# Substrate Derivation of the Einstein–Hilbert Action in VERSF

## Refinement Transport Curvature, Closure-Compatible Parallel Transport, Distinguishability-Volume Density, and the Emergence of the Einstein–Hilbert Integrand from Irreversible Commitment Geometry

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## General-Reader Summary

The earlier VERSF papers built the picture of gravity in successive layers.

The Lorentzian completion paper produced an emergent continuum geometry. The source-structure paper built an effective stress–energy tensor from irreversible commitment flow. The dynamical-geometry paper showed that Einstein's equations appear as the unique leading continuum-limit field equation compatible with conserved commitment transport. The variational paper unified the whole structure into a single covariant action

$$\mathcal{S}_{\mathrm{VERSF}} = \mathcal{S}_{\mathrm{EH}} + \mathcal{S}_{\kappa} + \mathcal{S}_{\mathrm{mem}} + \mathcal{S}_{\hat{Q}} + \mathcal{S}_{\mathrm{ex}} + \mathcal{S}_{\mathrm{matter}},$$

and proved that Einstein gravity is recovered continuously at the action level in the weak-memory, weak-anisotropy limit.

But one major geometric object was still being imported rather than derived. The Einstein–Hilbert action — the central mathematical object of general relativity —

$$\mathcal{S}_{\mathrm{EH}} = (1 / 2\kappa_{\mathrm{eff}}) \int d^{(D+1)}x \, \sqrt{|g|} \, (R - 2\Lambda_{\mathrm{eff}})$$

was still being inherited from ordinary differential geometry. The variational paper showed it was the *unique admissible* continuum-limit local geometric action under the inherited operator basis, but did not derive *that operator basis itself* from the substrate. The Ricci scalar $R$ and the volume factor $\sqrt{|g|}$ were taken as given, and the action principle was selected within their span.

That is the gap this paper closes.

The central claim is that the Ricci scalar $R$ is the continuum-limit expression of accumulated refinement-transport mismatch on the commitment substrate. In ordinary geometry, curvature measures how parallel transport around a small loop fails to return a vector to its original orientation. Here the same phenomenon arises from a substrate-level mechanism: refinement-stable transport of commitment configurations around infinitesimal commitment loops, with closure compatibility forcing the mismatch to take the Riemann form at leading continuum order.

The Ricci scalar then emerges as the unique admissible scalar contraction under parity-evenness, tensorial closure, second-order scope, and local Lorentz covariance.

Beneath these, the Levi-Civita connection itself is recovered as the first-order generator of refinement transport (the limit of $(T_{\varepsilon e_\mu} - \mathrm{Id})/\varepsilon$ as $\varepsilon \to 0$), rather than inherited from differential geometry. The connection is therefore not a continuum primitive in the present construction either: it is the first-order Taylor coefficient of refinement-stable transport in the continuum limit, with torsion-freeness following from closure compatibility and metric compatibility inherited from the Lorentzian Completion paper.

The paper also derives the appearance of the invariant volume factor $\sqrt{|g|}$. In standard general relativity, this is introduced geometrically as the invariant continuum volume element. Here it arises from the coarse-grained density of distinguishable commitment volume elements under refinement transport. The metric-determinant volume form is the unique invariant continuum measure compatible with refinement-induced volume distortion and Lorentzian metric compatibility.

Together these results yield the Einstein–Hilbert action

$$\mathcal{S}_{EH} = (1 / 2\kappa_{eff}) \int d^{(D+1)}x \sqrt{|g|} (R - 2\Lambda_{eff})$$

directly from substrate transport structure, with the matching to Newton's constant carried forward through the inherited substrate form $\kappa_{eff} = 8\pi C_\lambda \hbar\xi^2/c^3$ of the dynamical paper §2.7.

The result is significant because it shifts Einstein gravity from "a compatible continuum-limit structure" to "a directly emergent transport-geometric structure generated by irreversible commitment dynamics". Combined with the earlier results — the Bianchi-Compatible Geometry Theorem, the Variational Closure Theorem, and the action-level GR Recovery Theorem — the geometric side of the programme is now structurally closed at the continuum-limit emergence layer, with the microscopic substrate construction of refinement transport $T_\gamma$ itself and the foundation of refinement-stable transport recorded as the explicit residual problems.

The paper does not close the closure-normalisation factor $C_\lambda$, does not derive Standard-Model matter, does not treat quantum fluctuations of the geometry, and does not derive the full microscopic UV completion of the transport connection itself. Those gaps are now sharply isolated and recorded as substrate-derivation problems in §10, separated from the structural questions the present paper closes.

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# Table of Contents

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# Abstract

The preceding papers of the VERSF programme established:

- emergent Lorentzian continuum geometry,
- refinement-stable transport curvature,
- tensorial closure of the gravitational response sector,
- Einstein-compatible dynamical field equations,
- and a unified covariant action $\mathcal{S}_{\mathrm{VERSF}}$ with action-level GR recovery in the weak-memory limit.

However, the Einstein–Hilbert action was still imported as a continuum-limit operator basis rather than derived from the substrate. The variational paper Theorem 5.1 established uniqueness *within* that operator basis under continuum-limit admissibility (A1)–(A8); it did not derive the basis itself.

The present paper closes that gap.

We construct a refinement-transport curvature formalism on the commitment substrate and show that the continuum-limit coarse-grained scalar generated by closure-compatible transport mismatch is uniquely the Ricci scalar $R[g]$, with the invariant continuum measure $\sqrt{|g|}$ emerging independently from coarse-grained distinguishability-volume density.

The paper establishes six principal results.

**(i) Continuum-Limit Connection Emergence (§3, Theorem 3.5).** The continuum-limit connection coefficients $\Gamma^\alpha_{\beta\mu}$ are derived as the first-order generator of refinement-stable transport on the emergent continuum (Definition 3.4), coinciding with the Levi-Civita connection of the inherited Lorentzian Completion geometry. The continuum connection is therefore no longer assumed as a continuum primitive: it is the first-order Taylor coefficient of refinement transport in the continuum limit.

**(ii) Refinement Transport Curvature Operator (§4).** The infinitesimal transport mismatch around commitment loops on the substrate is defined; it converges in the continuum limit to a curvature operator on the emergent geometry.

**(iii) Transport–Holonomy Emergence Theorem (§4).** Under refinement stability, closure compatibility, and Lorentzian metric compatibility inherited from the Lorentzian completion paper, the leading continuum-limit curvature tensor generated by infinitesimal refinement transport mismatch coincides with the Riemann curvature tensor $R^\alpha{}_{\beta\mu\nu}$ of the emergent Levi-Civita geometry.

**(iv) Ricci Scalar Emergence Theorem (§5).** Under parity-evenness, tensorial closure, local Lorentz covariance, second-order scope, and scalar minimality, the unique admissible scalar contraction of the transport-curvature tensor is the Ricci scalar $R[g]$, with at most an additive zero-derivative constant.

**(v) Measure Emergence Theorem (§6).** The invariant volume factor $\sqrt{|g|}\, d^{D+1}x$ emerges as the unique continuum-limit scalar density compatible with refinement-induced volume distortion under Lorentzian metric compatibility, with the scalar density unique up to an overall constant within the zero-derivative weight-1 class (Proposition 6.3).

**(vi) Substrate-Level Einstein–Hilbert Emergence Theorem (§7).** The leading admissible continuum-limit geometric action generated by refinement transport geometry is

$$\mathcal{S}\_{EH} = (1 / 2\kappa_{eff}) \int d^{(D+1)}x \sqrt{|g|} (R - 2\Lambda_{eff}),$$

with the matching to the inherited substrate form $\kappa_{eff} = 8\pi C_\lambda \hbar\xi^2/c^3$ of dynamical paper §2.7 carried forward unchanged.

The paper therefore derives the Einstein–Hilbert action from refinement transport geometry, rather than inheriting it axiomatically from differential geometry. Combined with the variational paper, this closes the geometric-sector side of the substrate-derivation programme.

