the image (which now sits in a 2+-dim ambient). The image inherits the quadratic constant C_β via Theorem 6.1's argument applied within V_β (where (A2) is non-vacuous because $d_\beta \geq 2$). Hence $F_\alpha(r) = C_\beta r^2$ for the $d_\alpha = 1$ channel, and $C_\alpha = C_\beta$.

The symmetry of the argument gives $C_\alpha = C_\beta$ in all cases.

Caveat for the minimal URHG case. If all $d_\alpha = 1$ (minimal multiplicity, i.e. each non-trivial $\mathbb{Z}_7$-character occurs with multiplicity exactly one), no $d_\beta \geq 2$ channel exists to embed into and (B2)'s direct form becomes vacuous — see Remark 6.2.1. In that case (B2) must be supplemented by (B3) (which accesses $U(V_\alpha \oplus V_\beta)$ on 2-dim multi-channel subspaces) or replaced by (B1) (which operates on character labels and is dimension-blind).

Proof via (B3) (refinement-stability). The channel decomposition $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$ is one admissible orthogonal decomposition. Consider a unitary rotation W of $\mathcal{A}_\mathbb{C}$ that mixes two channels V_α and V_β of the *same* total dimension into a new decomposition $V_{\alpha'} \oplus V_{\beta'}$ of the combined subspace $V_\alpha \oplus V_\beta$. By (B3), $\mu(W\psi) = \mu(\psi)$ for all ψ.

Take $\psi = \psi_\alpha \in V_\alpha$ with $\|\psi_\alpha\|_\mathbb{C} = r$ and zero V_β-component. Then $W\psi$ has, in the new decomposition $(V_{\alpha'}, V_{\beta'})$, components $(W\psi)_{\alpha'}$ and $(W\psi)_{\beta'}$ with $\|(W\psi)_{\alpha'}\|^2_\mathbb{C} + \|(W\psi)_{\beta'}\|^2_\mathbb{C} = r^2$ (norm preservation by W). For W not preserving the channel decomposition, both new-channel components are non-zero. The measure under the new decomposition is

$$\mu(W\psi) = C_{\alpha'} \|(W\psi)_{\alpha'}\|^2_\mathbb{C} + C_{\beta'} \|(W\psi)_{\beta'}\|^2_\mathbb{C}.$$

This must equal $\mu(\psi) = C_\alpha r^2$ for all choices of W mixing V_α and V_β. To make the parametrization explicit: for $d_\alpha = d_\beta = 1$ (the minimal case where each channel is one-dimensional), pick unit-norm vectors $e_\alpha \in V_\alpha$ and $e_\beta \in V_\beta$, and consider the one-parameter family of unitaries

$$W_\theta : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}, \quad W_\theta e_\alpha = \cos(\theta) \cdot e_{\alpha'} + \sin(\theta) \cdot e_{\beta'},$$

for $\theta \in [0, \pi/2]$, where $(e_{\alpha'}, e_{\beta'})$ is a relabeling of (e_α, e_β) viewed as basis vectors of the new decomposition $(V_{\alpha'}, V_{\beta'})$. Applied to $\psi = r \cdot e_\alpha \in V_\alpha$:

$$\|(W_\theta \psi)_{\alpha'}\|^2_\mathbb{C} = r^2 \cos^2\theta, \quad \|(W_\theta \psi)_{\beta'}\|^2_\mathbb{C} = r^2 \sin^2\theta.$$

The measure under the new decomposition is

$$\mu(W_\theta \psi) = C_{\alpha'} \cdot r^2 \cos^2\theta + C_{\beta'} \cdot r^2 \sin^2\theta.$$

By (B3), this must equal $\mu(\psi) = C_\alpha r^2$ for all $\theta \in [0, \pi/2]$:

$C_{\alpha'} \cos^2\theta + C_{\beta'} \sin^2\theta = C_\alpha$ for all $\theta \in [0, \pi/2]$.

The only way this holds for all θ (in particular at $\theta = 0, \pi/4, \pi/2$) is $C_{\alpha'} = C_{\beta'} = C_\alpha$. For higher d_α, d_β the argument generalizes via parametrized one-parameter families of $U(d_\alpha + d_\beta)$ unitaries; the conclusion is the same. By (B3), the new-channel constants $C_{\alpha'}$ and $C_{\beta'}$ are the per-channel constants of the original decomposition under a relabeling, hence the constants of all channels are forced equal: $C_\alpha = C_\beta = ... = C$.

Remark 6.2.1 (Structural distinctness of the three bridging conditions). Each of (B1), (B2), (B3) suffices individually to force cross-channel uniformity, but the three conditions are structurally distinct and have different scope. The three bridging routes do not have identical strength in every structural regime.

- **(B1)** is the *compositional* statement, supplying uniformity via the multiplicative-homomorphism structure on the $\mathbb{Z}_7$-character lattice. It requires character-matching consistency between the tensor extension and the original carrier as part of its operational content. (B1) is robust in the minimal URHG case where each non-trivial $\mathbb{Z}_7$-character may occur with multiplicity one, because the $\mathbb{Z}_7$-multiplicative-homomorphism argument operates on character labels rather than on within-channel dimensions.
- **(B2)** is the *isosymmetric* statement: the measure of a state is intrinsic to its abstract Hilbert-geometric content, not to the channel-realization. (B2) is weaker in the minimal URHG case, because a one-dimensional channel contains no non-trivial complex-orthogonal pairs and cannot by itself generate the Cauchy-equation argument; (B2) requires the existence of a $d_\beta \geq 2$ channel to embed into. Therefore (B2) should be viewed as a valid bridging principle only when sufficient higher-dimensional channel multiplicity exists, or when supplemented by (B3) to access multi-channel subspaces of dimension ≥ 2.
- **(B3)** is the *refinement-stability* statement: the measure does not depend on the particular orthogonal decomposition used to compute it. (B3) is robust in the minimal URHG case because it elevates the within-channel $U(V_\alpha) \times U(V_\beta)$ to the full $U(V_\alpha \oplus V_\beta)$ on multi-channel subspaces, making the Cauchy-equation argument applicable to $(d_\alpha + d_\beta)$-dimensional subspaces even when individual channels are 1-dimensional.

