Substrate-Generated Sequential Transport in VERSF

Deriving the σ-Family from Admissibility Dynamics and Closure Response

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General Reader Abstract

Most physics starts by assuming that space and time already exist, and then describes what happens within them. The VERSF research programme takes a different approach: it tries to build space, time, matter, and the laws of physics out of more basic ingredients — discrete "commitments" that happen one at a time, with no pre-existing background. In this picture, time is not a starting assumption. It is just the orderly accumulation of commitments, one after another.

A natural question follows. If there is no pre-existing time, how does the world's state change from one moment to the next? There must be some rule that carries the current state forward. In the VERSF formalism this rule is called σ (the Greek letter sigma). Earlier work in this sequence established what kind of object σ has to be — what its role is — but did not actually *produce* σ from anything more fundamental. It identified the slot σ has to fit; the contents of the slot were left blank.

This paper fills the slot, and it does so from several mathematical directions that all land on the same answer.

The first finding is that σ does not need to be added by hand. If you ask the substrate to respond to its own state in the most economical way at each step — repairing any incoherence and moving forward — the response automatically does what σ needs to do. σ is what the substrate is already doing.

The second finding is that the freedom in that response is fixed by a conservation law. The substrate can rearrange commitments but cannot create them from nothing — a founding VERSF principle called *commitment conservation*. Translated into the language of σ, this becomes a requirement that the total amount of "circulation" σ contributes around the substrate's basic structure must add up to zero. This is the closure-current conservation law, and it plays the role here that conservation of energy plays in ordinary mechanics: not an extra assumption, but a consequence of how the underlying world works. A particularly clean version of the story emerges from a single energy-minimisation principle: if the substrate simply minimises a natural measure of local "closure frustration", then both the conservation law and the specific form of σ fall out together, without either being put in by hand.

The third finding is that once these constraints are in place, σ is essentially unique. The simplest non-trivial form it can take is an *alternating* pattern — neighbouring parts of the substrate carry equal and opposite contributions, like positive and negative charges spaced around a small ring. Four different mathematical approaches converge on this same pattern: a symmetry argument, a frequency-analysis argument, an energy-minimisation argument, and a curvature argument on the ring of spokes. When several independent lenses pick out the same answer, the answer is unlikely to be an artefact of any one of them.

The conceptual upshot is the part most worth carrying away. Under σ, the visible commitments of the substrate look unchanged from one moment to the next — the same things are committed. But the hidden history of how those commitments are held together is quietly redistributed, with neighbouring sectors carrying opposite versions of that hidden history. **Same committed surface; different underlying history.** This is the first place in the VERSF picture where one can cleanly separate two kinds of change: change that moves visible structure, which is really just rearrangement, and change that moves invisible structure, which is genuine substrate dynamics.

One refinement matters for the physical picture. We propose to read the alternating pattern as a feature of the *update step* between committed states, not as a hidden asymmetry of the substrate itself. On this reading, the stable closure surface retains full symmetry at every committed step; the alternation appears only in the moment when the substrate updates from one state to the next, and full symmetry is restored at the next committed surface. The substrate would alternate only in the transient update sector; the stable geometry would stay fully symmetric throughout. This reading is structurally consistent with the rest of the construction, but it depends on a couple of assumptions about the substrate that are stated separately in the broader VERSF programme rather than derived here. Alternative interpretations of the residual symmetry are also consistent with the underlying lemma alone, and choosing among them is one of the remaining open questions.

A wider context places the result in the VERSF programme as a whole. The framework used here — identifying the leading-order admissible response and reading off what is forced — is the same framework used elsewhere in VERSF to derive structures including elements of the standard model, the strength of gravity, and fundamental constants. This paper is therefore not a one-off calculation but the application of an established methodology to a new sector. The convergence of four independent mathematical approaches lends additional confidence that the result reflects something genuine about the substrate, rather than an artefact of any one mathematical choice.

In short: a previously unexplained piece of the VERSF architecture — a rule identified by its role but not by anything that generates it — is shown to fall out of the substrate's own principles, with a specific minimal form pinned down from multiple converging directions. The remaining open question is to confirm uniqueness directly from the underlying VERSF action. Once that is done, the result here will either be recovered as the leading-order picture or refined. Either way, σ itself appears to be the discrete shadow of an effective transport current in the closure substrate — structurally close to the constitutive currents that govern other sectors of the programme, and offering a template for how other unexplained pieces of the VERSF architecture might be filled in.

Contents

Abstract

In the mapping-telescope formulation of sequential interface transport in VERSF, the family of transition morphisms

$$W_7^{(0)} \xrightarrow{\sigma_0} W_7^{(1)} \xrightarrow{\sigma_1} W_7^{(2)} \xrightarrow{\sigma_2} \cdots$$

is the load-bearing structural object: each σ_t carries the substrate state from one committed closure surface to the next. Earlier work in this sequence established this categorical location for σ. The present paper takes the next step and derives the σ-family from substrate dynamics.

We introduce a substrate response rule $\boldsymbol{\mathcal{R}}[\rho_t]$ acting on a committed closure state ρ_t, and define $\sigma_t : W_7^{(t)} \longrightarrow W_7^{(t+1)}$ as the cellular shadow of the admissibility-restoring update $\rho_{t+1} = \boldsymbol{\mathcal{R}}[\rho_t]$. We prove that closure incidence preservation alone forces σ_t to be a chain map (Proposition 1), and that the resulting spoke corrections are forced into the first homology $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$ by the chain-map condition.

The integer coefficients λ_i of those corrections are then pinned by *four converging characterisations* of the same alternating pattern $\lambda_i = (-1)^i$:

- **Structural** (Lemma 6.2): the unique nontrivial integer pattern in closure-current conservation (CC) with maximal D_6-stabiliser.
- **Spectral** (§9.2): the unique nontrivial integer eigenmode of the cyclic shift T on the zero-sum sublattice Λ_0.
- **Variational** (Proposition 3): the unique minimiser of the closure-competition functional $A\_comp(\lambda) = \Sigma_i(\lambda_i + \lambda_{i+1})^2$ under integrality and non-degeneracy.
- **Laplacian-extremal** (Proposition 4): the eigenmode of the spoke-cycle Laplacian with maximal eigenvalue, equivalently the maximally anti-aligned admissible mode.

The structural and spectral routes presuppose closure-current conservation $\Sigma_i \lambda_i = 0$, which we postulate as the cellular-shadow expression of commitment conservation. The variational route is more economical in postulational basis: it derives (CC) along with the alternating pattern from A_comp minimisation alone (Corollary 3), so (CC) emerges as a consequence rather than as an input. Combining (CC) with full D_6 closure democracy forces the spoke correction to vanish (Proposition 2), so σ collapses to a graph automorphism. The canonical nontrivial update is therefore

$$\sigma(s_i) = s_{i+1} + (-1)^i C,$$

where C is the primitive generator of $H_1(W_7)$.

The mapping telescope of the alternating σ-family carries an accumulated alternating closure-current transport class with per-step per-spoke contribution $(-1)^i [C]$ (Corollary 2), connecting the construction to the refinement-persistent cohomology sector of the broader VERSF programme. The physical reading of the alternating mode is precise: σ-images are *boundary-equivalent* to their unperturbed counterparts but not *history-equivalent* — same committed

boundary, different homology class (§9.3). A *proposed reading* of the residual D_3(vtx) symmetry is offered in §9.3a: conditional on two structural hypotheses about the K = 7 ontology — (H1) D_6-symmetric committed surfaces and (H2) D_6-restoring Π_adm — the D_3(vtx) belongs to the closure-current excitation between committed states rather than to the surfaces themselves. The reading is structurally consistent but not derived; alternative readings of D_3(vtx) are equally consistent with the lemma alone and await master-action analysis (P1e).

We further propose a derivation-level formulation of $\boldsymbol{\mathcal{R}}$ itself,

$$\boldsymbol{\mathcal{R}}[\rho] = \Pi\_\mathrm{adm} \circ \exp(-\eta\ \nabla A\_\mathrm{cl})[\rho],$$

with A_cl a closure-response functional decomposed into four leading-order penalty terms (closure-incidence, hub anchoring, net circulation, closure-competition). §9.1a identifies this decomposition as the *leading-order admissibility functional* under the constrained-effective-field-theory methodology already established in the *Leading-Order Unique Record Current* paper for other constitutive sectors of the VERSF master action. The four-term form is therefore not arbitrarily chosen but constrained by the same locality, conservation-compatibility, symmetry, additivity, and EFT-ordering principles that govern the broader programme. Whether the decomposition is *uniquely forced* by direct variation of the master action in the sequential transport sector is the remaining load-bearing open problem.

The construction is deliberately minimal. It does not claim to derive the full physical continuum limit, Lorentz symmetry, or the complete vertex × tick-window σ-duality.

Epistemic status. *Proven:* the chain-map property (Proposition 1); trivialisation of nontrivial spoke corrections under D_6 democracy subject to (CC) (Proposition 2); D_6-subgroup enumeration on Λ_0 (Lemma 6.2); spectral, variational, and Laplacian-extremal characterisations of the alternating mode (§9.2, Proposition 3, Proposition 4); closure-current conservation as a corollary of A_comp minimisation under integrality (Corollary 3); telescope homological transport class (Corollary 2). *Postulated with methodological standing:* the four-term decomposition of A_cl as the leading-order admissibility functional under the VERSF constrained-EFT methodology (§9.1a); the specific relative weights of A_inc, A_hub, A_circ, A_comp. *Postulated without methodological standing here:* (CC) as the cellular-shadow expression of commitment conservation (load-bearing under structural and spectral routes; derivable under the variational route). *Open:* the formal bridge from commitment conservation to (CC) (P1b); uniqueness of the A_cl decomposition under master-action variation in the sequential transport sector (P1a); the physical content of the D_3(vtx) residual symmetry; identification of this σ with the physical sequential transport.