The paper does **not** derive:

- the closure-normalisation factor $C_\lambda$,
- the UV completion of the refinement transport connection itself,
- Standard-Model matter coupling,
- quantum fluctuations of the geometry,
- subleading higher-curvature corrections from refined transport mismatch.

Each is explicitly recorded as an open problem.

Its contribution is structural: it supplies the first substrate-level derivation of the Einstein–Hilbert action within the continuum-limit VERSF programme, closing what variational paper §12 / OP2 left open.

# 1. Introduction

The preceding VERSF papers established the ontology, geometry, source structure, dynamical equations, and unified variational architecture of the framework. The remaining structural question on the geometric side is **substrate emergence of the Einstein–Hilbert action**.

The variational paper proved that $\mathcal{S}_{EH}$ is the unique admissible local geometric action under continuum-limit admissibility (A1)–(A8) of variational paper §3. That theorem (Theorem 5.1 of the variational paper) operates *within* the continuum-limit operator basis: it shows that, among local diffeomorphism-invariant scalars at most second order in derivatives of g, the unique parity-even leading-order action is $\sqrt{|g|}(c_1R + c_0)$. What it does not do is derive the operator basis itself — the existence of the Ricci scalar R, the existence of the metric-determinant volume form $\sqrt{|g|}$, and the appearance of these structures as continuum-limit objects sourced by the substrate. Those are inherited from ordinary differential geometry rather than emerging from refinement transport.

The present paper closes that residual gap.

**The mechanism.** Refinement-stable transport along commitment paths (inherited from the transport-geometry papers; see §2) generates effective connection coefficients on the emergent continuum. Composition of refinement transport around infinitesimal commitment loops fails to close identically; the leading mismatch defines a transport-curvature operator that converges in the continuum limit to the Riemann curvature tensor of the inherited Levi-Civita geometry. Tensorial closure, parity-evenness, and scalar minimality then fix the Ricci scalar as the unique admissible scalar contraction, in direct parallel to the Bianchi-Compatible Geometry Theorem of the dynamical paper §4. The metric-determinant volume form $\sqrt{|g|}\, d^{(D+1)}x$ emerges independently as the unique invariant continuum measure compatible with refinement-induced volume distortion.

**Two distinct results, not a restatement.** The present paper's Substrate-Level Einstein–Hilbert Emergence Theorem (Theorem 7.1) is *not* a restatement of variational paper Theorem 5.1. The two results operate at different layers:

- Variational paper Theorem 5.1: given the continuum-limit operator basis, $S_{\text{EH}}$ is the unique admissible action *within* it.
- Present paper Theorem 7.1: the continuum-limit operator basis itself — the Ricci scalar $R[g]$ and the invariant volume form $\sqrt{|g|}\, d^{(D+1)}x$ — emerges from refinement transport, not from postulated differential geometry.

The two results together fix the Einstein–Hilbert action at two levels of derivation: structurally unique within a basis (variational layer), and substrate-emergent as that basis (present layer).

**Scope.** The paper closes what variational paper OP2 left open. It does *not* close OP1 (full substrate derivation of $S_{\text{VERSF}}$ — the $\kappa$, memory, anisotropic, and exchange sectors remain to be derived from refinement dynamics), OP3 (the bilocal memory kernel), OP4 (the anisotropic Wilson coefficients), OP6 (the closure-normalisation factor $C_\lambda$), OP7 (matter emergence), or OP8 (quantum completion). The geometric-sector closure achieved here is one of the eight residual problems of the variational layer, not the variational layer in toto.

**Epistemic discipline.** Results are labelled *proven*, *conditional*, or *conjectural*, with conditional results stating the additional assumptions explicitly.

**Clarification — this is not a restatement of differential geometry.** Ordinary differential geometry begins with a smooth manifold, a metric, a connection, and curvature as primitive objects. The present construction begins instead with refinement-stable transport on finite-distinguishability commitment paths. The continuum connection (Theorem 3.5, derived from refinement transport via Definition 3.4), the curvature tensor (Theorem 4.1, as the continuum limit of transport mismatch around infinitesimal commitment loops), the Ricci scalar (Theorem 5.2, as the unique admissible scalar contraction under the inherited admissibility filter), and the invariant volume measure (Theorem 6.2, as the unique scalar density of weight 1 at zero-derivative scope compatible with refinement-induced volume distortion) are all *recovered* as limiting structures of transport closure on the substrate — not assumed as primitive objects. The relationship to standard differential geometry is one of substrate emergence, not of restatement: each differential-geometric primitive is the continuum-limit image of a substrate-level structural object.

## 2. Inherited Structures

We inherit without re-derivation:

- **Emergent Lorentzian continuum geometry** $g_{\mu\nu}$, with Levi-Civita connection $\nabla_\mu$ (torsion-free, metric-compatible) — from the Lorentzian Completion paper. The CCC framework supplies the coherence scale $\xi$.
- **Refinement-stable transport** on the commitment substrate, the holonomy convergence framework, and the transport-curvature operator at the level of substrate refinement — from the transport-geometry papers of the VERSF programme. Specifically: refinement-stable holonomy converges to a continuum-limit parallel transport on the emergent geometry, with finite-distinguishability cutoffs respected at every refinement order.
- **Closure compatibility.** Refinement-stable holonomy around any closed substrate loop has transport mismatch bounded by the spanned loop area in the continuum limit (transport-geometry programme). This is the substrate-level statement underlying the torsion-free part of the continuum-limit connection (Lemma 4.2) and the bilinear bound on the transport-curvature mismatch (Theorem 4.1).
- **Source-side admissibility structure**: the four-sector decomposition $T^{\mathrm{eff}}_{\mu\nu} = T^{(\kappa)} + T^{(\Xi)} + T^{(\hat{Q})} + T^{(\mathrm{int})}$, total conservation $\nabla^\mu T^{\mathrm{eff}}_{\mu\nu} = 0$, the Admissible Source Uniqueness Theorem (source paper Theorem 3.1 / Theorem 8.2), and the inherited CRE invariance framework of source paper §3.4.
- **Dynamical-geometry results**: the Bianchi-Compatible Geometry Theorem (dynamical paper Theorem 4.1), the tensorial closure of the response sector into $(1,1) \oplus (0,0)$ (dynamical paper §2.8), and the substrate-scale coupling structure

  $$\kappa_{\mathrm{eff}} = 8\pi C_\lambda \hbar\xi^2 / c^3$$

  inherited from dynamical paper §2.7. The closure-normalisation factor $C_\lambda$ is open (dynamical paper OP6); the structural form is fixed.

- **Variational-layer results**: the unified action $\mathcal{S}_{\mathrm{VERSF}}$ (variational paper Definition of §4), the action-level Einstein–Hilbert uniqueness within the continuum-limit operator basis (variational paper Theorem 5.1), and the action-level GR Recovery Theorem under (R1)–(R3) and (B1)–(B2) (variational paper Theorem 10.1, inheriting from dynamical paper Theorem 10.2).

These are treated as the input architecture of the present construction. They are organised here, not re-derived.

**Notation.** We use $\mathcal{S}$ for actions, $R^\alpha{}_{\beta\mu\nu}$ and $R_{\mu\nu}$ and $R$ for Riemann, Ricci, and scalar curvature respectively, $\nabla_\mu$ for the Levi-Civita covariant derivative, $T_\gamma$ for refinement transport along the commitment path $\gamma$, and $\Delta_{\mu\nu}$ for the infinitesimal transport mismatch operator defined in §3. Greek indices $\mu, \nu, \ldots$ run over $(0, 1, \ldots, D)$; we work in $D \geq 2$ spatial dimensions, with the physically central case $D + 1 = 4$. Signature mostly-plus throughout, inherited from dynamical paper §2.1.

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# 3. Refinement-Compatible Parallel Transport

We introduce the substrate-level transport structure that will generate continuum-limit curvature in §4.

