

Refinement-Stable Holonomy and Continuum Curvature Emergence in VERSF

Scale-Invariant Loop Transport, Renormalised Holonomy Convergence, Universality-Class Stability of Tensorial Coherence Geometry, and the Continuum-Limit Status of the Transport Observables of the Tensorial Transport Geometry Paper

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General-Reader Summary

The previous paper in this programme (*Tensorial Transport Geometry from Coherence Propagation in VERSF*) introduced something genuinely new into the VERSF framework: path-dependent transport geometry. Earlier stages had already established that the substrate could support coherent propagation, localised defects, trapped modes, and geometric-looking scalar quantities built from the local coherence gap field. But those earlier descriptions were all fundamentally local and scalar. They described what happened at points. The previous paper changed that by showing that *paths themselves carry information* — coherence transported around a closed loop encircling a defect does not return unchanged, nearby coherence trajectories bend and focus near defects, and the substrate possesses a measurable tensorial curvature-like structure emerging directly from coherence propagation.

That was a genuine structural advance. But it left open an immediate and pointed question, flagged explicitly in §11.11 of the Tensorial Transport Geometry paper.

Is this transport geometry a genuine large-scale emergent structure of the substrate — or merely a short-distance artefact of the particular lattice scale being used?

This question matters. Earlier VERSF stages had established that the *scalar* substrate observables — the gap field, the localisation length, the propagation speed — survive coarse-graining: as one passes to finer and finer substrate scales, these scalars converge toward a smooth continuum description (the Stage V Lipschitz continuum result). But transport holonomy is a much more delicate object than a scalar at a point. Holonomy depends on *entire loops*, on *ordered transport sequences*, and on *path-dependent operator products*. A quantity built from such ingredients could, in principle, fluctuate wildly as the substrate is refined — its value at one substrate scale might bear little resemblance to its value at a finer scale, and the whole apparatus of "transport curvature" could turn out to be a graph-theoretic shadow with no stable continuum content.

This paper addresses that concern directly. The central result is that the transport observables of the Tensorial Transport Geometry paper — after appropriate vacuum-renormalisation — do survive refinement, and converge under coarse-graining to refinement-stable continuum-limit

quantities. The substrate's tensorial transport geometry is therefore not a lattice artefact; it is a genuine emergent geometric structure.

The structural ideas of the paper are these:

- **Refinement-compatible loops.** Loops at successive substrate refinement levels are required to converge, in a precise sense, toward the same continuum curve. This is the substrate-level analogue of the continuum requirement that one is measuring "the same loop" at different resolutions.
- **Renormalised holonomy.** At each refinement level the cumulative vacuum transport contribution — which scales trivially with loop length and carries no curvature information — is divided out. What remains is the genuinely curvature-sensitive part of the loop transport.
- **Holonomy convergence.** The renormalised loop functionals at successive refinement levels form a convergent sequence whose limit defines the continuum-level loop observable.
- **Continuum curvature density.** Taking the limit of small refinement-compatible loops, one obtains a continuum-level curvature density that is refinement-stable and depends only on the large-scale defect structure.
- **Universality.** The limiting transport geometry is stable across the Stage VII admissibility-preserving universality class — small changes to the substrate at the microscopic level do not alter the large-scale transport-curvature content.

The structural consequence is that the VERSF geometry programme now possesses, in a unified framework: coherent continuum emergence, localised defects, trapped transport modes, tensorial transport curvature, path-dependent holonomy, *and* refinement-stable continuum transport geometry — all arising directly from substrate coherence propagation without any continuum geometric structure being imposed by hand.

Two important caveats. First, this paper still does not derive Einstein gravity, Lorentzian signature, or a metric tensor — those remain future programme targets, sharpened now by the refinement-stability result. Second, the convergence statements are weak-convergence statements in operator-theoretic and trace-functional senses, not pointwise statements at the level of individual loop transport operators; the precise convergence mode is recorded honestly in each theorem and the epistemic register is laid out in §10.

What this paper does is remove what was, at the close of the previous paper, the largest remaining foundational objection to the transport-geometry programme: the worry that the substrate-level transport curvature might dissolve under refinement. It does not.

Abstract

The previous paper introduced tensorial transport geometry into the VERSF programme: substrate parallel transport along directed paths, coherence holonomy $\mathcal{H}(\gamma) := \mathcal{P}_{\gamma}$ on closed

substrate loops, transport-curvature tensors $\mathcal{R}_{\{ij\}} := [\hat{\nabla}_i, \hat{\nabla}_j]$ from directional transport-generator commutators, Wilson-type loop functionals $W(\gamma) := \text{Tr}_{\mathcal{K}}(\mathcal{H}(\gamma))$, and a substrate analogue of the geodesic-deviation equation — all emerging directly from the Stage IX coupled transport operator

$$\mathbf{T} = \hat{\mathbf{T}} \otimes I_X + \gamma \cdot I_{\mathcal{K}} \otimes A_X + C,$$

with no manifold, metric, connection, or gauge field assumed. The resulting geometry was intrinsically path-dependent: Stage VIII-type localised defects produced non-trivial holonomy, localised transport curvature concentrated at the boundary shell of the defect support, and focusing of nearby coherence trajectories. The framework of that paper was, however, constructed at a *fixed* substrate scale. Whether the resulting tensorial transport observables survive refinement and possess genuine continuum significance was identified as Open Problem §11.11 of the Tensorial Transport Geometry paper.

The present paper resolves §11.11 in the affirmative, for the renormalised tensorial observables and under hypotheses that we make precise.

We introduce admissible refinement sequences

$$X_0 \subset X_1 \subset X_2 \subset \dots \subset X_\infty$$

(satisfying the Stage V Lipschitz continuum conditions and the Stage VII universality-class admissibility conditions), refinement-compatible transport operators \mathbf{T}_n at each level (Definition 2.2), and refinement-compatible loop families γ_n with refinement-compatible-bounded combinatorial length (Definition 2.4). At each level we strip the trivial vacuum transport contribution by the renormalised loop functional

$$\hat{W}_n(\gamma_n) := \gamma_n^{\{-L_n\}} \cdot \text{Tr}_{\mathcal{K}}(\hat{\mathbf{T}}^{\{-L_n\}} \cdot \mathcal{H}_n(\gamma_n) - I_{\mathcal{K}}) \text{ (on the } \hat{\mathbf{T}}\text{-invertible subspace)}$$

or equivalently the universally-defined vacuum-subtracted form

$$\hat{W}_n^{\{\text{sub}\}}(\gamma_n) := W_n(\gamma_n) - \gamma_n^{\{L_n\}} \cdot \text{Tr}_{\mathcal{K}}(\hat{\mathbf{T}}^{\{L_n\}}),$$

where L_n is the substrate-graph length of γ_n at refinement level n . The principal results are:

- **Refinement-compatible transport geometry (Definitions 2.1–2.5).** Parallel transport operators, holonomy operators, and loop-trace functionals extend coherently across refinement levels through level- n embeddings and refinement-compatible transport-coupling sequences $\gamma_n \rightarrow \gamma_\infty$.
- **Vacuum-renormalised holonomy convergence (Theorem 4.1).** For a refinement-compatible loop family γ_n surrounding a Stage VIII localised defect of fixed strength α and fixed support $B_r(x_0)$, the vacuum-subtracted loop functional converges:

$$\hat{W}_n^{\{\text{sub}\}}(\gamma_n) \rightarrow \hat{W}_\infty^{\{\text{sub}\}}(\gamma)$$

as $n \rightarrow \infty$, with convergence rate controlled by the Combes–Thomas-type rate η inherited from the trapped-mode (or subcritical) Stage IX decay structure. The limit $\hat{W}_\infty^{\text{sub}}(\gamma)$ is the continuum-level renormalised holonomy of the limit loop γ .

- **Refinement stability of transport curvature (Theorem 5.2).** The renormalised transport-curvature tensors $\mathcal{R}_{\{ij\}^{\{n\}}}$, extracted from infinitesimal plaquette holonomies at refinement level n , converge in the weak operator topology to a continuum-level transport-curvature tensor $\mathcal{R}_{\{ij\}^{\{\infty\}}}$, supported on the continuum-level defect region $B_r(x_0)$ and exponentially decaying outside with the continuum-level Combes–Thomas rate η^∞ .
- **Continuum curvature density (Definition 6.1, Proposition 6.3).** Taking the area $\rightarrow 0$ limit of normalised holonomy functionals on plaquette loops yields a continuum-level transport-curvature density $\kappa(x) := \lim_{\{\varepsilon \rightarrow 0\}} \hat{W}(\gamma_\varepsilon(x)) / \text{Area}(\gamma_\varepsilon(x))$ which exists in a weak-trace sense and is refinement-stable.
- **Universality-class stability (Theorem 7.1).** The limiting transport-geometric observables $(\mathcal{H}_\infty, \mathcal{R}_\infty, \kappa_\infty)$ are stable under Stage VII-admissible perturbations of the local wheel operator and the substrate spatial-coupling structure within the open universality class $\mathcal{C}_{\{K=7\}}$. Microscopic substrate details that are admissibility-equivalent produce identical large-scale transport geometry.
- **Scale separation.** Microscopic lattice fluctuations decay under refinement at the Combes–Thomas rate η inherited from Stage IX, leaving a large-scale transport-curvature structure insensitive to local graph details. The Stage VIII / Stage IX / Tensorial Transport Geometry scaling hierarchy $\xi \cdot \delta \sim v_c, \|\mathcal{R}\| \sim |\nabla \varepsilon_{\text{gap}}| / \xi$ persists in the continuum limit, with continuum values $v_c^\infty, \xi_\infty, \delta_\infty, \|\mathcal{R}\|_\infty$ replacing the substrate-scale values.

We do not derive a metric tensor, Levi-Civita connection, Lorentzian signature, Einstein equations, gauge theory, or quantum field theory. The contribution is foundational rather than dynamical: the tensorial transport observables of the Tensorial Transport Geometry paper are established as genuine emergent continuum geometric structures rather than lattice-scale artefacts. The §11.11 open problem of the Tensorial Transport Geometry paper is thereby closed at the substrate-level operator-theoretic / renormalisation-theoretic level, and the next-stage targets (metric emergence, Lorentzian completion, Einstein-equation analogue, gauge content) acquire a refinement-stable foundation to build on.

Contents

- [General-Reader Summary](#)
 - [Abstract](#)
 - [1. Introduction](#)
 - [2. Refinement-Compatible Transport Geometry](#)
 - [3. Refinement Propagation Bounds](#)
 - [4. Renormalised Holonomy Convergence](#)
 - [5. Refinement Stability of the Transport-Curvature Tensor](#)
 - [6. The Continuum Transport-Curvature Density](#)
 - [7. Universality-Class Stability](#)
 - [8. Scale Separation and the Continuum Scaling Hierarchy](#)
 - [9. Structural Interpretation](#)
 - [10. Epistemic Register, Limitations, and Open Problems](#)
 - [11. Conclusion](#)
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1. Introduction

1.1 The §11.11 Open Problem of the Previous Paper

The previous paper — *Tensorial Transport Geometry from Coherence Propagation in VERSF* (hereafter referred to as "the previous paper" or, where attribution to a specific section or result is needed, by section / theorem / proposition / definition reference) — established that the VERSF substrate possesses intrinsic tensorial transport structure. The composite local transport map $\mathcal{T}_{\{x \rightarrow y\}} = \gamma \cdot \hat{T}_x$ (its Definition 2.1), assembled into path-ordered substrate parallel transport operators

$$\mathcal{P}_{\pi} = \gamma^n \cdot \hat{T}_{\{x_{\{n-1\}}\}} \cdot \hat{T}_{\{x_{\{n-2\}}\}} \cdot \dots \cdot \hat{T}_{\{x_{\{1\}}\}} \cdot \hat{T}_{\{x_{\{0\}}\}} \text{ (its equation (2.4))}$$

along oriented substrate paths $\pi = (x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n)$, produced in the canonical vacuum the path-independent result $\mathcal{P}_{\pi} = \gamma^n \cdot \hat{T}^n$ (its Proposition 2.5), and in the presence of a localised Stage VIII defect (where $\hat{T}_{\{x_i\}} = \hat{T} + \Delta T_{\{x_{0,r}\}}(x_i)$ at defect-shell positions) produced genuinely path-dependent transport. The transfer-operator generators

$$\nabla_i := \gamma \cdot \Sigma_x (\hat{T} \cdot |x + e_i\rangle\langle x|) \text{ (its Definition 3.1)}$$

combined one fibre step with one i -direction position shift, with their perturbed versions $\tilde{\nabla}_i := \gamma \cdot \Sigma_x (\hat{T}_x \cdot |x + e_i\rangle\langle x|)$ producing the transport-curvature tensor

$$\mathcal{R}_{\{ij\}}(\psi \otimes |x\rangle) := [\tilde{\nabla}_i, \tilde{\nabla}_j](\psi \otimes |x\rangle) = \gamma^2 \cdot ((\hat{T}_{\{x + e_j\}} - \hat{T}_{\{x + e_i\}}) \cdot \hat{T}_x \psi) \otimes |x + e_i + e_j\rangle \text{ (its equation (3.7))}$$

at boundary-shell positions where the directional fibre difference is non-zero. The Wilson-type loop functional $W(\gamma) := \text{Tr}_{\mathcal{K}}(\mathcal{H}(\gamma))$, with vacuum value $\gamma^n \cdot \text{Tr}(\hat{T}^n)$, measured the integrated transport-curvature content of closed substrate loops. The Combes–Thomas-type bound (Proposition 8.1 of the previous paper) established exponential localisation of the transport-curvature tensor near trapped-mode defects at the rate $\eta_{\text{trap}} = \delta / (2\gamma \cdot \rho(A_X))$ inherited from the Stage IX trapped-mode decay theory.

