Tensorial Transport Geometry from Coherence Propagation in VERSF

Substrate Parallel Transport, Coherence Holonomy, Transport-Curvature Tensors, and the Geodesic-Deviation / Scaling Structure of Localized Defects

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General-Reader Summary

The VERSF programme proposes that the large-scale structure we recognise as physical reality — geometry, fields, and ultimately matter — emerges from a much simpler underlying layer: a vast lattice of points, with each point carrying a small set of 7 internal states that evolve step by step according to a fixed local rule. Earlier papers established what this picture buys us. The local rule produces a smooth continuum at large scales (Stage V). The setup is robust: small changes to the rule don't break it (Stage VII). When the rule is disturbed in a small region — a "defect" — that disturbance produces four measurable effects, all controlled by a single local quantity called the coherence gap (Stage VIII). And when we let neighbouring points talk to each other in a controlled way, we get an explicit speed limit for how fast disturbances can spread, plus a precise test for when a defect is strong enough to permanently trap a piece of the substrate's activity around itself (Stage IX).

Throughout all of this, however, the picture had one important limitation. Every quantity that mattered — distances, speeds, the size of trapped regions, the strength of defects — was a single number attached to each point. The geometry was, in a precise sense, *flat in its descriptive vocabulary*: each location got a number, those numbers varied from place to place, but nothing in the framework recorded *how you got from one place to another*. In the geometry of curved surfaces — think of the surface of the Earth — this kind of information matters enormously. Walk a thousand miles north, then a thousand miles east, then a thousand miles south, and you don't end up where you started. The order of your moves matters; the path you took matters. Curvature is, fundamentally, the fact that paths matter.

The previous VERSF papers had no machinery to detect this. This paper adds it.

The central new idea is straightforward in outline. Given any path through the substrate — a sequence of neighbouring points — we can ask what happens to a piece of internal "coherence" carried along that path. In the undisturbed substrate (no defects), the answer is boring: the piece of coherence arrives at the destination unchanged, and it doesn't matter which path you took. This is the substrate's *vacuum* state, and it corresponds to a flat geometry. But once a defect is present, the answer becomes interesting. Different paths between the same two points now produce different results. And — most importantly — if you carry a piece of coherence around a closed loop that encircles a defect, it comes back transformed. The loop has *memory*. The defect has left a measurable imprint on the coherence that travelled around it.

This loop memory is the substrate's version of curvature. It is the first genuinely directional, path-aware structure the VERSF programme has produced, and it changes the character of the geometry from "a collection of local numbers" to "a setting where paths carry information." Once this structure is in place, several things that the earlier papers could only describe with scalars acquire richer content:

- **The candidate "curvature" identified in Stage VIII** — built from how the coherence gap varies in space — is closely related to the new path-dependent structure, but exactly *how* it sits inside the richer object is still an open question. Both quantities are derived from the same defect structure, both decay exponentially around defects in the same way, and both serve corresponding roles (Stage VIII's scalar as a curvature precursor, the new tensor as the full curvature). The precise mathematical projection that takes the new tensor and produces Stage VIII's scalar from it has not yet been pinned down, and we record this honestly rather than papering over it. What is established is the structural correspondence; what remains open is the precise translation map.
- **Trapped regions are not just localised activity — they are localised curvature.** A defect strong enough to trap coherence around itself is also a defect strong enough to bend the paths of coherence passing nearby. The two roles — "traps activity" and "bends nearby paths" — are inseparable in the new framework, with the same exponential reach controlling both. This is, structurally, the substrate-level echo of how concentrated mass in classical gravity both produces curvature *and* binds nearby orbits.
- **Nearby coherence trajectories focus toward defects.** Two paths starting close together and passing near a defect will be pushed together by the defect's transport-curvature, in much the same way that two beams of light passing near the Sun are bent toward it. The mathematics is structurally identical to the geodesic-deviation equation of general relativity — though here it is derived from substrate transport rather than from a gravitational metric, and the analogy should be taken as a structural correspondence, not a claim about gravity.
- **Loops around defects have measurable summary numbers** that work like the Wilson loops of lattice physics: they tell you how much "curvature" is enclosed inside the loop, in a way that depends only on the loop itself, not on the particular path used to compute it. This is, again, a structural correspondence — the substrate is not a gauge theory and we don't add gauge fields by hand.
- **A single scaling relationship now connects everything.** The earlier paper showed that three quantities — propagation speed, localisation size of trapped regions, and binding strength of trapped states — are tied together by one simple relation. The present paper adds curvature to that relation: how strong the path-dependence is, how steep the gap variations are, and how localised the trapped regions are all sit on the same scaling backbone. The Stage VIII picture of "one field controlling four effects" upgrades to "one field controlling four effects within one path-dependent geometry under one scaling relation."

What this paper does *not* do is also important. It does not derive Einstein's equations of general relativity. It does not show that the substrate's geometry is Lorentzian (the signature of relativistic spacetime). It does not introduce gauge fields, electromagnetism, or any of the other ingredients of the Standard Model of particle physics. It does not quantise. What it *does* do is supply the structural layer that any future attempt at any of those things would have to build on — the transition from a substrate where geometry is just "numbers attached to points" to a substrate where geometry is *intrinsically path-dependent*, with measurable curvature, loop memory, focusing of trajectories, and a unified scaling law. The epistemic register is honest about which pieces are rigorously proven (the basic transport machinery, the vanishing of curvature in the undisturbed substrate, the exponential decay of curvature around defects), which are conditional on identifications that worked well in earlier stages (the precise quantitative form of the scaling relations), and which are structural analogies whose deeper meaning is not yet established (the parallels with gravity, with gauge theory, and with the geodesic-deviation equation). The contribution is the bridge — not the territory on the other side.

Abstract

Stage IX established a coupled global refinement transport operator

$\mathbf{T} = \hat{T} \otimes I_X + \gamma \cdot I_{\mathcal{K}} \otimes A_X + C$,

acting on the substrate Hilbert space $\mathscr{H} = \mathbb{R}^{\mathcal{K}} \otimes \mathbb{R}^X$, with bulk transport bands, finite coherence propagation speed $v_c = \gamma \cdot \rho(A_X)$, the Birman–Schwinger criterion for trapped-mode existence, and Combes–Thomas-type exponential localization of trapped eigenvectors. The resulting geometry of Stage IX was fundamentally scalar, controlled by the local spectral-gap field $\varepsilon_{gap}(x)$ and its derivatives. The present paper develops the next structural layer: a tensorial transport geometry emerging directly from substrate coherence propagation, with no manifold, metric, connection, or gauge field assumed.

We define substrate-level local transport maps $\mathcal{T}_{\{x \to y\}} := P_y \cdot \mathbf{T} \cdot P_x$ between adjacent fibres of $\mathbf{T}$ and assemble them into the **substrate parallel transport operator** $\mathcal{P}_\pi$ along directed substrate paths π. In the canonical vacuum (uniform spatial fibres), parallel transport along homotopic paths agrees and the geometry is *flat*. Localized defects break this equivalence: transport becomes path-dependent and substrate loops accumulate non-trivial *coherence holonomy* $\mathscr{H}(\gamma) := \mathcal{P}_\gamma$. The principal results of the paper are:

- **Substrate parallel transport (Definition 2.3, Proposition 2.5).** The map $\pi \mapsto \mathcal{P}_\pi$ is a well-defined directional product of composite local transport maps (Definition 2.1, each combining one fibre step $\hat{T}_x$ with one spatial step γ), with $\mathcal{P}_\pi = \gamma^n \cdot \hat{T}^n$ in the canonical vacuum (uniform $\hat{T}_x \equiv \hat{T}$) for any path of length n between fixed endpoints, hence path-independent in the vacuum.
- **Directional transport generators and the transport-curvature tensor (Definitions 3.1, 3.4, 5.1; Proposition 3.3).** Any directional decomposition $A_X = \Sigma_i A_i$ induces transfer-operator generators $\nabla i := \gamma \cdot \Sigma_x (\hat{T} \cdot |x + e_i\rangle\langle x|)$, *combining one fibre step with*

one i-direction position shift. The perturbed generators in the presence of a Stage VIII defect are $\tilde{\nabla}_i := \gamma \cdot \Sigma_x (\hat{T}_x \cdot |x + e_i\rangle\langle x|)$ *with defect correction* $\Delta_i := \tilde{\nabla}_i - \nabla_i = \gamma \cdot \Sigma_x (\Delta T_{x_0,r}(x) \cdot |x + e_i\rangle\langle x|)$ *(Definition 3.5). The transport-curvature tensor is the commutator* $\mathscr{R}_{ij} := [\tilde{\nabla}_i, \tilde{\nabla}_j]$. For *commuting* directional decompositions of A_X (the canonical case for $\mathbb{Z}^d$-type substrates), $\mathscr{R}_{ij} \equiv 0$ in the bulk (Proposition 3.3) and curvature is generated by the boundary-shell fibre difference $\hat{T}_{x + e_i} - \hat{T}_{x + e_j} = \Delta T_{x_0,r}(x + e_i) - \Delta T_{x_0,r}(x + e_j)$ (Proposition 3.5, equation (3.7)), which is non-zero on substrate positions x where stepping in direction i enters the defect support and stepping in direction j does not (or vice versa).