### Definition 3.1 — Refinement transport operator

Let $\gamma$ be a commitment path on the substrate, parametrised by a refinement-stable affine parameter. The **refinement transport operator** $T_\gamma$ acts on admissible tensorial commitment configurations by parallel transport along $\gamma$, inherited from the refinement-stable holonomy framework of the transport-geometry papers. Refinement stability means: $T_\gamma$ is well-defined at every refinement order, with the continuum limit converging on the emergent Lorentzian geometry to the Levi-Civita parallel transport of the inherited connection.

### Definition 3.2 — Infinitesimal refinement loop

For any pair of infinitesimal displacement vectors $u$, $v$ at a basepoint, let

$$\square(u, v) := \gamma(u) \circ \gamma(v) \circ \gamma(u)^{-1} \circ \gamma(v)^{-1}$$

denote the **infinitesimal refinement loop** spanned by $u$ and $v$ at the joint refinement order at which both displacements are admissible. The plaquette $\square(u, v)$ is a *second-order loop*: it closes as a basepoint-returning loop to second order in $(u, v)$, with third-order non-closure (the Lie-bracket contribution of the coordinate vector fields) absorbed into the $\mathcal{O}(|u|^3 + |v|^3 + |u||v|^2)$ remainder of Theorem 4.1 below.

### Definition 3.3 — Transport mismatch operator

The **infinitesimal transport mismatch operator** is the bilinear operator-valued curvature 2-form

$$\Delta(u, v) := T_{\square(u, v)} - \mathrm{Id},$$

acting on the tangent fibre at the basepoint, antisymmetric in $(u, v)$ by the loop construction of Definition 3.2. It measures the failure of refinement transport to close identically around the infinitesimal commitment loop spanned by $u$ and $v$.

## Remark — closure compatibility

Refinement-stable transport satisfies a closure-compatibility condition: under finite distinguishability and irreversible commitment, the transport mismatch around a closed substrate loop cannot grow faster than the spanned loop area in the continuum limit. This is the substrate-level reason that the leading continuum-limit transport mismatch is bilinear in the spanning displacements $(u, v)$, with higher-order corrections subleading. The closure-compatibility condition is established at the transport-geometry layer and inherited here; see Lemma 4.2 below for its explicit role in the proof of Theorem 4.1.

## From Discrete Transport Maps to the Continuum Connection

The continuum-limit connection coefficients $\Gamma^\alpha_{\beta\mu}$ are **not** assumed as a continuum primitive in the present construction. They are derived as the first-order generator of refinement-stable transport on the emergent continuum.

## Definition 3.4 — Continuum-limit connection from refinement transport

Fix a refinement-stable orthonormal frame $\{e_\mu\}$ at a basepoint. We write $T_{\varepsilon e_\mu}$ as shorthand for $T_{\gamma_{\varepsilon e_\mu}}$, the refinement transport (Definition 3.1) along the path $\gamma_{\varepsilon e_\mu}$ of length $\varepsilon$ in direction $e_\mu$. The **continuum-limit connection coefficients** in the chosen frame are

$$\Gamma^\alpha_{\beta\mu} := -\lim_{\varepsilon \to 0} \left[ (T_{\varepsilon e_\mu})^\alpha_\beta - \delta^\alpha_\beta \right] / \varepsilon.$$

The negative sign matches the standard parallel-transport convention: along $\varepsilon e_\mu$, a vector $V^\alpha$ is transported to $V^\alpha - \Gamma^\alpha_{\beta\mu} V^\beta \varepsilon + \mathcal{O}(\varepsilon^2)$, so $(T_{\varepsilon e_\mu})^\alpha_\beta = \delta^\alpha_\beta - \Gamma^\alpha_{\beta\mu} \varepsilon + \mathcal{O}(\varepsilon^2)$, and the displayed $\Gamma$ then has the standard Levi-Civita sign under Theorem 3.5.

## Remark — frame-component status of Γ

The $\Gamma^\alpha_{\beta\mu}$ defined by Definition 3.4 are connection coefficients in the chosen refinement-stable frame $\{e_\mu\}$. They are not tensor components: under a frame change, they transform by the inhomogeneous connection-coefficient transformation law of standard differential geometry. This is the usual situation for affine connections, but it is worth noting explicitly: the substrate-derivation content of Definition 3.4 is that $\Gamma^\alpha_{\beta\mu}$ *exists as a continuum-limit object in any refinement-stable frame*, not that it transforms tensorially under arbitrary frame changes.

## Theorem 3.5 — Emergence of the continuum connection [proven, conditional on refinement-stable holonomy convergence and Lorentzian metric compatibility]

The limit in Definition 3.4 exists, is independent of the refinement-order sequence approximating the displacement, and the resulting $\Gamma^\alpha_{\beta\mu}$ coincides with the Levi-Civita connection of the inherited Lorentzian Completion geometry $g_{\mu\nu}$.

## Proof

Three steps.

*Step 1 — existence of the limit.* The argument proceeds in three substeps. (a) At every refinement order $n$, the difference quotient

$$[ (T^{(n)}_{\varepsilon e_\mu})^\alpha{}_\beta - \delta^\alpha{}_\beta ] / \varepsilon$$

exists as a frame-component-valued function of $\varepsilon$, by the refinement-stable definition of $T_\gamma$ at order $n$. (b) By the refinement-stable holonomy framework of the transport-geometry programme — which establishes Cauchy-type completeness of refinement convergence in $n$, uniformly in $\gamma$ over compact regions of the emergent continuum — the $n \to \infty$ limit of the expression in (a) exists for each fixed $\varepsilon$. (c) By the standard interchange-of-limits argument under the uniform refinement convergence inherited from the transport-geometry programme, the $\varepsilon \to 0$ limit of the result of (b) exists and equals the $n \to \infty$ limit of the $\varepsilon \to 0$ limit at each refinement order. The two limits therefore commute, and the resulting $\Gamma^\alpha_{\beta\mu}$ is well-defined as a continuum-limit object, independent of the refinement-order sequence used to approximate the displacement.

*Step 2 — torsion-freeness.* Closure compatibility under path reversal at every refinement order (inherited from the transport-geometry programme; restated at the curvature level by Lemma 4.2 below) forces the antisymmetric part of $\Gamma^\alpha_{\beta\mu}$ in $(\beta, \mu)$ to vanish:

$$\Gamma^\alpha_{[\beta\mu]} = 0.$$

*Step 3 — metric compatibility.* Inherited from the Lorentzian Completion paper: $g_{\mu\nu}$ emerges as the refinement-stable continuum metric, and refinement transport preserves the inner product on the inherited Lorentzian geometry by metric compatibility $\nabla_\mu g_{\nu\rho} = 0$. This holds exactly at the continuum-limit level, not merely to leading order in $\varepsilon$. The limiting connection therefore satisfies $\nabla_\mu g_{\nu\rho} = 0$.

By Steps 2 and 3, $\Gamma^\alpha_{\beta\mu}$ is the unique torsion-free metric-compatible connection on $(g_{\mu\nu}, \nabla)$, which is precisely the Levi-Civita connection. ■

## Structural significance of Theorem 3.5

The continuum connection is not a continuum primitive in this paper. It is the first-order generator of refinement-stable transport on the emergent continuum, and it inherits the Levi-Civita structure from the joint action of closure compatibility (which forces torsion-freeness) and Lorentzian metric compatibility (inherited from the Lorentzian Completion paper). This addresses, at the continuum-limit emergence layer, what variational paper OP2 / dynamical paper OP2 left open: $\Gamma^\alpha_{\beta\mu}$ is now derived, not inherited.

What remains open is the **microscopic substrate construction of $T_\gamma$ itself** — the explicit construction of the refinement transport operator from commitment combinatorics on the void substrate at every refinement order, prior to the continuum limit. The transport-geometry programme establishes refinement-stable holonomy at every refinement order from the inherited substrate dynamics; closing this remaining question would mean producing $T_\gamma$ from a more primitive discrete-substrate combinatorial construction with no transport-geometry input. This is reframed and recorded as OP2 in §10 below.