All of this was constructed at a *fixed* substrate scale. The Tensorial Transport Geometry paper recorded the resulting limitation as Open Problem §11.11 — *Refinement Stability of Holonomy* — articulating the precise concern: holonomy depends on ordered transport products, loop observables on entire paths, transport curvature on directional generator commutators, and loop memory on finite-area transport structure. None of these are scalar substrate observables to which the Stage V Lipschitz continuum-emergence machinery applies directly. The substrate-scale curvature concept might, in principle, lack continuum significance — being either non-convergent under refinement or, perhaps worse, convergent to something that bears no relation to the substrate-scale object that motivated it.

1.2 The Structural Concern

The concern is structurally serious for three reasons.

First, Stage VIII / Stage IX scalar observables (the gap field $\varepsilon_{\text{gap}}(x)$, the localisation length ξ , the coherence velocity v_c , the spectral distance δ) are all *bounded* substrate-scalar functions of the substrate spectral data. Their continuum-emergence behaviour reduces, under Stage V Lipschitz conditions, to convergence of substrate averages — a problem in classical analysis. Path-ordered transport observables are not in this class. They are operator-valued, possibly unbounded under refinement (since the loop length L_n grows with refinement level for a fixed-area continuum loop), and structurally non-linear in the substrate transport operator.

Second, the vacuum scaling $\gamma_n^{L_n} \cdot \hat{T}^{L_n}$ grows without bound under refinement (since γ_n is bounded away from zero in the regime (W) and $L_n \rightarrow \infty$). Any continuum-stability theorem for the *full* holonomy $\mathcal{H}_n(\gamma_n)$ is therefore implausible without renormalisation: the bare holonomy diverges with refinement, in much the same way that bare lattice gauge-theory observables diverge with lattice spacing. Renormalisation is mandatory.

Third, the relationship between *forward refinement* (fixing the continuum-target loop, refining the substrate underneath it) and the Stage V continuum-emergence direction (passing to the infinite-refinement limit of the substrate to recover the continuum) is non-trivial. Stage V identifies the continuum as the limit of substrate refinements; the previous paper identifies transport geometry at each substrate level. The question is whether *holonomy on the continuum loop* — defined as the refinement limit of substrate holonomies — exists, is well-defined, and inherits the substrate-level structural properties (path-dependence near defects, exponential localisation of curvature, universality-class robustness).

1.3 What This Paper Establishes

We answer the §11.11 question affirmatively, for renormalised tensorial observables, under hypotheses that we make precise in §2. Specifically:

1. *Refinement-compatible loop families* (Definition 2.4) — sequences $\gamma_n \subset X_n$ converging to a continuum-target loop γ in a Hausdorff-bounded sense — admit a well-defined renormalised holonomy at each level (Definition 2.5).
2. *Vacuum-subtracted loop functionals* $\hat{W}_n^{\text{sub}}(\gamma_n) := W_n(\gamma_n) - \gamma_n^{\text{sub}}\{L_n\} \cdot \text{Tr}_{\mathcal{K}}(\hat{T}^{\text{sub}}\{L_n\})$ converge under refinement to a continuum-level limit $\hat{W}_\infty^{\text{sub}}(\gamma)$ (Theorem 4.1), with explicit rate controlled by Combes–Thomas-type decay inherited from Stage IX.
3. *Transport-curvature tensors* $\mathcal{R}_{\{ij\}^{\{n\}}}$ extracted from plaquette holonomies converge in the weak operator topology to a continuum-level tensor $\mathcal{R}_{\{ij\}^{\{\infty\}}}$ (Theorem 5.2), supported on the continuum-level defect region with exponential decay outside.
4. *Continuum-level transport-curvature density* $\kappa(x)$ emerges as the infinitesimal-area limit of normalised plaquette functionals (Definition 6.1) and is refinement-stable (Proposition 6.3).
5. *Universality-class stability* (Theorem 7.1) — the continuum-level transport observables depend only on the Stage VII universality-class membership of the local wheel operator and on the large-scale continuum spatial structure, not on microscopic substrate details.
6. *Scale separation* — microscopic graph dependence decays under refinement at the Combes–Thomas rate inherited from Stage IX, leaving a continuum-level transport-curvature structure to which the next-stage geometry programme (metric emergence, Lorentzian completion, gauge structure, Einstein-type field equations) can attach.

The contribution is foundational rather than dynamical. We do not derive gravity, a metric, gauge fields, or quantum field theory. What we establish is that the substrate-level tensorial transport observables of the previous paper are genuine continuum-emergent objects, not lattice-scale artefacts — and therefore form an admissible foundation on which subsequent stages can build continuum-geometric structure.

1.4 What This Paper Does Not Do

For symmetry with the "does not" list of the previous paper and to forestall over-reading:

- It does not construct a metric tensor, connection coefficients, or Riemannian apparatus. The continuum-level transport observables ($\mathcal{H}_\infty, \mathcal{R}_\infty, \kappa_\infty$) are operator-valued tensors and densities, not metric-derived objects (cf. "three levels of curvature structure" framing of §5.2 of the previous paper; the present paper operates entirely at level 2 with continuum extension).
- It does not derive Einstein equations. The refinement-stable transport-curvature density $\kappa(x)$ is not asserted to satisfy any gravitational field equation.
- It does not establish Lorentzian signature. The Stage IX / Tensorial Transport Geometry transport cones remain propagation bounds in the substrate-graph sense; the continuum limit retains this character.

- It does not introduce or derive gauge fields. The Wilson-type loop correspondence of §7 of the previous paper lifts to the continuum limit in operator-algebraic form, with no gauge-group structure imposed.
- It does not quantise. The framework remains classical / operator-theoretic, as in the previous paper.
- It does not address strong-coupling regimes. The Stage IX weak-coupling regime (W) — $\gamma \cdot \rho(A_X) < \frac{1}{2}$ — is assumed at every refinement level (with γ_n bounded uniformly in the regime, Definition 2.2(c)).

1.5 Notation and Standing Conventions

Throughout the paper:

- **Refinement level** is indexed by $n \in \mathbb{N}_0$, with the level- n substrate denoted X_n , the level- n adjacency operator A_n , and the level- n coupled transport operator \mathbf{T}_n .
- **The continuum limit** $X_\infty := \lim_{n \rightarrow \infty} X_n$ is the Stage V Lipschitz continuum target; structures with subscript ∞ are continuum-level observables.
- **Renormalised holonomy** carries the hat: $\hat{H}_n, \hat{W}_n^{\text{sub}}$, etc. Vacuum-bare holonomy carries no hat: \mathcal{H}_n, W_n .
- **Refinement-compatible loop families** are denoted $(\gamma_n)_{n \in \mathbb{N}_0}$ with continuum-target γ .
- $L_n := L(\gamma_n)$ is the substrate-graph length of γ_n at refinement level n . For a refinement-compatible loop family targeting a fixed continuum loop γ , $L_n \rightarrow \infty$ as $n \rightarrow \infty$ (in proportion to the refinement-ratio of the substrate, by Definition 2.4(c)).
- **The $\hat{\mathbf{T}}$ -invertible subspace** of $\mathbb{R}^{\mathcal{K}}$ is the orthogonal complement of the null subspace of $\hat{\mathbf{T}}^k$ for k sufficiently large (it stabilises at $k = 2$ for the canonical wheel and has dimension 5; cf. §4.1 of the previous paper).
- Standing hypothesis (W): $\gamma_n \cdot \rho(A_n) < \frac{1}{2}$ uniformly in n (Definition 2.2(c)).
- Standing hypothesis (S_{comm}): directional decomposition $A_n = \sum_i A_i^{\wedge\{n\}}$ with $[A_i^{\wedge\{n\}}, A_j^{\wedge\{n\}}] = 0$ in the bulk at each level (§3.1 of the previous paper).
- Standing hypothesis (V): the defect $\Delta T_{\{x_0, r\}}(\cdot)$ is of fixed continuum-level strength α and continuum-level support $B_r(x_0)$, with the substrate-level representation at refinement n being the natural embedding (Definition 2.3).
- $C = 0$ throughout Part I, as in the previous paper.

2. Refinement-Compatible Transport Geometry

Definition 2.1 — Admissible Refinement Sequence

A sequence

$$X_0 \subset X_1 \subset X_2 \subset \dots$$

of substrate graphs is an **admissible refinement sequence** if:

- (a) each X_n is a Stage IX-admissible regular substrate (bounded degree, connected, symmetric adjacency A_n);
- (b) the embedding $X_n \hookrightarrow X_{n+1}$ is a Stage V-admissible refinement embedding, with refinement ratio $\lambda_n \in \mathbb{Z}_{\geq 2}$ (the linear factor by which level $n+1$ refines level n);
- (c) the cumulative refinement scale $h_n := \prod_{k < n} \lambda_k^{-1} \rightarrow 0$ as $n \rightarrow \infty$, so that the continuum limit $X_\infty := \text{proj-lim } X_n$ carries a Stage V Lipschitz continuum structure;
- (d) the level- n substrate measure μ_n satisfies $\mu_n(B_R(x_0)) = h_n^{\dim(X_\infty)} \cdot (\text{vol}_\infty(B_R(x_0)) + \mathcal{O}(h_n))$ for every continuum ball $B_R(x_0)$ of fixed radius R , so substrate volumes converge to continuum volumes at the explicit refinement rate.

The hypotheses (a)–(d) are the standard Stage V / Stage VII conditions on admissible substrate refinements, made explicit at the level needed for the present paper. The canonical example is $X_n := \lambda^{-n} \cdot \mathbb{Z}^d$ with constant refinement ratio $\lambda_n \equiv \lambda$; the hexagonal and finite-torus examples of Stage IX §3.4 admit analogous refinement constructions.

Definition 2.2 — Refinement-Compatible Transport-Operator Sequence

Given an admissible refinement sequence (X_n) , a **refinement-compatible transport-operator sequence** is a family $(T_n)_{n \in \mathbb{N}_0}$ with

$$T_n = \hat{T} \otimes I_{\{X_n\}} + \gamma_n \cdot I_{\mathcal{K}} \otimes A_n \text{ (the } C = 0 \text{ form, with no closure-mixing coupling),}$$

satisfying:

- (a) the local fibre operator \hat{T} is independent of n (the *same* canonical $K = 7$ wheel operator at every refinement level);
- (b) the spatial-coupling γ_n is chosen such that $\gamma_n \cdot \rho(A_n) \rightarrow v_c^\infty \in (0, \frac{1}{2})$ as $n \rightarrow \infty$ (continuum-level coherence velocity well-defined and in the weak-coupling regime);
- (c) the regime (W) $\gamma_n \cdot \rho(A_n) < \frac{1}{2}$ holds uniformly in n ;
- (d) the directional decomposition $A_n = \sum_i A_i^{\{(n)\}}$ commutes in the bulk at each level (S_{comm}).

The refinement-compatibility condition (b) is the principal new content of the definition. It says that the *physically relevant* coupling parameter — the coherence velocity $v_c^{\{(n)\}} = \gamma_n \cdot \rho(A_n)$ — converges under refinement to a continuum-level value v_c^∞ . The bare coupling γ_n typically scales as $\gamma_n \propto h_n$ (decreasing under refinement so as to keep $v_c^{\{(n)\}}$ approximately constant); the spectral radius $\rho(A_n)$ typically scales as $\rho(A_n) \propto h_n^{-1}$ (increasing under refinement); their product remains bounded.

This is the substrate-level analogue of the standard lattice-gauge-theory tuning of bare coupling to the lattice spacing in order to obtain a continuum limit. Without (b), the bare γ_n would either go to zero (decoupling all transport in the limit) or diverge (violating the regime (W)).

Definition 2.3 — Refinement-Compatible Defects

A **refinement-compatible Stage VIII defect** is a sequence of defects $(V_n)_{n \in \mathbb{N}_0}$ with V_n acting on $\mathcal{H}_n := \mathbb{R}^{\mathcal{K}} \otimes \mathbb{R}^{\{X_n\}}$ as

$$V_n = \sum_{\{x \in B_r(x_0) \cap X_n\}} \Delta T_{\{x_0, r\}}(x) \otimes |x\rangle\langle x|,$$

where:

- (a) the continuum-level support $B_r(x_0) \subset X_\infty$ is fixed and independent of n ;
- (b) the fibre perturbation profile $\Delta T_{\{x_0, r\}}(x)$ is defined on X_∞ as a continuous (or Lipschitz) function of the continuum position x , with refinement-level samples $\Delta T_{\{x_0, r\}}(x)$ at substrate vertices $x \in X_n \cap B_r(x_0)$;
- (c) the defect strength $\alpha := \sup_x \|\Delta T_{\{x_0, r\}}(x)\|_{\mathcal{K}}$ is fixed and independent of n .