Summary of structural regimes. In the minimal one-dimensional-channel regime (each non-trivial $\mathbb{Z}_7$-character occurring with multiplicity one), (B1) or (B3) should be regarded as the primary bridging route, with (B2) requiring supplementation. In higher-multiplicity regimes where some $d_\alpha \geq 2$, all three routes are directly available.

All three are natural operational properties of a probability measure, and the operational structure of admissibility supports each of them. The framework therefore has *three structurally distinct justifications* for the cross-channel bridging — robust against challenges to any single condition. The asymmetry of scope between them (particularly the comparative strength of (B1) and (B3) in the minimal-multiplicity case) is genuine and worth flagging: a clean substrate-level derivation of bridging might prefer (B1) or (B3) as the most direct routes, with (B2) as an alternative formulation requiring auxiliary support in the minimal regime.

Remark 6.2.2 (Status of the bridging condition). The cross-channel bridging condition is doing *real* structural work — it cannot be derived from the minimal core alone, and earlier versions of this paper quietly papered over this gap. The honest formulation: admissible unitary invariance is genuinely weaker than full-unitary invariance, and a separate structural input is required to bridge across $\mathbb{Z}_7$-channels. The three operational principles (B1, B2, B3) are the natural candidates; the substrate-level question of which is most primitive (or whether they all reduce to one) is Open Problem 2 of §13.

A further structural observation worth flagging: the three bridging arguments should be regarded as structurally distinct *operational formulations*, not necessarily as three fundamentally independent substrate principles. The coexistence of three successful bridging routes — compositional multiplicativity, isosymmetric embedding, refinement-stability — may indicate that they are different operational manifestations of a deeper substrate-level invariance not yet isolated explicitly. Candidate: a single "substrate-character-uniformity principle" stating that the operational measure on isotypic components depends only on the $\mathbb{Z}_7$-character label and not on carrier-specific multiplicity data — from which (B1), (B2), (B3) would all be derived consequences. Whether such a unifying principle exists, and what its substrate-level form would be, is the deeper structural question behind Open Problem 2. The existence of three convergent routes is therefore not a coincidence to be defended but a clue that the framework's bridging step has a more fundamental origin awaiting articulation.

6.3 Normalization

Theorem 6.3 (Normalization fixes C = 1). *Under the single-reference normalization axiom (A7), the constant C of Theorem 6.2 satisfies C = 1, so* $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$.

Proof. By Theorems 6.1–6.2 with all $C_\alpha = C$, $\mu(\psi) = C\,\|\psi\|^2_{\mathbb{C}}$. The single-reference normalization (A7) supplies one ψ_0 with $\|\psi_0\|_{\mathbb{C}} = 1$ and $\mu(\psi_0) = 1$, giving $C \cdot 1 = 1$, hence $C = 1$. The full unit-sphere statement $\mu(\psi) = 1$ for every $\psi \in S_1$ is then derived (not assumed): for any $\psi \in S_1$, $\mu(\psi) = C\,\|\psi\|^2_{\mathbb{C}} = 1 \cdot 1 = 1$.

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7. Born-Rule Recovery

The quadratic operational measure of Theorems 6.1–6.3 yields the Born rule as an immediate consequence.

Theorem 7.1 (Born-rule recovery). *Let* $\phi \in \mathcal{A}_{\mathbb{C}}$ *be a normalized admissible outcome state* ($\|\phi\|_{\mathbb{C}} = 1$)*, and let* $\psi \in \mathcal{A}_{\mathbb{C}}$ *be a normalized admissible input state* ($\|\psi\|_{\mathbb{C}} = 1$)*. The operational transition probability from* ψ *to* ϕ *is*

$$P(\psi \to \phi) = |\langle \phi, \psi \rangle_{\mathbb{C}}|^2.$$

Proof. The transition from ψ to ϕ is the operational refinement of ψ onto the one-dimensional admissible subspace span(ϕ) $\subseteq \mathcal{A}_\mathbb{C}$. By URHG Theorem 7.1 (admissibility-as-projection), this refinement is implemented by the orthogonal projection P_ϕ onto span(ϕ):

$$P_\phi \psi = \langle \phi, \psi \rangle_\mathbb{C} \cdot \phi.$$

The transition probability is the operational measure of the post-projection state:

$$P(\psi \to \phi) = \mu(P_\phi \psi).$$

By Theorems 6.1–6.3 ($\mu(\chi) = \|\chi\|^2_\mathbb{C}$) and the absolute homogeneity of the complex norm under complex scalars ($\|\lambda\psi\|_\mathbb{C} = |\lambda| \cdot \|\psi\|_\mathbb{C}$ for $\lambda \in \mathbb{C}$):

$$\mu(P_\phi \psi) = \|P_\phi \psi\|^2_\mathbb{C} = \|\langle \phi, \psi \rangle_\mathbb{C} \cdot \phi\|^2_\mathbb{C} = |\langle \phi, \psi \rangle_\mathbb{C}|^2 \cdot \|\phi\|^2_\mathbb{C} = |\langle \phi, \psi \rangle_\mathbb{C}|^2,$$

using $\|\phi\|_\mathbb{C} = 1$ in the last step. The complex absolute homogeneity used at the third equality is where the complex structure derived in URHG Theorem 3.5 explicitly enters the Born formula — if the carrier were a real Hilbert space (or a quaternionic Hilbert space), the analogous formula would take a different form.

Remark 7.1.1 (Sum rule). For any complete orthonormal set of outcomes $\{\phi_i\}$ (i.e. an admissible measurement basis), the sum rule

$$\sum_i P(\psi \to \phi_i) = \sum_i |\langle \phi_i, \psi \rangle_\mathbb{C}|^2 = \|\psi\|^2_\mathbb{C} = 1$$

follows from Parseval's identity on ($\mathcal{A}_\mathbb{C}$, $\langle \cdot,\cdot \rangle_\mathbb{C}$) and the normalization $\|\psi\|_\mathbb{C} = 1$.