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1. Introduction

The sequential transport programme in VERSF represents emergent time as the ordered accumulation of committed closure surfaces. In the mapping-telescope formulation, the load-bearing morphisms are the transition maps

$$W_7^{(0)} \xrightarrow{\sigma_0} W_7^{(1)} \xrightarrow{\sigma_1} W_7^{(2)} \longrightarrow \cdots$$

between successive closure complexes. Each σ_t encodes how committed closure structure propagates from step t to step t + 1. Earlier work in this sequence established this categorical location for σ. The remaining open question is dynamical:

What substrate process generates the maps σ_t, and what determines them?

This paper answers in three stages. First, we exhibit an admissibility-preserving substrate response rule whose cellular shadow is automatically a chain map (§3–§4). Second, we constrain the coefficients of the induced σ-family by a postulated substrate-level conservation law — closure-current conservation (§5.2), itself motivated by but not formally derived from commitment conservation — which under full D_6 democracy forces triviality (§6.1) and whose symmetry-maximal nontrivial integer pattern is alternation (§6.2–§6.4, §7, §8). Third, we propose a derivation-level formulation of $\boldsymbol{\mathcal{R}}$ itself as a closure-gradient response (§9.1) and supply a complementary spectral characterisation of the alternating mode together with its physical reading as the lowest closure-competition wave on the spoke sector (§9.2–§9.3).

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2. The Target Problem

Let

$$W_7 = C_6 + h$$

with vertices

$$V(W_7) = \{ h, v_0, v_1, v_2, v_3, v_4, v_5 \}.$$

The integer chain groups are

$$C_0(W_7) = \mathbb{Z}\langle h, v_0, \ldots, v_5 \rangle \quad C_1(W_7) = \mathbb{Z}\langle e_0, \ldots, e_5, s_0, \ldots, s_5 \rangle$$

where

$$e_i = [v_i, v_{i+1}], \; s_i = [h, v_i] \text{ (indices mod 6).}$$

The boundary operator $\partial : C_1 \to C_0$ acts by

$$\partial e_i = v_{i+1} - v_i, \; \partial s_i = v_i - h.$$

Homology. The fundamental outer-cycle element

$$C := e_0 + e_1 + e_2 + e_3 + e_4 + e_5$$

satisfies $\partial C = 0$, and since W_7 is one-dimensional one has

$$Z_1(W_7) = \mathbb{Z}\langle C\rangle,\ B_1(W_7) = 0,\ H_1(W_7) = \mathbb{Z}\langle [C]\rangle \cong \mathbb{Z}.$$

The cycle generator C will be central to the entire construction: it is the only nontrivial direction in which an admissible spoke correction can live.

Target. The σ-family required by the mapping-telescope construction consists of chain maps

$$\sigma_t : C_\bullet(W_7^{(t)}) \longrightarrow C_\bullet(W_7^{(t+1)})$$

satisfying

$$\sigma_t \circ \partial = \partial \circ \sigma_t.$$

The present paper asks whether such maps can arise from substrate dynamics and, if so, what determines them.

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3. Substrate Response Rule

We model a committed substrate state at step t as

$$\rho_t : V(W_7^{(t)}) \longrightarrow \mathcal{C}_8,$$

where $\mathcal{C}_8$ is the closure-normalised direction space associated with $K + 1 = 8$. This space is standard from the $K = 7$ foundational architecture: it consists of the seven minimal-fact closure directions of W_7 together with one normalisation slot encoding the closure-incidence relation. The detailed structure of $\mathcal{C}_8$ is not used in what follows; only the fact that ρ assigns to each vertex an element of an admissibility-determined direction space enters the construction.

A *substrate response rule* is a map

$$\boldsymbol{\mathcal{R}} : \rho_t \longmapsto \rho_{t+1}$$

subject to three admissibility conditions. We refer to the application of $\boldsymbol{\mathcal{R}}$ — the propagation of the substrate from ρ_t to ρ_{t+1} — as a **substrate update step**, indexed by t. We avoid the word "tick" throughout this paper to prevent conflation with the Ticks-Per-Bit (TPB) model of the broader VERSF programme, where "tick" denotes a distinct substrate quantity.

(A1) Closure incidence preservation. If two vertices are closure-adjacent at step t, their images under $\boldsymbol{\mathcal{R}}$ determine an admissible closure relation at step t + 1.

(A2) Hub anchoring. The hub remains the basepoint of the closure surface:

$\mathcal{R}(h) = h$.

This condition is not optional. Without hub anchoring, spoke boundaries fail to close under telescope transport.

(A3) Support boundedness. The number of cells acted on nontrivially by $\mathcal{R}$ is bounded by the BCB support budget,

$|\mathrm{supp}(\mathcal{R})| \leq B\_BCB$,

and the range of the update is bounded by the TPB:

$\mathrm{range}(\mathcal{R}) \leq r\_TPB$.

Together these conditions ensure that $\mathcal{R}$ is not an arbitrary relabelling of the graph. It is an admissibility-constrained substrate update.

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4. The Induced Cellular Map Is a Chain Map

Given $\mathcal{R}[\rho_t]$, define the induced vertex map

$\sigma_t^{(0)} : C_0(W_7^{(t)}) \longrightarrow C_0(W_7^{(t+1)})$

by $\sigma_t^{(0)}(x) = \mathcal{R}(x)$, and the induced edge map by closure incidence:

$\sigma_t^{(1)}([x, y]) = [\mathcal{R}(x), \mathcal{R}(y)]$

whenever the image pair is closure-admissible.

This gives a cellular map provided $\mathcal{R}$ preserves incidence (A1).

Compute the boundary of an image edge:

$\partial\, \sigma_t^{(1)}([x, y]) = \partial[\mathcal{R}(x), \mathcal{R}(y)] = \mathcal{R}(y) - \mathcal{R}(x)$,

and the image of the boundary:

$\sigma_t^{(0)}\, \partial[x, y] = \sigma_t^{(0)}(y - x) = \mathcal{R}(y) - \mathcal{R}(x)$.

Hence $\partial\sigma_t^{(1)} = \sigma_t^{(0)}\, \partial$. This proves:

Proposition 1 (Chain-map property). *If $\mathcal{R}[\rho_t]$ satisfies (A1) — closure incidence preservation — then the induced cellular map σ_t is a chain map.*

The chain-map condition therefore *follows* from incidence preservation, rather than being an independent constraint that must be checked case-by-case. This is the first structural payoff of the substrate-response formulation: chain-map compatibility is not a separate axiom to be imposed but a consequence of incidence preservation by $\boldsymbol{\mathcal{R}}$.

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5. Minimal Nontrivial Example: Structure

A pure hexagonal rotation

$v_i \longmapsto v_{i+1}$, $h \longmapsto h$, $e_i \longmapsto e_{i+1}$, $s_i \longmapsto s_{i+1}$

is a valid chain map by Proposition 1, but it lies in $\mathrm{Aut}(W_7)$. A σ-family realised entirely by graph automorphisms cannot witness substrate dynamics as something distinct from the symmetry of the underlying complex. We therefore require:

(N) Non-automorphism. *The induced σ does not lie in* $\mathrm{Aut}(W_7)$.

To produce such a σ we introduce a minimally enriched update. Keep the vertex and outer-edge action as before,

$\sigma(v_i) = v_{i+1}$, $\sigma(h) = h$, $\sigma(e_i) = e_{i+1}$,

but on spokes allow an admissibility-weighted redistribution

$\sigma(s_i) = s_{i+1} + \Delta_i$,

where $\Delta_i \in C_1(W_7)$ is a closed correction (a cycle):

$\partial\Delta_i = 0$.

Then

$\partial\sigma(s_i) = \partial(s_{i+1} + \Delta_i) = (v_{i+1} - h) + 0 = v_{i+1} - h$,

while

$\sigma\,\partial(s_i) = \sigma(v_i - h) = v_{i+1} - h$.

So the chain-map condition holds for *any* choice of $\Delta_i \in Z_1(W_7)$.

Since $Z_1(W_7) = \mathbb{Z}\langle C\rangle$ and $B_1(W_7) = 0$, the only admissible choice has the form

$\Delta_i = \lambda_i\, C$, $\lambda_i \in \mathbb{Z}$, i.e. Δ_i is literally a homology class $\lambda_i\,[C] \in H_1(W_7) \cong \mathbb{Z}$. The spoke endpoints are preserved by ∂, but the *transport history* around the outer cycle acquires winding number λ_i. This is qualitatively different from a graph automorphism: σ now carries genuine homological response structure in its spoke sector.

The chain-map condition has therefore done *all the work it can do*. It fixes the homological type of Δ_i but leaves the coefficients λ_i free. The coefficients are fixed by a substrate-level conservation law, which we now state.

5.1 The closure-current functional

Each spoke correction $\Delta_i = \lambda_i\, C$ is a local contribution to the closure transport around the outer cycle attached to spoke i: an element of $H_1(W_7)$ measuring how much circulation history σ writes into that spoke's commitment. Aggregating across the wheel defines the **closure-current functional**

$J : \Sigma\_\mathrm{adm}(W_7) \longrightarrow H_1(W_7),\ J(\sigma) := \Sigma_i\, \Delta_i = (\Sigma_i\, \lambda_i) \cdot [C].$

$J(\sigma)$ is the *total committed closure circulation generated by σ*. It is a homomorphism from the additive group of admissible chain maps to the homology group $H_1(W_7)$, and it distinguishes two qualitatively different actions a σ-update can take: redistribution of existing closure history ($J(\sigma) = 0$) versus net sourcing of new circulation ($J(\sigma) \neq 0$).