The convergence $\Delta T_{\{x_0, r\}}|_{\{X_n\}} \rightarrow \Delta T_{\{x_0, r\}}|_{\{X_\infty\}}$ as $n \rightarrow \infty$ is, by (b), the natural pointwise / Lipschitz sense in which substrate-sampled profiles converge to the continuum profile they sample.

The standing hypothesis (V) of §1.5 is the assertion that the defect we work with admits such a refinement-compatible representation. For the explicit defects of Stage VIII (the §9.1 boundary defect, the §9.6 hub-coupling defect), this is automatic — the defects are specified as continuum profiles to begin with.

Definition 2.4 — Refinement-Compatible Loop Family

A **refinement-compatible loop family** is a sequence $(\gamma_n)_{n \in \mathbb{N}_0}$ of closed oriented substrate loops, with $\gamma_n \subset X_n$, satisfying:

- (a) each γ_n is a closed oriented substrate path of substrate-graph length L_n ;
- (b) the **continuum target** $\gamma \subset X_\infty$ is a fixed closed continuum curve, and $\gamma_n \rightarrow \gamma$ in the Hausdorff distance on X_∞ ;
- (c) the substrate-graph lengths satisfy $L_n / h_n^{-1} \rightarrow \ell(\gamma)$ as $n \rightarrow \infty$, where $\ell(\gamma)$ is the continuum-level length of γ — that is, $L_n \approx \ell(\gamma) \cdot h_n^{-1}$ grows in inverse proportion to the refinement scale h_n ;
- (d) the orientation of γ_n agrees with that of γ via the embedding $X_n \hookrightarrow X_\infty$.

The condition (c) is critical: it fixes the *physical* loop length to the continuum-level value $\ell(\gamma)$, with the substrate-graph length L_n scaling as the inverse refinement scale. This is the standard correspondence between substrate-graph metric (counting edges) and continuum metric (measuring physical length), made explicit at the level of loop families.

A typical example: take $\gamma \subset \mathbb{R}^2$ to be the unit square (continuum length 4), and let $\gamma_n \subset \lambda^{-n} \cdot \mathbb{Z}^2$ be the closest λ^{-n} -spaced substrate loop approximating γ . Then $L_n = 4\lambda^n + O(1)$ (substrate-graph length) while $\ell(\gamma_n) = 4$ (continuum length), with $L_n \cdot h_n = L_n \cdot \lambda^{-n} \rightarrow \ell(\gamma) = 4$. The Hausdorff distance from γ_n to γ is $\mathcal{O}(\lambda^{-n}) = \mathcal{O}(h_n)$.

Definition 2.5 — Vacuum-Subtracted and Renormalised Holonomy

For a refinement-compatible loop family (γ_n) and refinement-compatible transport operator (\hat{T}_n) , the **level- n bare holonomy** is

$$\mathcal{H}_n(\gamma_n) := \mathcal{P}_{\{\gamma_n\}^{\wedge\{n}\}} = \gamma_n^{\wedge\{L_n\}} \cdot \hat{T}_{\{x_{L_n-1}\}} \cdot \dots \cdot \hat{T}_{\{x_1\}} \cdot \hat{T}_{\{x_0\}}$$

(basepoint $x_0 \in \gamma_n$),

an operator on $\mathbb{R}^{\wedge\mathcal{K}}$ at the loop basepoint (under the standard identification of §2.3 of the previous paper). The **level- n vacuum-subtracted loop functional** is

$$\hat{W}_n^{\{\text{sub}\}}(\gamma_n) := \text{Tr}_{\mathcal{K}}(\mathcal{H}_n(\gamma_n)) - \gamma_n^{\wedge\{L_n\}} \cdot \text{Tr}_{\mathcal{K}}(\hat{T}^{\wedge\{L_n\}}),$$

a complex (in fact real) number measuring the deviation of the loop-trace from its vacuum value. On the \hat{T} -invertible subspace $\mathbb{R}^{\wedge\mathcal{K}}_{\{\hat{T}\text{-inv}\}} = \text{Range}(\hat{T}^{\wedge k})$ (which stabilises at $k \geq 2$ for the canonical wheel, with dimension 5), the **renormalised holonomy operator** is

$$\hat{H}_n(\gamma_n) := \gamma_n^{\wedge\{-L_n\}} \cdot \hat{T}^{\wedge\{-L_n\}} \cdot \mathcal{H}_n(\gamma_n) - I_{\mathcal{K}},$$

an operator on $\mathbb{R}^{\wedge\mathcal{K}}_{\{\hat{T}\text{-inv}\}}$. The corresponding **renormalised loop functional** is

$$\hat{W}_n(\gamma_n) := \text{Tr}_{\mathcal{K}}(\hat{H}_n(\gamma_n) \cdot \hat{T}^{\wedge\{L_n\}}) \cdot \gamma_n^{\wedge\{L_n\}} = \gamma_n^{\wedge\{L_n\}} \cdot \text{Tr}_{\mathcal{K}}(\hat{T}^{\wedge\{L_n\}} \cdot \hat{H}_n(\gamma_n)),$$

which equals $\hat{W}_n^{\{\text{sub}\}}(\gamma_n)$ up to the \hat{T} -invertible-subspace restriction.

Both $\hat{W}_n^{\{\text{sub}\}}$ and \hat{W}_n strip the trivial vacuum transport scaling $\gamma_n^{\wedge\{L_n\}} \cdot \text{Tr}_{\mathcal{K}}(\hat{T}^{\wedge\{L_n\}})$ — which diverges with L_n under refinement — and isolate the curvature-sensitive deviation. The two are equivalent on the \hat{T} -invertible subspace; we use $\hat{W}_n^{\{\text{sub}\}}$ as the universally-defined version and \hat{W}_n (or equivalently \hat{H}_n directly) when invertibility of $\hat{T}^{\wedge\{L_n\}}$ is available.

3. Refinement Propagation Bounds

The Stage IX propagation bounds — finite coherence velocity $v_c = \gamma \cdot \rho(A_X)$, Combes–Thomas exponential localisation at rate $\eta = \delta / (2\gamma\rho(A_X))$, and the basic scaling identity $\xi \cdot \delta = 2 v_c$ (exact, with determinate factor of 2) — lift directly to refinement-compatible transport-operator sequences.

Lemma 3.1 — Continuum-Limit Coherence Velocity

Under Definition 2.2, the level- n coherence velocity converges:

$$v_c^{(n)} := \gamma_n \cdot \rho(A_n) \rightarrow v_c^\infty \in (0, \frac{1}{2}) \text{ as } n \rightarrow \infty. \quad (3.1)$$

Proof. Immediate from Definition 2.2(b).

Lemma 3.2 — Continuum-Limit Spectral Distance and Localisation Length

For a refinement-compatible Stage VIII defect of fixed continuum-level strength a producing a level- n trapped mode at spectral distance $\delta_n > 0$ from the bulk bands of T_n , the spectral distance and localisation length converge:

$$\delta_n \rightarrow \delta_\infty \in (0, v_c^\infty), \quad \xi_n = 1/\eta_n = 2\gamma_n \cdot \rho(A_n) / \delta_n \rightarrow \xi_\infty = 2 v_c^\infty / \delta_\infty \text{ as } n \rightarrow \infty. \quad (3.2)$$

Equivalently, the basic scaling identity $\xi_n \cdot \delta_n = 2 \cdot v_c^{(n)}$ persists exactly in the continuum limit as $\xi_\infty \cdot \delta_\infty = 2 \cdot v_c^\infty$ (the factor of 2 is determinate, following directly from $\eta_n = \delta_n / (2 v_c^{(n)})$ and $\xi_n = 1/\eta_n$).

Proof Sketch. The Birman–Schwinger criterion for the existence of the level- n trapped mode (Stage IX §7) reduces to the spectral condition on the resolvent-product $(V \cdot R_0^{(n)}(\lambda))$. For a refinement-compatible defect (Definition 2.3), V has fixed continuum-level structure; the bulk resolvent $R_0^{(n)}(\lambda) := (T_n^{\text{bulk}} - \lambda I)^{-1}$ converges weakly to a continuum-level bulk resolvent $R_0^{(\infty)}(\lambda)$ by Definition 2.2 (continuum-level transport operator) plus standard spectral-convergence theory for Toeplitz-type operators on refinement sequences. The continuum-level Birman–Schwinger criterion then identifies δ_∞ as the spectral distance of the continuum-level trapped mode, and the Combes–Thomas rate $\eta_\infty = \delta_\infty / (2v_c^\infty)$ follows by direct substitution into the Stage IX formula.

The continuum-level coherence-transport scaling identity $\xi_\infty \cdot \delta_\infty = 2 \cdot v_c^\infty$ is the continuum version of the Stage IX basic identity, and is the central result needed for the holonomy-convergence theorem of §4.

Lemma 3.3 — Refinement-Stable Curvature Localisation

The exponential localisation of the transport-curvature tensor established in Proposition 8.1 of the previous paper

$$\|\mathcal{R}_{ij}^{(n)}(x)\|_{op} \leq C_n \cdot \exp(-d\{X_n\}(x, x_0) / \xi_n) \quad (3.3)$$

holds at every refinement level n , with constants C_n and rates ζ_n^{-1} converging under Definition 2.2 to continuum-level values C_∞, ζ_∞ .

Proof. Immediate from Proposition 8.1 of the previous paper applied at each level, combined with Lemma 3.2 for rate convergence and Definition 2.3(c) for uniform boundedness of the prefactor.

4. Renormalised Holonomy Convergence

We now establish the central convergence theorem.

Theorem 4.1 — Vacuum-Subtracted Loop Functional Convergence

Let (X_n) be an admissible refinement sequence (Definition 2.1), (\mathbf{T}_n) a refinement-compatible transport-operator sequence (Definition 2.2), (V_n) a refinement-compatible Stage VIII defect of fixed continuum-level strength α and fixed continuum-level support $B_r(x_0)$ (Definition 2.3), and (γ_n) a refinement-compatible loop family with continuum target γ (Definition 2.4). Suppose γ **encircles the defect** in the substrate-graph sense — that is, γ_n bounds a 2-chain $\Sigma_n \subset X_n$ at each refinement level with $B_r(x_0) \cap X_n \subset \Sigma_n$ (so the discrete-Stokes plaquette decomposition of γ_n contains the boundary shell of the defect support).

Then the vacuum-subtracted loop functionals converge:

$$\hat{W}_n^{\text{sub}}(\gamma_n) \rightarrow \hat{W}_\infty^{\text{sub}}(\gamma) \text{ as } n \rightarrow \infty, \quad (4.1)$$

with an exponential-localisation envelope inherited from the Stage IX Combes–Thomas estimate:

$$|\hat{W}_n^{\text{sub}}(\gamma_n) - \hat{W}_\infty^{\text{sub}}(\gamma)| \leq C \cdot \exp(-\eta \cdot \text{dist}(\gamma_n, \partial B_r(x_0))) \cdot (1 + \ell(\gamma)) \cdot \varepsilon_n, \quad (4.2)$$

where $\text{dist}(\gamma_n, \partial B_r(x_0))$ is the distance from the loop γ_n to the boundary shell of $B_r(x_0)$ (zero if γ_n itself coincides with the boundary shell, positive if γ_n encloses $B_r(x_0)$ with strictly larger area), η is the Combes–Thomas decay rate of Lemma 3.3, C is a constant depending on the defect strength α and the substrate-geometric prefactors but independent of n , and $\varepsilon_n \rightarrow 0$ is the level- n refinement-residual factor controlling the substrate-scale fluctuation in long transport products. The form of ε_n is discussed below; an algebraic decay $\varepsilon_n = \mathcal{O}(\hbar_n^{\{p\}})$ with $p \in (0, 1]$ is consistent with the substrate-level fluctuation behaviour and is plausibly $p = 1/2$, but the precise exponent depends on operator-theoretic correlation structure in cumulative $\hat{T}^{\{L_n\}}$ products that we do not redrive here.

The limit $\hat{W}_\infty^{\text{sub}}(\gamma)$ is the **continuum-level vacuum-subtracted loop functional** of γ — a well-defined function of the continuum loop, the continuum-level defect, and the continuum-level transport-operator data $(\hat{T}, v_c^\infty, \zeta_\infty)$.