Remark 7.1.2 (Comparison with Gleason: a genuine dimensional advantage, honestly purchased). Gleason's theorem (1957) derives the Born rule from frame functions on the projection lattice of a complex Hilbert space of dimension ≥ 3. The dimensional restriction $d \geq 3$ is essential to Gleason's argument: in $d = 2$, frame functions on the projection lattice are not determined by their projection-lattice values alone, and Gleason's uniqueness fails. The present operational derivation, by contrast, works whenever $d_{op}^\mathbb{C} \geq 2$ (the dimensional requirement of Theorem 6.1 part (i)). The reason is structurally significant: axiom (A2) — orthogonal additivity *on vectors* — is strictly stronger than frame additivity *on projections*, because (A2) constrains the measure on every complex-orthogonal vector pair, whereas frame additivity constrains it only at the projection level. The extra constraint closes the $d = 2$ case that Gleason cannot.

The dimensional advantage does not come for free: it is purchased by the stronger structural assumption of vector-level orthogonal additivity rather than projection-frame additivity alone. This is an honest acknowledgment of where the advantage originates. Vector-level additivity is itself a non-trivial axiom — it asserts that probability content adds across complex-orthogonal vector pairs, which is a richer condition than frame additivity at the projection lattice. The substrate-level justification for vector-level additivity in the VERSF framework is given in Theorem 4.1: distinct orthogonal sectors correspond to operationally independent admissible distinguishability channels (Schur-block-diagonal under $\mathbb{Z}_7$-equivariance), so the measure must add across complex-orthogonal vector pairs by independence. This justification is structurally specific to the substrate; it is not available to a generic projection-lattice framework that lacks an underlying admissibility geometry. The present framework therefore beats Gleason in $d = 2$ not by getting something for nothing, but by exploiting the additional structural content the substrate supplies.

The $\mathbb{Z}_7$-architecture of VERSF supplies $d\_op^\wedge\mathbb{C} \geq 3$ in the physically realised case (URHG Theorem 3.5(i)), so the $d = 2$ advantage is not exploited in practice — but it demonstrates that the present axiomatic framework is strictly more constraining than Gleason's at the abstract Hilbert-space level, hence more structurally natural as a substrate-level derivation than a projection-lattice axiomatization would be.

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8. The Common Operational Measure Theorem

We now establish the synthesis result: the previously-distinct VERSF derivations of the Born rule are equivalent operational projections of the same underlying measure geometry.

Theorem 8.1 (Common Operational Measure Theorem). *Under the inherited operational structure of §2 (admissible Hilbert geometry, substrate-derived complex structure, orthogonal channel decomposition, projection structure, admissible unitary reversible transport, finite distinguishability packing), the following reconstruction routes are not independent axiom systems in strict biconditional equivalence. Rather, they are structurally distinct operational routes that converge on the same unique measure once their respective structural assumptions are imposed. In this precise sense, they share a common operational measure. The reconstruction routes are:*

(i) ***Axiomatic characterization (this paper).*** *μ satisfies the minimal-core axioms (A1)–(A3) + (A5) + (A7) of Definition 3.1 plus any one of the cross-channel bridging conditions (B1)–(B3).*

(ii) ***Gleason-type frame characterization (in $d\_op^\wedge\mathbb{C} \geq 3$).*** *$\mu$ extends to a frame function on the orthogonal-projection lattice of $\mathcal{A}\_\mathbb{C}$.*

(iii) ***Double Square pairwise characterization.*** *μ arises as the unique pairwise-correlation measure on admissible path pairs that is consistent with bilinear amplitude composition (a form of B1) and admissible reversible-transport invariance (a form of A3).*

(iv) ***Tensor-product bilinear characterization.*** *μ is the unique non-negative measure on the carrier satisfying the minimal core plus compositional multiplicativity (B1).*

(v) ***Entropic-unfolding characterization.*** *μ arises as the iso-entropic limit of a thermodynamic projection weighting whose pre-limit form $P\_i \propto |c\_i|^2 e^\wedge\{-\lambda\Delta S\_i\}$ already presupposes the quadratic operational measure $|c\_i|^2$ as the carrier of the projection weight; the iso-entropic limit (when $\Delta S\_i = \Delta S$, an operational isosymmetry of form B2) removes the entropic correction and recovers the bare quadratic form.*

(vi) ***TPB normalization characterization.*** *μ is the unique non-negative measure preserving normalization under admissible reversible transport for every complete orthogonal projection system, combined with the cross-channel bridging condition.*

(vii) ***Finite-packing stability characterization.*** *μ is the unique non-negative measure refinement-stable under packing decomposition: orthogonal-additive (A2) and invariant under refinement choice (B3), with continuity (A5).*

(viii) ***Isosymmetric-equivalence characterization.*** *μ is the unique non-negative measure invariant under sub-channel isosymmetric embeddings (B2), combined with within-channel admissible unitary invariance (A3) and continuity (A5).*

All eight routes converge on the same unique measure:

$\mu(\psi) = \|\psi\|^2\_\mathbb{C}$,

with the Born transition probability $P(\psi \to \phi) = |\langle\phi, \psi\rangle\_\mathbb{C}|^2$ *as a corollary (Theorem 7.1).*

Proof. We show (i) ⟹ $\mu(\psi) = \|\psi\|^2\_\mathbb{C}$ (Theorems 5.2, 6.1, 6.2, 6.3 combined), and that each of (ii)–(viii) is equivalent to (i) under the inherited operational structure.

(ii) ⟺ (i) (in d_op^ℂ ≥ 3): Gleason's theorem in finite dimension $d \geq 3$ states that every frame function on the orthogonal projection lattice of a complex Hilbert space of dimension ≥ 3 has the form $f(P) = \mathrm{tr}(\rho P)$ for some density operator ρ. Restricting to pure states ψ ($\rho = |\psi\rangle\langle\psi|$), this gives $f(P\_\phi) = |\langle\phi, \psi\rangle\_\mathbb{C}|^2$ — the Born expression. The frame-additivity condition for f is the projection-level analogue of orthogonal additivity (A2); frame functions are automatically basis-independent (covering A3 plus A8 jointly); the bridging across irrep classes is supplied by the connectedness of the projection lattice in $d \geq 3$, which plays the structural role of (B3). Hence (ii) is equivalent to (i) for d_op^ℂ ≥ 3. In d_op^ℂ = 2, (ii) is strictly weaker than (i) (as noted in Remark 7.1.2). Precise reference: Gleason (1957) *J. Math. Mech.* 6, 885; standard textbook treatments in Varadarajan, *Geometry of Quantum Theory*, Chapter VII.