- **Holonomy from defects (Theorem 4.2).** For a localized defect V supported in $B_r(x_0)$ with non-zero fibre commutator $[\hat{T}, \Delta T_{x_0,r}(x)]$ and non-zero boundary-shell asymmetry, there exist substrate loops γ surrounding the defect region with non-trivial normalised holonomy $\hat{H}(\gamma) := \gamma^{-n} \cdot \hat{T}^{-n} \cdot \mathscr{H}(\gamma) - I_{\mathcal{K}}$ on the $\hat{T}$-invertible subspace (the operator-norm form, 4.2a); the corresponding loop-trace functional $W(\gamma) := \mathrm{Tr}_{\mathcal{K}}(\mathscr{H}(\gamma))$ deviates generically from its vacuum value $\gamma^n \cdot \mathrm{Tr}(\hat{T}^n)$ (4.2b), subject to the caveat that symmetric defect configurations can produce trace coincidences while leaving the operator-norm form intact. The holonomy magnitude is bounded by an integrated commutator estimate and decays exponentially with the loop's distance from the defect, with rate governed by the trapped-mode Combes–Thomas localization length ξ of Stage IX (Proposition 8.1) or, for subcritical defects, by the subcritical rate of Remark 8.2.
- **Geodesic deviation equation (Theorem 6.3).** Coherence trajectories near defects satisfy a substrate analogue of the geodesic-deviation equation, $\mathcal{D}^2\xi/ds^2 \sim \mathscr{R}(\xi)$, with focusing for $\mathscr{R} > 0$ and defocusing for $\mathscr{R} < 0$. The derivation is via second variation of the Stage IX transport action and inherits the §9.1 heuristic register of Stage IX (the $v_c^{local}(x) \propto \varepsilon_{gap}(x)$ identification).
- **Wilson-type loop functionals (§7).** The trace functional $W(\gamma) := \mathrm{Tr}(\mathscr{H}(\gamma))$ measures integrated transport curvature, with vacuum value $\gamma^n \cdot \mathrm{Tr}(\hat{T}^n)$ and deviation $\hat{W}(\gamma) \approx \gamma^n \cdot \oint_\gamma \mathrm{Tr}(\mathscr{R} \cdot \hat{T}^n) \cdot dA^{ij} + \mathcal{O}(\mathscr{R}^2)$ to leading order. The structural correspondence with lattice-gauge Wilson loops is exact at the level of operator algebra; the absence of gauge content is explicit.
- **Curvature localization at trapped modes (Proposition 8.1, Remark 8.2).** For defects *producing* a trapped mode of Stage IX with localization length ξ_{trap}, the transport curvature satisfies $\|\mathscr{R}(x)\| \leq C \cdot e^{-d_X(x, x_0)/\xi_{trap}}$, inheriting the Combes–Thomas decay rate. For subcritical defects (no trapped mode), the curvature decay rate is governed instead by distance to the nearest bulk-spectrum point of $\mathbf{T}_{bulk}$ (Remark 8.2). Under the present construction, trapped-mode configurations are structurally distinguished by the property that the curvature-localization rate matches the eigenvector-localization rate — a coexistence and rate-coincidence statement, not a claim of strict equivalence (cf. §8.2).
- **Scalar curvature — open identification problem (Discussion 5.4).** The Stage VIII scalar candidate $R(x) = \nabla^2\varepsilon_{gap}(x)$ and the present transport-curvature tensor $\mathscr{R}_{ij}(x)$ are both ε_{gap}-derived geometric objects from the same defect structure and share their qualitative localization behaviour. The precise scalar projection $\mathscr{R} \to R$ that recovers Stage VIII's R(x) is, however, *not* the naive Kretschmann-style contraction $\mathrm{Tr}(\mathscr{R}_{ij}\mathscr{R}^{ij})$ (which has the wrong α-order and sign behaviour) and has not yet been established; the diagonal-sum $\Sigma_i \mathscr{R}_{ii}$ vanishes by antisymmetry, and

divergence-type contractions remain unexplored. We record this honestly as an open identification problem, with a Born-series expansion of the perturbed transport generators as the natural next-stage target.

- **Extended Coherence Transport Scaling Identity (Proposition 9.2).** The Stage IX identity $\xi \cdot \delta \sim v_c$ extends to include curvature: $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}| / \xi \sim \delta \cdot |\nabla\varepsilon_{gap}| / v_c$. Localization, curvature, transport velocity, and gap gradients are governed by one underlying scaling structure. The Stage VIII Defect-Coherence Principle upgrades to "one field, four functionals, one transport geometry, one scaling identity".

We do *not* derive Einstein equations, metric tensors, Lorentzian signature, gauge fields, or quantization. The contribution is the explicit substrate-level construction of parallel transport, holonomy, curvature tensors, and geodesic deviation as operator-theoretic structures emerging from the Stage IX transport operator **T** — the genuinely tensorial geometric layer Stage VIII §12.1 and Stage IX §14.6 identified as the principal next-stage target.

Contents

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1. Introduction

Stage IX established that the refinement substrate is not merely a collection of independent local wheel operators but a globally coupled transport system: the operator

$$\mathbf{T} = \hat{T} \otimes I_X + \gamma \cdot I_{\mathcal{K}} \otimes A_X + C \quad (1.1)$$

acting on $\mathscr{H} = \mathbb{R}^{\mathcal{K}} \otimes \mathbb{R}^X$ produces transport bands ($\mathrm{spec}(\mathbf{T}_{bulk}) = \{\lambda_i + \gamma \cdot \mu : \lambda_i \in \mathrm{spec}(\hat{T}), \mu \in \mathrm{spec}(A_X)\}$, organising into 5 distinct bands for the canonical wheel), finite coherence propagation speed $v_c = \gamma \cdot \rho(A_X)$ (the substrate-level Lieb–Robinson bound), the Birman–Schwinger criterion for trapped-mode existence outside the bulk bands, Combes–Thomas exponential localization of trapped eigenvectors with rate $\eta = \min(1, \delta/(2\gamma \cdot \rho(A_X)))$, and the Coherence Transport Scaling Identity $\xi \cdot \delta \sim v_c$. The Stage IX framework supplied the four engineering pieces identified in Stage VIII §12.2 as prerequisites for trapped-mode analysis: substrate spatial coupling, bulk resolvent structure, Birman–Schwinger criteria, and Combes–Thomas decay.

Across Stages V–IX, however, the emergent geometric structure remained fundamentally **scalar**. The local spectral-gap field $\varepsilon_{gap}(x)$, the candidate-curvature indicator $R(x) := \nabla^2 \varepsilon_{gap}(x)$, the localization length ξ, the coherence velocity v_c, the spectral distance δ, and the transport bands themselves were all scalar properties of either the substrate at a point or of the spectral / propagation structure as a whole. The Coherence Transport Scaling Identity $\xi \cdot \delta \sim v_c$ linked these scalars but did not produce tensorial content. The Stage VIII Defect-Coherence Principle unified four functionals of one scalar field; the Stage IX transport reading explained their common physical origin as different aspects of local coherence-transport capacity, also scalar.

This was, retrospectively, a genuine structural gap. In ordinary differential geometry, curvature is fundamentally *transport-theoretic*: vectors parallel-transported around closed loops fail to return unchanged, transport along distinct paths to the same endpoint produces distinct images, and the infinitesimal failure of transport-operator commutativity defines the Riemann curvature tensor. None of this structure was visible in the Stage V → IX framework, because the substrate transport operator **T** had only been studied in its action on individual states — not in the path-ordered products that define parallel transport.

The present paper supplies that missing structure. We define **local transport maps** $\mathcal{T}_{x \to y}$ between adjacent substrate fibres, **parallel transport operators** $\mathcal{P}_\pi$ along directed substrate paths π, **directional transport generators** ∇*i from a directional decomposition of the spatial-coupling operator A_X,* ***coherence holonomy*** *$\mathscr{H}(\gamma) := \mathcal{P}_\gamma$ along substrate loops, and the* ***transport-curvature tensor*** *$\mathscr{R}$*{ij} := $[\nabla_i, \nabla_j]$ from generator commutators. We then establish that:

1. In the canonical vacuum (uniform spatial fibres $\hat{T}_x \equiv \hat{T}$ at every position), parallel transport is path-independent and curvature vanishes — the substrate is *flat*.
2. Localized defects (the Stage VIII admissibility-preserving perturbations) generically produce directional anisotropy of the perturbed transport generators, generating non-trivial holonomy on loops surrounding the defect region and non-zero transport curvature at the defect core.
3. Trapped modes of Stage IX, lifted to the present setting, appear as localized concentrations of transport curvature with the same Combes–Thomas exponential decay rate as the trapped eigenvectors themselves.
4. Coherence trajectories near defects satisfy a substrate analogue of the geodesic-deviation equation, with focusing / defocusing behaviour governed by the sign of the transport curvature.
5. The Stage VIII scalar candidate curvature $R(x) = \nabla^2 \varepsilon_{gap}(x)$ and the present transport-curvature tensor $\mathscr{R}_{ij}(x)$ are both ε_{gap}-derived geometric objects from the same

Stage VIII defect structure, sharing exponential localization near defects and structurally corresponding roles. The precise scalar projection $\mathscr{R} \to R$ that recovers Stage VIII's R(x) is, however, *not* the naive Kretschmann-style contraction $Tr(\mathscr{R}_{ij} \mathscr{R}^{ij})$ (which has wrong α-order and sign behaviour), and remains an open identification problem (Discussion 5.4). The structural correspondence is established; the precise translation map awaits a Born-series-rigorous derivation.

6. The Stage IX Coherence Transport Scaling Identity extends to include curvature, producing a single scaling structure governing localization, curvature, velocity, and gap gradients.

What this paper does *not* do:

- It does not construct a metric tensor, connection, Levi-Civita compatibility, or any other piece of Riemannian apparatus. The transport geometry is *operator-theoretic and combinatorial*, built from products of substrate transport operators along graph paths.
- It does not derive Einstein equations. No gravitational field equation is proposed, no stress-energy tensor is identified, and the connection between transport curvature and gravity-like dynamics remains conjectural.
- It does not assume or derive Lorentzian signature. The Stage IX transport cones are propagation bounds, not light cones in any Minkowski sense; the temporal direction in the present framework is the refinement-step index, not an emergent Lorentzian time.
- It does not introduce gauge fields. The Wilson-loop analogy of §7 is structural — the operator algebra of loop transport is mathematically isomorphic to lattice-gauge Wilson loops, but no gauge group action is imposed and no gauge-covariant derivative structure is derived.
- It does not quantize. The framework remains classical / operator-theoretic, in the Stage VIII / IX sense; quantization of the transport curvature is one of the principal open targets.

The contribution is the explicit operator-theoretic construction of the tensorial geometric layer that any future tensorial completion (metric, Lorentzian, gauge, quantized, or gravitational) must build upon.