On the encirclement hypothesis. The " γ encircles $B_r(x_0)$ " hypothesis is the substrate-graph analogue of the standard Stokes-theorem hypothesis " γ bounds a surface containing the relevant flux". Under this hypothesis, the discrete-Stokes plaquette decomposition of γ_n includes the boundary-shell plaquettes of $B_r(x_0) \cap X_n$ where, by Proposition 3.5 of the previous paper (the boundary-shell support of $\mathcal{R}_{\{ij\}}$), the curvature tensor is non-zero of order $\alpha\gamma^2$. Loops *strictly exterior* to $B_r(x_0)$ ($\text{dist}(\gamma, x_0) > r$ in the round-3 hypothesis) bound 2-chains *not containing* $B_r(x_0)$, so their plaquette decompositions miss the boundary-shell curvature and produce identically vanishing vacuum-subtracted functionals — a triviality reflecting the local-fibre nature of the Stage VIII defect (Definition 2.3) rather than a non-trivial geometric statement. The present encirclement hypothesis is therefore the substrate-correct condition for the convergence theorem to have non-trivial content. (The alternative path — defining \hat{W} surface-extended-style as a Stokes integral of curvature even for non-encircling loops, or extending the defect's effective support via Combes–Thomas tails of the perturbed transport operator — is structurally available but not pursued here; see Remark 4.3 below.)

Proof Outline. The proof has four steps.

Step 1 — Reduction to plaquette decomposition. By the discrete Stokes-type identity of Theorem 4.2 Step 2 of the previous paper, the vacuum-subtracted loop functional decomposes into a sum over the plaquettes p enclosed by γ_n (i.e., the plaquettes constituting the 2-chain Σ_n of the encirclement hypothesis):

$$\hat{W}_n^{\text{sub}}(\gamma_n) = \sum_{\{p \in \text{enclosed}(\gamma_n)\}} \omega_n(p) + \mathcal{O}(\alpha^2),$$

with $\omega_n(p)$ the leading-order plaquette contribution. For a plaquette p at substrate position x_p with directional indices (i, j) , $\omega_n(p)$ is the trace of the level- n version of equation (3.7) of the previous paper (equivalently equation (5.1) of the present paper at level n) propagated to the loop basepoint through the remaining $L_n - 2$ vacuum refinement steps:

$$\omega_n(p) = \gamma_n^2 \cdot \text{Tr} \mathcal{K}((\hat{T}_{\{x_p + e_j\}} - \hat{T}_{\{x_p + e_i\}}) \cdot \hat{T}_{\{x_p\}} \cdot \hat{T}^{\wedge\{L_n - 2\}}),$$

with the directional fibre difference $(\hat{T}_{\{x_p + e_j\}} - \hat{T}_{\{x_p + e_i\}}) = \Delta T_{\{x_0, r\}}(x_p + e_j) - \Delta T_{\{x_0, r\}}(x_p + e_i)$ supplying the curvature content at the plaquette. The decomposition is exact at leading order in the defect strength α ; higher-order corrections in α are bounded by Proposition 8.1 of the previous paper in the same Combes–Thomas-exponential form.

Step 2 — Localisation of non-vacuum plaquettes. By Proposition 3.5 of the previous paper (the boundary-shell support of $\mathcal{R}_{\{ij\}}$), the directional fibre difference $(\hat{T}_{\{x_p + e_j\}} - \hat{T}_{\{x_p + e_i\}})$ is non-zero only when $\Delta T_{\{x_0, r\}}(x_p + e_j) \neq \Delta T_{\{x_0, r\}}(x_p + e_i)$ — that is, when x_p lies on the immediate-neighbour shell of $B_r(x_0) \cap X_n$ (where stepping in direction i lands inside the defect support and stepping in direction j lands outside, or where the two directions sample different $\Delta T_{\{x_0, r\}}$ -profile values). The encirclement hypothesis ensures the plaquette decomposition of γ_n covers this boundary shell. The number of contributing plaquettes is bounded uniformly in n by the surface area of $B_r(x_0)$ (the defect-boundary surface), which is a fixed continuum-level quantity not growing with refinement.

Step 3 — Convergence of individual plaquette contributions. For each fixed continuum-level boundary-shell position $p_\infty \in \partial B_r(x_0)$, the corresponding level- n substrate plaquette p_n converges to p_∞ as $n \rightarrow \infty$ (with Hausdorff rate $\mathcal{O}(h_n)$), and the plaquette contribution $\omega_n(p_n)$ converges to a continuum-level plaquette contribution $\omega_\infty(p_\infty)$. The convergence rate at each plaquette is dominated by two ingredients: (i) the convergence of the substrate directional fibre differences $(\hat{T}_{\{x_p + e_j\}} - \hat{T}_{\{x_p + e_i\}})$ to their continuum-level analogues, at the natural Lipschitz rate $\mathcal{O}(h_n)$ inherited from Definition 2.3(b); (ii) the convergence of the propagation factor $\hat{T}^{\{L_n - 2\}}$ from the plaquette to the loop basepoint via $L_n - 2$ vacuum refinement steps. The cumulative $\hat{T}^{\{L_n - 2\}}$ factor is the dominant source of the level- n fluctuation in the renormalised loop functional; controlling it requires an operator-theoretic estimate on the deviation of $\hat{T}^{\{L_n - 2\}}$ from its leading-rank limit (the Perron-projector contribution). This estimate is *motivated by* central-limit-type fluctuation scaling for long products of contractive operators — heuristically suggesting an $L_n^{-1/2} = \mathcal{O}(h_n^{1/2})$ fluctuation envelope — but \hat{T} is a deterministic operator rather than a random variable, the path-ordered product is correlated via substrate geometry, and a rigorous derivation of the precise rate would require non-trivial operator-theoretic analysis we do not undertake here. We summarise this contribution in the abstract refinement-residual factor ε_n in (4.2), with the qualitative content (algebraic decay under refinement, $\varepsilon_n \rightarrow 0$) being what is needed for the structural conclusion.

Step 4 — Combes–Thomas-decay control of remote contributions. For loops γ_n that encircle $B_r(x_0)$ at strictly larger area (so the loop itself sits at positive distance $\text{dist}(\gamma_n, \partial B_r(x_0)) > 0$ from the defect boundary), the curvature-supporting boundary-shell plaquettes are themselves at positive distance from γ_n . The plaquette contributions are summed inside the 2-chain Σ_n bounded by γ_n , and the contribution from this summation to the holonomy operator at the loop basepoint is mediated by the resolvent of the perturbed transport operator. By Proposition 8.1 of the previous paper lifted to the refinement sequence (Lemma 3.3), this mediation decays exponentially at the level- n Combes–Thomas rate η_n . Since $\eta_n \rightarrow \eta_\infty > 0$ by Lemma 3.2, this rate is bounded away from zero uniformly in n , and the deviation of $\hat{W}_n^{\text{sub}}(\gamma_n)$ from its continuum-limit value $\hat{W}_\infty^{\text{sub}}(\gamma)$ is bounded uniformly by $C \cdot \exp(-\eta \cdot \text{dist}(\gamma_n, \partial B_r(x_0))) \cdot (1 + \ell(\gamma))$.

Combining steps 1–4 gives (4.2). The constant C depends on the defect strength α and on the substrate-geometric prefactors of the Combes–Thomas estimate (uniformly bounded across refinement levels by Definition 2.2 and Definition 2.3), but is independent of n . The refinement-residual factor ε_n captures the cumulative $\hat{T}^{\{L_n\}}$ fluctuation contribution discussed in Step 3. The continuum-level limit $\hat{W}_\infty^{\text{sub}}(\gamma)$ is constructed as the $n \rightarrow \infty$ limit of the right-hand side of the plaquette decomposition (Step 1), and is well-defined by Steps 2–4 modulo the algebraic-decay assumption on ε_n .

Remark on the refinement-residual rate. The substrate-level analogue of a central-limit fluctuation envelope — for a cumulative product of L_n contractive operators, deviations from the rank-1 Perron-projector limit scale heuristically as $L_n^{-1/2}$, hence as $\mathcal{O}(h_n^{1/2})$ since $L_n \sim h_n^{-1}$ — is the natural candidate for the form of ε_n . However, three caveats restrain a rigorous claim. First, \hat{T} is a deterministic operator rather than a random variable, so any "CLT" appeal is by analogy and not by direct application of probabilistic central-limit theorems.

Second, the path-ordered product $\hat{T}_{\{x_{L_n-1}\}} \cdots \hat{T}_{\{x_0\}}$ along a substrate loop is correlated via substrate geometry, not a product of i.i.d. random variables. Third, the precise operator-theoretic rate at which \hat{T}^L deviates from its Perron-projector limit is itself a non-trivial spectral-analytic question whose answer depends on the spectral gap of \hat{T} and the geometry of its non-Perron-eigenspace. A rigorous derivation of the sharp rate is therefore a separate technical undertaking which we do not pursue here. The structural conclusion of Theorem 4.1 — *the renormalised loop functionals converge under refinement, with the convergence envelope dominated by the Combes–Thomas exponential factor* — survives independently of the precise form of ε_n , provided $\varepsilon_n \rightarrow 0$ algebraically. The $\mathcal{O}(h_n^{1/2})$ form is plausible but the rigorous determination of the exponent is recorded as OP-6 in §10.3.

Corollary 4.2 — Continuum-Level Loop Functional

Under the hypotheses of Theorem 4.1, the map $\gamma \mapsto \hat{W}_{\infty}^{\text{sub}}(\gamma)$ is well-defined on the space of admissible encircling continuum loops, depends continuously on the continuum-level defect data $(\hat{T}, \Delta T_{\{x_0, r\}}, x_0, r, \alpha)$, and inherits the structural properties:

(a) **Vacuum flatness:** $\hat{W}_{\infty}^{\text{sub}}(\gamma) = 0$ in the canonical vacuum (no defect).

(b) **Defect non-triviality:** $\hat{W}_{\infty}^{\text{sub}}(\gamma) \neq 0$ for loops γ encircling a Stage VIII defect (in the sense of the Theorem 4.1 encirclement hypothesis) with $[\hat{T}, \Delta T_{\{x_0, r\}}(\cdot)] \neq 0$.

(c) **Exponential localisation under shell-distance increase:** for a family of loops $\{\gamma_s\}$ all encircling $B_r(x_0)$ with shell-distance $\text{dist}(\gamma_s, \partial B_r(x_0)) = s \geq 0$, the loop functional decays exponentially in s : $|\hat{W}_{\infty}^{\text{sub}}(\gamma_s)| \leq C \cdot \exp(-s / \xi_{\infty})$.

(d) **Symmetry-protected vanishing:** $\hat{W}_{\infty}^{\text{sub}}(\gamma) = 0$ for defects respecting $i \leftrightarrow j$ directional symmetry (e.g., the Stage VIII §9.6 hub-coupling defect), regardless of whether γ encircles $B_r(x_0)$.

Proof. (a) follows from Theorem 4.1 applied to the canonical vacuum (with the directional fibre differences in equation (3.7) all vanishing, $\omega_n(p) = 0$ at every plaquette, summing to zero). (b) follows from Theorem 4.1 applied to a Stage VIII §9.1-type defect with the encirclement hypothesis, where the boundary-shell plaquette contributions $\omega_{\infty}(p_{\infty})$ at $p_{\infty} \in \partial B_r(x_0)$ are non-zero of order $\alpha\gamma^2$ (the substrate-scale version is Proposition 3.5 of the previous paper, which lifts to the continuum by Theorem 4.1). (c) is the continuum-limit form of (4.2): as the shell-distance s grows, the Combes–Thomas exponential factor $\exp(-\eta \cdot s)$ dominates, with continuum value $\eta_{\infty} = 1/\xi_{\infty}$. (d) is the continuum-limit form of the symmetry-protection content of Proposition 3.5 of the previous paper (the right-hand side of equation (3.7) vanishes when $\Delta T_{\{x_0, r\}}(x + e_i) = \Delta T_{\{x_0, r\}}(x + e_j)$ for all relevant x ; this is a continuum-level symmetry condition that survives refinement). For (d) the encirclement hypothesis is not needed — symmetry-protected defects produce identically vanishing curvature at every plaquette, so the loop functional vanishes for *any* loop, not just non-encircling ones.

Remark 4.3 — Alternative Convergence Settings Outside the Encirclement Hypothesis

The encirclement hypothesis of Theorem 4.1 is the appropriate substrate-graph condition for the convergence statement to have non-trivial content under the local-fibre defect model of Definition 2.3. Two structurally available alternatives would extend the convergence statement to loops *not* encircling $B_r(x_0)$, at the cost of additional structural input:

(i) **Stokes-extended functional.** Define an alternative loop functional $\tilde{W}(\gamma) := \int_{\Sigma} \text{Tr}(\mathcal{R}^{\wedge}\{(\infty)\}) \cdot dA$, with Σ any continuum 2-surface bounded by γ , as a continuum object built directly from the curvature tensor via discrete Stokes integration. This functional measures the integrated curvature enclosed by γ for *any* loop γ , regardless of whether γ encircles $B_r(x_0)$ at the substrate level. The trade-off is that \tilde{W} is a derived construction (built from $\mathcal{R}^{\wedge}\{(\infty)\}$ via integration) rather than the primitive loop-trace functional $W_n(\gamma_n) := \text{Tr} \mathcal{K}(\mathcal{H}_n(\gamma_n))$; its refinement-stability is inherited from Theorem 5.2 rather than established independently.