(iii) ⟺ (i): Double Square pairwise geometry (developed in the *Born Rule Overdetermination* sequence of the VERSF corpus; see versf-eos.com for the dedicated paper on Double Square geometry) establishes that the unique pairwise-correlation measure on admissible path pairs satisfying pairwise-correlation symmetry and reversible-transport invariance is the Hermitian-bilinear form $\langle\phi, \psi\rangle\_\mathbb{C} \cdot \langle\psi, \phi\rangle\_\mathbb{C} = |\langle\phi, \psi\rangle\_\mathbb{C}|^2$. The structural content of the Double Square route — pairwise correlation symmetry across admissible path pairs — is operationally distinct from compositional multiplicativity (B1), although both terminate at the same quadratic measure. The precise axiom-by-axiom correspondence between the Double Square framework and the minimal core of this paper is in the dedicated paper. For the purposes of Theorem 8.1, we identify (iii) ⟺ (i) at the level of "both derive the same quadratic operational measure under operationally equivalent (though distinctly formulated) structural inputs," and refer to the dedicated paper for the precise structural mapping rather than forcing the correspondence into a specific bridging condition.

(iv) ⟺ (i): This is the content of bridging condition (B1) combined with the minimal core. So (iv) is essentially the (B1) form of (i), and the equivalence is the content of Theorem 6.2 via (B1).

(v) ⟺ (i): The thermodynamic entropic-unfolding programme writes admissible projection weightings in the form $P_i = (1/Z) \cdot |c_i|^2 \cdot e^{-\lambda\Delta S_i}$. The prefactor $|c_i|^2$ is *already* the bare quadratic operational measure — supplied by the same minimal-core arguments as in §§5–6 — and the exponential factor $e^{-\lambda\Delta S_i}$ is a thermodynamic correction. In the iso-entropic limit $\Delta S_i = \Delta S$ across outcomes, the exponential becomes a normalization constant and the measure reduces to $P_i = |c_i|^2$. The structural content: the entropic-unfolding programme *presupposes* the quadratic operational measure (in the $|c_i|^2$ prefactor); the iso-entropic limit removes the thermodynamic decoration. Route (v) is therefore not an independent derivation of the Born rule but a thermodynamic refinement of the same underlying derivation — its operational measure inputs are those of (i), with iso-entropy serving as a regime-selection condition (equal-cost outcomes) rather than as an additional structural axiom. The mapping is not to a specific bridging condition like (B2); it is structurally that (v) presupposes (i) and adds thermodynamic decoration. Detailed derivation: see the dedicated entropic-unfolding papers at versf-eos.com.

(vi) ⟺ (i): A non-negative measure μ preserves normalization $\sum_i \mu(P_i \psi) = \mu(\psi)$ under admissible reversible transport for every complete orthogonal projection system $\{P_i\}$ iff μ has the form $\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_{\mathbb{C}})$ (Theorem 5.2). The per-channel form is quadratic by the continuity argument of Theorem 6.1; cross-channel coefficients match by any of (B1)–(B3). Hence (vi) ⟺ (i).

(vii) ⟺ (i): Refinement-stability under packing decomposition is structurally (A2) plus (B3). Combined with continuity (A5) and within-channel admissible unitary invariance (A3), this is the input set of (i) with (B3) as the bridging condition. Hence (vii) ⟺ (i).

(viii) ⟺ (i): Sub-channel isosymmetric invariance is (B2), combined with (A3) and (A5). This is the input set of (i) with (B2) as the bridging condition. Hence (viii) ⟺ (i).

All eight routes converge on the unique measure $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$.

Remark 8.1.1 (What the Common Operational Measure Theorem says, and at what level). The eight characterizations (i)–(viii) are *not* eight independent derivations of one rule. They are eight operational projections of one underlying measure geometry: each axiom system supplies a partial structural characterization, and the inherited operational structure of §2 supplies the additional inputs needed to close each route to the same unique measure.

A clarification on the meaning of "operationally equivalent" here. In axiomatic mathematics, "equivalence" between axiom systems usually means biconditional implication between the axiomatic statements themselves. The equivalences asserted in Theorem 8.1 are weaker than that: they are *conclusion-identity equivalences under the inherited operational structure of §2*. That is, given the §2 structure (carrier, complex structure, channel decomposition, projection structure, unitary reversible transport, finite packing), each of (i)–(viii) is sufficient to derive $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$, and the resulting measure is the same across all routes. The axiom systems themselves are not all biconditional translations of one another — they highlight different operational principles (compositionality, isosymmetry, refinement-stability, frame-additivity, etc.) that, when combined with §2, all yield the same conclusion. Routes (iii) and (v) in particular refer to detailed derivations in the prior VERSF corpus (available at versf-eos.com); for these the conclusion-identity equivalence holds, but the precise axiom-level correspondence with (i) requires reading the dedicated papers. The Born rule is therefore not over-determined in the sense of "multiple proofs converging on the same answer"; it is one result with multiple operationally co-derivative formulations, each capturing a different aspect of the underlying geometric structure.

Remark 8.1.2 (Why this matters for reconstruction). Reconstruction programmes that derive the Born rule from any single axiom system (Gleason, Hardy, Deutsch–Wallace, Masanes–Müller) capture one operational projection of the measure geometry. The Common Operational Measure Theorem shows that, given the inherited operational structure of §2 (which the VERSF programme supplies from substrate primitives), all these projections converge on the same unique measure. The programme-wide implication: any reconstruction programme that establishes the inherited operational structure of §2 will recover the Born rule through any of the eight routes — the choice of route is a matter of perspective rather than derivational priority.

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9. Probability as Operational Distinguishability Volume

The finite packing structure of OG (Theorem 10.1) provides the geometric interpretation of probability. The relation between operational measure and operational volume is one of *definitional identification* under the operational interpretation: probability is normalized distinguishability volume, with the squared-norm form supplied by the uniqueness arguments of §§5–6.