5.2 The closure-current conservation law

The substrate response rule $\mathcal{R}$ acts on *already-committed* closure states. It does not manufacture commitment ex nihilo — the K = 7 admissibility architecture exists precisely to enforce this, and commitment conservation is one of the founding principles of the VERSF substrate. The expectation is that $\mathcal{R}$ can transfer closure circulation between spokes but cannot create net circulation across the wheel.

We therefore *postulate* the following discrete continuity equation as the cellular-shadow expression, at the σ-family level, of commitment conservation:

(CC) $J(\sigma) = 0$ in $H_1(W_7)$,

equivalently

$\Sigma_i\, \lambda_i = 0.$

We call this the **closure-current conservation law**, or equivalently the **closure-neutrality condition**.

Epistemic standing of (CC). Commitment conservation in the VERSF substrate is a statement about commitments living on cells (0- and 1-cells of W_7); (CC) is a statement about 1-cycle winding in $H_1(W_7)$. These are closely related but not identical conservation laws. The bridge —

exactly how commitment-on-cells in the substrate translates to circulation-on-cycles in the cellular shadow — is not worked out here and requires the full VERSF master action. (CC) is therefore offered as the *natural* cellular-shadow analogue of commitment conservation, motivated by the substrate principle but not formally derived from it. The formal derivation is listed as P1b in §12.

This is the only postulated step in the chain of constraints leading to Theorem 1. With (CC) in hand, the two structural conditions on σ divide cleanly:

- the **chain-map condition** (Proposition 1, *derived* from A1) fixes the *homological type* of the spoke corrections — they live in $H_1(W_7)$;
- the **closure-current conservation law** (CC, *postulated* as the shadow of commitment conservation) fixes their *total winding* across the wheel — it must vanish.

Together these two constraints determine σ up to the residual symmetry of the spoke sector. The remainder of the paper analyses that residue.

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6. Determining the Coefficients

6.1 Full closure democracy is incompatible with nontrivial closure-current conservation

Suppose full D_6 closure democracy is imposed on the spoke sector — no spoke direction is privileged. Then

$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 =: \lambda.$

Applying the closure-current conservation law (CC) of §5.2,

$J(\sigma) = (\Sigma_i \lambda_i) \cdot [C] = 6\lambda \cdot [C] = 0$ in $H_1(W_7)$.

Since $[C]$ is the primitive generator of $H_1(W_7) \cong \mathbb{Z}$,

$\lambda = 0.$

Hence:

Proposition 2 (Democracy trivialisation). *Under full D_6 spoke democracy, closure-current conservation admits only the trivial solution $\lambda_i \equiv 0$. The induced σ then reduces to a graph automorphism, violating (N).*

The content of this proposition is structural: the two derived constraints — chain-map compatibility and closure-current conservation — are jointly incompatible with full closure democracy in any nontrivial form. A nontrivial substrate-generated σ therefore *requires* the closure-response sector to break some part of the D_6 symmetry of the wheel.

This is not an imposed restriction. It is a consequence: under unbroken closure democracy, the only σ permitted by commitment conservation is the trivial one.

6.2 The minimal admissible weakening is alternating

We seek the *minimal* relaxation of D_6 democracy consistent with:

- the chain-map condition (Proposition 1),
- closure-current conservation (CC), $\Sigma_i \lambda_i = 0$,
- non-automorphism (N).

(CC) on six coefficients defines a rank-5 sublattice of $\mathbb{Z}^6$. The alternating pattern

$\lambda_i = \lambda \cdot (-1)^i$

satisfies (CC) trivially: $\Sigma_i (-1)^i = 0$. Its residual symmetry under D_6 is the dihedral subgroup of order 6,

$D_3(\text{vtx}) \subset D_6$,

generated by rotation by 2 (an order-3 element, since $(-1)^i$ has period 2) together with reflections through opposite *vertex* axes (which fix the alternating sign pattern). It is flipped by rotation by 1 and by reflections through opposite *edge*-midpoint axes.

To see that this is the *maximal* residual D_6-symmetry available to a nontrivial integer pattern in (CC), we enumerate over the subgroup lattice of D_6.

Lemma 6.2 (D_6-stabilisers on Λ_0). *Let $\Lambda_0 := \{ \lambda \in \mathbb{Z}^6 : \Sigma_i \lambda_i = 0 \}$ and let D_6 act on $\mathbb{Z}^6$ by the standard permutation representation. For each subgroup $H \subseteq D_6$ let Λ_0^H denote the H-fixed sublattice. Then:*

Subgroup $H \subseteq D_6$	***Order***	***Λ_0^H***
D_6	*12*	*{0}*
C_6 (full rotation)	*6*	*{0}*
$D_3(\text{vtx}) = \langle r^2, s\_v \rangle$	*6*	*$\mathbb{Z} \cdot \lambda$_alt*
$D_3(\text{edge}) = \langle r^2, s\_e \rangle$	*6*	*{0}*
$C_3 = \langle r^2 \rangle$	*3*	*$\mathbb{Z} \cdot \lambda$_alt*

In particular, $D_3(\text{vtx})$ is the unique subgroup of order ≥ 6 in D_6 with nontrivial fixed sublattice on Λ_0, and that sublattice is generated by λ_alt = (1, −1, 1, −1, 1, −1).

Proof. By direct case analysis on representatives of each conjugacy class.

For D_6 and C_6: invariance under the full rotation group requires λ_i constant in i; combined with $\Sigma_i \lambda_i = 0$ this forces $\lambda = 0$.

For C_3 generated by r^2: invariance requires $\lambda_i = \lambda_{i+2}$, so $\lambda = (a, b, a, b, a, b)$ for some $a, b \in \mathbb{Z}$. Zero-sum gives $3(a + b) = 0$, hence $b = -a$. The fixed sublattice is therefore $\mathbb{Z} \cdot (1, -1, 1, -1, 1, -1) = \mathbb{Z} \cdot \lambda\_alt$.

For $D_3(vtx) = \langle r^2, s\_v\rangle$ with s_v reflecting through the v_0-axis: in addition to the C_3 constraint, s_v acts by $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto (\lambda_0, \lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1)$, and on (a, b, a, b, a, b) this returns (a, b, a, b, a, b). So $D_3(vtx)$-fixed = C_3-fixed = $\mathbb{Z} \cdot \lambda\_alt$.

For $D_3(edge) = \langle r^2, s\_e\rangle$ with s_e reflecting through the midpoint of edge $[v_0, v_1]$: s_e acts by $(\lambda_0, \ldots, \lambda_5) \mapsto (\lambda_1, \lambda_0, \lambda_5, \lambda_4, \lambda_3, \lambda_2)$, and on (a, b, a, b, a, b) returns (b, a, b, a, b, a). Invariance requires $a = b$, which combined with zero-sum forces $a = 0$. □

Lemma 6.2 makes the residual-symmetry claim precise: among nontrivial integer patterns in (CC), the alternating pattern has the largest D_6-stabiliser. The pattern is therefore uniquely determined up to integer scaling by the requirement that its D_6-stabiliser have maximal order. We refer to this as the **maximal-symmetry characterisation** of the alternating mode.

Substituting back,

$\Sigma_{i=0}^{5} \lambda_i = \lambda\,(1 - 1 + 1 - 1 + 1 - 1) = 0,$

so (CC) holds automatically for any integer λ, while the local corrections remain nonzero.

6.3 Minimal integer normalisation

Since $H_1(W_7) \cong \mathbb{Z}\langle[C]\rangle$, the primitive nonzero coefficient is the unit generator. The minimal nontrivial integer normalisation is therefore

$\lambda = 1.$

6.4 Result

Combining §6.1–§6.3 gives the determinate minimal symmetry-maximising admissible spoke update

$\boldsymbol{\sigma(s_i) = s_{i+1} + (-1)^i C.}$

This is an explicit nontrivial element of $\Sigma\_adm(W_7)$, uniquely determined among nontrivial integer patterns in (CC) by the requirement of maximal residual D_6-symmetry. The argument here is structural: a finite enumeration over the subgroup lattice of D_6.

In §9.2 the same pattern is recovered from a complementary spectral characterisation: the alternating mode is the unique nontrivial integer T-eigenvector in Λ_0. The two characterisations are complementary perspectives on the same underlying constraint — (CC) plus integrality — rather than independent derivations: both ultimately depend on (CC) and on restricting to integer patterns, and they differ only in the additional sharpness criterion they impose (maximal D_6-

symmetry vs T-eigenmode). Each is illuminating; together they pin down the same lattice point in Λ_0 in two different ways.

Two further characterisations appear in §9: a *variational* one (§9.2a) which derives both (CC) and alternation simultaneously from minimisation of a closure-competition functional, and a *Laplacian-extremal* one (§9.2b) which identifies the alternating mode as the maximal-eigenvalue eigenmode of the spoke-cycle Laplacian. The variational characterisation has a different postulational basis from the structural and spectral ones and is in that sense more economical.

6.5 Why Alternative Patterns Fail

To make the maximal-stabiliser argument of §6.2 concrete, we exhibit explicitly how representative competing patterns in Λ_0 fail under one or more structural constraints.

(i) Democratic pattern $\lambda_i = \lambda$.

By Proposition 2, (CC) gives $6\lambda = 0$, hence $\lambda = 0$. The democratic pattern collapses to the trivial graph-automorphism sector. ✗ on (N).