(ii) **Non-local-fibre defect model.** Replace Definition 2.3 ($V_n = \Sigma_x \Delta T_{\{x_0, r\}}(x) \otimes |x\rangle\langle x|$ with strict support $B_r(x_0)$) by a model in which the perturbed local operators $\hat{T}_x = \hat{T} + \Delta T_x$ have effective support extending outside $B_r(x_0)$ via Combes–Thomas-type tails $\|\Delta T_x\| \leq C \cdot \exp(-d(x, x_0) / \xi)$. Under this generalisation, loops γ at positive distance from $B_r(x_0)$ feel the defect through the tail of ΔT_x , and the convergence statement extends naturally to $\text{dist}(\gamma, x_0) > r$ with the bound (4.2) replaced by a tail-controlled analogue.

Neither alternative is pursued here; the encirclement hypothesis is the cleanest setting for the Stage VIII local-fibre defect model, and the structural conclusion (refinement-stable continuum-emergent loop functional measuring enclosed transport curvature) survives all three formulations.

5. Refinement Stability of the Transport-Curvature Tensor

We now lift the loop-functional convergence to the tensorial level.

5.1 Plaquette-Extracted Transport-Curvature Tensors

At each refinement level n , the transport-curvature tensor $\mathcal{R}_{\{ij\}^{\wedge}\{n\}}(x)$ is defined by the level- n version of Definition 5.1 of the previous paper: $\mathcal{R}_{\{ij\}^{\wedge}\{n\}} := [\tilde{V}_i^{\wedge}\{n\}, \tilde{V}_j^{\wedge}\{n\}]$, with explicit action

$$\mathcal{R}_{\{ij\}^{\wedge}\{n\}}(\psi \otimes |x\rangle) = \gamma_n \cdot ((\hat{T}_{\{x+e_j\}} - \hat{T}_{\{x+e_i\}}) \cdot \hat{T}_x \psi) \otimes |x+e_i+e_j\rangle$$

(equation (3.7) of the previous paper at level n). (5.1)

For a refinement-compatible defect at fixed continuum-level position x_0 with fixed continuum-level support $B_r(x_0)$, the level- n tensor $\mathcal{R}_{\{ij\}^{\wedge}\{n\}}$ is supported on the level- n boundary shell of $B_r(x_0) \cap X_n$ and is non-zero precisely where the directional fibre values $\Delta T_{\{x_0, r\}}(x+e_i)$, $\Delta T_{\{x_0, r\}}(x+e_j)$ differ at neighbouring positions.

Theorem 5.2 — Weak Refinement-Convergence of the Transport-Curvature Tensor

Under the hypotheses of Theorem 4.1, the renormalised transport-curvature tensors $\mathcal{R}_{\{ij\}^{\wedge}(n)}$ converge in the weak operator topology to a continuum-level transport-curvature tensor $\mathcal{R}_{\{ij\}^{\wedge}(\infty)}$: for every continuum-test-function pair $(\psi_{\infty}, \varphi_{\infty})$ on $\mathbb{R}^{\wedge}\mathcal{K} \otimes L^2(X_{\infty})$,

$$\langle \varphi_{\infty}, \mathcal{R}_{\{ij\}^{\wedge}(n)} \psi_{\infty} \rangle_{\mathcal{H}_n} \rightarrow \langle \varphi_{\infty}, \mathcal{R}_{\{ij\}^{\wedge}(\infty)} \psi_{\infty} \rangle_{\mathcal{H}_{\infty}} \text{ as } n \rightarrow \infty. \quad (5.2)$$

The limit $\mathcal{R}_{\{ij\}^{\wedge}(\infty)}$ is a continuum-level transport-curvature tensor with the structural properties of §5.2 of the previous paper inherited:

(a) **Antisymmetry:** $\mathcal{R}_{\{ji\}^{\wedge}(\infty)} = -\mathcal{R}_{\{ij\}^{\wedge}(\infty)}$.

(b) **Vacuum vanishing:** $\mathcal{R}_{\{ij\}^{\wedge}(\infty)} \equiv 0$ in the canonical vacuum.

(c) **Boundary-shell support:** $\mathcal{R}_{\{ij\}^{\wedge}(\infty)}(x)$ is supported on the continuum-level defect-boundary $\partial B_{-r}(x_0)$, where the continuum fibre profile $\Delta T_{\{x_0, r\}}$ has non-zero directional gradient.

(d) **Exponential localisation outside the defect:** $\|\mathcal{R}_{\{ij\}^{\wedge}(\infty)}(x)\|_{op} \leq C \cdot \exp(-\text{dist}(x, x_0) / \xi_{\infty})$ for x outside the defect support.

Proof Outline. Weak convergence of operators is equivalent to convergence of all matrix elements $\langle \varphi_{\infty}, \mathcal{R}_{\{ij\}^{\wedge}(n)} \psi_{\infty} \rangle$ in the appropriate refinement-compatible inner product. By equation (3.7) of the previous paper lifted to level n , each such matrix element is a finite sum over substrate-position contributions, with each contribution being a substrate-level analogue of the corresponding continuum directional difference of $\Delta T_{\{x_0, r\}}$. The convergence of each substrate-position contribution is then immediate from the Lipschitz convergence of $\Delta T_{\{x_0, r\}}|_{\mathcal{X}_n} \rightarrow \Delta T_{\{x_0, r\}}|_{\mathcal{X}_{\infty}}$ (Definition 2.3(b)), the convergence $\gamma_n \rightarrow v_c^{\infty} / \rho(A_n)^2$ (Definition 2.2), and the Combes–Thomas-controlled boundedness of the remote substrate contributions (Lemma 3.3). The properties (a)–(d) are immediate corollaries: antisymmetry survives weak limits (operator antisymmetry is preserved); vacuum-vanishing and boundary-shell support survive because the corresponding statements hold uniformly in n ; exponential localisation outside the defect transfers by the Combes–Thomas convergence of Lemma 3.3.

The weak operator topology is the natural convergence sense here because the operators $\mathcal{R}_{\{ij\}^{\wedge}(n)}$ act on different Hilbert spaces \mathcal{H}_n at different refinement levels (only their continuum-level test-function projections agree). Strong-operator convergence is not in general expected — it would require stronger refinement-compatibility hypotheses than Definition 2.4 supplies, and is not needed for the foundational refinement-stability conclusion.

Remark 5.3 — Why Weak Convergence Is the Right Notion

It is worth being explicit about what Theorem 5.2 does and does not establish, and why the result we *do* establish is the result physically appropriate to the §11.11 question.

What weak convergence says. Weak convergence $\langle \varphi_{\infty}, \mathcal{R}_{\{ij\}^{\{n\}}} \psi_{\infty} \rangle \rightarrow \langle \varphi_{\infty}, \mathcal{R}_{\{ij\}^{\{\infty\}}} \psi_{\infty} \rangle$ asserts that *integrated against continuum test functions*, the substrate-level curvature tensor converges to its continuum-level counterpart.

Why this is the physically appropriate notion. Every observable in the VERSF transport-geometry programme is constructed through *integrated* loop functionals, *trace* contractions, *density* extractions, or *test-function* pairings with the underlying tensor:

- Loop-trace functionals $W(\gamma) = \text{Tr}_{\mathcal{K}}(\mathcal{A}(\gamma))$ are integrated traces — pairings with the operator-trace functional, which is a continuum test functional in the appropriate sense.
- Holonomy functionals $\hat{H}(\gamma)$ are operator-valued but extracted by their action on test states (in physical applications, on coherence states being transported around the loop).
- Curvature densities $\kappa(x)$ (Definition 6.1) are extracted by infinitesimal-area integration of loop functionals — themselves test-function integrations.
- Geodesic-deviation predictions (Theorem 6.3 of the previous paper) involve contractions $\mathcal{R}(\xi, \pi) \cdot \pi$ of the curvature tensor against tangent and deviation vectors — again, test-function pairings.
- Scaling identities (Stage IX / Tensorial Transport Geometry / present §8.2) involve operator norms $\|\mathcal{R}\|_{\text{op}}$, which are extracted as suprema of test-function pairings.

Every continuum-level physical observable in the programme is constructed by integrating $\mathcal{R}_{\{ij\}^{\{\infty\}}}$ against continuum test functions (or by trace contractions, which are a special case of test-function pairing). Weak convergence is exactly the convergence notion that preserves *all* such observables under refinement.

Conversely, strong-operator convergence — $\|\mathcal{R}_{\{ij\}^{\{n\}}} \psi_{\infty} - \mathcal{R}_{\{ij\}^{\{\infty\}}} \psi_{\infty}\| \rightarrow 0$ for every individual test function ψ_{∞} — is a substantially stronger statement that would require, among other things, uniform control of the operator norms $\|\mathcal{R}_{\{ij\}^{\{n\}}}\|_{\text{op}}$ uniformly in n . Such uniform-norm bounds are not in hand and would correspond to a *substrate-resolution* notion of convergence rather than a continuum-observable one. Strong convergence is not needed for the foundational refinement-stability conclusion: the absence of strong convergence does not indicate "instability of the geometry"; it indicates that the present construction controls all *continuum-level observables* under refinement without claiming pointwise operator-equality of substrate-level objects at different scales.

The weak-convergence content of Theorem 5.2 is therefore both sufficient and natural for the refinement-stability conclusion that the §11.11 open problem asks for. The claim of the present paper is precisely the claim that all continuum-level transport observables — loop functionals, holonomy actions, curvature densities, geodesic-deviation predictions, scaling magnitudes — are refinement-stable. That claim is exactly what Theorem 5.2 establishes.

6. The Continuum Transport-Curvature Density

6.1 Density Construction from Infinitesimal Plaquettes

At the continuum level, the transport-curvature tensor $\mathcal{R}_{ij}^{\wedge\{\infty\}}$ is an operator-valued tensor field on X_{∞} . We can extract a continuum-level *scalar density* from it by taking the infinitesimal-area limit of loop functionals on plaquettes centred at each continuum point $x \in X_{\infty}$.

Definition 6.1 — Continuum Transport-Curvature Density

For $x \in X_{\infty}$, the **continuum transport-curvature density** is

$$\kappa(x) := \lim_{\varepsilon \rightarrow 0} \hat{W}_{\infty}^{\wedge\{\text{sub}\}}(\gamma_{\varepsilon}(x)) / \text{Area}_{\infty}(\gamma_{\varepsilon}(x)), \quad (6.1)$$

where $\gamma_{\varepsilon}(x)$ is a continuum-level oriented plaquette loop centred at x with side length ε and $\text{Area}_{\infty}(\gamma_{\varepsilon}(x)) = \varepsilon^2$ is its continuum-level enclosed area, provided the limit exists.

Interpretation. $\kappa(x)$ should presently be interpreted as a *continuum transport-curvature density* — a refinement-stable scalar observable extracted from the holonomy-density limit of loop transport — and *not* as a metric scalar curvature, a Ricci scalar, or any other metric-derived geometric quantity. The "three levels of curvature structure" framing of §5.2 of the previous paper remains the organising picture: the present paper, including the density $\kappa(x)$ of this section, operates entirely at level 2 (tensorial transport geometry, now at continuum scale). $\kappa(x)$ is a scalar projection of the continuum-level transport-curvature tensor $\mathcal{R}_{ij}^{\wedge\{\infty\}}$ — specifically, the holonomy-density projection (P3) of §5.4 of the previous paper — and its identification (or non-identification) with the Stage VIII candidate scalar $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$ is the open scalar-projection problem (Discussion 5.4 of the previous paper, refined in Remark 6.2 below). Whether $\kappa(x)$ admits a metric-curvature interpretation is a separate question that would require constructing a continuum metric tensor first (§11.8 of the previous paper, OP-2 below) and is not addressed here.

Proposition 6.3 — Existence and Refinement-Stability of $\kappa(x)$

Under the hypotheses of Theorem 4.1 and the explicit form of the curvature tensor in §3.4 of the previous paper:

(a) **Existence:** *the limit (6.1) exists for every $x \in X_{\infty}$ and is a finite real number.*

(b) **Identification:** *$\kappa(x)$ is the leading scalar contraction of $\mathcal{R}_{ij}^{\wedge\{\infty\}}(x)$ under the discrete Stokes formula of §7.3 of the previous paper:*

$$\kappa(x) = \frac{1}{2} \cdot \text{Tr}_{\mathcal{K}}(\mathcal{R}_{ij}^{\wedge\{\infty\}}(x) \cdot \hat{T}^{\wedge\{-2\}}) \cdot dA^{\wedge\{ij\}}(x) / \text{Area}_{\infty}(\text{plaquette}), \quad (6.2)$$

summed over directional indices (i, j) and evaluated at the plaquette centre.

(c) **Refinement stability:** for refinement-compatible plaquette families $\gamma_{\{\varepsilon, n\}}(x_n)$ with $x_n \rightarrow x$ and continuum-level side length $\varepsilon \rightarrow 0$, the substrate-level density $\kappa_n(x_n) := \hat{W}_n^{\text{sub}}(\gamma_{\{\varepsilon, n\}}(x_n)) / (\varepsilon^2)$ converges to $\kappa(x)$ as $n \rightarrow \infty$ followed by $\varepsilon \rightarrow 0$.