Conditional status of the volume identification. Paralleling the conditional structure described in the Scope and Conditional Status section, the Born-rule-as-volume interpretation developed below is conditional on the OG identification $\mathrm{Vol\_op}(\chi) := \|\chi\|^2_{\mathbb{C}}$, which is at present a definitional convention within OG rather than a derived theorem. Open Problem 10 of §13 identifies the strengthening of this identification to a derived theorem as a future-work target. The mathematical content of §9 — that the Born transition probability equals a ratio of operational volumes — is therefore conditional on this convention; the structural uniqueness of the quadratic measure itself (§§5–6) does not depend on it.

Proposition 9.1 (Probability as normalized distinguishability volume). *Under the operational measure $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$ derived in §§5–6, and under the operational identification (definitional in OG §2.6 and §8) of the squared norm $\|\psi\|^2_{\mathbb{C}}$ with the operational distinguishability volume of the admissible state ψ, the Born transition probability $P(\psi \to \phi) = |\langle \phi, \psi \rangle_{\mathbb{C}}|^2$ is the ratio of operational distinguishability volumes:*

$$P(\psi \to \phi) = \mathrm{Vol\_op}(P\_\phi\, \psi) / \mathrm{Vol\_op}(\psi),$$

where $Vol\_op(\chi) := \|\chi\|^2\_\mathbb{C}$ is the operational distinguishability volume of admissible state χ.

Justification. Under the definitional identification $Vol\_op(\chi) := \|\chi\|^2\_\mathbb{C}$ (an operational-volume convention internal to OG: the operational volume of the distinguishability content concentrated in admissible state χ is taken to be the squared operational norm, consistent with the packing-volume proportionality of OG §2.6):

$$P(\psi \to \phi) = \mu(P\_\phi\ \psi) = \|P\_\phi\ \psi\|^2\_\mathbb{C} = Vol\_op(P\_\phi\ \psi).$$

For a normalized base state ψ ($\|\psi\|\_\mathbb{C} = 1$, $Vol\_op(\psi) = 1$), this equals $Vol\_op(P\_\phi\ \psi)$ / $Vol\_op(\psi)$.

Remark 9.1.1 (Status of this proposition). This is stated as a proposition rather than a theorem because its content is largely an interpretive identification of the squared-norm operational measure with the operational volume measure of OG. The structural work has already been done in §§5–6 (uniqueness of the quadratic measure) and in OG (identification of squared norm with operational distinguishability volume); §9 is the interpretive synthesis. Strengthening the OG identification $Vol\_op(\chi) = \|\chi\|^2\_\mathbb{C}$ to a derived theorem rather than a definitional convention is Open Problem 10 below.

Remark 9.1.2 (Interpretation). The Born rule is therefore not a subjective uncertainty, not a hidden-variable weighting, not an epistemic ignorance measure — it is *operational distinguishability volume on the admissible Hilbert geometry*. Probability measures the geometric content of admissible distinguishability concentrated in the outcome subspace, normalized by the total admissible content of the input state.

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10. Relation to Prior VERSF Routes

The present paper unifies the following prior VERSF derivational routes into one structure. The table maps each route to its operational interpretation and to the corresponding characterization (i)–(viii) of Theorem 8.1, with explicit identification of which structural axioms it supplies when the route does not cleanly match a single characterization.

Prior route	Operational interpretation	Equivalence in Theorem 8.1
Double Square Rule	Pairwise projection geometry on admissible path pairs	(iii) directly
Entropic Unfolding	Thermodynamic projection weighting in iso-entropic limit	(v) directly

Prior route	Operational interpretation	Equivalence in Theorem 8.1
Physical Necessity	Admissibility elimination under operational scrutiny	Supplies a structurally redundant collection of conditions — including (A3), continuity-related content of (A5), and bridging content overlapping with (B2) and (B3) — by ruling out non-Born measures under operational stress-testing. The full route is broader than any single axiom mapping; structurally it amounts to negative-form equivalence with (i) (ruling out alternatives rather than positively deriving the quadratic measure).
Structural Forcing	Constraint-intersection robustness	Supplies (A2) + (A3) + (A5) + (B3); equivalent to (vii)
Isosymmetry	Operational task-class equivalence	(viii) directly, via (B2)
Tensor-product Derivation	Bilinear compositional geometry on assemblies	(iv) directly, via (B1)
Entanglement-through-the-Void	Global projection consistency on entangled subspaces	Equivalent to (ii) restricted to entangled-projection subspaces (in $d\_op^\mathbb{C} \geq 3$)
TPB Consistency	Norm preservation under reversible transport plus bridging	(vi) directly

The paper therefore synthesizes reconstruction, geometry, admissibility, thermodynamics, and operational structure into one measure framework. The two routes (Physical Necessity and Structural Forcing) that do not correspond to a single characterization (i)–(viii) supply *combinations* of structural axioms that, under the inherited operational structure of §2, are equivalent to one of the eight characterizations. This is a slightly weaker equivalence than the direct mapping enjoyed by the other six routes, but the structural content — that all eight routes terminate at the same unique measure — is preserved.

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11. Interpretation

The standard quantum formalism may now be re-interpreted within the VERSF framework as follows:

Standard quantum object	Operational meaning
Complex Hilbert space $\mathscr{H}$	Substrate-derived admissible Hilbert geometry ($\mathcal{A}_\mathbb{C}$, $\langle\cdot,\cdot\rangle_\mathbb{C}$), with complex structure from $\mathbb{Z}_7$-equivariance
State vector ψ	Admissible distinguishability configuration
Amplitudes $c_i = \langle e_i, \psi\rangle_\mathbb{C}$	Operational transport coordinates in a channel-adapted basis
Unitary evolution U(τ)	Reversible distinguishability transport on $\mathcal{A}_\mathbb{C}$
Orthogonal projection P	Irreversible commitment refinement onto Im(P)
Born probability $\lvert\langle\phi, \psi\rangle_\mathbb{C}\rvert^2$	Normalized operational distinguishability volume of the projected state
Hamiltonian H	Self-adjoint generator of refinement-parameter τ-evolution
Schrödinger equation $i\hbar\, \partial_\tau \psi = H\psi$	Operational refinement-flow equation on the complex carrier

Within this interpretation:

- **Hilbert space is not a postulate** — it is the substrate-derived carrier of admissible distinguishability content (OG, URHG).
- **Complex structure is not a postulate** — it is forced by $\mathbb{Z}_7$-equivariance of the substrate (URHG Theorem 3.5).
- **Unitary evolution is not a postulate** — it is the reversible factor in the canonical polar decomposition of admissible refinement transport (URHG §8).
- **Projection is not a postulate** — it is the canonical commitment regime of refinement (URHG Theorem 7.1).
- **The Born rule is not a postulate** — it is the unique refinement-stable distinguishability measure on the admissible operational Hilbert geometry (this paper).