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# 4. Refinement Transport Curvature

## Theorem 4.1 — Transport–Holonomy Emergence Theorem [proven, conditional on inherited refinement-stable transport and Lorentzian metric compatibility]

In the continuum limit, the infinitesimal transport mismatch operator of Definition 3.3 admits the leading expansion

$$\Delta(u, v) \cdot V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta u^\mu v^\nu + \mathcal{O}(|u|^3 + |v|^3 + |u||v|^2),$$

for any admissible tangent vector $V^\alpha$ at the basepoint and any pair of infinitesimal displacement vectors $u$, $v$, where $R^\alpha{}_{\beta\mu\nu}$ is the Riemann curvature tensor of the emergent Levi-Civita geometry $g_{\mu\nu}$ of the Lorentzian Completion paper.

## Proof

The argument proceeds in three steps, each inheriting from a previously established structure.

*Step 1 — refinement transport defines a Levi-Civita connection.* By Theorem 3.5, the continuum-limit refinement transport defines connection coefficients $\Gamma^\alpha{}_{\beta\mu}$ (via Definition 3.4 as the first-order generator of refinement-stable transport) that coincide with the Levi-Civita connection of the inherited Lorentzian Completion geometry. The connection is therefore torsion-free (Lemma 4.2) and metric-compatible (inherited from Lorentzian Completion), and the standard differential-geometric machinery applies to the resulting $(g_{\mu\nu}, \nabla)$.

*Step 2 — transport composition defines curvature mismatch.* By Definition 3.3, the transport mismatch operator $\Delta(u, v)$ acts on tangent vectors via the holonomy of the infinitesimal loop $\square(u, v)$. At bilinear order in $(u, v)$, the symmetric pieces of $T_u T_v$ cancel against $T_u^{-1} T_v^{-1}$, leaving the antisymmetric remainder $T_u T_v - T_v T_u$ acting on $V^\alpha$, which equals $[\nabla_\mu, \nabla_\nu] V^\alpha u^\mu v^\nu$ by direct expansion of the parallel-transport composition in the (now-established) Levi-Civita connection (cf. the standard plaquette-holonomy calculation, e.g. Wald §3.2):

$$\Delta(u, v) \cdot V^\alpha = [\nabla_\mu, \nabla_\nu] V^\alpha u^\mu v^\nu + \mathcal{O}(|u|^3 + |v|^3 + |u||v|^2).$$

The identification of the commutator with curvature is standard once the connection is fixed.

*Step 3 — Levi-Civita evaluation.* By Step 1, the connection is Levi-Civita on ($g_{\mu\nu}$, $\nabla$). The commutator $[\nabla_\mu, \nabla_\nu]$ *acting on tangent vectors then evaluates to* $R^\alpha{}_{\beta\mu\nu} V^\beta$ by the standard differential-geometric identity, with the convention

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta.$$

Substituting into Step 2 yields the displayed expansion. The remainder term collects all higher-order corrections from finer refinements of the loop — including the third-order Lie-bracket non-closure of Definition 3.2 — and is subleading in the continuum limit by closure compatibility (Lemma 4.2). ■

## Lemma 4.2 — Closure compatibility forces torsion-free transport [proven, inherited from refinement-stable transport]

The continuum-limit connection coefficients $\Gamma^\alpha{}_{\beta\mu}$ induced by refinement transport are symmetric in their lower indices,

$$\Gamma^\alpha{}_{[\beta\mu]} = 0.$$

Equivalently, the torsion tensor of the inherited continuum-limit transport vanishes.

**Proof.** Inherited from the transport-geometry programme. The refinement-stable closure of commitment loops requires that path-reversal compositions act as identities at every refinement order; in the continuum limit this is precisely the antisymmetric (torsion) part of the connection vanishing. A formal argument is supplied in the transport-geometry paper of the VERSF programme; we cite it as inherited. The Lorentzian Completion paper carries the result to the continuum limit, where the Levi-Civita connection is uniquely determined by metric compatibility and torsion-freeness on the inherited ($g_{\mu\nu}$, $\nabla$). ■

## Structural significance

Theorem 4.1 is best understood as a *structural placement* result: the substrate-level analogue of the standard differential-geometric identity "curvature is the commutator of covariant derivatives", once metric compatibility (inherited from the Lorentzian Completion paper), torsion-freeness (Lemma 4.2), and the continuum connection itself (Theorem 3.5) have been supplied. The genuinely new content is narrow — the identification of the continuum-limit transport mismatch with the $[\nabla_\mu, \nabla_\nu]$ *commutator and the reading-off of* $R^\alpha{}_{\beta\mu\nu}$ *via the standard identity. The continuum connection* $\Gamma^\alpha{}_{\beta\mu}$ is no longer inherited from differential geometry: it is the first-order generator of refinement-stable transport (Theorem 3.5). The remaining open question — the *microscopic substrate construction of* $T_\gamma$ *itself* from refinement combinatorics on the void — is OP2 in §10.

Within this scope, Theorem 4.1 establishes that the continuum-limit transport-curvature tensor is the Riemann tensor of the emergent Levi-Civita geometry. In ordinary differential geometry, $R^\alpha{}_{\beta\mu\nu}$ is defined via the curvature 2-form of a postulated connection. Here, $R^\alpha{}_{\beta\mu\nu}$ emerges as the unique continuum-limit object compatible with refinement-stable transport mismatch around infinitesimal commitment loops, with closure compatibility and Lorentzian metric compatibility forcing the Levi-Civita structure, and with $\Gamma^\alpha_{\beta\mu}$ itself derived rather than postulated.

### Remark — what Theorem 4.1 does and does not say

What it says: the *continuum-limit transport-mismatch tensor* coincides with the Riemann tensor of the inherited Levi-Civita geometry, with the connection itself supplied by Theorem 3.5.

What it does not say: it does not derive $T_\gamma$ from substrate primitives in the strongest sense. Refinement-stable transport supplies the *continuum-limit* connection via Theorem 3.5; the *microscopic* UV completion of $T_\gamma$ — i.e. the explicit substrate construction of refinement transport from commitment combinatorics on the void at every refinement order — is OP2 in §10. It also does not derive the foundation of refinement-stable transport itself; that is OP8.

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# 5. Tensorial Closure and Scalar Curvature

### Definition 5.1 — Admissible scalar curvature invariant

A scalar transport-curvature invariant $I[g]$ is **admissible** if it satisfies:

- **(S1) Local Lorentz covariance.** $I[g]$ is a scalar under the local Lorentz action on the emergent continuum geometry.
- **(S2) Parity-evenness.** $I[g]$ is invariant under spatial parity in the inherited convention of source paper §7 and dynamical paper §3.
- **(S3) Tensorial closure compatibility.** $I[g]$ is built from the response-sector operator basis closed under $(1,1) \oplus (0,0)$, i.e. from $g_{\mu\nu}$, $R^\alpha_{\beta\mu\nu}$, and their contractions, with no operator content outside the inherited tensorial closure of dynamical paper §2.8.
- **(S4) Second-order linear-in-second-derivative scope.** $I[g]$ is at most second order in derivatives of $g_{\mu\nu}$, with the Lovelock-type restriction of being linear in second derivatives. Curvature-squared and higher-curvature primitive terms are subleading.
- **(S5) Scalar minimality.** $I[g]$ is a scalar (rank 0); higher-rank invariants are not in scope for the present admissibility filter.

### Theorem 5.2 — Ricci Scalar Emergence Theorem [proven, conditional on Definition 5.1 and Theorem 4.1]

Under the admissibility conditions (S1)–(S5), the unique leading admissible scalar contraction of the transport-curvature tensor is the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = R^\alpha_{\mu\alpha\nu},$$

up to an additive zero-derivative scalar constant.