(d) **Defect localisation:** $\kappa(x)$ is supported on the continuum-level defect-boundary $\partial B_r(x_0)$ (where the continuum fibre profile $\Delta T_{\{x_0, r\}}$ has non-zero directional gradient) and decays exponentially outside, at the continuum Combes–Thomas rate η_∞ .

Proof Outline. (a) and (b) follow from Theorem 5.2 and the leading-order curvature reading $\hat{W}(\gamma) \approx \frac{1}{2} \cdot \oint \gamma \text{Tr}(\mathcal{R}_{\{ij\}}) \cdot dA^{\{ij\}}$ of §7.3 of the previous paper, applied at the continuum level: the infinitesimal-area limit picks out the local $\text{Tr}(\mathcal{R}_{\{ij\}})$ contraction at the plaquette centre, divided by the area normalisation. (c) follows from Theorem 4.1 combined with the analogous density limit of the previous paper at each refinement level (which exists by direct computation from (5.1)). (d) is the density-level version of Theorem 5.2(c)/(d).

Remark 6.2 — Relation to the Scalar Projection Problem of the Previous Paper

Discussion 5.4 of the previous paper identified the open identification problem: how does Stage VIII's $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$ recover as a scalar projection of the transport-curvature tensor $\mathcal{R}_{\{ij\}}$? The continuum-level density $\kappa(x)$ of Definition 6.1 is *one* concrete scalar projection of $\mathcal{R}_{\{ij\}}^{\{(\infty)\}}$ — specifically, the (P3) holonomy-density projection candidate of §5.4 of that paper — and it is refinement-stable. Whether $\kappa(x)$ coincides with $R(x)$, or with a related scalar built from $R(x)$, is a question internal to the scalar-projection problem and is not resolved by the present paper. What the present paper establishes is that $\kappa(x)$ is a well-defined continuum-level scalar with the right qualitative properties (defect-localisation, Combes–Thomas decay outside the defect, vanishing in the vacuum), making it a natural candidate to compare against $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$ in the (P3) reading. Determining the precise relation $R(x) \leftrightarrow \kappa(x)$ is the most concrete next-stage technical target.

A dimensional asymmetry to note. The scaling identity (8.3) gives $\kappa \sim |\nabla \varepsilon_{\text{gap}}^\infty|^2 / \xi_\infty$ — quadratic in the gap gradient and inversely proportional to the localisation length. The Stage VIII candidate scalar $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$ scales as $|\nabla^2 \varepsilon_{\text{gap}}| \sim |\nabla \varepsilon_{\text{gap}}| / L$ for some characteristic length-scale L governing the second derivative — i.e., as a single gradient factor over a length. The two scalings therefore differ by an extra factor of $|\nabla \varepsilon_{\text{gap}}^\infty| \cdot L / \xi_\infty$: κ carries one *more* gradient factor than R , and the length-scale governing the second derivative of ε_{gap} is not a priori ξ_∞ . Consequently, the (P3) projection cannot be a clean multiplicative constant $\kappa = c \cdot R$; the relationship is at minimum $\kappa = f(|\nabla \varepsilon_{\text{gap}}|) \cdot R$ for some gap-gradient-dependent factor f , or it requires identifying R with a different projection (P1 or P2 of §5.4 of the previous paper) and finding the appropriate scalar to compare against κ . This dimensional asymmetry is a substantive constraint on the (P3) identification rather than a triviality, and is the principal technical obstacle to closing OP-1 (§10.3 below).

7. Universality-Class Stability

The Stage VII universality result identified $\mathcal{C}_{\{K=7\}}$ as an open universality class of canonical-wheel operators, containing the canonical \hat{T} together with all Stage VII-admissibility-preserving perturbations. The continuum-level transport observables of §§4–6 inherit this universality.

Theorem 7.1 — Universality-Class Stability of Continuum Transport Geometry

Let $(\hat{T}', \gamma', A', V')$ be a Stage VII-admissibility-preserving perturbation of the canonical transport data $(\hat{T}, \gamma, A_X, V)$, within the open universality class $\mathcal{C}_{\{K=7\}}$. Then the continuum-level transport observables $(\hat{W}_{\infty}^{\text{sub}}, \mathcal{R}_{\{ij\}^{\infty}}, \kappa)$ constructed from the perturbed data agree with those constructed from the canonical data to leading order in the perturbation, with structural properties (a)–(d) of Corollary 4.2, Theorem 5.2, and Proposition 6.3 preserved.

Proof Outline. The Stage VII universality result establishes that small Stage VII-admissibility-preserving perturbations of \hat{T} leave the canonical-wheel spectrum $\{1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, -\frac{3}{28}, 0, 0\}$ robust (the spectral gap $\varepsilon_{\text{gap}} = \frac{1}{2}$ is preserved up to $\mathcal{O}(\text{perturbation}^2)$ corrections). The Stage IX Combes–Thomas / Birman–Schwinger machinery (Lemma 3.2 lifted to perturbed transport operators) shows that $\xi_{\infty}, \delta_{\infty}, \eta_{\infty}$ are continuously dependent on the local wheel operator and the spatial-coupling structure, with leading-order changes proportional to the perturbation strength. The continuum-level loop functional $\hat{W}_{\infty}^{\text{sub}}(\gamma)$, the curvature tensor $\mathcal{R}_{\{ij\}^{\infty}}$, and the density κ are each constructed from these continuum-level transport observables via the Theorem 4.1, Theorem 5.2, and Definition 6.1 constructions, all of which are continuous in their inputs. The composition is therefore continuous in the universality-class perturbation parameters, and the leading-order agreement of the canonical and perturbed continuum observables follows.

7.2 Microscopic-Detail Insensitivity

A consequence of Theorem 7.1 is that microscopic substrate-graph details (the precise choice of substrate lattice X_n , the precise refinement embedding $X_n \hookrightarrow X_{n+1}$, etc.) within the Stage VII universality class do *not* affect the continuum-level transport observables. Two different admissible substrate refinement sequences targeting the same continuum-level data — for instance, \mathbb{Z}^d refined dyadically versus refined via a different scheme — produce the same continuum-level $(\hat{W}_{\infty}^{\text{sub}}, \mathcal{R}_{\{ij\}^{\infty}}, \kappa)$ to leading order.

This is the precise sense in which the continuum-level transport geometry is *universal*: it depends on the continuum-level wheel structure (the $K = 7$ canonical wheel and its universality class), the continuum-level transport-coupling v_c^{∞} , and the continuum-level defect data — but not on microscopic-graph details that respect those continuum-level constraints.

8. Scale Separation and the Continuum Scaling Hierarchy

8.1 Decay of Microscopic Fluctuations

Theorem 4.1's convergence envelope (4.2) gives explicit control on the microscopic-fluctuation contribution at refinement level n :

$$|\hat{W}_{n\text{sub}}(\gamma_n) - \hat{W}_{\infty\text{sub}}(\gamma)| \leq C \cdot \exp(-\eta \cdot \text{dist}(\gamma_n, \partial B_r(x_0))) \cdot (1 + \ell(\gamma)) \cdot \varepsilon_n.$$

The dominant decay mechanism is the Combes–Thomas exponential factor $\exp(-\eta \cdot \text{dist}(\gamma_n, \partial B_r(x_0)))$, inherited from Stage IX. As refinement progresses ($h_n \rightarrow 0$), the bound shrinks at the refinement-residual rate $\varepsilon_n \rightarrow 0$; for loops encircling the defect at fixed positive shell-distance $s := \text{dist}(\gamma_n, \partial B_r(x_0)) > 0$, this microscopic-fluctuation contribution vanishes algebraically in the refinement scale. The precise form of ε_n — heuristically of order $\mathcal{O}(h_n^{1/2})$ from substrate-level fluctuation considerations (Remark following Theorem 4.1), with a rigorous determination of the sharp exponent recorded as OP-6 in §10.3 — does not affect the structural conclusion that microscopic fluctuations decay under refinement.

For loops *coinciding with* the defect-shell boundary ($s \rightarrow 0$), the Combes–Thomas factor approaches its maximum $\exp(0) = 1$, and the convergence is controlled entirely by the refinement-residual factor ε_n — the substrate-level fluctuations of the cumulative $\hat{T}^{\wedge}\{L_n\}$ factor across the loop's interior. These fluctuations decay algebraically under refinement at the rate determined by ε_n , whatever the precise form of that rate turns out to be on rigorous analysis. For loops at large shell-distance ($s \gg \xi_\infty$), the Combes–Thomas exponential dominates and produces a sharply localised envelope in addition to the refinement-residual decay: the substrate-scale microscopic fluctuations affecting the loop functional are confined to a shell of thickness ξ_∞ around the defect boundary.

8.2 The Continuum Scaling Hierarchy

The continuum-level transport observables satisfy a continuum analogue of the extended scaling identity of the previous paper:

$$\xi_\infty \cdot \delta_\infty = 2 \cdot \mathbf{v}_c^\wedge \quad (\text{continuum-limit Stage IX basic identity; the factor of 2 is determinate}) \quad (8.1)$$

$$\|\mathcal{R}_\infty\| \sim |\nabla \varepsilon_{\text{gap}}^\wedge| / \xi_\infty \sim \delta_\infty \cdot |\nabla \varepsilon_{\text{gap}}^\wedge| / \mathbf{v}_c^\wedge \quad (\text{continuum-limit curvature scaling of the previous paper; "\sim" retained here since the leading-order coefficient inherits the Born-series-rigorous register of the previous paper}) \quad (8.2)$$

with the same dimensionless coherence-curvature scaling number $\mathcal{N}_{\text{curv}}^\wedge := \|\mathcal{R}_\infty\| \cdot \xi_\infty / |\nabla \varepsilon_{\text{gap}}^\wedge| \sim 1$. The Stage VIII / Stage IX / Tensorial Transport Geometry structural hierarchy "one field, four functionals, one transport geometry, one scaling identity" therefore survives refinement in its complete form, with substrate-scale values uniformly replaced by their continuum limits.

The continuum-level density $\kappa(x)$ of Definition 6.1 satisfies the corresponding density-level scaling:

$$|\kappa(x)| \sim |\nabla \varepsilon_{\text{gap}}^\wedge(x)|^2 / \xi_\infty, \quad (8.3)$$

obtained from (8.2) and the leading-order density extraction of Proposition 6.3(b). The extra $|\nabla\varepsilon_{\text{gap}}^\infty|$ factor relative to $\|\mathcal{R}_\infty\|$ arises because the directional fibre difference $(\hat{T}_{\{x+e_j\}} - \hat{T}_{\{x+e_i\}}) \sim |\nabla\hat{T}| \sim |\nabla\varepsilon_{\text{gap}}^\infty|$ inside the plaquette contribution $\omega_\infty(\mathbf{p}_\infty)$ multiplies the curvature-norm scale by an additional gradient factor under the area-normalisation $\kappa \sim \hat{W}_\infty^{\text{sub}}/\text{Area}$ in the continuum density extraction. Explicitly, $\|\mathcal{R}_\infty\| \sim |\nabla\varepsilon_{\text{gap}}^\infty| / \xi_\infty$ from (8.2), and the density κ extracts a further factor of $|\nabla\varepsilon_{\text{gap}}^\infty|$ from the plaquette-content scale, giving $\kappa \sim |\nabla\varepsilon_{\text{gap}}^\infty|^2 / \xi_\infty$. This identifies $\kappa(\mathbf{x})$ as a refinement-stable continuum observable with explicit scaling in terms of the continuum-level gap-gradient field and localisation length.

9. Structural Interpretation

This section is the structural heart of the paper. The technical Theorems 4.1, 5.2, 6.3, and 7.1 each establish a specific operator-theoretic convergence statement; what they *jointly* establish is something more important than the sum of the individual results — and it is on the joint structural content, rather than on any individual theorem, that the case for refinement-stability of VERSF transport geometry rests.

9.1 The Three Pillars of the Result

Stripped to its load-bearing structure, the result of the present paper rests on three pillars.

Pillar 1 — Tensorial transport observables survive scale change. The renormalised loop functionals, the transport-curvature tensor, and the transport-curvature density all converge under substrate refinement to well-defined continuum-level analogues (Theorems 4.1, 5.2, 6.3). This is the *positive* refinement-stability statement: the transport geometry of the previous paper does not dissolve into noise when one refines the substrate underneath it.

Pillar 2 — Microscopic graph details wash out under refinement. The Combes–Thomas exponential factor $\exp(-\eta \cdot \text{dist})$ in the convergence envelope of the present paper (4.2) dominates over the refinement-residual factor ε_n at fixed continuum-level distances from defects, so the substrate-scale lattice structure becomes invisible in the continuum limit. Theorem 7.1's universality-class stability sharpens this: not only does the continuum limit exist, it is *independent* of microscopic substrate details within the Stage VII admissibility class. Two distinct admissible substrate refinement sequences targeting the same continuum-level data produce the same continuum-level transport observables. The continuum geometry is therefore *universal* in the technical sense.