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2. Parallel Transport on the Coherence Substrate

2.1 Setting

Throughout this paper we work with the Stage IX coupled global refinement transport operator

$\mathbf{T} = \hat{T} \otimes I_X + \gamma \cdot I_{\mathcal{K}} \otimes A_X + C$, (2.1)

acting on the substrate Hilbert space $\mathscr{H} = \mathbb{R}^{\mathcal{K}} \otimes \mathbb{R}^X$, with the standing hypotheses and notation conventions of Stage IX §§2.1–2.4:

- X is a regular substrate graph with bounded degree z_{max}, symmetric adjacency operator A_X, and connected substrate graph.
- $\mathcal{K} = \{\kappa_h, \kappa_{b_1}, \ldots, \kappa_{b_6}\}$ is the closure catalogue (7 states).
- $\hat{T}$ is the canonical K = 7 wheel operator with $\mathrm{spec}(\hat{T}) = \{1, ½, ½, -¼, -3/28, 0, 0\}$.
- $\gamma > 0$ is the coherence-transport coupling, working in the weak-coupling regime $\gamma \cdot \rho(A_X) < ½$ (the regime (W) of Stage IX §2.4) throughout.
- C is the closure-label-mixing coupling. **Throughout Part I of this paper we work in the minimal-coupling regime C = 0**, indicating where $C \neq 0$ generalisations are straightforward and where they require additional Duhamel-style analysis (cf. Stage IX §4 "Note on $C \neq 0$").
- The inner product on $\mathcal{H}$ is $L^2(\pi \otimes \mu_X)$, where π is the stationary measure of $\hat{T}$ on $\mathcal{K}$ and μ_X is the substrate measure with respect to which A_X is self-adjoint (Stage IX §2.1 / §8). When spatial fibres preserve reversibility, $\hat{T}$ is self-adjoint in $L^2(\pi)$ and A_X is self-adjoint in $L^2(\mu_X)$, so $\mathbf{T}_{bulk}$ is self-adjoint in $L^2(\pi \otimes \mu_X)$ — the default setting for the Combes–Thomas / Birman–Schwinger machinery we invoke from Stage IX.

For each substrate position $x \in X$, let $P_x : \mathcal{H} \to \mathbb{R}^\mathcal{K} \otimes |x\rangle$ denote the orthogonal projection onto the position-x fibre. By Stage IX Theorem 10.1 (the fibrewise projection theorem), the spatial fibre $\mathbf{T}|_{\{\mathcal{K},x\}} := P_x \cdot \mathbf{T} \cdot P_x$ *equals exactly the local refinement operator* $\hat{T}_x$ *(which reduces to* $\hat{T}$ *in the canonical vacuum and to* $\hat{T} + \Delta T_{\{x_0,r\}}(x)$ in the presence of a Stage VIII localized defect).

2.2 Local Transport Maps (Composite One-Step Form)

A naive definition of the local transport map as $\mathcal{T}_{\{x\to y\}}^{\{naive\}} := P_y \cdot \mathbf{T} \cdot P_x$ captures only the spatial-coupling matrix element of **T** between fibres x and y: in the minimal-coupling regime, $P_y \cdot (\hat{T} \otimes I_X) \cdot P_x = 0$ for $y \neq x$ (since $\hat{T} \otimes I_X$ is diagonal in X), so the naive map reduces to $\gamma \cdot I_\mathcal{K}$ and contains no information about the local fibre dynamics $\hat{T}_x$. For Stage VIII diagonal-in-X defects $\Delta T_{\{x_0,r\}}(x) \otimes |x\rangle\langle x|$, which modify $\hat{T}_x$ but not the off-diagonal-in-X spatial coupling, the naive map is *insensitive to the defect entirely* — and the corresponding path-ordered product $\gamma^n \cdot I_\mathcal{K}$ is identically equal to its vacuum value, no matter what defect is present.

This is structurally inadequate for parallel transport: a transport map that does not see diagonal-in-X defects cannot produce path-dependence from such defects. The fix is to define the local transport map as a *composite* of one fibre step ($\hat{T}_x$ acting on the source fibre) followed by one spatial step ($\gamma \cdot A_X$ propagating to the neighbour):

Definition 2.1 — Composite Local Transport Map

For each pair of substrate positions (x, y) with $y \sim x$, the **local transport map** from x to y is

$\mathcal{T}_{\{x\to y\}} := \gamma \cdot (\hat{T}_x \otimes \langle y|x'\rangle)$, (Definition 2.1)

acting on $\mathbb{R}^\mathcal{K} \otimes |x\rangle$ as: first apply the local fibre operator $\hat{T}_x$ to the closure-catalogue state, then propagate one spatial step to y with coupling γ. Concretely, on a state $\psi \otimes |x\rangle$,

$\mathcal{T}_{x\to y}\ (\psi \otimes |x\rangle) = \gamma \cdot (\hat{T}_x\ \psi) \otimes |y\rangle$. (2.2)

Under the natural identification $\mathbb{R}^\mathcal{K} \otimes |y\rangle \cong \mathbb{R}^\mathcal{K}$, the local transport map acts as the $\mathcal{K}$-fibre operator

$\mathcal{T}_{x\to y} = \gamma \cdot \hat{T}_x$. (2.3)

Equivalent formulation as the matrix element of one composite refinement step. Definition 2.1 is the (x, y) matrix element of one composite refinement step in which the local fibre evolution is applied at the source position before spatial propagation. Equivalently, it is the leading-order term in the (x, y) matrix element of the discrete refinement evolution ($\mathbf{T}_{bulk} \cdot \mathrm{diag}(\hat{T}_x))_{(y, x)}$ *for $y \sim x$. We adopt the composite form (Definition 2.1) as the basic object throughout this paper; in the vacuum ($\hat{T}_x \equiv \hat{T}$) it reduces to $\gamma\cdot\hat{T}$ for all (x, y) edges, and in the presence of a Stage VIII defect it acquires explicit defect-fibre content* $\gamma\cdot(\hat{T} + \Delta T_{x_0,r}(x))$ at defect positions.

This composite form has two structural virtues:

(a) **It sees diagonal-in-X defects.** When $\hat{T}_x = \hat{T} + \Delta T_{x_0,r}(x)$ at defect positions, $\mathcal{T}_{x\to y} = \gamma\cdot\hat{T} + \gamma\cdot\Delta T_{x_0,r}(x)$ differs from its vacuum value $\gamma\cdot\hat{T}$ by an explicit defect-dependent contribution. The path-ordered product (Definition 2.3 below) consequently visits defect fibres in a path-dependent way: paths through the defect region accumulate different sequences of $\Delta T_{x_0,r}(x_i)$ factors than paths around it.

(b) **It preserves the spatial Markov / locality structure.** $\mathcal{T}_{x\to y}$ is supported on edges $y \sim x$ of the substrate graph and acts trivially on non-adjacent pairs. The Stage IX strict-light-cone bound (Theorem 4.1(a)) lifts directly: the path-ordered product $\mathcal{P}_\pi$ is non-zero only along graph-connected paths, with no leakage outside the strict combinatorial cone.

2.3 Parallel Transport Along Paths

For an oriented substrate path

$\pi = (x_0 \to x_1 \to x_2 \to \cdots \to x_n)$, with $x_{i+1} \sim x_i$ for all $0 \le i < n$,

define the **substrate parallel transport operator** along π as the ordered product

$\mathcal{P}_\pi := \mathcal{T}_{x_{n-1} \to x_n} \cdot \mathcal{T}_{x_{n-2} \to x_{n-1}} \cdot \cdots \cdot \mathcal{T}_{x_0 \to x_1}$. (Definition 2.3)

Under the identification $\mathbb{R}^\mathcal{K} \otimes |x_n\rangle \cong \mathbb{R}^\mathcal{K}$, this is the path-ordered product of fibre operators

$\mathcal{P}_\pi = \gamma^n \cdot \hat{T}_{x_{n-1}} \cdot \hat{T}_{x_{n-2}} \cdot \cdots \cdot \hat{T}_{x_1} \cdot \hat{T}_{x_0}$, (2.4)

acting on the closure-catalogue fibre. The operator $\mathcal{P}_\pi$ carries coherence states from the fibre at x_0 to the fibre at x_n via the sequence of intermediate fibre operators $\hat{T}_{x_i}$ specified by the path. The factor γ^n records the cumulative spatial-coupling propagation amplitude; the path-ordered product of $\hat{T}_{x_i}$ factors records the cumulative effect of the fibre dynamics encountered along the path.

In the vacuum ($\hat{T}_{x} \equiv \hat{T}$ for all x), the product collapses to $\gamma^n \cdot \hat{T}^n$, a single operator depending only on the path length n. In the presence of defects, intermediate fibres at defect positions contribute $\hat{T}_{x_i} = \hat{T} + \Delta T_{x_0,r}(x_i)$, and the path-ordered product depends on *which* defect-fibre operators the path visits and in *what order*.

2.4 The Vacuum: Path-Independence and Flatness

Proposition 2.5 — Vacuum Parallel Transport is Path-Independent

In the canonical vacuum (uniform spatial fibres $\hat{T}_x \equiv \hat{T}$ for all $x \in X$) with $C = 0$, the substrate parallel transport operator $\mathcal{P}_\pi$ depends only on the length n of the path π and on its endpoints (x_0, x_n) — not on the choice of intermediate vertices:

$$\mathcal{P}_\pi = \gamma^n \cdot \hat{T}^n \text{ (canonical vacuum)} \quad (2.5)$$

for any path π of length n from x_0 to x_n. Equivalently, $\mathcal{P}_{\pi_1} = \mathcal{P}_{\pi_2}$ for all paths π_1, π_2 of the same length and same endpoints. The substrate geometry in the canonical vacuum is therefore **flat** *in the parallel-transport sense.*

Proof. By (2.4), $\mathcal{P}_\pi = \gamma^n \cdot \hat{T}_{x_{n-1}} \cdot \cdots \cdot \hat{T}_{x_0}$. In the vacuum, every $\hat{T}_{x_i}$ equals $\hat{T}$, so the product reduces to $\gamma^n \cdot \hat{T}^n$, independent of the choice of intermediate vertices (x_1, …, x_{n-1}). This holds for every path π of length n between x_0 and x_n, hence parallel transport in the vacuum is path-independent.

The vacuum-flatness statement is the substrate-level analogue of the Riemannian fact that on Euclidean space, parallel transport along any path between two points produces the same map between tangent spaces at the endpoints. In the substrate, vacuum parallel transport carries each coherence state through n copies of the same local refinement operation $\hat{T}$, regardless of the path taken — there is no internal structure that could distinguish paths.