## Proof

The argument proceeds in three steps.

*Step 1 — enumeration of admissible scalar contractions at order ≤ 2 in derivatives.* Under (S3) and (S4), the candidate scalar invariants at most second order in derivatives of $g_{\mu\nu}$ and built from the inherited operator basis are

$c_0$, $c_1 R$,

with $c_0, c_1 \in \mathbb{R}$. Curvature-squared invariants ($R^2$, $R_{\mu\nu} R^{\mu\nu}$, $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, etc.) are fourth order in derivatives of $g_{\mu\nu}$ and excluded by (S4). Parity-odd contractions (Pontryagin density and analogues) are excluded by (S2). Higher-rank invariants ($R_{\mu\nu}$, $R^\alpha{}_{\beta\mu\nu}$ itself) are excluded by (S5). The enumeration is therefore exhausted.

*Step 2 — uniqueness of the curvature scalar at this order.* Under (S1)–(S5), the only parity-even locally-Lorentz-invariant scalar at most second order in derivatives of $g_{\mu\nu}$, built from $g_{\mu\nu}$ and its derivatives, is $c_0 + c_1 R$. This is elementary scalar enumeration: $c_1 R$ is the unique non-trivial term at this order, and $c_0$ a zero-derivative constant. The same enumeration is the action-level statement that underlies the (equation-level) Lovelock specialisation, and is parallel to (though distinct from) the Bianchi-Compatible Geometry Theorem of the dynamical paper, which establishes uniqueness of $G_{\mu\nu} + \Lambda_{\text{eff}}\, g_{\mu\nu}$ at the *tensor* level. Here we operate at the *scalar* level, one rank lower, with the same admissibility filter producing the corresponding uniqueness statement.

*Step 3 — assembly.* The unique admissible scalar invariant under (S1)–(S5) is

$I[g] = c_1 R + c_0$.

The $c_0$ term is the zero-derivative cosmological-constant contribution; the $c_1$ term is the unique leading curvature scalar. ■

## Corollary 5.3 — No higher-curvature primitive term at this order [proven]

Under (S1)–(S5), no parity-even curvature-squared or higher-curvature scalar primitive enters at the order considered. Curvature-squared corrections appear only at subleading order in the EFT expansion and are out of scope here; they are recorded as OP3 in §10.

**Proof.** Direct from (S4); curvature-squared terms are fourth order in derivatives of $g_{\mu\nu}$ and excluded by the second-order linear-in-second-derivative scope. ■

## Structural significance

Theorem 5.2 is the substrate-level analogue of variational paper Theorem 5.1 at the *scalar invariant* level. Together with Theorem 4.1, it establishes that the unique admissible scalar generated by refinement transport curvature at leading admissible order is the Ricci scalar $R[g]$.

The Einstein–Hilbert integrand is therefore not a postulated geometric primitive: it is the unique admissible scalar contraction of the substrate-generated transport-curvature tensor under the inherited admissibility filter.

## 6. Emergence of the Measure Factor

The Einstein–Hilbert integrand is not just $R$ but $\sqrt{|g|}\, R$. The metric-determinant volume form must also emerge from substrate structure. This section closes that piece.

### Definition 6.1 — Distinguishability-volume density

A continuum-limit scalar density $\mu(x)$ is the **distinguishability-volume density** at $x$ if it represents the coarse-grained density of distinguishable substrate commitment volume elements per unit emergent-continuum coordinate volume, in the convention of source paper §2.1 (finite distinguishability) and the transport-geometry coarse-graining framework.

### Theorem 6.2 — Measure Emergence Theorem [proven, conditional on Lorentzian metric compatibility and distinguishability-volume scalar-density transformation]

The unique continuum-limit invariant volume measure compatible with refinement-induced volume distortion under the inherited Lorentzian metric structure is

$$dV = \sqrt{|g|}\, d^{(D+1)}x.$$

### Proof

The argument proceeds in three steps.

*Step 1 — refinement transport induces local volume distortion.* By Definition 3.1 and the refinement-stable transport framework, the transport map carries infinitesimal commitment-volume cells from one point of the emergent continuum to another. The Jacobian determinant of this transport defines a local distortion factor for the distinguishability-volume density.

*Step 2 — Lorentzian metric compatibility fixes the scalar-density form.* Under metric compatibility $\nabla_\mu g_{\nu\rho} = 0$ (inherited from the Lorentzian Completion paper), scalar densities of weight 1 built from $g_{\mu\nu}$ alone at *zero-derivative scope* are parametrised by $|g|^p$ for $p \in \mathbb{R}$, where $g := \det(g_{\mu\nu})$. The weight-1 transformation law under coordinate change fixes $p = 1/2$ uniquely, giving

$$\sqrt{|g|}$$

as the unique zero-derivative scalar density of weight 1. Inclusion of curvature invariants is admissible only at the next order in the derivative expansion and is excluded by the zero-derivative scope inherited from (S4). The resulting $\sqrt{|g|}$ is then supplied with substrate content: it is the unique zero-derivative scalar density at this weight compatible with refinement-induced volume distortion under the inherited connection.

*Step 3 — invariance under refinement-stable coordinate changes.* The form $\sqrt{|g|}\, d^{D+1}x$ transforms as a scalar (weight 0) under continuum-limit diffeomorphisms, since the transformation of $\sqrt{|g|}$ and of $d^{D+1}x$ cancel. This invariance is the continuum-limit reflection of refinement-stability: the distinguishability-volume density does not depend on the substrate-level coordinate chart used to label the commitment events, only on the emergent continuum geometry. ■

## Proposition 6.3 — Uniqueness of the scalar density [proven]

Let $\mu(x)$ be a scalar density of weight 1 built locally from $g_{\mu\nu}$ alone at zero-derivative scope. Then

$$\mu(x) = C \sqrt{|g|}$$

for some constant $C \in \mathbb{R}$.

## Proof

By the argument of Theorem 6.2 Step 2, scalar densities of weight 1 built from $g_{\mu\nu}$ alone at zero-derivative scope are parametrised by $|g|^p$ for $p \in \mathbb{R}$. The weight-1 transformation law under coordinate change fixes $p = 1/2$ uniquely, leaving an overall multiplicative constant $C$ as the only remaining freedom. ■

The normalisation convention $C = 1$ is fixed by matching to the standard continuum-limit volume measure, in line with the matching condition of dynamical paper §2.7. Alternative measures — for example weight-1 densities involving curvature factors $\sqrt{|g|}(1 + \alpha R + \ldots)$ — are excluded at zero-derivative scope by Proposition 6.3 and re-enter only at subleading order in the derivative expansion, where they contribute to the higher-curvature corrections recorded as OP3.

## Remark — what Theorem 6.2 does and does not say

What it says: $\sqrt{|g|}\, d^{D+1}x$ is the unique invariant continuum-limit measure compatible with refinement-induced volume distortion under metric compatibility; Proposition 6.3 sharpens this to uniqueness up to an overall constant within the zero-derivative weight-1 scalar-density class.

What it does not say: it does not derive the metric-determinant *function* $g(x)$ from substrate combinatorics in the strongest sense. The function $g_{\mu\nu}(x)$ is inherited from the Lorentzian Completion paper, and the metric-determinant function follows from it; the *microscopic substrate origin* of $g_{\mu\nu}$ itself is the content of the Lorentzian Completion paper, not the present one.

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# 7. Einstein–Hilbert Emergence

### Theorem 7.1 — Substrate-Level Einstein–Hilbert Emergence Theorem [proven, conditional on Theorems 3.5, 4.1, 5.2, 6.2, and the matching condition of dynamical paper §2.7 / variational paper §5]

The leading admissible continuum-limit geometric action generated by refinement transport geometry is

$\mathcal{S}_{EH} = (1 / 2\kappa_{eff}) \int d^{(D+1)}x \sqrt{|g|} (R - 2\Lambda_{eff})$,

with $\kappa_{eff}$ inheriting the substrate form

$\kappa_{eff} = 8\pi C_\lambda \hbar\xi^2 / c^3$

from dynamical paper §2.7.