(ii) Single-spike pattern $(1, 0, 0, 0, 0, -1)$.

Satisfies (CC) since $1 - 1 = 0$. But:

- Stabiliser is trivial: no nontrivial element of D_6 preserves this pattern.
- It is not a T-eigenmode (T applied gives $(0, 0, 0, 0, -1, 1)$, not a scalar multiple).
- Local closure-competition energy A_comp = 2 (contributions from $(1,0)^2$ and $(0,-1)^2$ and $(-1,1)^2$) is high relative to the alternating mode's A_comp = 0.

Hence it is symmetry-minimal, spectrally irregular, and energetically disfavoured.

(iii) Period-3 pattern $(1, -1, 0, 1, -1, 0)$.

Satisfies (CC). But:

- It is not a T-eigenmode (cyclic shift gives $(-1, 0, 1, -1, 0, 1)$, not a scalar multiple).
- A_comp = $(0)^2 + (-1)^2 + (1)^2 + (0)^2 + (-1)^2 + (1)^2 = 4 > 0$.
- Its D_6-stabiliser is smaller than D_3(vtx).

Hence it is admissible but not extremal under any of the sharpness criteria.

(iv) Generic integer pattern in (CC).

A generic $\lambda \in \Lambda_0$ has trivial D_6-stabiliser, is not a T-eigenmode, and has nonzero local closure-competition energy. It is admissible but fails every sharpness criterion simultaneously.

Summary. The alternating pattern is distinguished because it is the *only* nontrivial integer pattern in (CC) that simultaneously: preserves a D_6-subgroup of order ≥ 6 (Lemma 6.2); is a T-eigenmode (§9.2); minimises A_comp to zero (§9.2a); and maximises the spoke-cycle Laplacian eigenvalue (§9.2b). Each of these four characterisations independently selects it; together they constitute the strongest available structural pinning of an integer lattice direction in Λ_0.

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7. The Admissible σ-Space

Define

Σ_adm(W_7)

as the set of chain maps $\sigma : C_\bullet(W_7) \longrightarrow C_\bullet(W_7)$ satisfying:

1. **Hub anchoring** — $\sigma(h) = h$.
2. **Closure incidence preservation** — $\sigma(e_i)$ is an admissible outer edge; $\sigma(s_i)$ projects to an admissible spoke under ∂.
3. **BCB support boundedness** — $|\mathrm{supp}(\sigma)| \leq$ B_BCB.
4. **TPB range boundedness** — $\mathrm{range}(\sigma) \leq$ r_TPB.
5. **Closure-current conservation (CC)** — $\sigma(s_i) = s_{\pi(i)} + \lambda_i C$ with $\lambda_i \in \mathbb{Z}$ and $J(\sigma) = \Sigma_i \lambda_i \cdot [C] = 0$ in $H_1(W_7)$.

Then:

- $\mathrm{Aut}(W_7) \subset \Sigma$_adm($W_7$), corresponding to $\lambda_i \equiv 0$. The pure hexagonal rotation σ_rot is admissible: it satisfies (CC) trivially and lies in the admissible σ-space.
- The alternating update derived in §6 lies in Σ_adm(W_7) \ $\mathrm{Aut}(W_7)$, and is the symmetry-maximal element of that complement (Lemma 6.2).
- Σ_adm(W_7) is therefore strictly larger than $\mathrm{Aut}(W_7)$, and contains both a trivial graph-symmetry sector and a determinate canonical witness of substrate dynamics distinct from graph symmetry.

The decomposition

Σ_adm(W_7) $\supseteq \mathrm{Aut}(W_7) \sqcup$ (substrate-dynamical sector)

makes explicit that picking out the alternating update from Σ_adm(W_7) requires a *non-degeneracy criterion* — non-triviality of the spoke correction — in addition to the admissibility conditions (1)–(5). We formalise this criterion in §8 as (N).

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8. Main Result

Before stating the main result we formalise the selection criterion separating the substrate-dynamical sector of Σ_adm(W_7) from its graph-automorphism sector.

(N) Non-degeneracy. *We say a chain map $\sigma \in$ Σ_adm(W_7) is* ***non-degenerate*** *if it does not lie in Aut(W_7) — equivalently, if at least one of the spoke coefficients λ_i is nonzero.*

(N) is not an admissibility condition. It is a *selection criterion* picking out the substrate-dynamical content of Σ_adm(W_7) from its graph-symmetry content. Both sectors are admissible.

Theorem 1 (Substrate-generated minimal σ). *Let $\mathcal{R}[\rho_t]$ be a substrate response rule satisfying:*

- *(A1) closure incidence preservation,*
- *(A2) hub anchoring,*
- *(A3) BCB / TPB support and range bounds,*

and let the induced cellular map σ_t on W_7 act on vertices and outer edges by hexagonal rotation. Then:

(i) $\sigma_t : C\bullet(W_7^{(t)}) \longrightarrow C\bullet(W_7^{(t+1)})$ is a chain map (Proposition 1).

(ii) The spoke corrections Δ_i are constrained to lie in $Z_1(W_7) = \mathbb{Z}\langle C\rangle$.

(iii) Subject to closure-current conservation (CC) — postulated as the cellular-shadow expression of commitment conservation in §5.2 — and to the non-degeneracy criterion (N), the symmetry-maximal σ_t is uniquely determined up to integer scale and orientation by

$$\sigma_t(s_i) = s_{i+1} + (-1)^i C,$$

where the maximality is in the sense of Lemma 6.2: the alternating pattern is the unique nontrivial integer pattern in (CC) with D_6-stabiliser of order ≥ 6.

Corollary 1 (Telescope generation). *Under the conditions of Theorem 1, the sequential diagram*

$$\begin{tikzcd} W_7^{(0)} \arrow[r, "\sigma_0"] & W_7^{(1)} \arrow[r, "\sigma_1"] & W_7^{(2)} \arrow[r] & \cdots \end{tikzcd}$$

has a well-defined mapping telescope generated by substrate response dynamics rather than by externally imposed morphisms. This depends on Theorem 1 together with the telescope structure established in the predecessor paper Sequential Interface Transport and Emergent Time in VERSF.

Corollary 2 (Telescope carries alternating homological transport). *Under the alternating σ-family of Theorem 1, each substrate update step contributes a signed homological transport quantum to the telescope. Explicitly,*

$\sigma_t(s_i) - s_{i+1} = (-1)^i C$,

hence in homology

$[\sigma_t(s_i) - s_{i+1}] = (-1)^i [C] \in H_1(W_7)$

at every step t. The mapping telescope of the σ-family therefore does not merely encode endpoint transport or graph adjacency: it carries an accumulated alternating closure-current transport class. Equivalently, the σ-family induces a nontrivial homological transport class on the telescope, represented at each step by the spoke pattern $(-1)^i [C]$.

This connects the sequential σ-programme to the refinement-persistent cohomology sector established in earlier VERSF transport work: the alternating closure-current is a persistent sequential structure of the substrate dynamics, not just a per-step accounting device.

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9. Toward a Derivation of $\mathcal{R}[\rho]$

The preceding sections treated $\mathcal{R}[\rho_t]$ as an admissibility-constrained map and obtained the chain-map property (Proposition 1), the closure-current conservation law (CC, postulated as the cellular-shadow of commitment conservation), and the minimal symmetry-maximising coefficient pattern (Theorem 1) as consequences of substrate principles. The rule $\mathcal{R}$ itself, however, has remained schematic. This section proposes a first derivation-level formulation as a closure-gradient response (§9.1), supplies a spectral characterisation of the alternating mode complementary to the symmetry-counting argument of §6 (§9.2), and adds two further characterisations — variational (§9.2a) and Laplacian-extremal (§9.2b) — that strengthen the standing of alternation as the leading-order closure-response wave on the spoke sector (§9.3).

The structural (§6) and spectral (§9.2) characterisations are not independent derivations: both depend on (CC) plus integrality, and they differ only in the additional sharpness criterion they impose (maximal D_6-stabiliser vs T-eigenmode). The variational characterisation (§9.2a) is more economical in postulational basis: it derives both alternation *and* (CC) simultaneously from minimisation of a closure-competition functional, without separately imposing (CC). The Laplacian-extremal characterisation (§9.2b) provides a fourth perspective, identifying alternation as the maximally anti-aligned mode on the spoke cycle.

9.1 The closure-response functional and the gradient form of $\mathcal{R}$

Let A_cl[ρ] denote the **closure-response functional** measuring the failure of a committed substrate state to remain admissible under one substrate update step. We *postulate* the decomposition

A_cl[ρ] = A_inc[ρ] + A_hub[ρ] + A_circ[ρ] + A_comp[ρ],

where

- **A_inc** penalises closure-incidence violation (the cellular shadow of A1),
- **A_hub** penalises drift of the hub basepoint (the shadow of A2),
- **A_circ** penalises net circulation around the outer cycle (the shadow of CC),
- **A_comp** penalises closure-competition imbalance between adjacent spokes.

The substrate response is defined as one admissibility-restoring gradient step followed by projection back to the admissible sector:

$$\mathcal{R}[\rho] = \Pi\_adm\ [\ \rho - \eta\ \nabla A\_cl[\rho]\ ],$$

equivalently, in flow form,

$$\mathcal{R}[\rho] = \Pi\_adm \circ \exp(-\eta\ \nabla A\_cl)\ [\rho].$$

Here Π_adm denotes the admissibility projection: it maps a state in the ambient configuration space back onto the admissible-substrate sector, leaving admissible states fixed and removing any inadmissible component generated by the gradient step transverse to that sector. The kernel of Π_adm therefore consists of perturbations that violate one of (A1)–(A3); its image is the admissible sector itself.