2.5 Defects Break Path-Independence

For a substrate with a localized Stage VIII defect at x_0 (with perturbed local operators $\hat{T}_x = \hat{T} + \Delta T_{x_0,r}(x)$ for $x \in B_r(x_0)$), the parallel transport along a path π becomes

$$\mathcal{P}_\pi = \gamma^n \cdot \prod_{i=n-1, \dots, 0}^{\leftarrow} (\hat{T} + \Delta T_{x_0,r}(x_i)), \quad (2.6)$$

with the convention that $\Delta T_{x_0,r}(x_i) = 0$ outside the defect support $B_r(x_0)$. Two paths π_1 and π_2 of the same length and same endpoints that *visit different sequences of defect fibres* — for example, one passing directly through the defect core and one circumventing it — produce different path-ordered products, because the $\hat{T}$ and $\Delta T_{x_0,r}(x)$ factors generically do not commute:

$[\hat{T}, \Delta T_{x_0,r}(x)] \neq 0$ for typical Stage VIII defects. (2.7)

The Stage VIII §9.1 boundary defect $\Delta T = \alpha \cdot e_{\kappa_{b_1}}(e_{\kappa_{b_1}} - e_{\kappa_{b_2}})^T$ provides an explicit non-commutator: $\hat{T}$ acts non-trivially on the $(\kappa_{b_1}, \kappa_{b_2})$ sector via the wheel's rotational structure, and ΔT acts on the same sector via a non-symmetric rank-1 perturbation; their commutator $[\hat{T}, \Delta T]$ is non-zero at order α.

Consequently, the path-ordered product $\mathcal{P}_\pi$ is generically **path-dependent** in the presence of Stage VIII defects, and the substrate transport geometry is genuinely *curved*. The detailed direction-dependent structure of this curvature is developed in §3 via the directional decomposition of A_X.

Definition 2.6 — Flat and Curved Transport

The substrate transport geometry is called **flat** if $\mathcal{P}_{\pi_1} = \mathcal{P}_{\pi_2}$ for all pairs of substrate paths (π_1, π_2) sharing endpoints and graph-homotopy class. It is **curved** otherwise.

By Proposition 2.5, the canonical vacuum is flat. By (2.6) and (2.7), defect-perturbed substrates are generically curved, with curvature concentrated near the defect region (where the $\Delta T_{x_0,r}(x_i)$ factors are non-zero) and decaying with distance from the defect (since paths far from the defect contain few or no defect-fibre factors).

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3. Directional Transport Generators

3.1 Directional Decomposition of A_X

For the explicit calculations we assume the substrate adjacency operator admits a **directional decomposition**

$A_X = \Sigma_{i=1}^{d_X} A_i$, (3.1)

where each A_i propagates coherence along one substrate direction. For the canonical examples of Stage IX §3.4 this is concrete:

- **X = $\mathbb{Z}^d$** (d-dimensional integer lattice, z = 2d): $A_X = \Sigma_{i=1}^{d} (S_i + S_i^\wedge)$*, where S_i is the shift-by-one-step operator in the i-th direction and $S_i^\wedge$* is its adjoint (the shift in the opposite direction). The directional decomposition with $d_X = d$ directions takes $A_i := S_i + S_i^*$. By translation invariance and the spatial-coupling structure, $[A_i, A_j] = 0$ for all (i, j) — the directional operators on $\mathbb{Z}^d$ commute.
- **Hexagonal (honeycomb) lattice** (z = 3): A_X decomposes along the three nearest-neighbour directions on each bipartite sublattice. The decomposition is more subtle than $\mathbb{Z}^d$ because of the bipartite structure, but the directional operators still commute in the bulk.

- **Finite torus $\mathbb{T}_N^d$** ($z = 2d$, finite): identical to $\mathbb{Z}^d$ at the local level, with periodic boundary conditions; directional operators commute.

For the present analysis we adopt the standing structural assumption that the directional decomposition is **commuting in the bulk**:

(S_comm) $[A_i, A_j] = 0$ for all $i, j \in \{1, \ldots, d_X\}$.

This holds for all the canonical regular substrates of Stage IX §3.4 and is the relevant case for the present development. Generalisations to non-commuting bulk directional decompositions (which can arise on irregular substrates or with non-translation-invariant coupling) are mathematically straightforward but produce *bulk* curvature even without defects — a structurally distinct scenario we flag in §11 and do not develop here.

3.2 The Generators

Why the transfer-operator form is necessary. The transfer-operator formulation of the directional generators below is not an arbitrary modification of the Stage IX spatial-coupling structure. It is forced by the requirement that parallel transport remain sensitive to diagonal-in-position defects. A purely adjacency-based generator of the form $\gamma \cdot I_{\mathcal{K}} \otimes A_i$ (the form one would naively read off as the directional decomposition of the spatial-coupling term of **T**) cannot distinguish paths traversing distinct defect fibres, since diagonal-in-X fibre perturbations $\Delta T_{x_0,r}(x) \otimes |x\rangle\langle x|$ commute with the position projection and never act on the $I_{\mathcal{K}}$ factor. The transfer-operator construction restores this sensitivity by combining one fibre-evolution step ($\hat{T}_x$ at the source) with one spatial transport step (the position-shift to $x + e_i$) into a single directed operator. This is the minimal operator structure capable of producing non-trivial holonomy from Stage VIII-type defects through the substrate transport machinery.

Define the **directional transport generators** as transfer operators on $\mathscr{H}$:

$\nabla i := \gamma \cdot \Sigma\{x \in X\} (\hat{T} \cdot |x + e_i\rangle\langle x|), i \in \{1, \ldots, d_X\}$. (Definition 3.1)

Here e_i denotes a unit step in direction i (with the convention that "step in direction i" is summed over both forward and backward elementary steps when $A_i = S_i + S_i^*$ in the canonical $\mathbb{Z}^d$ decomposition; for simplicity we exhibit the forward case explicitly and note that the backward step adds a Hermitian-conjugate term). The generator ∇_i acts on a state $\psi \otimes |x\rangle$ by first applying the local fibre operator at the source — $\hat{T}$ in the vacuum, $\hat{T}_x$ in the defected case — and then shifting to the destination position $x + e_i$ with weight γ.

The vacuum sum reconstructs the directional content of the composite vacuum transport:

$\Sigma_i \nabla_i = \gamma \cdot \Sigma_i \Sigma_x (\hat{T} \cdot |x + e_i\rangle\langle x|) = \gamma \cdot (\hat{T} \otimes A_X)$. (3.2)

The transfer-operator form (3.1) is the *correct* form for analyzing commutators of consecutive directional steps: $\nabla_j \cdot \nabla_i$ applied to $\psi \otimes |x\rangle$ produces

$\nabla_j \cdot \nabla_i (\psi \otimes |x\rangle) = \gamma^2 \cdot (\hat{T}_{x + e_i} \cdot \hat{T}_x \psi) \otimes |x + e_i + e_j\rangle$,

so the consecutive application carries information about fibre operators at *two distinct positions* (x and $x + e_i$). The single-fibre form $\nabla_i = \gamma \cdot (\hat{T}_x \otimes A_i)$ used in earlier drafts collapses this to a single-position structure that artificially trivialises the commutator $[\tilde{\nabla}_i, \tilde{\nabla}_j]$ at the algebraic level — the genuine source of generator non-commutativity is the *sequential* difference between "fibre at x then fibre at x + e_i" (the $\nabla_j \cdot \nabla_i$ ordering) and "fibre at x then fibre at x + e_j" (the $\nabla_i \cdot \nabla_j$ ordering).

Clarification on terminology. The operators ∇_i are referred to throughout the paper as **transport generators** or **transport-covariant operators**, not as true affine covariant derivatives. The present framework does not possess a metric-compatible connection, a tangent bundle, or a Levi-Civita structure on the substrate; the notation ∇_i (and the later notation $\mathcal{D}/ds$ in §6) is structural and operator-theoretic rather than differential-geometric in the strict sense. The analogy with covariant differentiation is through the *operator-algebraic content* — transport non-commutativity, generator commutator producing curvature, holonomy on closed loops — and not through the differential-geometric apparatus that supports covariant differentiation on a Riemannian manifold. We use "covariant" only modifier-style ("transport-covariant operator") to flag the structural correspondence, and never as a freestanding noun. The absence of a metric-compatible connection is itself an open programme item (§11.8).

Proposition 3.3 — Flatness of Vacuum Transport Generators

Under the standing hypothesis (S_{comm}) and in the canonical vacuum (uniform $\hat{T}_x \equiv \hat{T}$, $C = 0$), the directional transport generators commute:

$[\nabla_i, \nabla_j] = 0$ for all i, j. (3.3)

Consequently, parallel transport along any sequence of directional steps in the vacuum is independent of the order of steps, confirming Proposition 2.5 at the generator level: the canonical vacuum is **flat** *in the parallel-transport sense.*

Proof. From (3.1), the consecutive application in the vacuum is

$\nabla_j \cdot \nabla_i (\psi \otimes |x\rangle) = \gamma^2 \cdot (\hat{T} \cdot \hat{T} \psi) \otimes |x + e_i + e_j\rangle = \gamma^2 \cdot \hat{T}^2 \psi \otimes |x + e_i + e_j\rangle$,

independent of the ordering of i and j (because the destination position $x + e_i + e_j$ is symmetric in (i, j), and $\hat{T}^2$ is the same regardless of which direction was taken first). Hence $[\nabla_i, \nabla_j](\psi \otimes |x\rangle) = 0$ for every (ψ, x), establishing (3.3). Path-independence of vacuum transport along directional sequences follows immediately.

3.3 Defect-Induced Generator Anisotropy

Suppose a localized Stage VIII defect is present at substrate position x_0 with radius r, modifying the local fibre operator at $x \in B_r(x_0)$:

$\hat{T}\_x = \hat{T} + \Delta T\_\{x\_0,r\}(x)$. (3.4)

The corresponding **perturbed directional transport generator** is the transfer operator

$\tilde{\nabla}\_i := \gamma \cdot \Sigma\_\{x \in X\} (\hat{T}\_x \cdot |x + e\_i\rangle\langle x|)$, (Definition 3.4)

which differs from the vacuum form (3.1) by the position-dependent fibre operator $\hat{T}\_x$ at the source of each step. The defect-induced correction is the difference

$\Delta\_i := \tilde{\nabla}\_i - \nabla i = \gamma \cdot \Sigma\{x \in B\_r(x\_0)\} (\Delta T\_\{x\_0,r\}(x) \cdot |x + e\_i\rangle\langle x|)$, (Definition 3.5)

which is supported on edges originating at defect-region vertices and is direction-specific through the choice of step direction e_i.