The full quantum-mechanical formalism therefore emerges as the unique fixed point of admissible operational structure under refinement, projection, transport, composition, and isosymmetry. No quantum-mechanical input is imposed; the formalism is derived from substrate-level admissibility.

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12. Predictions and Falsifiability

We distinguish *generic* predictions that follow from orthogonal additivity alone (and so are not specific to VERSF) from *substrate-specific* predictions that test the deeper structural content of the VERSF derivation.

12.1 Generic predictions from orthogonal additivity

These predictions follow from axiom (A2) alone and are shared with any framework imposing orthogonal additivity:

- **Sorkin third-order interference vanishes.** The Sorkin third-order interference quantity

  $$I_3 := P(A \vee B \vee C) - P(A \vee B) - P(B \vee C) - P(A \vee C) + P(A) + P(B) + P(C)$$

  vanishes for any complete set of orthogonal admissible outcomes A, B, C, as a direct consequence of orthogonal additivity. Detection of non-vanishing I_3 in admissible operational experiments would falsify (A2). (Standard Sorkin test, experimentally verified to high precision in optical interferometry; see Sinha et al. 2010, *Science* 329, 418.)

- **Quadratic probability rather than higher-order norms.** No stable non-quadratic norm (e.g. $\|\psi\|^p$ for $p \neq 2$) can satisfy the minimal core plus a bridging condition. Demonstration of a stable non-quadratic admissible probability assignment would falsify Theorem 6.1.

12.2 Substrate-specific VERSF predictions

These predictions follow from the specific substrate-level structure of the VERSF derivation, distinguishing it from other reconstruction programmes:

- **Failure of cross-channel uniformity — the central VERSF-specific falsifier.** The most direct VERSF-specific falsifier is not merely failure of the Born rule in general, but *failure of cross-channel coefficient uniformity*. The cross-channel bridging condition (B1, B2, or B3) — derived from the $\mathbb{Z}_7$-equivariant structure plus tensor-product compositional, sub-channel isosymmetric, or refinement-stable arguments — forces the per-channel quadratic coefficients to be exactly equal. If admissible systems could be prepared whose probabilities were empirically described by

  $$\mu(\psi) = \sum_\alpha C_\alpha \|\psi_\alpha\|^2_{\mathbb{C}}$$

  with stable, reproducible $C_\alpha \neq C_\beta$ across distinct $\mathbb{Z}_7$-channels, then the minimal-core structure would still survive but the VERSF bridging principle would fail. This is a sharper falsifier than generic non-Born probability, because it targets the exact additional step required to move from channel-wise quadraticity (which the minimal core supplies, by Theorem 6.1) to the universal Born rule (which requires bridging, by Theorem 6.2 and Lemma 6.2A). Detection of stable channel-dependent probability-amplitude weightings in admissible operational experiments — different per-channel prefactors for the squared-norm contributions, persisting under controlled refinements — would falsify the cross-channel bridging while leaving the rest of the operational measure structure intact. This is the cleanest experimental discriminator between the present framework and any alternative reconstruction that accepts the minimal core but supplies a different bridging principle (or none).

- **Refinement-stable probability assignments.** Probability assignments must be invariant under admissible refinement of the operational decomposition. Demonstration of refinement-dependent probability assignments — i.e. two admissible refinements of the same operational situation yielding different total probabilities — would falsify condition (B3) and Theorem 6.2.
- **Operational isosymmetry of probability across substrate realizations.** Two admissible systems related by an operational isosymmetry must agree on all probability assignments. Demonstration of substrate-dependent probability differences between operationally isosymmetric systems would falsify condition (B2).
- **Compositional multiplicativity for independent admissible subsystems.** $\mu(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi)$ for independent admissible subsystems. Demonstration of a deviation from compositional multiplicativity in admissible composites would falsify condition (B1).
- **Continuity of probability under finite-packing refinement.** Probability assignments must be continuous in the $\|\cdot\|_{\mathbb{C}}$ topology, with continuity supplied by the finite-packing structure of OG Theorem 10.1. Demonstration of a discontinuous admissible probability assignment (a substrate-discrete artifact at scales $\geq \Delta_{op}$) would falsify the operational interpretation of finite packing as supplying continuity.

12.3 Potential general falsifiers

Each of the following would falsify the framework:

- demonstration of stable non-Born probabilities in admissible operational systems;
- demonstration of persistent higher-order interference (non-vanishing I_3) in admissible experiments;
- demonstration of refinement-dependent probability assignments inconsistent with axiom (A2) or condition (B3);
- demonstration of admissible non-quadratic normalization-preserving measures;
- demonstration of substrate-dependent probability differences between operationally isosymmetric systems;
- demonstration of cross-channel quadratic-coefficient inequalities under controlled admissible refinements;
- demonstration of non-multiplicative measure on tensor-product composites of independent admissible subsystems;
- demonstration that finite-packing stability admits non-quadratic continuous solutions of the orthogonal-additivity functional equation (which would contradict Kuczma's theorem on continuous additive functions on $[0, \infty)$).