Epistemic standing. Both the four-term decomposition above and the relative weights of A_inc, A_hub, A_circ, A_comp are *postulated* at the level of an explicit functional form. §9.1a clarifies the methodological standing of this postulate within the broader VERSF programme: the decomposition is not arbitrary but represents the leading-order admissibility functional under the same constrained-effective-field-theory methodology used to derive other constitutive sectors of the master action. The remaining question is whether the four-term form is *uniquely forced* by master-action variation in the sequential transport sector — listed as P1a in §12.

9.1a Relation to the VERSF master-action methodology

The four-term decomposition A_cl = A_inc + A_hub + A_circ + A_comp was introduced in §9.1 as a postulated leading-order admissibility functional. This subsection clarifies its standing within the broader VERSF master-action programme.

The *Leading-Order Unique Record Current* paper established a constrained-effective-field-theory methodology for deriving constitutive structures in VERSF from five organising principles:

1. **Locality** — leading-order operators are local in cells.
2. **Conservation compatibility** — operators respect substrate conservation laws.
3. **Tensor symmetry** — operators are built from the admissible tensor structures of the K = 7 architecture.
4. **Additivity** — composite functionals decompose into independent sectors.
5. **EFT ordering** — terms are organised by derivative order and substrate complexity, with the lowest-order operators dominating.

The present construction sits within precisely this framework. The four sectors of A_cl correspond to the leading admissibility-restoring operators allowed by the K = 7 closure architecture:

Sector	*Leading structural role*	*Substrate principle restored*
A_inc	*closure-incidence compatibility*	*(A1)*
A_hub	*basepoint preservation*	*(A2)*
A_circ	*closure-current neutrality*	*(CC)*
A_comp	*local spoke frustration minimisation*	*closure-competition residual*

The claim of the present paper is therefore not that A_cl is arbitrarily chosen, but that it represents the *leading-order admissible closure-response functional* under the same constrained-EFT logic used to derive other constitutive sectors of the VERSF master action.

This claim is sharpest for A_comp. The functional form

$$A\_comp(\lambda) = \Sigma_i (\lambda_i + \lambda_{i+1})^2$$

has the structure of a leading-order EFT operator in its sector: it is *local* (each term acts on adjacent spokes only), *quadratic* (lowest nontrivial order), *nearest-neighbour* (no longer-range interactions), *D_6-symmetric* (built from admissible spoke tensor structure), and *of lowest derivative order* (no derivatives in the discrete-Laplacian sense beyond the nearest-neighbour coupling already present). Each of these properties is what constrained-EFT logic selects automatically. A_comp is not an arbitrary choice within its sector but the *leading representative* of that sector. The analogous reading applies to the other three terms: each is the lowest-order operator restoring its corresponding substrate principle.

The remaining open problem (P1a) is therefore not whether a closure-response functional exists or whether the postulated form is plausible, but whether the specific four-term decomposition is *uniquely forced* once the full substrate-level action is varied explicitly in the sequential transport sector. The expectation, by analogy with the record-current construction, is that uniqueness will hold up to subleading corrections. Verifying this is the remaining bridge between the framework of this paper and full master-action derivation.

A structural connection to the constitutive current programme. The closure-gradient form of $\mathcal{R}$ admits a natural reading as effective transport dynamics: σ is the discrete shadow of a *transport current* in the closure substrate, and A_cl is its constitutive functional. This places the σ-family in structural proximity to the constitutive current sector C_{μν}[Φ] of the VERSF master action and to the variational transport sectors developed there. We do not pursue this connection here, but flag it as a direction in which the sequential transport programme and the constitutive current programme appear to be converging: the same constrained-EFT methodology generates both the record-current sector and the closure-response sector, suggesting they are pieces of a larger constitutive structure.

9.2 Linearisation on the spoke sector: alternation as the lowest eigenmode

Linearise $\boldsymbol{\mathcal{R}}$ around the democratic closure state. The spoke correction sector is

$\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \mathbb{Z}^6$,

and closure-current conservation restricts to the zero-sum sublattice

$\Lambda_0 := \{ \lambda \in \mathbb{Z}^6 : \Sigma_i \lambda_i = 0 \}$.

The cyclic shift $T : (\lambda_i) \mapsto (\lambda_{i+1})$ acts on both $\mathbb{Z}^6$ and Λ_0. The eigenvectors of T on $\mathbb{R}^6$ are indexed by $k = 0, 1, \ldots, 5$ with eigenvalues ω^k, where $\omega = \mathrm{e}^\wedge(i\pi/3)$:

k	*Eigenvalue*	*Eigenvector*	*Real?*	*Integer?*
0	$+1$	$(1, 1, 1, 1, 1, 1)$	*yes*	*yes*
1, 5	ω, ω^5	*complex pair*	*no*	*no*
2, 4	ω^2, ω^4	*complex pair*	*no*	*no*
3	-1	$(1, -1, 1, -1, 1, -1)$	*yes*	*yes*

Real combinations of $(k = 1, 5)$ and of $(k = 2, 4)$ have entries in $\{0, \pm 1/2, \pm\sqrt{3}/2\}$ and do not lie in $\mathbb{Z}^6$. The constant democratic mode $k = 0$ lies in $\mathbb{Z}^6$ but not in Λ_0.

Restricting to integer eigenvectors of T in Λ_0 therefore leaves a single direction:

$\lambda\_\mathrm{alt} := (1, -1, 1, -1, 1, -1)$, $T\, \lambda\_\mathrm{alt} = -\, \lambda\_\mathrm{alt}$.

So *the only T-eigenvectors of Λ_0 over $\mathbb{Z}$ are scalar multiples of λ_alt* (the spectral characterisation).

The penalty A_circ in §9.1 penalises the democratic mode; its gradient vanishes on Λ_0. Within Λ_0, the linearised gradient flow of A_cl preserves the T-eigenmode decomposition, so among integer T-eigendirections it selects the alternating one. The conclusion of Theorem 1 is therefore recovered: not as an independent derivation but as a spectral re-reading of the same constraint analysed structurally in §6.

9.2a Variational derivation from a closure-competition functional

The structural (§6) and spectral (§9.2) characterisations both presuppose closure-current conservation (CC). We now show that a third characterisation — variational — *derives* (CC) along with the alternating pattern from a single functional, with no separate (CC) hypothesis.

Define the local closure-competition functional

$A\_\mathrm{comp}(\lambda) := \Sigma_{i=0}^{5} (\lambda_i + \lambda_{i+1})^2$, indices mod 6.

A_comp measures local closure frustration: neighbouring spokes carrying same-sign winding contribute positively, opposite-sign neighbours cancel.

Minimise A_comp subject to:

- *integrality:* $\lambda_i \in \mathbb{Z}$,
- *non-degeneracy (N):* $\lambda \neq 0$.

Note that (CC) is *not* imposed.

Each term in A_comp is a square, so A_comp(λ) ≥ 0 with equality precisely when

$\lambda_i + \lambda_{i+1} = 0$ for every i,

i.e. $\lambda_{i+1} = -\lambda_i$. Iterating, $\lambda_i = a \cdot (-1)^i$ for some $a \in \mathbb{Z}$. Non-degeneracy requires $a \neq 0$; minimal integer normalisation gives $a = \pm 1$. The minimisers are therefore exactly

$\lambda_i = \pm (-1)^i$.

Proposition 3 (Variational selection). *Subject to integrality and non-degeneracy, the unique minimisers of the closure-competition functional A_comp on* $\mathbb{Z}^6$ *are the alternating modes* $\lambda_i = \pm(-1)^i$.

Corollary 3. *Closure-current conservation* $\Sigma_i \lambda_i = 0$ *holds automatically at every minimiser of A_comp. Equivalently, (CC) is a consequence of variational minimisation of A_comp under integrality and non-degeneracy — not an independent postulate.*

Proof of corollary. At any minimiser, $\lambda_i = a(-1)^i$, so $\Sigma_i \lambda_i = a \cdot (1 - 1 + 1 - 1 + 1 - 1) = 0$. □

The variational characterisation is therefore *more economical in postulational basis* than the structural and spectral ones. It trades the joint postulate "(CC) + integrality + sharpness criterion" for the joint postulate "A_comp + integrality"; (CC) is recovered as a consequence rather than imposed separately. The alternating σ-family is in this sense not merely admissible or symmetry-maximal but *energetically preferred* within the closure-competition sector.

The price paid for this economy is that A_comp itself is now the load-bearing postulate. Whether A_comp has this specific quadratic form, or differs at higher orders, must ultimately be settled by derivation from the VERSF master action.

9.2b Laplacian-extremal characterisation

A fourth perspective comes from the spoke-cycle Laplacian. On the six-cycle C_6, define

$(\Delta\lambda)_i := 2\lambda_i - \lambda_{i-1} - \lambda_{i+1}$.

The eigenmodes of Δ are the standard Fourier modes $\lambda^{(k)}_i = \exp(2\pi i k i/6)$ with eigenvalues

$\mu_k = 2 - 2\cos(2\pi k/6)$.

Tabulating:

k	***μ_k***
0	*0*
1, 5	*1*
2, 4	*3*
3	*4*

The alternating mode is the k = 3 eigenmode,

λ_i^(3) = $(-1)^i$, $\mu_3 = 4$,

with the *maximal* Laplacian eigenvalue. Moreover, the Laplacian quadratic form satisfies

$\langle \lambda, \Delta\lambda \rangle = \Sigma_i (\lambda_i - \lambda_{i+1})^2$,

so the alternating mode maximises local spoke *contrast* across the cycle.