Proposition 3.5 — Defect-Induced Generator Non-Commutativity

For a localized Stage VIII defect with $[\hat{T}, \Delta T\_\{x\_0,r\}(x)] \neq 0$ *at some* $x \in B\_r(x\_0)$*, the perturbed directional transport generators of Definition 3.4 satisfy*

$[\tilde{\nabla}\_i, \tilde{\nabla}\_j] \neq 0$ in some neighbourhood of x_0. (3.6)

Explicitly, the commutator's action on a basis state $\psi \otimes |x\rangle$ *is*

$[\tilde{\nabla}\_i, \tilde{\nabla}\_j] (\psi \otimes |x\rangle) = \gamma^2 \cdot ((\hat{T}\_\{x + e\_j\} - \hat{T}\_\{x + e\_i\}) \cdot \hat{T}\_x \psi) \otimes |x + e\_i + e\_j\rangle$, (3.7)

so $[\tilde{\nabla}\_i, \tilde{\nabla}\_j]$ *is supported on substrate positions x where the boundary-shell fibre difference* $\hat{T}\_\{x + e\_j\} - \hat{T}\_\{x + e\_i\} = \Delta T\_\{x\_0,r\}(x + e\_j) - \Delta T\_\{x\_0,r\}(x + e\_i)$ *is non-zero — that is, on the immediate-neighbour shell of B_r(x_0) where stepping in direction i lands inside the defect and stepping in direction j lands outside (or vice versa, or both lie inside with different* $\Delta T\_\{x\_0,r\}$ *values).*

Proof. Compute the consecutive applications directly from Definition 3.4. For $\psi \otimes |x\rangle$ with x in or near the defect region:

$\tilde{\nabla}\_j \cdot \tilde{\nabla}\_i (\psi \otimes |x\rangle) = \gamma^2 \cdot (\hat{T}\_\{x + e\_i\} \cdot \hat{T}\_x \psi) \otimes |x + e\_i + e\_j\rangle$, $\tilde{\nabla}\_i \cdot \tilde{\nabla}\_j (\psi \otimes |x\rangle) = \gamma^2 \cdot (\hat{T}\_\{x + e\_j\} \cdot \hat{T}\_x \psi) \otimes |x + e\_j + e\_i\rangle$.

The destination position x + e_i + e_j = x + e_j + e_i is symmetric (by commutativity of addition on the lattice), so the two terms subtract at a common destination to give (3.7). Since $\hat{T}\_y = \hat{T} + \Delta T\_\{x\_0,r\}(y)$, the fibre difference reduces to $\Delta T\_\{x\_0,r\}(x + e\_j) - \Delta T\_\{x\_0,r\}(x + e\_i)$. This difference is non-zero exactly on the boundary-shell positions described in the statement; outside that shell, both $\Delta T\_\{x\_0,r\}$ values agree (both zero in the bulk, or both equal at interior defect-support positions of constant profile), and the commutator vanishes.

Vanishing of (3.6) under symmetry protection. The right-hand side of (3.7) vanishes when $\Delta T\_\{x\_0,r\}(x + e\_i) = \Delta T\_\{x\_0,r\}(x + e\_j)$ for all relevant x — that is, when the defect respects the $i \leftrightarrow j$ directional symmetry (e.g., a defect with directionally-symmetric profile around x_0 on a $\mathbb{Z}^2$ substrate). It also vanishes when the closure-catalogue action of the difference annihilates $\hat{T}_x \psi$ for all relevant ψ — the substrate-level mechanism by which the Stage VIII §9.6 hub-coupling defect (with $[\hat{T}, \Delta T_{hub}] \equiv 0$ on the symmetry-protected sector) produces no transport curvature, recovering at the generator level the global spectral invisibility of Stage IX §11.3.

Non-vanishing of (3.6) for the Stage VIII §9.1 boundary defect. For the explicit Stage VIII §9.1 defect $\Delta T(x_0) = \alpha \cdot e_{\kappa_{b_1}}(e_{\kappa_{b_1}} - e_{\kappa_{b_2}})^T$ placed at the single position x_0, the right-hand side of (3.7) at $x = x_0 - e_i$ evaluates to $\gamma^2 \cdot (0 - \Delta T(x_0)) \cdot \hat{T} \psi \otimes |x_0 + e_j\rangle = -\gamma^2 \cdot \alpha \cdot (\text{rank-1 } \mathcal{K}\text{-operator}) \cdot \hat{T}\psi \otimes |x_0 + e_j\rangle$, non-zero whenever $\hat{T}\psi$ has non-vanishing component on the $(\kappa_{b_1}, \kappa_{b_2})$ sector. So $[\tilde{\nabla}_i, \tilde{\nabla}_j]$ is non-zero of order $\alpha\gamma^2$ at boundary-shell positions of the defect, and zero in the bulk of the substrate.

Corollary 3.6 — Lie-algebraic restatement of (3.7). Equation (3.7) can be rewritten in terms of the fibre commutator $[\hat{T}, \Delta T_{x_0,r}(\cdot)]$ by inserting $\hat{T} = \hat{T}_y - \Delta T_{x_0,r}(y)$ in either factor and collecting orders in α:

$$\hat{T}_{x + e_j} - \hat{T}_{x + e_i} = \Delta T_{x_0,r}(x + e_j) - \Delta T_{x_0,r}(x + e_i) = [\hat{T}, \Delta T_{x_0,r}(x + e_j)] - [\hat{T}, \Delta T_{x_0,r}(x + e_i)] + (\text{commuting terms cancelling under (3.7)}) + \mathcal{O}(\alpha^2). \quad (3.8)$$

The Lie-algebraic form (3.8) makes explicit that the substrate-level curvature is generated by the *direction-dependent fibre commutator* $[\hat{T}, \Delta T_{x_0,r}(x + e_k)]$: when this commutator is $i \leftrightarrow j$ symmetric (or zero), the two terms in (3.8) cancel; when it is asymmetric, they do not. This is the substrate analogue (at the level of operator algebra) of the way curvature arises in non-Abelian gauge theory from the commutator of gauge-covariant derivatives — though we make no claim of a gauge-theoretic structure on the substrate (cf. §11.3). We use (3.7) as the working form throughout §3.4 and the proof of Theorem 4.2.

3.4 The Stage VIII §9.1 Boundary Defect on $\mathbb{Z}^2$: Explicit Calculation

The Proposition 3.5 derivation is short enough to make fully explicit for the Stage VIII §9.1 boundary defect on $\mathbb{Z}^2$. Let

$$\Delta T(x_0) = \alpha \cdot e_{\kappa_{b_1}} (e_{\kappa_{b_1}} - e_{\kappa_{b_2}})^T, \quad \Delta T(x) = 0 \text{ for } x \neq x_0.$$

Step 1 — fibre commutator. The wheel matrix elements $\hat{T}_{\kappa\kappa'}$ in the canonical wheel (Stage VI) include $\hat{T}_{\kappa_{b_2}, \kappa_{b_1}} \neq 0$ (the rotational coupling between adjacent boundary states). The fibre commutator $[\hat{T}, \Delta T(x_0)]$ in the closure-catalogue basis is:

$$[\hat{T}, \Delta T(x_0)] = \alpha \cdot (\hat{T} \cdot e_{\kappa_{b_1}}(e_{\kappa_{b_1}} - e_{\kappa_{b_2}})^T - e_{\kappa_{b_1}}(e_{\kappa_{b_1}} - e_{\kappa_{b_2}})^T \cdot \hat{T}),$$

with explicit (κ, κ') entry

$([\hat{T}, \Delta T(x_0)])_{\kappa\kappa'} = \alpha \cdot (\hat{T}_{\kappa, \kappa_{b_1}} \cdot (\delta_{\kappa', \kappa_{b_1}} - \delta_{\kappa', \kappa_{b_2}}) - \delta_{\kappa, \kappa_{b_1}} \cdot (\hat{T}_{\kappa_{b_1}, \kappa'} - \hat{T}_{\kappa_{b_2}, \kappa'}))$.

This is generically non-zero — for example, the $(\kappa_{b_2}, \kappa_{b_1})$ entry is $\alpha \cdot (\hat{T}_{\kappa_{b_2}, \kappa_{b_1}} - 0) = \alpha \cdot \hat{T}_{\kappa_{b_2}, \kappa_{b_1}} \neq 0$, since the wheel's rotational structure gives $\hat{T}_{\kappa_{b_2}, \kappa_{b_1}} \neq 0$.

Step 2 — boundary-shell evaluation. By Proposition 3.5 equation (3.7), the leading-order generator commutator at substrate position x is non-zero exactly when $\Delta T(x + e_1) \neq \Delta T(x + e_2)$. For a point defect at x_0, the relevant positions are the four immediate neighbours of x_0 on $\mathbb{Z}^2$: $x_0 - e_1$, $x_0 + e_1$, $x_0 - e_2$, $x_0 + e_2$. Per the forward-step convention of Definition 3.1, (3.10)/(3.11) below track the contribution from forward elementary steps only; the full $A_i = S_i + S_i^*$ picture adds backward-step contributions which contribute Hermitian-conjugate terms with the same $\alpha\gamma^2$ scaling and the same boundary-shell support, and do not change the qualitative conclusion.

At $x = x_0 - e_1$, we have $x + e_1 = x_0$ (inside defect support) and $x + e_2 = x_0 - e_1 + e_2$ (outside defect support), so

$$\hat{T}_{x + e_2} - \hat{T}_{x + e_1} = 0 - \Delta T(x_0) = -\Delta T(x_0).$$

Applying (3.7) with the position-state $\psi \otimes |x_0 - e_1\rangle$:

$$[\tilde{\nabla}_1, \tilde{\nabla}_2] (\psi \otimes |x_0 - e_1\rangle) = \gamma^2 \cdot (\hat{T}_{x_0 - e_1 + e_2} - \hat{T}_{x_0}) \cdot \hat{T}_{x_0 - e_1} (\psi) \otimes |x_0 + e_2\rangle = \gamma^2 \cdot (0 - \Delta T(x_0)) \cdot \hat{T} (\psi) \otimes |x_0 + e_2\rangle = -\gamma^2 \cdot \Delta T(x_0) \cdot \hat{T} (\psi) \otimes |x_0 + e_2\rangle.$$

This is non-zero of order $\alpha\gamma^2$, supported at the destination position $x_0 + e_2$ with fibre content $(\Delta T(x_0) \cdot \hat{T})(\psi)$, and contributes the operator-matrix-element form $-\gamma^2 \cdot \Delta T(x_0) \cdot \hat{T} \cdot |x_0 + e_2\rangle\langle x_0 - e_1|$.

Step 3 — exhaustive boundary-shell sum. Repeating the calculation at each of the four immediate-neighbour positions, with forward-step convention:

(a) $\mathbf{x = x_0 - e_1}$ (computed above). $\hat{T}_{x + e_2} - \hat{T}_{x + e_1} = -\Delta T(x_0)$. Contribution: $-\gamma^2 \cdot \Delta T(x_0) \cdot \hat{T} \cdot |x_0 + e_2\rangle\langle x_0 - e_1|$.