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13. Open Problems

The operational measure framework is now in place. The following problems remain open within the broader VERSF programme:

1. **Full Born-rule reconstruction from finite packing alone.** Whether the finite-packing structure of OG Theorem 10.1 alone is sufficient to derive the Born rule, supplying both orthogonal additivity (A2) and continuity (A5) as derived consequences rather than separate axioms. The structural form of such an argument: orthogonal additivity would emerge from the operational-independence of admissible packing sectors (sectors corresponding to non-overlapping admissible volumes contribute independently to the total measure); continuity would emerge from the smoothness of packing density at scales $\geq \Delta\_op$ (the operational density of states is smooth, hence so is the measure derived from it). If both can be derived from packing alone, the minimal axiom set reduces to (A1) + finite packing + (A7) + cross-channel bridging — a substantial structural simplification.
2. **Universal Measure Principle from coherence-sector commensurability.** Whether (B1), (B2), (B3) are three operational manifestations of a deeper single substrate-level principle — a *Universal Measure Principle* asserting that any operational theory admitting mutually composable coherence sectors requires a universal probability assignment across sectors. The three bridging conditions of this paper are mutually consistent and each suffices; the deeper structural question is whether they all derive from one underlying *commensurability principle* on coherence sectors. Candidate formulation: "the probability measure on isotypic components depends only on the substrate-character label and not on carrier-specific multiplicity data, because compositional, isosymmetric, and refinement-stable operational structure all demand commensurability across sectors of the same admissible geometry." If such a unifying principle can be made precise and substrate-derived, (B1)/(B2)/(B3) become derived consequences and the cross-channel bridging step has a single fundamental origin rather than three operationally distinct sources. This is the natural next paper of the programme; the present paper makes the structural gap explicit without yet closing it at this deepest level.
3. **Infinite-dimensional operational geometry.** Extension of the operational measure framework to infinite-dimensional Hilbert carriers, including the technical complications of unbounded self-adjoint generators (Stone's theorem proper), unbounded admissible refinement maps, and the Cheeger–Gromov compactness arguments of OC §11 in the infinite-dimensional limit.
4. **Lorentzian operational measure structure.** Whether the operational measure structure of this paper survives the Riemannian-to-Lorentzian signature transition conjectured in OC and URHG. Lorentzian signature changes the inner-product structure (one negative direction) and may require modification of axioms (A2)–(A3).
5. **Quantum-field-theoretic operational packing.** Extension of finite distinguishability packing to second-quantized admissible carriers (Fock-space-like structures on the operational Hilbert geometry), and derivation of QFT probability rules from operational packing in this regime.
6. **Coupling between operational curvature and admissible measure.** Whether the conformal-deformation curvature of OC interacts with the admissible measure in a structurally non-trivial way — e.g. whether the operational measure on a curved admissible Hilbert geometry differs structurally from the flat-reference case (this paper is implicitly flat-reference; the curved case is a natural extension).

7. **Derivation of decoherence directly from packing instability.** Whether decoherence — the loss of admissible quantum coherence under environment-coupling — admits an operational characterization as packing instability of admissible sectors under refinement transport coupled to inadmissible degrees of freedom.
8. **Probability beyond pure states: mixed-state operational measure.** Extension of the operational measure framework from pure states $\psi \in \mathcal{A}_\mathbb{C}$ to mixed admissible states (density operators ρ on $\mathcal{A}_\mathbb{C}$), with the resulting measure $\mathrm{tr}(\rho P_\phi)$ recovered as the operational distinguishability volume of mixed admissible configurations.
9. **Operationally distinctive VERSF predictions vs. generic reconstruction predictions.** Of the substrate-specific predictions of §12.2, identify which would be experimentally distinguishable from the predictions of other reconstruction programmes (Gleason, Hardy, Masanes–Müller). The cross-channel uniformity prediction (B1/B2/B3-derived) is the most natural candidate, but identifying a clean experimental discriminator would substantially strengthen the empirical case for the VERSF route.
10. **Strengthening Proposition 9.1 to a theorem.** Establish the identification $\mathrm{Vol}_{op}(\chi) = \|\chi\|^2_\mathbb{C}$ as a derived theorem within the OG framework rather than a definitional convention. The structural step required: a substrate-level argument that the OG packing-volume measure on admissible state vectors must take the squared-norm form (rather than being defined to do so). This would upgrade Proposition 9.1 to a fully rigorous structural identification of probability with normalized distinguishability volume.
11. **Inevitability of the $\mathbb{Z}_7$ substrate symmetry.** Why $\mathbb{Z}_7$ specifically, rather than some other finite cyclic group $\mathbb{Z}_n$? The present paper accepts $\mathbb{Z}_7$ as inherited from URHG and the broader VERSF corpus, where the K = 7 closure architecture is established as a separate structural result. The deeper substrate-origin question — whether $\mathbb{Z}_7$ is *inevitable* (forced by substrate-level structural constraints) or merely *selected* (one possible choice among several admissible ones) — is outside the scope of a measure-reconstruction paper. Structurally, several aspects of the present derivation depend on the cyclic group order being prime: the $\mathbb{Z}_7$-multiplicative-homomorphism argument in Theorem 6.2 (B1) uses the fact that $\mathbb{Z}_7$ is finite cyclic of prime order with no non-trivial subgroups, which forces $C_\alpha^7 = 1$ and hence (under positivity) $C_\alpha = 1$. Prime cyclic order is also what gives URHG Theorem 3.5 its clean conjugate-pair character structure with $k \in \{1, 2, 3\}$. Whether *exactly* 7 — as opposed to some other small prime — is forced by deeper substrate principles is the long-term target. A future paper establishing this would make $\mathbb{Z}_7$ itself appear inevitable rather than selected, closing the deepest scope gap of the programme.

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14. Conclusion

Prior VERSF work established the admissible operational Hilbert geometry (OG), the conformal-deformation curvature it carries under distinguishability-density gradients (OC), and the canonical polar decomposition of admissible refinement transport with substrate-derived complex structure from $\mathbb{Z}_7$-equivariance (URHG). These supplied the carrier, the inner product, the complex structure, the reversible dynamics, the self-adjoint generator, and the projection structure — six of the seven structural ingredients required for quantum reconstruction.

The present paper closes the seventh: the probability measure layer. The central result is the **Common Operational Measure Theorem** (Theorem 8.1): under the inherited operational structure, any admissible probability assignment satisfying the minimal-core axioms (A1)–(A3) + (A5) + (A7) plus any one of three cross-channel bridging conditions (B1, B2, or B3) is uniquely the quadratic operational measure

$\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$,

with the Born transition probability

$P(\psi \to \phi) = |\langle \phi, \psi \rangle_{\mathbb{C}}|^2$

as a corollary (Theorem 7.1). The chain of reasoning has four structural steps:

(A) Channel-wise norm-only dependence. Admissible ($\mathbb{Z}_7$-equivariant) unitary invariance acts transitively on within-channel norm-spheres (Lemma 5.1), forcing $\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_{\mathbb{C}})$ for continuous per-channel functions F_α (Theorem 5.2).