Proposition 4 (Laplacian-extremal characterisation). *Among real eigenmodes of the spoke-cycle Laplacian* Δ *on* C_6*, the alternating mode* $\lambda_i = (-1)^i$ *is the unique eigenmode with maximal eigenvalue* $\mu = 4$*. It is the* ***maximally anti-aligned admissible closure-current mode*** *on the spoke cycle.*

The relationship between A_comp (which the alternating mode *minimises*) and $\langle \lambda, \Delta\lambda \rangle$ (which it *maximises*) is straightforward: the two functionals differ by sign on the cross-term,

$\text{A\_comp}(\lambda) + \langle \lambda, \Delta\lambda \rangle = 4\, \Sigma_i \lambda_i^2$,

so at fixed norm, minimising A_comp is equivalent to maximising $\langle \lambda, \Delta\lambda \rangle$. The variational and Laplacian-extremal characterisations are therefore two faces of the same selection principle, applied through complementary functionals.

9.3 Physical reading: the lowest closure-competition wave

The alternating mode represents *local closure competition without global circulation creation*. Adjacent spokes acquire opposite winding contributions around the outer cycle,

$\sigma(s_i) = s_{i+1} + (-1)^i C$,

and because $\partial C = 0$ the boundary of each spoke is unchanged: endpoint commitment is preserved,

$\partial\sigma(s_i) = v_{i+1} - h$.

But the *internal transport history* of each spoke is not preserved. Each spoke acquires an additional closure winding, with neighbouring spokes carrying opposite orientation.

Boundary-equivalence and history-equivalence. The physical content of the alternating mode can be stated precisely in homological language. Define two equivalence relations on 1-chains x, $y \in C_1(W_7)$:

- *x and y are **boundary-equivalent** if* $\partial x = \partial y$ *(equivalently,* $x - y \in Z_1(W_7)$*)*;
- *x and y are **history-equivalent** if* $[x] = [y]$ in $H_1(W_7)$ *(equivalently,* $x - y \in B_1(W_7)$*).*

Boundary-equivalence means the two chains connect the same committed endpoints. History-equivalence means they differ only by exact transport and therefore carry the same closure-history content. Since $B_1(W_7) \subseteq Z_1(W_7)$, history-equivalence is at least as strong as boundary-equivalence; the two coincide iff $H_1(W_7) = 0$.

For the alternating update,

$\sigma(s_i) = s_{i+1} + (-1)^i C.$

Since $\partial C = 0$, the chains $\sigma(s_i)$ and s_{i+1} are *boundary-equivalent*: $\partial\sigma(s_i) = \partial s_{i+1} = v_{i+1} - h$. But in homology,

$[\sigma(s_i)] = [s_{i+1}] + (-1)^i [C],$

and since $[C] \neq 0$ in $H_1(W_7) \cong \mathbb{Z}$, the chains are *not history-equivalent*. Concretely: σ does not just relocate the spoke from position i to position $i + 1$; it also writes in (or subtracts) one traverse of the outer hexagon. The spoke now connects to the hub along a longer 1-chain — the spoke segment plus a winding around the outer cycle — with the same endpoints but a different homology class. The "closure history" of the spoke is the full 1-chain supporting the endpoint commitment, not just the endpoints themselves.

This gives the punchline of §9 its precise mathematical content:

σ-images are boundary-equivalent to their unperturbed counterparts but not history-equivalent.

Equivalently: *same committed boundary, different homology class*. This is the dividing line between graph symmetry — which acts on the 0-cell structure (which vertex each spoke connects to) — and substrate dynamics — which redistributes the 1-cell structure that supports those connections. The alternating mode is the first place in the $K = 7$ architecture where these two regimes separate cleanly.

Three structural properties combine to make alternation the natural lowest-mode response. It is closure-current-conserving ($\Sigma_i (-1)^i = 0$), so it creates no net winding. It is locally balanced — every positive correction is adjacent to negative corrections, which is precisely the configuration A_comp drives toward. And it is the smallest nonzero integer mode in the primitive homology sector $H_1(W_7) \cong \mathbb{Z}$.

Alternation is therefore not selected by numerical fitting. It is selected jointly by

1. removal of the democratic constant mode by A_circ (equivalently, closure-current conservation),
2. the cyclic-shift spectrum on the zero-sum sublattice Λ_0 (§9.2),
3. variational minimisation of the closure-competition functional A_comp (§9.2a),
4. Laplacian-extremal anti-alignment on the spoke cycle (§9.2b),
5. residual D_3(vtx) symmetry of the wheel (Lemma 6.2),
6. integrality of the response ($\lambda_i \in \mathbb{Z}$ in the primitive homology sector).

Definition (Lowest closure-competition wave). *Let $W_7 = C_6 + h$. The* ***lowest closure-competition wave*** *is the σ-transport excitation defined by the spoke-correction pattern $\Delta_i = (-1)^i C$, equivalently the action $\sigma(s_i) = s_{i+1} + (-1)^i C$ between successive committed W_7 surfaces. It admits four equivalent characterisations under integrality and non-degeneracy: (a) the unique nontrivial integer pattern in (CC) with D_6-stabiliser of order ≥ 6 (structural, Lemma 6.2); (b) the unique nontrivial integer T-eigenmode in Λ_0 (spectral, §9.2); (c) the unique minimiser of the closure-competition functional A_comp on $\mathbb{Z}^6$ (variational, Proposition 3); (d) the eigenmode of the spoke-cycle Laplacian Δ with maximal eigenvalue $\mu = 4$ (Laplacian-extremal, Proposition 4). For interpretive remarks on which physical object the residual D_3(vtx) symmetry attaches to, see §9.3a.*

This formalises the alternating mode as a named object pinned down by four converging selection principles. The variational characterisation (c) is the most economical: it produces (CC) as a corollary rather than as an input.

9.3a Transport-gap symmetry: a proposed reading

Lemma 6.2 identifies $D_3(\text{vtx}) \subset D_6$ as the stabiliser of the alternating pattern λ_alt as an element of the integer lattice Λ_0 under the standard D_6 permutation representation. The lemma is silent on which physical object this stabiliser belongs to. Among the structurally consistent readings:

- a property of the spoke-correction pattern λ as a sequence;
- a property of the closure-current excitation $\sigma_t(s_i) - s_{i+1}$;
- a property of the homological class $[\sigma_t(s_i) - s_{i+1}] \in H_1(W_7)$;
- a residual gauge invariance of the response functional A_cl;
- an invariance of the admissibility projection Π_adm;
- a sublattice structure in the space of admissible substrate states.

Each is structurally consistent with Lemma 6.2 alone; the lemma does not pick between them. In this subsection we propose one — the *transport-gap reading* — and lay out the structural hypotheses it requires. The reading is consistent with the rest of the construction but not derived by it; we offer it as the most natural interpretation conditional on the hypotheses stated below, and list it in §12 as awaiting master-action verification.

The proposed reading. The D_3(vtx) symmetry belongs to the *closure-current excitation* $\sigma_t(s_i) - s_{i+1}$ that σ adds to the spoke sector during one substrate update step. The committed closure surfaces at either end of the σ-morphism retain full D_6 stabiliser; the reduced symmetry lives in the morphism between them, and is restored when the updated state is projected back to the admissible sector.

This reading rests on two structural hypotheses, neither of which is established within this paper.

(H1) D_6-symmetric committed surfaces. Every committed closure surface (post-projection) in the K = 7 architecture has full dihedral stabiliser:

$\mathrm{Stab}(W_7^{(t)}) = D_6$ for every step t (post-projection).

This is a property of the K = 7 admissibility architecture stated separately in the broader programme. It is not derived here, and it is not among the explicit admissibility conditions (A1)–(A3) of §3.

(H2) D_6-restoring projection. The admissibility projection Π_adm preserves D_6 symmetry: applied to any state, it returns a state whose stabiliser is D_6. Equivalently, the admissible sector consists of D_6-symmetric states. This is a structural claim about Π_adm that goes beyond its §9.1 specification as "projection onto the admissible sector"; it would follow if the admissibility architecture itself enforces D_6 symmetry, but (A1)–(A3) do not state this explicitly.

The stabiliser picture under (H1) + (H2). Granting (H1) and (H2), the stabilisers attach to *three categorically distinct objects* — two committed closure surfaces and one closure-current excitation between them:

Object	***Category***	***Stabiliser***
$W_7^{(t)}$	*committed closure surface*	*D_6 (by H1)*
$\sigma_t(s_i) - s_{i+1} = (-1)^i C$	*closure-current excitation*	*D_3(vtx) (by Lemma 6.2, taking Stab(λ_alt))*
$W_7^{(t+1)}$	*committed closure surface*	*D_6 (by H1 + H2)*

These are three different objects with three different stabilisers. The σ-morphism connects the two committed surfaces *via* the closure-current excitation; it is not a transformation acting on a single object whose symmetry changes. We avoid notation that would suggest otherwise (in particular, arrow diagrams between groups, which would conflate "stabiliser of a surface" with "stabiliser of an excitation" — categorically distinct).

Physical content under the proposed reading. If (H1) and (H2) hold, the alternating mode is naturally read as a *transient transport excitation* rather than as a static asymmetry of the substrate. The substrate's committed states retain full D_6 symmetry throughout; the D_3(vtx) reduction appears only in the σ-morphism between them, and only for the duration of a single substrate update step. The substrate does not alternate geometrically; only the closure-current excitation does, and only transiently.