### Proof

The argument assembles the three previous theorems.

By Theorem 5.2, the unique leading admissible scalar invariant generated by refinement transport curvature under (S1)–(S5) is $c_1 R + c_0$.

By Theorem 6.2, the unique invariant continuum-limit measure is $\sqrt{|g|}\, d^{(D+1)}x$.

The leading parity-even scalar geometric action is therefore

$\int d^{(D+1)}x \sqrt{|g|} (c_1 R + c_0)$,

up to total derivatives and subleading higher-curvature corrections (Corollary 5.3, OP3). Identifying $c_1 = 1/(2\kappa_{eff})$ and $c_0 = -\Lambda_{eff}/\kappa_{eff}$ via the matching condition of dynamical paper §2.7 / variational paper Theorem 5.1 (where $\kappa_{eff} = 8\pi G$ fixes the closure-normalisation factor $C_\lambda$ in the substrate form) yields

$\mathcal{S}_{EH} = (1 / 2\kappa_{eff}) \int d^{(D+1)}x \sqrt{|g|} (R - 2\Lambda_{eff})$. ■

### Corollary 7.2 — Einstein tensor recovery at the substrate-derived level [proven]

Variation of the emergent $\mathcal{S}_{EH}$ with respect to $g_{\mu\nu}$ yields

$(G_{\mu\nu} + \Lambda_{eff} g_{\mu\nu}) / \kappa_{eff} = 0$

in vacuum, i.e. the homogeneous Einstein equation with cosmological constant. Coupled to the inherited source structure of variational paper §4, this reproduces the dynamical-geometry field equation of dynamical paper Definition 6.1 with the geometric sector now substrate-derived rather than inherited.

**Proof.** Standard metric variation of the displayed $\mathcal{S}_{\mathrm{EH}}$; cf. variational paper Theorem 5.1 proof. The geometric content is unchanged from the variational layer; what has changed is the layer at which the action is derived. ■

### Structural significance

Theorem 7.1 closes the substrate-derivation problem for the geometric sector of $\mathcal{S}_{\mathrm{VERSF}}$. Combined with the variational paper's Theorem 5.1, it establishes Einstein–Hilbert at two distinct layers:

- *Variational layer (variational paper Theorem 5.1):* $\mathcal{S}_{\mathrm{EH}}$ is the unique admissible local geometric action within the continuum-limit operator basis under (A1)–(A8).
- *Substrate layer (present Theorem 7.1):* the continuum-limit operator basis itself — the Ricci scalar $R[g]$, the volume form $\sqrt{|g|}\, d^{D+1}x$ — emerges from refinement transport geometry, with $R$ from transport-curvature mismatch (Theorem 5.2) and $\sqrt{|g|}$ from distinguishability-volume density (Theorem 6.2).

Einstein gravity is therefore not a postulated geometric law at any layer of the VERSF construction. It is the unique leading admissible action generated by refinement-compatible irreversible commitment transport, structurally constrained at both the variational and substrate layers.

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# 8. Action-Level Recovery and Consistency with the Variational Layer

We now record the consistency between the present substrate derivation and the variational-layer treatment.

### Proposition 8.1 — Layered consistency [proven]

The substrate-derived Einstein–Hilbert action of Theorem 7.1 coincides identically with the continuum-limit Einstein–Hilbert sector $\mathcal{S}_{\mathrm{EH}}$ of variational paper Definition 4.1, up to inherited boundary terms and Wilson-coefficient ambiguities of the form recorded in variational paper Theorem 4.1.

**Proof.** Both expressions are

$$\mathcal{S}_{\mathrm{EH}} = (1 / 2\kappa_{\mathrm{eff}}) \int d^{D+1}x \, \sqrt{|g|}\, (R - 2\Lambda_{\mathrm{eff}}),$$

with identical κ_eff, Λ_eff, R, and √|g|. The present derivation supplies a substrate origin for the operator basis (R from Theorem 5.2, √|g| from Theorem 6.2); the variational derivation supplies the uniqueness statement within that basis. The two are coherent layers of one and the same action. ■

### Theorem 8.2 — Substrate-level GR Recovery [proven, conditional on the regime (R1)–(R3) of dynamical paper Theorem 10.2 and the boundary-data assumptions (B1)–(B2)]

Under (R1)–(R3) and (B1)–(B2) of the dynamical paper, the full 𝓢_VERSF reduces continuously to the substrate-derived Einstein–Hilbert action plus matter coupling:

𝓢_VERSF → 𝓢_EH + 𝓢_matter,

with the 𝓢_EH on the right-hand side now understood at the *substrate* layer, via Theorem 7.1.

**Proof.** By variational paper Theorem 10.1, 𝓢_VERSF → 𝓢_EH + 𝓢_matter in the regime (R1)–(R3) ∩ (B1)–(B2), with 𝓢_EH the continuum-limit action of variational paper §5. By Proposition 8.1 above, this continuum-limit 𝓢_EH coincides with the substrate-derived 𝓢_EH of Theorem 7.1. The recovery therefore operates at the substrate layer once Proposition 8.1 is invoked. ■

### Structural significance

Theorem 8.2 establishes that GR recovery operates not only at the equation-of-motion level (dynamical paper Theorem 10.2) and the action level (variational paper Theorem 10.1) but at the substrate-derivation level: in the recovery regime, the full 𝓢_VERSF reduces to the *substrate-derived* Einstein–Hilbert action, with the geometric operator basis itself sourced from refinement transport rather than postulated. This is the deepest layer at which GR recovery has been established in the VERSF programme.

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## 9. Structural Interpretation

The present paper substantially deepens the interpretation of gravity in VERSF.

The variational paper §11 established the inverted reading: irreversible commitment is primary, and geometry / memory / source structure / the admissible action emerge from closure constraints. The Einstein tensor appears as the unique admissible local curvature structure compatible with conserved irreversible commitment transport.

The present paper carries that inversion one layer deeper. The Einstein–Hilbert *action itself* — not just the Einstein tensor as a field-equation object — emerges from refinement transport geometry. The Ricci scalar is the unique admissible scalar contraction of transport-curvature mismatch; the volume form is the unique invariant continuum measure compatible with refinement-induced volume distortion. The full Einstein–Hilbert integrand $\sqrt{|g|}(R - 2\Lambda_{eff})$ is therefore substrate-emergent in both its curvature content and its measure factor.

The structural progression of the geometric side of the programme is now:

commitment ontology → refinement transport → continuum-limit connection (Theorem 3.5) → transport-curvature mismatch → Riemann structure (Theorem 4.1) → Ricci scalar (Theorem 5.2) → metric-determinant measure (Theorem 6.2, Proposition 6.3) → Einstein–Hilbert action (Theorem 7.1) → action-level GR recovery at the substrate layer (Theorem 8.2).

## Derivational chain through the present paper

The internal derivational structure of the present paper assembles as follows:

| Layer | Substrate-Derived Result | Theorem |
|---|---|---|
| Refinement transport on commitment paths | Continuum-limit Levi-Civita connection $\Gamma^{\alpha}_{\beta\mu}$ | Theorem 3.5 |
| Transport loop mismatch | Riemann curvature $R^{\alpha}_{\beta\mu\nu}$ | Theorem 4.1 |
| Scalar contraction at second-order scope | Ricci scalar $R$ | Theorem 5.2 |
| Distinguishability-volume density at zero-derivative scope | $\sqrt{|g|}$ (unique up to constant) | Theorem 6.2, Proposition 6.3 |
| Leading admissible scalar action | $\int d^{(D+1)}x \, \sqrt{|g|} \, (R - 2\Lambda_{eff})$ | Theorem 7.1 |
| Metric variation | Einstein tensor $G_{\mu\nu} + \Lambda_{eff} \, g_{\mu\nu}$ | Corollary 7.2 |
| Recovery regime (R1)–(R3), (B1)–(B2) | Standard general relativity $\mathcal{S}_{EH} + \mathcal{S}_{matter}$ | Theorem 8.2 |

Each row is a substrate-level structural object on the left, the continuum-limit emergent quantity in the middle, and the theorem doing the derivational work on the right. The chain reads top-to-bottom as a *substrate-emergence sequence*, with every standard differential-geometric primitive — connection, curvature, volume form, action, field equations, GR recovery — appearing as the continuum-limit image of an underlying transport-geometric structure.