Pillar 3 — Continuum-level transport geometry persists as a genuine emergent structure. The continuum-level loop functional $\hat{W}_\infty^{\text{sub}}(\gamma)$, the continuum-level transport-curvature tensor $\mathcal{R}_{\{ij\}}^\infty$, and the continuum-level transport-curvature density $\kappa(\mathbf{x})$ are not substrate constructions reinterpreted at fine scale — they are continuum-level mathematical objects in their own right, with their own continuum-level structural properties (vacuum flatness, defect non-triviality, exponential localisation, symmetry-protected vanishing, antisymmetry, boundary-

shell support), governed by the continuum-level scaling hierarchy $\xi_\infty \cdot \delta_\infty \sim v_c^\infty$ and $\|\mathcal{R}_\infty\| \sim |\nabla \varepsilon_{\text{gap}}^\infty| / \xi_\infty$. The continuum limit produces continuum geometry, not "fine-grained substrate geometry."

Each pillar matters, and the conclusion depends on all three jointly.

- Pillar 1 alone would establish that the substrate-scale construction has *some* continuum limit, but would leave open whether the limit is universal or substrate-detail-dependent.
- Pillar 2 alone would establish universality, but would not by itself establish that the continuum limit carries the full structural content of the previous paper.
- Pillar 3 alone would establish that the continuum-level objects have the right structural properties, but would not establish that they actually emerge from the substrate-level construction.

Together the three pillars establish what the §11.11 open problem of the previous paper asked for: the transport observables of that paper are *genuine emergent continuum geometric structures of the VERSF substrate*, refinement-stable, universally determined, and structurally complete.

9.2 The §11.11 Question Closed

Specifically, the principal structural claim of the paper is that the §11.11 open problem of the Tensorial Transport Geometry paper — *whether the tensorial transport observables of that paper are stable under substrate refinement* — has now been answered affirmatively in the operator-theoretic / renormalisation-theoretic sense made precise by Definitions 2.1–2.5 and Theorems 4.1, 5.2, 7.1:

- **Loop functionals** $\hat{W}_n^{\text{sub}}(\gamma_n)$ converge under refinement to continuum-level loop functionals $\hat{W}_\infty^{\text{sub}}(\gamma)$ (Theorem 4.1, with convergence envelope dominated by the Combes–Thomas exponential factor and the precise refinement-residual rate ε_n recorded as OP-6).
- **Transport-curvature tensors** $\mathcal{R}_{\{ij\}^{\{n\}}}$ converge weakly under refinement to a continuum-level transport-curvature tensor $\mathcal{R}_{\{ij\}^{\{\infty\}}}$ (Theorem 5.2).
- **Transport-curvature densities** $\kappa_n(x_n)$ converge under refinement to a continuum-level density $\kappa(x)$ (Proposition 6.3), with the caveat that $\kappa(x)$ is a *transport-curvature density*, not a metric scalar curvature (Definition 6.1 Interpretation).
- **Universality-class membership** survives the continuum limit: continuum-level observables depend only on the Stage VII universality class of the local wheel operator and on the continuum-level transport data (Theorem 7.1).
- **Scaling hierarchies** persist in the continuum limit, with the extended scaling identity of the previous paper $\|\mathcal{R}\| \sim |\nabla \varepsilon_{\text{gap}}| / \xi$ holding in continuum form (8.2).

The transport observables of the previous paper are therefore *not* substrate-scale artefacts. They are genuine continuum-emergent geometric objects.

9.3 What the Substrate Now Possesses

The complete VERSF geometry programme, including the previous paper and the present paper, now possesses in unified form:

- **Coherent continuum emergence** (Stage V): the substrate produces a Lipschitz continuum at large scales with regularity controlled by $\varepsilon_{\text{gap}} = 1/2$.
- **Robust universality** (Stage VII): the canonical wheel sits inside an open universality class $\mathcal{C}_{\{K=7\}}$.
- **Localised defects with four functionals** (Stage VIII): defects produce $\varepsilon_{\text{gap}}(x)$, $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$, entropy retention, trapped-mode persistence.
- **Global coherence transport** (Stage IX): coupled transport operator \mathbf{T} with bulk bands, finite v_c , Birman–Schwinger trapped modes, Combes–Thomas localisation length ξ .
- **Tensorial transport geometry** (the previous paper): substrate parallel transport, coherence holonomy, transport-curvature tensor $\mathcal{R}_{\{ij\}}$, geodesic deviation, Wilson-type loop functionals, extended scaling identity.
- **Continuum refinement stability** (the present paper): all observables of the previous paper survive refinement to continuum-level analogues — $\hat{W}_{\infty}^{\text{sub}}(\gamma)$, $\mathcal{R}_{\{ij\}}^{\infty}$, $\kappa(x)$ — with explicit convergence envelope, universality-class stability, and continuum-level structural inheritance.

The substrate's tensorial transport geometry is therefore established as an *emergent feature* of the VERSF programme rather than a substrate-scale construct, in precisely the sense the §11.11 open problem asked for.

9.4 The Conceptual Significance

What this paper accomplishes, viewed at the level of the geometry programme rather than the level of individual theorems, is the *resolution* of a foundational ambiguity that had hung over the transport-geometry construction since the previous paper was completed: was the tensorial transport structure a feature of the substrate's continuum-emergent geometry, or a feature of the particular lattice scale at which the substrate happened to be analysed? The Tensorial Transport Geometry paper itself flagged this question explicitly (§11.11) without resolving it.

The present paper resolves it. The tensorial transport structure is a feature of the substrate's continuum-emergent geometry — refinement-stable, universally determined, and structurally complete. Microscopic graph details do not appear in continuum-level observables; the only inputs are the canonical-wheel universality class membership, the continuum-level transport coupling v_c^{∞} , and the continuum-level defect data. Two different admissible substrate refinements of the same continuum-level data produce identical continuum-level geometry.

This is the conceptual significance of the result: it converts a *fine-scale substrate construction* into a *coarse-scale emergent geometry*, with explicit refinement-stability guarantees. The tensorial transport observables of the previous paper join the Stage V Lipschitz continuum and the Stage VII universality class as established continuum-emergent features of the substrate, rather than substrate-scale technical machinery that might or might not have continuum significance.

Future stages of the programme — metric emergence (OP-2), Lorentzian completion (OP-3), Einstein-equation analogue (OP-4), gauge content (OP-9), quantisation (OP-10) — now have a refinement-stable continuum foundation to build on. They do not need to re-establish refinement stability at each stage; they can take it as given for the transport-geometric inputs ($\hat{W}_{\infty}^{\text{sub}}$, $\mathcal{R}_{\{ij\}^{\infty}}$, κ) and focus on the additional continuum-level structural content each subsequent target requires. This is the foundational role of the present paper in the geometry programme: it converts substrate-scale transport geometry into refinement-stable continuum-emergent transport geometry, supplying the foundation on which downstream programme stages can build.

9.5 What the Substrate Still Does Not Possess

The continuum-level transport observables ($\hat{W}_{\infty}^{\text{sub}}$, $\mathcal{R}_{\{ij\}^{\infty}}$, κ) are operator-valued tensors and densities on the continuum target \bar{X}_{∞} — they are *not* a metric tensor, a Levi-Civita connection, a Lorentzian causal structure, a stress-energy tensor, an Einstein-equation analogue, a gauge structure, or a quantum field theory. The "three levels of curvature structure" framing of §5.2 of the previous paper remains the correct organising picture: the present paper establishes level 2 (tensorial transport geometry) at continuum scale; level 3 (metric curvature with derived Riemann tensor, Levi-Civita connection, Einstein-type field equations) remains future work.

The most concrete next-stage targets, sharpened by the present refinement-stability result, are:

1. **Scalar projection problem revisited (§5.3–§5.4 of the previous paper, refined here as §6.2):** does the continuum-level density $\kappa(x)$ coincide with Stage VIII's $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$, or with a related projection? The (P3) candidate of §5.4 of the previous paper (holonomy-density projection) now has a refinement-stable instantiation in $\kappa(x)$; comparing $\kappa(x)$ and $R(x)$ directly is the most concrete technical target.
2. **Metric emergence (§11.8 of the previous paper):** does the Hessian of the coherence-transport distance g_{coh} , or some related construction, define a refinement-stable continuum metric tensor $g_{\{ij\}^{\infty}}$? If so, does its associated Levi-Civita connection coincide with the continuum-level transport generators $\hat{\nabla}_i^{\infty}$?
3. **Lorentzian completion (§11.2 of the previous paper):** does the refinement-step index n acquire continuum-level temporal interpretation, with the strict propagation cone becoming a Minkowski light cone? Note that the present paper's continuum limit is *spatial* refinement on a fixed-time-index substrate; a separate temporal-refinement programme is needed for Lorentzian content.
4. **Einstein-type field equation (§11.1 of the previous paper):** does the refinement-stable continuum density $\kappa(x)$ and the refinement-stable continuum-level tensor $\mathcal{R}_{\{ij\}^{\infty}}$ satisfy any continuum-level field equation? This is the principal open question of the geometry programme.

10. Epistemic Register, Limitations, and Open Problems

10.1 Epistemic Status of the Main Results

We summarise the epistemic status of the principal results:

- **Theorem 4.1 (Renormalised loop-functional convergence):** *proven* at the operator-theoretic level under the hypotheses of Definitions 2.1–2.5, modulo the substrate-level operator-theoretic estimate on the deviation of cumulative $\hat{T}^{\wedge}\{L_n\}$ products from their leading-rank Perron-projector limit (the refinement-residual factor ε_n in (4.2)), whose qualitative content (algebraic decay $\varepsilon_n \rightarrow 0$) is what is needed for the structural conclusion. The precise form of ε_n — heuristically $\mathcal{O}(h_n^{\wedge}\{1/2\})$ by analogy with central-limit fluctuation behaviour, but a rigorous determination is OP-6 — does not enter the structural-correspondence content of the theorem.
- **Theorem 5.2 (Weak-operator-topology convergence of $\mathcal{R}_{ij}^{\wedge}\{n\}$):** *proven* in the weak operator topology under the same hypotheses; strong-operator-topology convergence is not asserted and would require additional uniform-norm bounds.
- **Definition 6.1 / Proposition 6.3 (Continuum curvature density $\kappa(x)$):** *proven existence and refinement stability* of the density limit at points where $\mathcal{R}_{ij}^{\wedge}\{\infty\}$ has the explicit form (3.7) of the previous paper (boundary-shell of the defect support); we do not address pathological cases where the infinitesimal-area limit fails to exist.
- **Theorem 7.1 (Universality-class stability):** *proven* to leading order in the universality-class perturbation; higher-order corrections inherit the Stage VII universality regularity.
- **The continuum scaling identities of §8.2:** inherit the *heuristic* register from Stage IX and the previous paper at the level of the leading-order coefficient (the $v_c^{\wedge}\text{local}(x) \propto \varepsilon_{\text{gap}}(x)$ identification of Stage IX §9.1 propagates to continuum scale). The functional form $\|\mathcal{R}_{\infty}\| \sim |\nabla\varepsilon_{\text{gap}}^{\wedge\infty}| / \xi_{\infty}$ is robust; the leading prefactor remains a Born-series-rigorous target as in §11.9 of the previous paper.

10.2 Limitations

10.2.1 Spatial refinement only. The present paper refines only the *spatial* substrate X_n ; the refinement-step index n on which the Stage IX transport operator acts is kept fixed throughout. A genuine four-dimensional continuum limit would require simultaneous refinement of the temporal direction (the refinement-step index of the substrate's discrete evolution), which we do not address. This is the principal limitation: the continuum X_{∞} obtained here is a continuum *spatial* structure on the substrate, not a four-dimensional Lorentzian-signature geometry.

10.2.2 No metric tensor. The continuum-level transport observables of §§4–6 are operator-valued tensors and densities, not metric-derived objects. The construction of a continuum metric tensor — with required algebraic / differential properties (symmetry, positive definiteness, Levi-Civita compatibility) — remains future work (cf. §11.8 of the previous paper).

10.2.3 Weak rather than strong convergence. Theorem 5.2 establishes weak operator convergence $\mathcal{R}_{ij}^{\wedge}\{n\} \rightarrow \mathcal{R}_{ij}^{\wedge}\{\infty\}$. The corresponding strong-operator convergence would require uniform-norm control we have not established. For foundational refinement-stability purposes, weak convergence is sufficient; for downstream applications (e.g., direct algebraic manipulation of continuum-level transport operators), strong-norm control may eventually be required.

10.2.4 Refinement-residual rate not rigorously determined. The convergence envelope in Theorem 4.1 includes a refinement-residual factor $\varepsilon_n \rightarrow 0$ that controls the substrate-level fluctuation in cumulative $\hat{T}^{\{L_n\}}$ factors. We have argued heuristically (Remark following Theorem 4.1) that ε_n is plausibly of order $\mathcal{O}(h_n^{\{1/2\}})$, by analogy with central-limit fluctuation behaviour for long products of contractive operators — but \hat{T} is a deterministic operator, the path-ordered product is correlated via substrate geometry, and a rigorous derivation of the sharp rate is a separate operator-theoretic undertaking. The structural conclusion of Theorem 4.1 (renormalised loop functionals converge under refinement) does not depend on the precise form of ε_n ; what depends on it is the *rate* of convergence at fixed loop position. The naive Lipschitz $\mathcal{O}(h_n)$ rate is what one would obtain in the absence of cumulative-product fluctuation considerations, and is plausibly an upper bound on the true rate; the heuristically-motivated $\mathcal{O}(h_n^{\{1/2\}})$ form is plausibly the correct rate; the rigorous determination is OP-6 in §10.3.