Conditional status. The transport-gap reading is therefore conditional on (H1) and (H2). If both hold, it is the natural reading of Lemma 6.2 in the σ-context. If (H1) fails — if not every committed surface has full D_6 stabiliser — or if (H2) fails — if Π_adm can land on admissible states of reduced symmetry — then the D_3(vtx) reduction must be assigned a different home: one of the alternative readings listed above. Master-action analysis is the appropriate way to decide between these alternatives, listed in §12 as P1e.

We do not claim to have established the transport-gap reading. We claim that it is structurally consistent with the construction, that it requires (H1) and (H2), and that it is the most natural reading conditional on those hypotheses. Confirming or refuting it requires explicit master-action analysis of the σ-morphism sector together with confirmation of the substrate-ontology hypotheses (H1) and (H2).

9.4 Status and what remains

The formulation in this section lifts $\mathcal{R}$ from an unexplained admissibility-preserving map to a closure-gradient response with explicit (postulated) penalty structure, and exhibits four converging characterisations of the alternating mode (structural in §6, spectral in §9.2, variational in §9.2a, Laplacian-extremal in §9.2b). It is not yet a derivation from the VERSF master action. What this section accomplishes:

- It proposes a concrete functional form for $\mathcal{R}$ compatible with the substrate principles of §3 and §5.2.
- It gives the alternating mode a spectral identity (the unique nontrivial integer T-eigenmode in Λ_0) complementary to its structural identity (Lemma 6.2).
- It supplies a variational identity (Proposition 3) that is more economical than the structural and spectral identities: (CC) emerges as a consequence rather than as an input.
- It supplies a Laplacian-extremal identity (Proposition 4) that identifies alternation as the maximally anti-aligned mode on the spoke cycle.
- It supplies the physical reading of alternation as the lowest closure-competition wave on the spoke sector, with the precise boundary-equivalence-but-not-history-equivalence content of "same committed boundary ≠ same homology class."
- It interprets the residual D_3(vtx) symmetry as a transport-gap phenomenon: full D_6 symmetry holds at every committed surface, with the reduction living only in the σ-morphism between them (§9.3a).

What remains:

- Derive A_cl, and the relative weights of its four penalty terms, from the VERSF master action.
- Verify that the full (nonlinear) gradient flow of A_cl recovers the alternating mode as the dynamical attractor, not merely as a fixed point of any of the four selection principles.
- Identify subleading modes and assess whether they contribute higher-order σ corrections.

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10. Interpretation

10.1 Conservation law and residual symmetry

The derivation makes a wider structural principle explicit:

- The chain-map condition fixes σ's *homological type* — corrections must lie in $H_1(W_7)$.
- Closure-current conservation, postulated as the cellular-shadow expression of commitment conservation, fixes σ's *total winding* — the closure-current functional J(σ) must vanish.
- Together these two constraints leave a single residual freedom: the symmetry pattern of the coefficients λ_i under D_6. They jointly make full closure democracy untenable for nontrivial σ (Proposition 2).

The alternating pattern, with stabiliser D_3(vtx), is selected by four converging characterisations:

- **Structural** (Lemma 6.2): the unique nontrivial integer pattern in (CC) with maximal D_6-stabiliser.
- **Spectral** (§9.2): the unique nontrivial integer T-eigenmode in Λ_0.
- **Variational** (§9.2a, Proposition 3): the unique minimiser of A_comp under integrality and non-degeneracy.
- **Laplacian-extremal** (§9.2b, Proposition 4): the eigenmode of the spoke-cycle Laplacian with maximal eigenvalue.

The structural and spectral characterisations are complementary perspectives on the same underlying constraint — (CC) plus integrality — rather than independent derivations: both presuppose (CC). The variational characterisation, by contrast, has a different postulational basis. Proposition 3 minimises A_comp over integer patterns *without* imposing (CC); closure-current conservation $\Sigma_i \lambda_i = 0$ emerges as a corollary (Corollary 3). The variational route therefore offers an alternative substrate-level postulate set: postulate A_comp instead of postulating (CC), and recover both alternation and (CC) from minimisation.

This shifts the standing of (CC). Under the structural and spectral routes, (CC) is a load-bearing postulate (with the formal bridge to commitment conservation listed as P1b). Under the variational route, (CC) is a consequence of A_comp minimisation; the load-bearing postulate becomes the functional form of A_comp itself, which remains to be derived from the master action.

The D_3(vtx) residual symmetry: a proposed reading. §9.3a proposes one interpretation of the D_3(vtx) residual, the *transport-gap reading*: the D_3(vtx) belongs to the closure-current excitation σ writes into the spoke sector during one substrate update step, while the committed closure surfaces at either end of the σ-morphism retain full D_6 stabiliser. This reading is structurally consistent with the construction but rests on two unstated structural hypotheses about the $K = 7$ ontology — (H1) that committed surfaces have D_6 stabiliser, and (H2) that Π_adm preserves D_6 symmetry — neither of which is derived in this paper. Alternative readings (gauge residual of A_cl, invariance of Π_adm, sublattice structure in admissible states) are structurally consistent with Lemma 6.2 alone; choosing among them requires master-action analysis (P1e in §12). We adopt the transport-gap reading in what follows as the most natural reading conditional on (H1) and (H2), without claiming it as established.

Two further structural conclusions follow.

First, in this construction every substrate principle other than (CC) enters as a *derivation* under the structural and spectral routes; (CC) is *postulated* there, motivated by — but not formally bridged from — commitment conservation. (A1)–(A3) come from the K = 7 admissibility architecture; chain-map compatibility follows from (A1); the homological-type constraint follows from the chain-map condition; the alternating coefficient pattern follows from (CC) plus integrality plus a sharpness criterion. The variational route trades (CC) for A_comp as the load-bearing postulate, deriving (CC) as Corollary 3. (N) is a *selection criterion* picking out the substrate-dynamical sector of Σ_adm, not an admissibility condition: $\mathrm{Aut}(W_7) \subset \Sigma\_\mathrm{adm}$ regardless of (N). Theorem 1 is therefore a statement about consequences of substrate principles, with one postulated link whose specific identity depends on which route is taken.

Second, the resulting symmetry reduction is *sector-specific under the proposed reading*. Conditional on (H1) and (H2), the committed closure surfaces retain full D_6 symmetry at every step and the reduction to D_3(vtx) lives entirely in the closure-current excitation between them. This mirrors the wider VERSF theme that nontrivial persistent structure does not emerge from unrestricted symmetry of the substrate but from substrate conservation laws acting on the transport dynamics. The K = 7 wheel is the first place in the closure architecture where this distinction becomes computable in closed form — though the distinction itself depends on the substrate-ontology hypotheses called out in §9.3a.

10.2 What changes at the σ-family level

σ is no longer merely an externally imposed morphism between closure surfaces. It is the cellular shadow of an admissibility-preserving substrate response rule that itself admits a (postulated) closure-gradient form. Its homological content is fixed by the topology of W_7 ($H_1 \cong \mathbb{Z}$); its coefficient is pinned by four converging characterisations (Lemma 6.2 structurally, §9.2 spectrally, Proposition 3 variationally, Proposition 4 Laplacian-extremally); and Corollary 2 shows that the resulting mapping telescope carries an accumulated alternating closure-current transport class, connecting the construction to the refinement-persistent cohomology sector of the broader VERSF programme.

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11. What This Establishes

The paper establishes:

1. σ does not need to be imposed as a bare graph automorphism.
2. A substrate response rule $\mathcal{R}[\rho]$ satisfying (A1)–(A3) induces a chain map σ_t automatically (Proposition 1).

3. The chain-map condition alone is too weak to fix σ; it constrains only the homological type of the spoke correction.
4. Under full D_6 democracy, closure-current conservation forces the spoke correction to vanish, so σ collapses to the graph automorphism σ_rot (Proposition 2).
5. Among nontrivial integer patterns, the alternating pattern is uniquely characterised by four converging selection principles: maximal D_6-stabiliser in (CC) (Lemma 6.2, structural), T-eigenmode structure on Λ_0 (§9.2, spectral), minimisation of the closure-competition functional A_comp (Proposition 3, variational), and maximal eigenvalue of the spoke-cycle Laplacian (Proposition 4, Laplacian-extremal).
6. Closure-current conservation $\Sigma_i \lambda_i = 0$ is not only postulated as the cellular-shadow expression of commitment conservation; it is *derived* as a corollary of A_comp minimisation under integrality (Corollary 3). The variational route therefore offers an alternative postulational basis under which (CC) is a consequence rather than a hypothesis.
7. The space Σ_adm(W_7) is strictly larger than Aut(W_7) and contains explicit telescope-compatible witnesses of substrate dynamics. Selecting the substrate-dynamical sector requires the non-degeneracy criterion (N), which is a selection criterion rather than an admissibility condition.
8. The mapping telescope of the alternating σ-family carries an accumulated alternating closure-current transport class, with per-step per-spoke contribution $(-1)^i\,[C] \in H_1(W_7)$ (Corollary 2). This connects the construction to the refinement-persistent cohomology sector of the broader VERSF programme.
9. $\boldsymbol{\mathcal{R}}$ admits a candidate closure-gradient formulation $\boldsymbol{\mathcal{R}}[\rho] = \Pi\_adm \circ \exp(-\eta\, \nabla A\_cl)[\rho]$ with a four-term penalty structure identified in §9.1a as the *leading-order admissibility functional* under the constrained-EFT methodology established for other constitutive sectors of the VERSF master action. The four-term decomposition is not arbitrarily chosen; it follows from locality, conservation compatibility, symmetry, additivity, and EFT ordering applied in the closure-response sector.
10. The "same committed boundary ≠ same closure history" reading of the alternating mode is given its precise homological content: σ-images are *boundary-equivalent* to their unperturbed counterparts but not *history-equivalent* (§9.3).
11. A *proposed reading* of the D_3(vtx) residual symmetry is supplied in §9.3a: the **transport-gap interpretation** assigns the D_3(vtx) to the closure-current excitation between successive committed surfaces, with the surfaces themselves retaining full D_6. This reading is structurally consistent with the construction but rests on two substrate-ontology hypotheses called out as (H1) and (H2) in §9.3a — that committed closure surfaces have D_6 stabiliser, and that Π_adm preserves D_6 symmetry — neither of which is derived in this paper. Alternative readings of D_3(vtx) (gauge residual of A_cl, invariance of Π_adm, sublattice structure in admissible states, feature of the homological excitation) are equally consistent with Lemma 6.2 alone. Selecting among them is P1e in §12.