## Correspondence with standard general relativity

The substrate-derivation perspective produces a structurally cleaner correspondence with standard general relativity than was available at the variational layer alone:

| Standard General Relativity | VERSF Substrate Origin |
|---|---|
| Levi-Civita connection $\Gamma^{\alpha}_{\beta\mu}$ | First-order generator of refinement-stable transport (Theorem 3.5, Definition 3.4) |

| Standard General Relativity | VERSF Substrate Origin |
|---|---|
| Riemann tensor $R^\alpha_{\beta\mu\nu}$ | Continuum limit of refinement transport mismatch around infinitesimal commitment loops (Theorem 4.1) |
| Ricci scalar $R$ | Unique admissible scalar contraction of transport-curvature tensor (Theorem 5.2) |
| Volume form $\sqrt{\lvert g\rvert}\, d^{D+1}x$ | Coarse-grained distinguishability-volume density under refinement transport, unique up to constant (Theorem 6.2, Proposition 6.3) |
| Einstein–Hilbert action | Leading admissible substrate-derived geometric action (Theorem 7.1) |
| Newton's constant $G$ | Substrate form $C_\lambda\, \hbar\xi^2/c^3$ (inherited from dynamical paper §2.7) |
| Gravitational coupling $8\pi G = \kappa_{\text{eff}}$ | Substrate form $8\pi\, C_\lambda\, \hbar\xi^2/c^3$ (inherited from dynamical paper §2.7) |
| Cosmological constant $\Lambda$ | Structurally decomposable $\Lambda_{\text{eff}}$ (dynamical paper §9) |

In each row, the right-hand entry is the substrate object whose continuum-limit coarse-graining produces the left-hand entry. Standard general relativity therefore emerges as the leading continuum-limit field-theory shadow of irreversible commitment transport, with each of its geometric primitives traceable to a substrate-level structural object.

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# 10. Limitations and Open Problems

The paper does **not** derive:

- the closure-normalisation factor $C_\lambda$ inherited from the fold-density programme,
- the microscopic substrate construction of the refinement transport operator $T_\gamma$ itself (its continuum-limit first-order generator $\Gamma^\alpha_{\beta\mu}$ is now established by Theorem 3.5),
- subleading higher-curvature corrections from finer transport mismatch,
- Standard-Model matter coupling,
- quantum fluctuations of the geometry,
- the analogous substrate derivations for the $\kappa$, memory, anisotropic, and exchange sectors of $\mathcal{S}_{\text{VERSF}}$ (these are OP1, OP3, OP4 of the variational paper, still open).

Specific open problems for the present geometric layer:

## OP1 — Substrate derivation of the closure-normalisation factor $C_\lambda$

The structural form $\kappa_{\text{eff}} = 8\pi\, C_\lambda\, \hbar\xi^2/c^3$ is inherited from dynamical paper §2.7. Closing $C_\lambda$ substrate-derivatively from fold energetics, $K = 7$ binary suppression, refinement transport closure, and the hexagonal interface structure of the fold-density programme would constitute a substrate-level derivation of Newton's constant. This is the same open problem as variational paper OP6 / dynamical paper OP6 and is recorded here unchanged.

## OP2 — Microscopic substrate construction of the refinement transport operator $T_\gamma$

Theorem 3.5 establishes that the continuum-limit connection $\Gamma^\alpha_{\beta\mu}$ emerges via Definition 3.4 as the first-order generator of refinement-stable transport $T_\gamma$ on the emergent continuum, and Theorem 4.1 establishes that the continuum-limit transport-curvature tensor is the Riemann tensor of the inherited Levi-Civita geometry. What remains open is the explicit substrate construction of $T_\gamma$ itself from refinement combinatorics on the void substrate at every refinement order, prior to the continuum limit. The transport-geometry programme establishes refinement-stable holonomy at every refinement order from the inherited substrate dynamics; closing this problem would mean producing $T_\gamma$ from a more primitive discrete-substrate combinatorial construction with no transport-geometry input.

## OP3 — Subleading higher-curvature corrections

Corollary 5.3 excludes curvature-squared and higher-curvature primitives at the leading admissible order. Subleading orders of refinement transport mismatch generate, in principle, higher-curvature corrections: Gauss–Bonnet at the next admissible second-order Lovelock level (in $D + 1 > 4$), and $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ at the next order in derivative count. The structural form of these subleading corrections from substrate refinement transport, and their relationship to the OP10 of the dynamical paper (higher-derivative / Lovelock extensions), is open.

## OP4 — Boundary-term structure and Gibbons–Hawking-type contributions

Theorem 7.1 establishes $S_{EH}$ up to boundary terms. The substrate origin of the Gibbons–Hawking–York boundary term in finite-region variational problems — and the corresponding substrate question of how refinement transport mismatch is bounded across the boundary of a finite continuum region — is open. This may be subordinate to OP2 (microscopic substrate construction of $T_\gamma$).

## OP5 — Matter coupling

The present paper does not derive Standard-Model matter. Matter enters only through the inherited recovery limit $T^{(\kappa)}_{\mu\nu} \to T^{(\mathrm{matter})}_{\mu\nu}$ of dynamical paper Theorem 10.2 / variational paper Theorem 10.1. The major remaining open question is the same as variational paper OP7 / source paper OP7: how do stable matter sectors emerge from irreversible commitment dynamics? Candidate routes include stable transport-localised excitations, topological commitment defects, persistent closure modes, and refinement-stable cohomological structures. Inherited unchanged.

## OP6 — Quantum completion of refinement transport geometry

The present treatment is classical / effective. Quantum fluctuations of $g_{\mu\nu}$, of the refinement transport operator $T_\gamma$, of the curvature mismatch operator $\Delta(u, v)$, and of the distinguishability-volume density remain untreated. The major unresolved question is: what is the quantum completion of refinement transport, and how does it relate to standard gravitational quantisation? Inherited from variational paper OP8 / source paper OP8 and lifted to the substrate layer.

### OP7 — Wilson-coefficient closure for the κ, memory, and anisotropic sectors

The present paper closes the substrate derivation for the geometric sector only. The corresponding substrate derivations for the κ-sector action S_κ, the memory-sector action S_mem with its bilocal kernel 𝒦(x, x′), and the anisotropic transport-curvature action S_Q̂ with its Wilson coefficients (α_Q̂, β_Q̂, γ_Q̂) remain open. These are variational paper OP1, OP3, and OP4 respectively, inherited here unchanged.

### OP8 — Refinement-stable transport from below

Theorem 4.1 inherits refinement-stable transport from the transport-geometry programme. The substrate-level *foundation* of refinement-stable transport itself — i.e. the explicit construction of refinement stability from the underlying commitment dynamics, with no transport-geometry input — is the deepest residual question of the geometric-substrate layer. This is subordinate to OP2 but is recorded separately because closing it would constitute a complete bottom-up reconstruction of the geometric side of the framework.

The OP2 / OP8 distinction is sharp: **OP2 closes the existence question** — what is T_γ as an explicit construction from void combinatorics at every refinement order — while **OP8 closes the stability question** — why does the so-constructed T_γ converge under refinement to a well-defined continuum-limit object. The two questions are logically distinct (existence does not imply stability, and a stable continuum-limit object could in principle exist without an explicit substrate construction), and both must be resolved to close the geometric side from below.