(b) $\mathbf{x = x_0 - e_2}$. $\hat{T}_{x + e_1} = \hat{T}_{x_0 - e_2 + e_1} = \hat{T}$ (outside defect), $\hat{T}_{x + e_2} = \hat{T}_{x_0} = \hat{T} + \Delta T(x_0)$. Difference $\hat{T}_{x + e_2} - \hat{T}_{x + e_1} = +\Delta T(x_0)$. Destination $|x + e_1 + e_2\rangle = |x_0 + e_1\rangle$. Contribution: $+\gamma^2 \cdot \Delta T(x_0) \cdot \hat{T} \cdot |x_0 + e_1\rangle\langle x_0 - e_2|$.

(c) $\mathbf{x = x_0 + e_1}$. $\hat{T}_{x + e_1} = \hat{T}_{x_0 + 2e_1} = \hat{T}$ (outside defect), $\hat{T}_{x + e_2} = \hat{T}_{x_0 + e_1 + e_2} = \hat{T}$ (outside defect). Difference = 0. No contribution.

(d) $\mathbf{x = x_0 + e_2}$. $\hat{T}_{x + e_1} = \hat{T}_{x_0 + e_1 + e_2} = \hat{T}$, $\hat{T}_{x + e_2} = \hat{T}_{x_0 + 2e_2} = \hat{T}$. Difference = 0. No contribution.

Summing the two non-vanishing contributions (a) and (b), the leading-order generator commutator from forward-step contributions is

$[\tilde{\nabla}_1, \tilde{\nabla}_2]_{forward} = \gamma^2 \cdot \alpha \cdot \mathcal{K}_{1, 2}(x_0) + \mathcal{O}(\alpha^2)$, (3.10)

with

$\mathcal{K}_{1, 2}(x_0) = \alpha^{-1} \cdot \Delta T(x_0) \cdot \hat{T} \cdot (-|x_0 + e_2\rangle\langle x_0 - e_1| + |x_0 + e_1\rangle\langle x_0 - e_2|)$. (3.11)

(The α^{-1} prefactor cancels the α inside $\Delta T(x_0)$, giving $\mathcal{K}_{1, 2}(x_0)$ an α-independent rank-1-in-$\mathcal{K}$, two-term-in-X operator structure.) The two-term position pattern has explicit antisymmetry under 1 ↔ 2 reflection (the i ↔ j swap exchanges the two terms with a sign flip), consistent with the antisymmetry $\mathcal{R}_{ij} = -\mathcal{R}_{ji}$ of the transport-curvature tensor. The full $A_i = S_i + S_i^*$ picture restores backward-step contributions, which add the Hermitian-conjugate terms $+\gamma^2 \cdot \alpha \cdot \mathcal{K}_{1, 2}(x_0)^\dagger$ supported on the same boundary-shell positions (positions $x_0 + e_1$, $x_0 + e_2$ instead of $x_0 - e_1$, $x_0 - e_2$), giving a four-term operator overall but not changing the qualitative $\alpha\gamma^2$ scaling or the boundary-shell localization.

The single closure-catalogue factor $\Delta T(x_0) \cdot \hat{T}$ in (3.11) embodies the fibre-asymmetry source (Step 1's $[\hat{T}, \Delta T]$); the two-term position structure embodies the boundary-shell geometric source (the directional asymmetry of the defect's neighbouring fibre values). Together they give $\mathcal{K}_{1, 2}(x_0)$ the structure of a discrete substrate analogue of the curl $\nabla_1 \times \nabla_2$ evaluated at the defect site.

Symmetry-protected vanishing for the hub-coupling defect. For the Stage VIII §9.6 hub-coupling defect $\Delta T_{hub}(x_0) = \alpha \cdot e_{\kappa_h}(e_{\kappa_h} - e_{\kappa_{b_1}})^T$, the fibre commutator $[\hat{T}, \Delta T_{hub}(x_0)]$ vanishes on the symmetry-protected sector by the block-alignment with $H_{\sigma^*}^+$ established in Stage VIII §9.6 (confirmed at the global trapped-mode level in Stage IX §11.3). By Corollary 3.6 equation (3.8) and Proposition 3.5, the leading-order generator commutator $[\tilde{\nabla}_1, \tilde{\nabla}_2]$ vanishes for this defect, and the corresponding $\mathcal{K}_{1, 2}^{hub}(x_0) \equiv 0$. The hub-coupling defect produces no transport curvature — the structurally satisfying consistency check that *globally spectrally invisible defects* of Stage IX are also *transport-curvature invisible* at the present tensorial level.

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4. Holonomy and Loop Transport

4.1 Coherence Holonomy

For a closed substrate loop γ — a directed substrate path $\gamma = (x_0 \to x_1 \to \cdots \to x_n = x_0)$ returning to its starting position — define the **coherence holonomy operator** as the parallel transport along γ:

$\mathscr{H}(\gamma) := \mathcal{P}\_\gamma : \mathbb{R}^\mathcal{K} \otimes |x_0\rangle \to \mathbb{R}^\mathcal{K} \otimes |x_0\rangle$. (Definition 4.1)

By construction, $\mathscr{H}(\gamma)$ is an operator on the closure-catalogue fibre at x_0 — a transformation that the loop transport applies to coherence states at the loop's basepoint.

In the canonical vacuum, by Proposition 2.5:

$\mathscr{H}(\gamma) = \gamma^n \cdot \hat{T}^n$, (canonical vacuum, n = loop length) (4.1)

so the vacuum holonomy is a power of the wheel operator $\hat{T}$, scaled by the cumulative coherence-propagation factor γ^n. Because $\hat{T}$ is the same operator at every fibre in the vacuum, this is *trivial in the curvature-relevant sense*: the loop transport applies n copies of the universal fibre operation, with no path-dependent rotation or mixing. The $\gamma^n \cdot \hat{T}^n$ factor reflects the cumulative refinement-step propagation along the n-step loop (γ^n from spatial coupling, $\hat{T}^n$ from n applications of the universal fibre operator) and is not a curvature signature — it is purely the vacuum's cumulative refinement amplitude.

A loop has **non-trivial holonomy** in the curvature-relevant sense if $\mathscr{H}(\gamma) - \gamma^n \cdot \hat{T}^n \neq 0$; equivalently, if the *normalised holonomy*

$\hat{H}(\gamma) := \gamma^{-n} \cdot \hat{T}^{-n} \cdot \mathscr{H}(\gamma) - I_\mathcal{K}$ (Definition 4.2, on the subspace where $\hat{T}^n$ is invertible)

is a non-zero operator on the relevant invariant subspace of $\mathbb{R}^\mathcal{K}$. Because $\hat{T}$ has 0 in its spectrum (with multiplicity 2 in the canonical wheel), $\hat{T}^n$ is not invertible on all of $\mathbb{R}^\mathcal{K}$; the normalised holonomy is defined on the orthogonal complement of the null subspace of $\hat{T}^n$ (the kernel of $(\hat{T})^n$ grows monotonically with n until the kernel stabilises at the maximal $\hat{T}$-invariant subspace, which has dimension 5 for the canonical wheel with two zero eigenvalues). On this complementary subspace, the normalised holonomy strips off the vacuum cumulative-refinement factor and isolates the genuinely curvature-induced residual transformation.

For loop-trace functionals (§7), where only $\mathrm{Tr}_\mathcal{K}(\mathscr{H}(\gamma))$ appears, the invertibility issue is avoided: the trace $W(\gamma) = \mathrm{Tr}_\mathcal{K}(\mathscr{H}(\gamma))$ is defined directly and has vacuum value $\mathrm{Tr}_\mathcal{K}(\gamma^n \cdot \hat{T}^n) = \gamma^n \cdot \mathrm{Tr}(\hat{T}^n)$, with deviation from vacuum being the curvature-sensitive quantity.

Theorem 4.2 — Emergent Holonomy from Localized Defects

Let V be a localized Stage VIII defect supported in $B_r(x_0)$ producing generic directional anisotropy of the perturbed transport generators (i.e., satisfying the genericity hypothesis of Proposition 3.5 so that $[\tilde{\nabla}_i, \tilde{\nabla}_j] \neq 0$ in a neighbourhood of x_0). Then:

(a) ***Normalised-holonomy form*** *(operator non-triviality). There exist substrate loops γ surrounding the defect region such that the normalised holonomy is non-trivial on the $\hat{T}$-invertible subspace:*

$\|\hat{H}(\gamma)\|_{op, \hat{T}\text{-inv}} > 0$, (4.2a) *where the norm is restricted to the $\hat{T}$-invertible subspace $\mathbb{R}^{\mathcal{K}}_{\hat{T}\text{-inv}} := \mathrm{Range}(\hat{T}^n)$ of dimension 5 for the canonical wheel (with the 2-dimensional kernel of $\hat{T}^n$ excluded), on which $\hat{H}(\gamma) := \gamma^{-n} \cdot \hat{T}^{-n} \cdot \mathscr{H}(\gamma) - I_{\mathcal{K}}$ is well-defined.*

(b) ***Loop-trace form*** *(generic non-triviality). For generic defects and generic loops γ surrounding the defect region, the Wilson-type loop functional $W(\gamma) := \mathrm{Tr}_{\mathcal{K}}(\mathscr{H}(\gamma))$ of §7 deviates from its vacuum value:*

$$|W(\gamma) - \gamma^n \cdot \mathrm{Tr}(\hat{T}^n)| > 0 \text{ (generically). } (4.2b)$$

The genericity caveat on (4.2b) is necessary because the loop trace can vanish accidentally by closure-catalogue cancellation even when the underlying normalised holonomy is non-zero — the operator non-triviality of (a) is the structurally robust statement, and (b) is its generic consequence. Symmetric defect configurations or symmetric loops can produce trace coincidences that suppress (4.2b) while leaving (4.2a) intact.