10.2.5 Bounded defect strength. The defect strength α is assumed fixed (i.e., not refinement-dependent). Refinement-compatible families with refinement-dependent α_n (e.g., $\alpha_n \rightarrow 0$ or $\alpha_n \rightarrow \infty$ under refinement) would produce qualitatively different continuum limits — a defect "smeared out under refinement" versus a defect "concentrated under refinement"; we do not analyse these regimes.

10.2.6 Closure-mixing coupling $C = 0$. As in the previous paper, the present paper works in the minimal-coupling regime $C = 0$. Extension to $C \neq 0$ is straightforward in qualitative content but requires explicit Duhamel-style tracking (cf. Stage IX §4, §11.7 of the Tensorial Transport Geometry paper).

10.2.7 Refinement-compatible loop family hypothesis. Definition 2.4 imposes Hausdorff-distance convergence $\gamma_n \rightarrow \gamma$. Loop families violating this hypothesis (e.g., loops that wander randomly under refinement, or loops whose orientation flips under refinement) are not covered by the present analysis. For physically motivated loop families targeting fixed continuum curves, Definition 2.4 is the natural condition.

10.2.8 The convergence results depend on admissibility hypotheses. The refinement-stability conclusions of §§4–7 depend on the full stack of admissibility hypotheses laid out in §§2.1–2.5: admissible substrate refinement sequences (Definition 2.1), refinement-compatible transport-operator sequences with $v_c^{(n)} \rightarrow v_c^\infty$ tuning (Definition 2.2), refinement-compatible defects with fixed continuum support and Lipschitz fibre profiles (Definition 2.3), and refinement-compatible loop families with Hausdorff-bounded convergence and preserved orientation (Definition 2.4). These hypotheses are not vacuous, and *non-admissible* refinement procedures could in principle destroy convergence in concrete ways:

- A refinement sequence with $\gamma_n \cdot \rho(A_n) \rightarrow 0$ (failing the regime (W) tuning of Definition 2.2(b)) decouples all transport in the continuum limit, and the renormalised loop functional $\hat{W}_n^{\{\text{sub}\}}(\gamma_n)$ loses curvature content.
- A refinement-dependent defect profile with $\alpha_n \rightarrow \infty$ violates the bounded-defect hypothesis (V), and the continuum limit would no longer be a Stage VIII-type localised perturbation.

- A loop family with γ_n failing to converge in Hausdorff distance (e.g., γ_n that "wanders" with successive refinement) targets no well-defined continuum loop, and the convergence statement of Theorem 4.1 simply does not apply — $\hat{W}_n^{\subscript{(\gamma_n)}}$ might oscillate without limit.
- A refinement embedding that fails to preserve the Stage V Lipschitz continuum structure (Definition 2.1(c)/(d)) breaks the spectral-convergence machinery of Lemmas 3.1–3.3 and the bulk-resolvent convergence on which Theorem 4.1 depends.

The admissibility hypotheses are the substrate-level analogues of the standard continuum-limit hypotheses in any refinement-stability analysis (e.g., compatible lattice spacings, bounded coupling, well-defined target curves). They are *not* automatic; they constrain the class of refinement procedures for which the continuum-emergence conclusion of the present paper applies. This is appropriate and indeed necessary: a refinement-stability theorem with no admissibility hypotheses would be either trivial or false. What the present paper establishes is that *within* the natural admissibility class — the same class for which Stage V Lipschitz continuum emergence and Stage VII universality-class robustness hold — the transport observables of the Tensorial Transport Geometry paper are refinement-stable continuum-emergent objects.

10.3 Open Problems

The present paper closes the refinement-stability open problem §11.11 of the previous paper. The following open problems remain (a subset of §11.1–§11.10 of the previous paper, sharpened by the present refinement-stability result):

- **OP-1: Scalar projection identification.** Does the refinement-stable continuum density $\kappa(x)$ coincide with the Stage VIII candidate scalar curvature $R(x) = \nabla^2 \varepsilon_{\text{gap}}(x)$? If not, what is the precise relation? (The (P3) holonomy-density projection of §5.4 of the previous paper is now refinement-stably realised as $\kappa(x)$; the open question is whether $\kappa(x) = c \cdot R(x)$ for some explicit constant c , or whether the projection map involves a more elaborate functional dependence.)
- **OP-2: Metric tensor emergence.** Does the substrate transport-distance Hessian, or a related construction, define a refinement-stable continuum metric tensor? Is its associated Levi-Civita connection equal to the continuum-level transport generators?
- **OP-3: Lorentzian completion.** Does simultaneous spatial-and-temporal refinement of the substrate yield a Lorentzian continuum structure? Does the strict propagation cone of Stage IX become a Minkowski light cone in the temporally-refined limit?
- **OP-4: Einstein-equation analogue.** Does $\mathcal{R}_{\{ij\}^{\infty}}$ or $\kappa(x)$ satisfy any continuum-level field equation? What is the substrate-level stress-energy source?
- **OP-5: Strong-operator convergence.** Can Theorem 5.2 be strengthened to strong-operator convergence with explicit uniform norm bounds?
- **OP-6: Sharp determination of the refinement-residual rate ε_n .** The convergence envelope in Theorem 4.1 includes a refinement-residual factor $\varepsilon_n \rightarrow 0$ whose precise form is recorded heuristically as $\mathcal{O}(h_n^{\{1/2\}})$ by analogy with central-limit fluctuation behaviour. A rigorous operator-theoretic determination of the exponent (and of the

leading-order coefficient) — taking proper account of the deterministic, correlated, path-ordered structure of cumulative $\hat{T}^{\wedge}\{L_n\}$ products on substrate loops — is open.

- **OP-7: Refinement-dependent defects.** What continuum-level transport observables emerge from refinement-compatible defect families with refinement-dependent strength α_n ?
- **OP-8: $C \neq 0$ closure-mixing.** Systematic extension of the refinement-stability results to the $C \neq 0$ regime via Duhamel expansion.
- **OP-9: Gauge content.** Does the continuum-level loop-trace functional $\hat{W}_{\infty}^{\wedge}\{\text{sub}\}(\gamma)$ admit a gauge-theoretic interpretation, with continuum-level perturbed generators playing the role of gauge-covariant derivatives? This would connect to Stage VIII's Standard-Model-gauge-derivation work via the continuum limit established here.
- **OP-10: Quantisation.** Does the continuum-level transport geometry admit a natural quantisation, possibly via path-integration over continuum-level substrate loops weighted by $\hat{W}_{\infty}^{\wedge}\{\text{sub}\}(\gamma)$?

11. Conclusion

Programme map. The VERSF geometry programme has now developed across a sequence of structural stages: Stage V (coherent continuum emergence), Stage VII (open universality class), Stage VIII (localised coherence defects with four scalar ε_{gap} -functionals), Stage IX (global coupled transport operator with bands, finite propagation, Birman–Schwinger trapped modes, Combes–Thomas localisation, and the basic scaling identity $\xi \cdot \delta \sim v_c$), the *Tensorial Transport Geometry* paper (parallel transport, coherence holonomy, transport-curvature tensors, geodesic deviation, Wilson-type loops, curvature concentration at trapped modes, and the extended scaling identity $\|\mathcal{R}\| \sim |\nabla\varepsilon_{\text{gap}}| / \xi$), and now the present paper (refinement stability of those tensorial transport observables: renormalised holonomy convergence, weak refinement-convergence of the transport-curvature tensor, continuum transport-curvature density, universality-class stability of the continuum-limit transport geometry).

The structural advance. The shift from the previous paper to the present one is the transition from *substrate-scale tensorial transport structure* to *refinement-stable continuum-emergent tensorial transport structure*. The transport observables of the Tensorial Transport Geometry paper, established at fixed substrate scale, now have continuum-level analogues that survive refinement at explicit rates with explicit universality-class stability. The §11.11 open problem of the Tensorial Transport Geometry paper — the worry that the substrate-level transport curvature might be a lattice-scale artefact with no continuum significance — is closed affirmatively at the operator-theoretic / renormalisation-theoretic level.

Principal results. The paper establishes:

- **Refinement-compatible transport geometry.** Admissible substrate refinement sequences $X_0 \subset X_1 \subset \dots$, refinement-compatible transport operators \mathbf{T}_n with continuum-level $v_c^{\infty} \in (0, 1/2)$, refinement-compatible Stage VIII defects with fixed

continuum-level support and strength, and refinement-compatible loop families $\gamma_n \rightarrow \gamma$ — Definitions 2.1–2.5.

- **Vacuum-subtracted loop-functional convergence.** $\hat{W}_n^{\text{sub}}(\gamma_n) \rightarrow \hat{W}_\infty^{\text{sub}}(\gamma)$ under refinement, with convergence envelope dominated by the Combes–Thomas exponential factor and the refinement-residual factor $\varepsilon_n \rightarrow 0$ (precise rate recorded as OP-6) — Theorem 4.1.
- **Continuum-level loop-functional inheritance.** $\hat{W}_\infty^{\text{sub}}$ inherits the structural properties (vacuum flatness, defect non-triviality, exponential localisation, symmetry-protected vanishing) at continuum scale — Corollary 4.2.
- **Weak refinement-convergence of the transport-curvature tensor.** $\mathcal{R}_{ij}^{\{n\}} \rightarrow \mathcal{R}_{ij}^{\{\infty\}}$ in the weak operator topology, with the continuum-level $\mathcal{R}_{ij}^{\{\infty\}}$ inheriting antisymmetry, vacuum vanishing, boundary-shell support, and exponential localisation — Theorem 5.2.
- **Continuum transport-curvature density.** The infinitesimal-area limit $\kappa(x) := \lim_{\varepsilon \rightarrow 0} \hat{W}_\infty^{\text{sub}}(\gamma_\varepsilon(x)) / \varepsilon^2$ exists, is refinement-stable, and provides the (P3) holonomy-density projection of §5.4 of the previous paper in continuum form — Definition 6.1, Proposition 6.3.
- **Universality-class stability.** Continuum-level transport observables depend only on the Stage VII universality-class membership of the canonical wheel and on continuum-level transport data, not on microscopic substrate details — Theorem 7.1.
- **Continuum scaling hierarchy.** The Stage IX / Tensorial Transport Geometry scaling identities $\xi \cdot \delta \sim v_c$ and $\|\mathcal{R}\| \sim |\nabla \varepsilon_{\text{gap}}| / \xi$ persist at continuum scale with continuum values $v_c^\infty, \xi_\infty, \delta_\infty, \|\mathcal{R}\|_\infty$ — §8.2.

Scope clarification. The present framework establishes *refinement-stable continuum-emergent tensorial transport geometry*, not Lorentzian, metric, gauge, or gravitational structure. The continuum-level transport observables ($\hat{W}_\infty^{\text{sub}}$, $\mathcal{R}_{ij}^{\{\infty\}}$, κ) are operator-valued tensors and densities on a continuum spatial structure X_∞ ; they are not the Riemann tensor of a metric, not the field strength of a gauge field, and not a stress-energy tensor of an Einstein-type field equation. The "three levels of curvature structure" framing of §5.2 of the previous paper remains the organising picture: the present paper extends level 2 (tensorial transport geometry) from substrate-scale to continuum-scale, with level 3 (metric curvature with associated Riemannian / Lorentzian apparatus) as the principal future programme target.

The honest summary. This paper does not derive gravity. It establishes that the tensorial transport observables of the Tensorial Transport Geometry paper are genuine continuum-emergent geometric structures rather than lattice-scale artefacts — and therefore form a refinement-stable foundation on which subsequent programme stages can attempt the construction of metric geometry, Lorentzian completion, gauge structure, and (eventually) Einstein-type field equations and gravitational dynamics. The §11.11 open problem of the Tensorial Transport Geometry paper is closed in the affirmative; the path forward is sharpened, with concrete next-stage targets (OP-1 through OP-10) made explicit.

Final position. Through this sequence of structural stages, the VERSF geometry programme has progressed from *coherence* (V) to *robust coherence* (VII) to *defected coherence* (VIII) to *globally transported coherence* (IX) to *tensorially structured transported coherence* (the

previous paper) to *refinement-stable continuum-emergent tensorial transported coherence* (the present paper). The substrate is not merely a coherent dynamical system with tensorial transport structure at the substrate scale — it is a substrate whose tensorial transport structure has been *established as continuum-emergent*, with refinement-stable observables, universality-class robustness, and explicit convergence envelopes. The Stage VIII Defect-Coherence Principle's "one field, four functionals" has, through Stage IX's "one scaling identity" and the previous paper's "one tensorial transport geometry, one extended scaling identity", reached the form "*one field, four functionals, one tensorial transport geometry, one extended scaling identity, all refinement-stable in the continuum limit*". The transition from this refinement-stable tensorial-transport setting to a full continuum-geometric / gravitational / gauge / quantum setting remains the principal target of the programme's subsequent stages — but those subsequent stages now have a refinement-stable foundation to build on.