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## 11. Conclusion

The previous VERSF papers established:

- irreversible commitment ontology,
- emergent continuum structure,
- refinement-stable transport geometry,
- Lorentzian causal completion,
- non-Markovian memory,
- effective stress–energy sourcing,
- Einstein-compatible dynamics,
- and a unified covariant action with action-level GR recovery.

The present paper closes the residual problem of variational paper OP2: the substrate-level emergence of the Einstein–Hilbert action.

The central results are:

- **Theorem 3.5.** The continuum-limit connection $\Gamma^{\alpha}_{\beta\mu}$ is derived as the first-order generator of refinement-stable transport via Definition 3.4, coinciding with the Levi-Civita connection of the inherited Lorentzian Completion geometry. The connection is therefore no longer inherited from differential geometry.
- **Theorem 4.1.** Continuum-limit refinement transport mismatch coincides with the Riemann curvature tensor of the inherited Levi-Civita geometry.
- **Theorem 5.2.** The unique leading admissible scalar contraction of the transport-curvature tensor under parity-evenness, tensorial closure, local Lorentz covariance, second-order scope, and scalar minimality is the Ricci scalar $R[g]$.
- **Theorem 6.2 and Proposition 6.3.** The unique invariant continuum-limit measure compatible with refinement-induced volume distortion under metric compatibility is $\sqrt{|g|}\, d^{(D+1)}x$, with the scalar density unique up to an overall constant within the zero-derivative weight-1 class.
- **Theorem 7.1 (Substrate-Level Einstein–Hilbert Emergence Theorem).** The leading admissible substrate-derived continuum-limit geometric action is

  $$\mathcal{S}_{\mathrm{EH}} = (1 / 2\kappa_{\mathrm{eff}}) \int d^{(D+1)}x \sqrt{|g|}\, (R - 2\Lambda_{\mathrm{eff}}),$$

with $\kappa_{\mathrm{eff}}$ inheriting the substrate form $8\pi\, C_{\lambda}\, \hbar\xi^{2}/c^{3}$.

- **Theorem 8.2.** Under the recovery regime (R1)–(R3) and boundary-data assumptions (B1)–(B2) of the dynamical paper, the full $\mathcal{S}_{\mathrm{VERSF}}$ reduces continuously to the substrate-derived $\mathcal{S}_{\mathrm{EH}}$ plus matter coupling.

The result therefore shifts the standing of the Einstein–Hilbert action in the VERSF framework from

*"a compatible continuum-limit operator basis"*

— that was variational paper Theorem 5.1 — to

*"a substrate-emergent geometric structure generated by refinement-compatible irreversible commitment transport"*

— which is the present paper. The two statements are layered consistently (Proposition 8.1): Theorem 5.1 establishes uniqueness within the operator basis, Theorem 7.1 establishes the substrate origin of the basis itself, and the two together fix $\mathcal{S}_{\mathrm{EH}}$ at both layers.

**Honest limitation.** The present paper closes the continuum-limit geometric operator-basis problem — the connection, the curvature tensor, the Ricci scalar, the invariant measure, the Einstein–Hilbert action, and the action-level GR recovery now all carry explicit substrate origins through Theorems 3.5, 4.1, 5.2, 6.2, 7.1, and 8.2 respectively. What this paper does *not* close is the microscopic UV construction of refinement transport itself: $T_{\gamma}$ at every refinement order is inherited from the transport-geometry programme rather than constructed from discrete substrate combinatorics (the existence question, OP2), and the convergence of refinement-stable transport under refinement is inherited rather than derived from below (the stability question, OP8). These are the genuinely deeper questions that the present paper sharpens rather than resolves.

The structural progression of the geometric side of the VERSF programme is now:

commitment ontology → continuum emergence → refinement transport → Lorentzian completion → effective stress–energy → Einstein-compatible dynamics → variational closure → substrate emergence of the Einstein–Hilbert action.

The geometric side is now structurally closed at the continuum-limit emergence layer for the Einstein–Hilbert sector specifically, with the microscopic substrate construction of T_γ (OP2) and the foundation of refinement-stable transport (OP8) explicitly recorded as remaining. What also remains is the corresponding substrate derivation of the κ, memory, anisotropic, and exchange sectors (OP7 above, inherited from variational paper OP1 / OP3 / OP4), the closure of the closure-normalisation factor C_λ (OP1 above), and the higher-curvature, matter-coupling, and quantum-completion problems (OP3, OP5, OP6 above). Each is sharply isolated, each has a clear next-paper home, and none remains a structural existence question.

The picture this leaves is: gravity is not a primitive geometric law at any layer of the VERSF construction. It is the unique leading admissible action generated by refinement-compatible irreversible commitment transport, structurally constrained at the substrate layer (where the operator basis emerges), the variational layer (where uniqueness within the basis is established), and the dynamical layer (where the field equations are derived and GR is recovered in the appropriate effective limit). The Einstein–Hilbert action is now the substrate-emergent attractor of a deeper commitment-based dynamics, recovered exactly when memory, anisotropy, and exchange sectors are negligible.

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# 12. References to Inherited VERSF Papers

The present paper carries inline citations to several earlier VERSF papers. For a stand-alone read, each is identified below by title and the specific results invoked.

- **Source-Structure Paper** — *Effective Source Structure and Admissibility Closure in VERSF.* Inherited: the four-sector decomposition T^eff_μν, the CRE invariance framework of §3.4, the finite-distinguishability cutoff of §2.1, the Admissible Source Uniqueness Theorem (Theorem 3.1 / Theorem 8.2), and total conservation (Theorem 8.3).
- **Lorentzian Completion Paper** — *Lorentzian Causal Completion of Emergent Commitment Geometry in VERSF.* Inherited: the emergent Lorentzian continuum geometry g_μν, the Levi-Civita connection structure (metric-compatible, torsion-free) on the inherited continuum, the CCC framework, the coherence scale ξ, and the causal-propagation structure.

- **Transport-Geometry Papers** — refinement-stable holonomy framework, the convergence of refinement transport to the continuum-limit Levi-Civita parallel transport, and the closure-compatibility condition underlying Lemma 4.2 above.
- **Dynamical-Geometry Paper** — *Bianchi-Compatible Geometry and Einstein-Compatible Dynamics in VERSF.* Inherited: the Bianchi-Compatible Geometry Theorem (Theorem 4.1, the equation-level uniqueness counterpart of Theorem 5.2 above), the tensorial closure structure of §2.8 (response sector into $(1,1) \oplus (0,0)$), the substrate-scale coupling form $\kappa\_eff = 8\pi C\_\lambda \hbar\xi^2/c^3$ (§2.7), the admissibility uniqueness theorem at leading order (Theorem 6.4), the GR Recovery Theorem with (R1)–(R3) and (B1)–(B2) (Theorem 10.2), and the H^k(K) Sobolev continuity bound (Theorem 10.3).
- **Variational Paper** — *Unified Covariant Action for Emergent Commitment Geometry in VERSF.* Inherited: the unified covariant action $\mathcal{S}$_VERSF (Definition 4.1), the Variational Closure Theorem (Theorem 4.1), the action-level Einstein–Hilbert uniqueness theorem within the continuum-limit operator basis (Theorem 5.1, the variational-layer counterpart of Theorem 7.1 above), the constrained-exchange action $\mathcal{S}$_ex with Lagrange-multiplier framing (Definition 8.1, Theorem 8.2), and the action-level GR Recovery Theorem (Theorem 10.1). Variational paper OP2 (the open problem the present paper closes) is reframed here as Theorem 7.1.

Numerical results and structural conventions inherited from these papers — including $m^2 = (4/3)\xi^{-2}$, $\kappa\_eff = 8\pi C\_\lambda \hbar\xi^2/c^3$, and the K = 7 closure architecture — are treated throughout the present paper as established inputs.