(B) Per-channel quadratic form. Within-channel orthogonal additivity on complex-orthogonal vectors (for $d_\alpha \geq 2$), via change of variable $G_\alpha(t) := F_\alpha(\sqrt{t})$, reduces to the additive Cauchy equation $G_\alpha(s + t) = G_\alpha(s) + G_\alpha(t)$ on $[0, \infty)$, with continuity supplied by axiom (A5) / finite packing. Kuczma's theorem on continuous additive functions forces $G_\alpha(t) = C_\alpha t$, hence $F_\alpha(r) = C_\alpha r^2$ (Theorem 6.1).

(C) Cross-channel coefficient equality. Any one of three structurally distinct conditions — (B1) compositional multiplicativity with character matching, (B2) sub-channel isosymmetric embedding, (B3) refinement-stability across channel decompositions — forces $C_\alpha = C$ uniformly across channels (Theorem 6.2). For (B1) the bridging proceeds via the $\mathbb{Z}_7$-multiplicative-homomorphism argument on the character lattice; for (B2) via direct dimension-respecting embedding; for (B3) via $U(d_\alpha + d_\beta)$-invariance on multi-channel subspaces. The three conditions have distinct scope: (B3) is structurally strongest in the minimal $d_\alpha = 1$ case (where within-channel orthogonal additivity is vacuous); (B2) requires the existence of $d \geq 2$ channels to embed into; (B1) works in all cases via the character-multiplication structure on $\mathbb{Z}_7$. The three structurally distinct bridging arguments give the framework robustness against challenges to any single condition.

(D) Normalization fixing. Axiom (A7) fixes $C = 1$ (Theorem 6.3).

The minimal axiomatic core is therefore (A1) + (A2) + (A3) + (A5) + (A7) + (any one of B1, B2, B3). The other axioms (A4 projection consistency, A8 full-unitary measure invariance) are *automatic consequences* of the quadratic solution and serve as structural confirmations rather than independent constraints (Remarks 6.1.2, 3.1.2). This is a cleaner axiom system than earlier formulations, which presented all axioms on equal footing and obscured both the proof structure and the role of the bridging condition.

The Born rule is therefore not an independent postulate of quantum theory but the unique refinement-stable operational distinguishability measure on the admissible Hilbert geometry that the VERSF substrate already supplies. The previously-partially-independent VERSF routes (Double Square pairwise geometry, entropic unfolding, isosymmetric reconstruction, tensor-product bilinearity, Gleason-type frame analysis, TPB normalization, finite-packing stability, isosymmetric equivalence) are convergent operational projections of the same measure geometry, made explicit by the Common Operational Measure Theorem.

The framework has one genuine structural advantage over Gleason's theorem: it works in $d_{op}^{\mathbb{C}} \geq 2$ (whereas Gleason requires $d \geq 3$), because axiom (A2) — orthogonal additivity on *vectors* — is strictly stronger than frame additivity on *projections*. The $d = 2$ case that Gleason cannot close is closed by the present approach (Remark 7.1.2).

Within this framework, probability is reinterpreted as **normalized operational distinguishability volume on the admissible Hilbert geometry** — not subjective uncertainty, not hidden-variable weighting, not epistemic ignorance, but the geometric measure of admissible distinguishability content concentrated in an outcome subspace.

The standard quantum formalism emerges as the unique fixed point of:

- admissible refinement structure (OG);
- substrate-derived complex structure (URHG Theorem 3.5);
- canonical polar decomposition of transport (URHG Theorem 8.1);
- orthogonal projection geometry (URHG Theorem 7.1);
- finite distinguishability packing (OG Theorem 10.1);
- conformal-deformation curvature (OC Theorem 5.2);
- and the operational measure derived here (Theorems 5.2, 6.1, 6.2, 6.3).

The Born rule is therefore not freely imposed on the operational geometry. Once the operational geometry and a cross-channel bridging principle are fixed, the quadratic measure is forced uniquely.

Essential novelty. The essential contribution of this paper is not merely another derivation of the Born rule. It is the identification of the *precise structural gap* that remains after admissible Hilbert geometry has been derived. Admissible unitary invariance gives only channel-wise invariance (by URHG Definition 3.2's $\mathbb{Z}_7$-equivariance constraint). It does not, by itself, justify full unitary invariance across inequivalent substrate channels (Lemma 6.2A). The Born rule therefore requires one further bridge: compositional multiplicativity with character matching (B1), sub-channel isosymmetric embedding (B2), or refinement-stability (B3). Once that bridge is made explicit, the quadratic measure follows uniquely.

This strengthens the programme rather than weakening it. The paper does not pretend that the Born rule follows automatically from Hilbert space alone — it shows exactly what additional operational principle is needed, why that principle is natural in the VERSF setting (the substrate-only principle on $\mathbb{Z}_7$-character labels), and how the probability rule becomes unique once it is accepted. The conditional nature of the result is structurally informative: it identifies the minimal additional input beyond the operational Hilbert geometry that determines the probability layer, separates this input cleanly from the other reconstruction ingredients, and supplies three structurally distinct operational formulations of it. The bridging condition is now an explicit, testable element of the framework rather than an unrecognized hidden assumption — with the cross-channel uniformity test of §12.2 supplying a direct experimental discriminator.

Admissibility supplies a single layered operational Hilbert geometry with substrate-derived complex structure (OG + URHG). Polar decomposition supplies the canonical transport split between reversible and irreversible content (URHG). Conformal deformation supplies curvature (OC). The unique operational measure compatible with all of this — orthogonal-additive on complex-orthogonal pairs, admissibly-unitarily invariant, continuous under finite packing, single-reference normalized, and cross-channel uniform via compositional/character-matching, isosymmetric, or refinement-stable bridging — is the quadratic norm, and the Born rule is its transition-probability corollary. The previously-distinct VERSF routes to the Born rule are equivalent projections of this single measure geometry. Probability is operational distinguishability volume; the Born rule is the unique fixed point that distinguishability volume forces.