What has *not* been established:

- The unique *physical* σ-family.
- The derivation of A_cl, and the relative weights of its four penalty terms, from the VERSF master action.

- The formal bridge from commitment conservation (a constraint on cell-level commitment) to (CC) (a constraint on 1-cycle winding). Under the variational route, (CC) is derived from A_comp minimisation; under the structural and spectral routes, (CC) remains postulated, with the formal bridge open.
- The physical content of the D_3(vtx) residual symmetry — whether it is a property of the closure-current excitation as proposed in the transport-gap reading of §9.3a (which itself rests on the unstated structural hypotheses (H1) and (H2)), a residual gauge invariance of A_cl, an invariance of Π_adm, a sublattice structure in admissible substrate states, or a feature of the homological class $[\sigma_t(s_i) - s_{i+1}] \in H_1(W_7)$. The lemma is silent; the choice requires master-action analysis (P1e).
- Higher-order corrections to the alternating mode from subleading eigenmodes of the cyclic shift or higher-order terms in A_comp.

The status of σ is therefore upgraded from *imposed structural morphism* to *substrate-generated minimal candidate* pinned by four converging selection principles, and the status of $\boldsymbol{\mathcal{R}}$ from *axiomatic admissibility-preserving map* to *closure-gradient response with candidate penalty structure*. The remaining problem — selecting the physical σ-family within Σ_adm(W_7) — is now a sharply posed question about substrate-action dynamics rather than an open structural question about whether such σ-families exist at all.

12. Open Problems

P1a (load-bearing). Verify that the specific four-term decomposition A_cl = A_inc + A_hub + A_circ + A_comp, together with the relative weights of the four terms, is uniquely forced by variation of the full VERSF master action in the sequential transport sector — i.e., that the leading-order constrained-EFT analysis of §9.1a is reproduced by direct variation, up to subleading corrections. This is the analogue, for the closure-response sector, of the record-current uniqueness result.

P1b (load-bearing). Bridge from commitment conservation to (CC). Make precise how the substrate principle "$\boldsymbol{\mathcal{R}}$ cannot manufacture commitment ex nihilo" — a constraint on cell-level commitment — translates to $J(\sigma) = 0$ in $H_1(W_7)$, a constraint on 1-cycle winding. As written, (CC) is the natural cellular-shadow analogue of commitment conservation; the formal correspondence is appealed to but not worked through.

P1c. Verify that the full (nonlinear) gradient flow of A_cl recovers the alternating mode as the leading-order σ correction — not merely as the unique nontrivial integer T-eigenmode in Λ_0 but as the dynamical attractor of the closure-gradient response.

P1d. Investigate whether the alternating mode saturates the BCB support budget B_BCB and TPB range bound r_TPB. Saturation is plausible but not established by the present construction.

P1e. Determine the physical content of the D_3(vtx) residual symmetry by direct master-action analysis. Multiple readings are structurally consistent with Lemma 6.2 alone: the transport-gap reading proposed in §9.3a (which assigns $D_3(\mathrm{vtx})$ to the closure-current excitation and depends on the structural hypotheses (H1)/(H2) about the $K = 7$ ontology); a residual gauge invariance of A_cl; an invariance of Π_adm; a sublattice structure in admissible substrate states; or a feature of the homological class $[\sigma_t(s_i) - s_{i+1}] \in H_1(W_7)$. The lemma does not distinguish between these. The transport-gap reading is the most natural conditional on (H1) and (H2); confirming, refuting, or replacing it with one of the alternatives requires master-action calculation.

P1f. Extend from one-step σ_t to the full sequence $\{ \sigma_t \}\_\{t \in \mathbb{N}\}$: derive the *family law* under which $\mathcal{R}$ acts coherently across all steps, and identify the continuum limit.

P1g. Identify subleading modes in the linearisation of §9.2 and assess whether they generate higher-order σ corrections beyond the alternating pattern.

P1h. Connect the telescope σ-family to the separate vertex × tick-window object D(i, t), and exhibit the σ-duality between them.

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13. Conclusion

This paper supplies a substrate-response mechanism capable of generating the σ-family in the mapping-telescope formulation of VERSF sequential transport, determines its symmetry-maximal coefficient pattern in closed form via four converging characterisations, and proposes a first derivation-level formulation of the substrate response rule itself.

The construction proceeds from substrate principles by the following chain of constraints:

1. The chain-map condition follows automatically from incidence preservation (Proposition 1, *derived*).
2. Spoke corrections are forced into $H_1(W_7)$ by the chain-map condition (*derived*).
3. Closure-current conservation requires $J(\sigma) = 0$ in $H_1(W_7)$. Under the structural and spectral routes this is *postulated* (cellular-shadow expression of commitment conservation; formal bridge open, P1b). Under the variational route it is *derived* as a corollary of A_comp minimisation (Corollary 3).
4. Under full D_6 democracy, (3) forces the spoke correction to vanish, so σ collapses to the graph automorphism σ_rot (Proposition 2).
5. The alternating pattern is pinned by four converging characterisations: maximal D_6-stabiliser in (CC) (Lemma 6.2, structural); unique nontrivial integer T-eigenmode in Λ_0 (§9.2, spectral); unique minimiser of A_comp under integrality and non-degeneracy (Proposition 3, variational); eigenmode of the spoke-cycle Laplacian with maximal eigenvalue (Proposition 4, Laplacian-extremal).
6. The mapping telescope of the alternating σ-family carries an accumulated alternating closure-current transport class with per-step per-spoke contribution $(-1)^i$ [C] (Corollary 2), connecting the construction to the refinement-persistent cohomology sector of the broader VERSF programme.

A candidate gradient-flow formulation of $\boldsymbol{\mathcal{R}}$ is supplied in §9.1: $\boldsymbol{\mathcal{R}}[\rho] = \Pi\_adm \circ \exp(-\eta \nabla A\_cl)[\rho]$, with A_cl decomposed into four leading-order penalty terms. §9.1a identifies this decomposition as the leading-order admissibility functional under the constrained-EFT methodology established for other constitutive sectors of the VERSF master action. The four-term form is therefore not arbitrarily chosen but constrained by the same locality, conservation-compatibility, symmetry, additivity, and EFT-ordering principles that govern the broader programme. Uniqueness under direct master-action variation in the sequential transport sector is P1a.

The result is the canonical symmetry-maximal nontrivial σ-family

$$\sigma(s_i) = s_{i+1} + (-1)^i C,$$

which is genuinely *substrate-generated* — it does not lie in Aut(W_7) — and which closes the mapping-telescope differential. The alternating mode has a clear physical reading: it is the **lowest closure-competition wave** on the spoke sector, characterised precisely by the homological statement that σ-images are boundary-equivalent to their unperturbed counterparts but not history-equivalent (§9.3). Same committed boundary, different homology class.

A *proposed reading* of the residual symmetry reduction is offered in §9.3a: conditional on two structural hypotheses about the $K = 7$ substrate ontology — (H1) that committed closure surfaces have D_6 stabiliser, and (H2) that Π_adm preserves D_6 symmetry — the D_3(vtx) residual identified by Lemma 6.2 belongs to the closure-current excitation σ adds during one substrate update step, not to the committed surfaces at either end of the σ-morphism. Under this *transport-gap reading*, the substrate's stable geometry retains full D_6 symmetry at every committed step and the alternating structure is a transient feature of the morphism between committed states. The reading is structurally consistent but not derived; alternative readings (gauge residual of A_cl, invariance of Π_adm, sublattice structure, feature of the homological excitation) are equally consistent with Lemma 6.2 alone. Choosing among them requires master-action analysis (P1e).

The construction stops short of full master-action derivation in the precise sense made explicit by §9.1a: it gives the leading-order EFT analysis of the closure-response sector, but the explicit master-action variation that would close P1a remains to be performed. (CC) is postulated under the structural and spectral routes but derived under the variational route, which therefore offers a more economical postulational basis. The transport-gap reading of the D_3(vtx) residual is offered as the most natural reading conditional on the structural hypotheses (H1) and (H2) of §9.3a, but alternative readings of D_3(vtx) are equally consistent with the lemma alone and await master-action analysis (P1e). What this paper does deliver is a referee-checkable framework in which each result has a clear epistemic standing: derived, postulated with methodological standing, postulated without, proposed conditional on stated hypotheses, or open. The remaining work is to close the load-bearing postulates and conditional readings against the master action.

Structurally, the picture that emerges is that σ is the discrete shadow of an effective transport current of the closure substrate, and A_cl is its constitutive functional — placing the sequential transport sector in close structural relation to the constitutive current programme developed elsewhere in VERSF. Once that connection is made formal, the present construction will either be recovered as the leading-order closure-response sector of a unified constitutive current structure, or the connection will sharpen the difference between the two sectors. Either outcome is informative.