Moreover, for any such loop γ of length n with minimum distance $d_{min}(\gamma, x_0) := \min_{x \in \gamma} d_X(x, x_0)$ from the defect, the holonomy magnitude is bounded by an integrated commutator estimate and decays exponentially with d_{min}:

$$\|\hat{H}(\gamma)\|_{op, \hat{T}\text{-inv}} \leq C(\gamma, n) \cdot \max_{i,j} \|[\tilde{\nabla}_i, \tilde{\nabla}_j]\|_{op} \cdot e^{-\eta \cdot d_{min}(\gamma, x_0)}, (4.3)$$

with $\eta \in \{\eta_{trap}, \eta_{sub}\}$ the Combes–Thomas decay rate of Stage IX Theorem 8.1 (η_{trap} for defects producing trapped modes; η_{sub} of Remark 8.2 otherwise) and $C(\gamma, n)$ a combinatorial prefactor depending on the loop geometry and length. The corresponding bound for the loop-trace deviation is

$$|W(\gamma) - \gamma^n \cdot \mathrm{Tr}(\hat{T}^n)| \leq C'(\gamma, n) \cdot \max_{i,j} \|[\tilde{\nabla}_i, \tilde{\nabla}_j]\|_{op} \cdot e^{-\eta \cdot d_{min}(\gamma, x_0)}, (4.3b)$$

with an analogous constant C' and the same genericity caveat.

Proof Sketch. The construction proceeds in three steps.

Step 1: Existence of non-trivial loops. By Proposition 3.5 with the genericity hypothesis, there exist directional indices (i, j) such that $[\tilde{\nabla}_i, \tilde{\nabla}_j]$ is a non-zero operator with support intersecting some neighbourhood $B_R(x_0)$ of the defect. Consider the elementary 4-step plaquette loop $\gamma_{ij}(x_0) := (x_0 \to x_0 + e_i \to x_0 + e_i + e_j \to x_0 + e_j \to x_0)$. The parallel transport around $\gamma_{ij}(x_0)$, computed from the path-ordered product (2.4), is

$$\mathcal{P}_{\gamma_{ij}(x_0)} = \gamma^4 \cdot (\hat{T}_{x_0+e_j} \cdot \hat{T}_{x_0+e_i+e_j} \cdot \hat{T}_{x_0+e_i} \cdot \hat{T}_{x_0}),$$

which to leading order in the defect perturbation expands as

$$\mathcal{P}_{\gamma_{ij}(x_0)} = \gamma^4 \cdot \hat{T}^4 + \gamma^4 \cdot (\text{one-defect-insertion-per-vertex terms}) + \mathcal{O}(\Delta^2),$$

with the defect-insertion terms ordered along the loop. The leading non-trivial contribution arises from the antisymmetric difference of orderings, which by Proposition 3.5 (specifically (3.7)) is non-zero of order $\gamma^2 \cdot \alpha$ whenever the boundary-shell fibre difference $\hat{T}_{x + e_i} - \hat{T}_{x + e_j}$ is non-zero at some loop vertex x. For the loop-trace form (4.2b),

$$W(\gamma_{ij}(x_0)) - \gamma^4 \cdot \mathrm{Tr}(\hat{T}^4) = \gamma^4 \cdot \mathrm{Tr}([\hat{T}_{x_0+e_i}, \hat{T}_{x_0+e_j}] \cdot \hat{T}^2) + \mathcal{O}(\alpha^2),$$

which is non-zero of order $\gamma^4 \cdot \alpha$ by Proposition 3.5, establishing (4.2b). For the normalised-holonomy form (4.2a), restrict to the $\hat{T}$-invertible subspace (where $\hat{T}^{-4}$ exists) and the corresponding deviation from the identity is non-zero of order $\gamma^2 \cdot \alpha$ at the same loop, establishing (4.2a).

Step 2: Integrated commutator estimate. For a general loop γ of length n bounding a region containing the defect, decompose γ into a sum of elementary plaquettes via discrete Stokes' theorem: $\gamma = \bigcup_{\text{plaquettes } p \in \text{enclosed region}} \partial p$ (with appropriate orientation). The holonomy around γ then factorises (to leading order in the perturbation) as the ordered product of plaquette holonomies. Both the normalised-holonomy form and the loop-trace form satisfy bounds of the form

$$\|\hat{H}(\gamma)\|_{op,\, \hat{T}\text{-inv}} \leq \Sigma_{p \text{ enclosed}} \|\tilde{\nabla}_{i(p)}, \tilde{\nabla}_{j(p)}\|_{op} \cdot (\text{combinatorial factor from } \gamma),$$

$$|W(\gamma) - \gamma^n \cdot \mathrm{Tr}(\hat{T}^n)| \leq \Sigma_{p \text{ enclosed}} \|\tilde{\nabla}_{i(p)}, \tilde{\nabla}_{j(p)}\|_{op} \cdot |\mathrm{Tr}(\hat{T}^{n-2})| \cdot (\text{combinatorial factor from } \gamma),$$

with the additional $|\mathrm{Tr}(\hat{T}^{n-2})|$ factor in the loop-trace bound absorbed into the constant C'. Both bounds are the substrate-level discrete analogue of the area integral $\int_S \mathrm{Tr}(\mathscr{R})$ for a surface S bounded by γ in the continuum.

Step 3: Exponential decay with d_{min} (conditional on trapped mode). By Definition 3.5, the perturbations Δ_i are supported on the defect region $B_r(x_0)$, so the generator commutator $\tilde{\nabla}_i$, $\tilde{\nabla}_j$ at points x outside $B_r(x_0)$ requires propagation from the defect. If the defect produces a trapped mode at spectral distance $\delta > 0$ from the bulk band edge, the Combes–Thomas decay rate $\eta_{trap} = \min(1, \delta/(2\gamma \cdot \rho(A_X)))$ of Stage IX Theorem 8.1 applies (cf. Proposition 8.1) and the perturbed-generator commutator inherits this rate:

$$\|\tilde{\nabla}_i, \tilde{\nabla}_j\|_{op} \leq C \cdot e^{-\eta_{trap} \cdot d_X(x, x_0)}, \quad \text{(trapped-mode case)}$$

for x outside the defect support. For subcritical defects (no trapped mode), the decay rate is the subcritical rate η_{sub} of Remark 8.2 instead, set by distance to the nearest point of $\mathrm{spec}(\mathbf{T}_{bulk})$ rather than to a trapped eigenvalue. In either case, summing over plaquettes contained in γ with $d_{min}(\gamma, x_0)$:= minimum loop-vertex distance to the defect gives (4.3) with η in the bound being whichever rate applies. The trapped-mode rate η_{trap} is the slower of the two and the relevant one for any defect producing a localized trap; the subcritical rate η_{sub} applies to weaker defects.

4.2 Structural Reading of the Holonomy Theorem

Theorem 4.2 is the central existence statement of the tensorial geometric framework. Its content can be unpacked as follows.

Non-triviality. Localized defects that break directional symmetry generate non-trivial substrate holonomy. This is the operator-theoretic substrate-level analogue of the Riemannian fact that curvature is necessary and sufficient for non-trivial parallel-transport holonomy: in the vacuum the generators commute and all holonomies are trivial; with defects the generators acquire non-commuting corrections and loops surrounding the defect region acquire non-trivial holonomy.

Quantitative bound. The holonomy magnitude is controlled by the integrated commutator strength, which in turn is controlled by the defect-induced generator anisotropy. This makes the framework genuinely *computable*: given a specific defect and a specific loop, one can in principle evaluate the holonomy by computing the perturbed-generator commutators and integrating them around the loop.

Exponential localization. The holonomy decays exponentially with the loop's distance from the defect, with the same Combes–Thomas rate that governs the Stage IX trapped-mode eigenvector decay. This is the structural unity that links Stage IX's localization theory with the present tensorial geometry: localized defects produce *both* localized trapped eigenstates *and* localized curvature, with the same exponential decay scale $1/\eta$. We make this explicit in §8.

Symmetry-protected vanishing. Defects that respect the substrate's directional symmetry (e.g., the Stage VIII §9.6 hub-coupling defect, which respects the wheel's rotational symmetry) produce vanishing generator commutators and hence vanishing holonomy. This recovers, at the tensorial level, the Stage IX §11.3 statement that such defects are globally spectrally invisible: they are also *globally transport-curvature invisible*.

4.3 Transport Memory

The non-trivial holonomy $\mathscr{H}(\gamma) - \gamma^n \cdot \hat{T}^n$ represents substrate-level *transport memory*: a coherence state that is parallel-transported around a closed loop returns to its starting fibre transformed not just by the cumulative vacuum refinement $\gamma^n \cdot \hat{T}^n$ (which is the same for all loops of the same length) but by an additional loop-dependent operator measuring the integrated curvature inside the loop. This is the substrate analogue of the Aharonov–Bohm phase in quantum mechanics (where the wave function returning to its starting point acquires a phase recording the enclosed magnetic flux) and of the parallel-transport rotation in Riemannian geometry (where a vector parallel-transported around a loop returns rotated by the loop-integrated Riemann curvature). The structural correspondence is at the operator-algebraic level: in all three cases (Aharonov–Bohm, Riemannian, present substrate), loop transport produces residual structure on the basepoint state that is computable from an integrated curvature.

The substrate, in this picture, possesses *path-sensitive transport structure* — a genuine departure from the scalar geometry of Stages V–IX. Once parallel transport, holonomy, and curvature are recognised as the relevant structural objects, the framework becomes irreducibly tensorial: scalar fields on the substrate (e.g., $\varepsilon_{gap}(x)$) capture only the *contracted* content of richer underlying structure, in the way the Ricci scalar captures only the contracted content of the full Ricci tensor.

5. The Transport-Curvature Tensor

5.1 Definition

The infinitesimal failure of transport commutativity, as identified in §3, defines a tensorial object.

Definition 5.1 — Transport-Curvature Tensor

For the perturbed directional transport generators $\tilde{\nabla}_i = \nabla_i + \Delta_i$, the **transport-curvature tensor** is the commutator

$\mathscr{R}_{ij} := [\tilde{\nabla}_i, \tilde{\nabla}_j] = \tilde{\nabla}_i \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_i$. (5.1)

$\mathscr{R}_{ij}$ is an operator on $\mathscr{H}$ (or equivalently, by the fibrewise projection of Stage IX Theorem 10.1, a position-indexed family of operators on $\mathbb{R}^{\mathcal{K}}$). The indices $(i, j) \in \{1, \ldots, d_X\}^2$ are substrate-directional, with $\mathscr{R}_{ij} = -\mathscr{R}_{ji}$ antisymmetric in (i, j) by the commutator structure.

Caveat — commutator vs directional-difference structure. Although $\mathscr{R}_{ij}$ is defined as an algebraic commutator $[\tilde{\nabla}_i, \tilde{\nabla}_j]$ of transport generators, its leading-order content (3.7) is the *directional finite difference* $(\hat{T}_{x + e_j} - \hat{T}_{x + e_i})\cdot\hat{T}_x$ rather than a Yang-Mills-style algebraic commutator $[\hat{T}_{x + e_i}, \hat{T}_{x + e_j}]$ of fibre operators (the latter enters only at order α^2, not at leading order — cf. Corollary 3.6, equation (3.8)). The substrate transport-curvature tensor is therefore a **substrate transport-curvature candidate** whose commutator structure emerges from directional asymmetry of *sequential* transport at adjacent fibres, not from a true connection-curvature commutator. The structural correspondence with Yang–Mills / Riemann curvature is operator-algebraic and at the level of antisymmetry, indexing, and qualitative role — not at the level of the underlying mechanism that generates the curvature. A future programme stage that derives a genuine algebraic-commutator structure (e.g., through a Born-series expansion identifying α^2-order contributions, or through emergence of a connection 1-form at a coarser scale) would sharpen this; the present paper claims only the candidate status.

Proposition 5.2 — Vacuum Flatness and Defect-Induced Curvature

Under the standing hypothesis (S_{comm}) of §3.1:

(a) In the canonical vacuum (uniform $\hat{T}_x \equiv \hat{T}$, $C = 0$), the transport-curvature tensor vanishes identically:

$\mathscr{R}_{ij} \equiv 0$ (canonical vacuum). (5.2)

(b) In the presence of a localized Stage VIII defect that breaks the substrate's directional symmetry (generic case of Proposition 3.5), the transport-curvature tensor is non-zero of order

$\alpha\gamma^2$ (leading order in the defect strength α) in a neighbourhood of the defect, with exponential decay outside.

Proof. Part (a) is Proposition 3.3 restated as a curvature statement. Part (b) is Proposition 3.5 and the §3.4 worked example restated in tensorial language; the order-$\alpha\gamma^2$ estimate comes from the explicit form of the perturbed-generator commutator (the leading defect-coupling-squared term in the expansion).

5.2 Geometric Interpretation

Three levels of curvature structure. Before describing the substrate-specific properties of $\mathscr{R}_{ij}$, we situate it within the broader programme. The VERSF programme now contains three distinct geometric layers, in order of structural depth:

1. **Scalar coherence curvature (Stage VIII).** The gap-field Laplacian $R(x) = \nabla^2\varepsilon_{gap}(x)$. A position-indexed scalar field built from second discrete differences of the local coherence gap, derivable from the Stage VIII Defect-Coherence Principle. This is the *scalar precursor* layer.
2. **Tensorial transport curvature (present paper).** The transport-generator commutators $\mathscr{R}_{ij} = [\tilde{\nabla}_i, \tilde{\nabla}_j]$. A two-index tensor on substrate directions taking operator values on the closure-catalogue fibre, derivable from path-ordered products of perturbed transport generators. This is the *tensorial transport* layer.
3. **Metric curvature (future target).** Would require emergence of a genuine metric tensor $g_{\mu\nu}$, an affine connection compatible with the metric (Levi-Civita or otherwise), and continuum covariance under change of coordinate frame. This layer is *not* constructed in the present paper; whether and how the Stage X transport-curvature tensor projects onto such a metric structure is open (cf. §11.8).

The present paper operates entirely at level 2 and establishes its existence, non-triviality near defects, and exponential localization. Level 1 is partially recovered (with the caveat of the open scalar-projection problem of Discussion 5.4); level 3 remains future work.

It is important not to overread $\mathscr{R}_{ij}$ as a Riemann tensor analogue in any strict sense: it has substrate-direction indices rather than spacetime indices, takes operator values on a closure-catalogue space rather than acting on tangent vectors, and is not derived from a metric. The structural correspondences identified throughout the paper (geodesic-deviation form, Wilson-type loop functionals, holonomy on closed loops) are *operator-algebraic*, not differential-geometric. With this scope clarification in place:

The transport-curvature tensor $\mathscr{R}_{ij}$ plays the substrate-level role that the Riemann curvature tensor $R^a{}_{bcd}$ plays in Riemannian geometry — but with important structural differences:

- **What it measures.** $\mathscr{R}_{ij}$ measures the infinitesimal failure of parallel transport along the i-direction to commute with parallel transport along the j-direction. This is structurally identical to the role of the Riemann tensor in measuring infinitesimal-loop holonomy.

- **What it is built from.** $\mathscr{R}_{ij}$ is built from substrate transport operators and their commutators, *not* from a metric tensor, connection coefficients, or any classical-geometric apparatus. It is a purely operator-theoretic construction.
- **Indices.** $\mathscr{R}_{ij}$ carries two substrate-direction indices (the directions of the two infinitesimal transport steps) and acts on the closure-catalogue fibre $\mathbb{R}^{\mathcal{K}}$ (rather than on tangent vectors). The closure-catalogue role is structurally similar to that of internal-gauge indices in lattice gauge theory.
- **Antisymmetry.** $\mathscr{R}_{ij} = -\mathscr{R}_{ji}$ by the commutator structure, consistent with the antisymmetry-in-(i,j) of the Riemann tensor.
- **No symmetry in the gauge / closure-catalogue indices.** Unlike the Riemann tensor, which carries additional symmetry properties (e.g., $R_{abcd} = R_{cdab}$), the substrate $\mathscr{R}_{ij}$ is a generic operator on $\mathbb{R}^{\mathcal{K}}$ and need not satisfy further index symmetries. This is structurally analogous to non-Abelian gauge curvature, which similarly lacks the additional Riemannian symmetries.

5.3 The Stage VIII Scalar Curvature and the Transport Tensor: An Open Identification

The Stage VIII candidate scalar curvature $R(x) = \nabla^2\varepsilon_{gap}(x)$ and the present transport-curvature tensor $\mathscr{R}_{ij}(x)$ are both ε_{gap}-derived geometric objects from the same underlying defect structure, but the *precise scalar projection of $\mathscr{R}_{ij}$ that recovers Stage VIII's $R(x)$* has not yet been established. The natural-looking Kretschmann-style contraction $\mathrm{Tr}(\mathscr{R}_{ij}\mathscr{R}^{ij})$ does *not* recover $\nabla^2\varepsilon_{gap}$, and we record this honestly.

Discussion 5.4 — The Scalar Projection Problem

A naive expectation, drawing on the Riemannian analogy where the Ricci scalar arises by contraction from the Riemann tensor, would be that $R(x) = \nabla^2\varepsilon_{gap}(x)$ appears as a scalar contraction of $\mathscr{R}_{ij}$. The most direct Kretschmann-like contraction is

$$K(x) := \mathrm{Tr}_{\mathcal{K}}(\mathscr{R}_{ij}(x) \cdot \mathscr{R}^{ij}(x)), \quad (5.3)$$

with the trace over closure-catalogue indices and summation over (i, j). But by Proposition 5.2 and §3.4, $\mathscr{R}_{ij}(x)$ is of order $\alpha\gamma^2$ at the defect core, so $K(x)$ is of order $\alpha^2\gamma^4$, while $\nabla^2\varepsilon_{gap}(x)$ is of order α (the second discrete difference of a function with $O(\alpha)$ depth in a region of size $\mathcal{O}(1)$). These have *different powers of α*, different sign behaviour ($K(x) \geq 0$ by construction; $\nabla^2\varepsilon_{gap}(x)$ is sign-indefinite), and different vanishing loci ($K(x) = 0$ iff all $\mathscr{R}_{ij}(x) = 0$; $\nabla^2\varepsilon_{gap}(x) = 0$ at zeros of the gap-field Laplacian). A constant prefactor cannot bridge these gaps: the Kretschmann scalar $K(x)$ is not the leading scalar projection that recovers $R(x)$.

Other contractions present similar obstructions or remain unexplored:

(a) **Ricci-like single-trace contraction.** The Riemannian Ricci tensor arises from a *single* index contraction $R^k{}_{ikj}$ of the four-index Riemann tensor. The substrate analogue would be $\Sigma_i \mathscr{R}_{ii}(x)$ — a sum of "diagonal" curvature components. By the antisymmetry $\mathscr{R}_{ij} =$

$-\mathscr{R}_{ji}$, the diagonal entries vanish identically: $\mathscr{R}_{ii}(x) = 0$ for all i. So the diagonal-sum contraction is identically zero and cannot recover $R(x)$.

(b) **Divergence-type contraction.** Another candidate is $\Sigma_i \nabla^i \mathscr{R}_{ij}(x)$ — a discrete divergence of $\mathscr{R}$ along one direction index. This is of order $\alpha\gamma^3$ (one extra spatial derivative) and is sign-indefinite, but its scalar reduction $\Sigma_j \Sigma_i \nabla^i \mathscr{R}_{ij}(x)$ involves contractions across both indices. Whether this reduces to $\nabla^2\varepsilon_{gap}(x)$ in the small-α limit is a Born-series-rigorous question that we have not resolved.

(c) **Trace-of-square-root or modulus-trace contractions.** Functions like $\mathrm{Tr}(|\mathscr{R}_{ij}|)$ carry the right order in α (linear, matching $\nabla^2\varepsilon_{gap}$) but lack a clean operator-algebraic justification from the substrate structure. They are dimensional-analysis candidates, not derivational ones.

We therefore record:

Open identification problem. The Stage VIII scalar candidate curvature $R(x) = \nabla^2\varepsilon_{gap}(x)$ and the Stage X transport-curvature tensor $\mathscr{R}_{ij}(x)$ are both geometric objects derived from the same Stage VIII defect structure, with shared ε_{gap}-dependence and shared exponential localization near defects. The precise scalar projection $\mathscr{R} \to R$ that maps the tensor to Stage VIII's scalar is not the naive Kretschmann contraction and has not yet been established. A Born-series expansion of the perturbed transport generators around the vacuum, with explicit tracking of how $\nabla^2\varepsilon_{gap}$ arises from the second variation of the path-ordered transport product, is the natural next-stage technical target (cf. §11.9).

The structural-correspondence content survives this open problem. The two objects share their physical origin (Stage VIII defects), share their qualitative behaviour (localization, ε_{gap}-dependence), and serve corresponding roles (R as a candidate scalar curvature precursor in Stage VIII; $\mathscr{R}_{ij}$ as a tensorial curvature in Stage X). What remains open is the *precise operator-theoretic projection map* between them.

Candidate Routes Toward the Scalar Projection

The failure of the naive Kretschmann-style contraction is informative: it suggests that the Stage VIII scalar curvature does *not* arise from a quadratic invariant of $\mathscr{R}_{ij}$, but instead from a lower-order, linear-in-α projection of substrate-transport structure. Three candidate projection mechanisms now appear structurally plausible as concrete next-stage technical targets.

(P1) Divergence-type contractions. The candidate $\Sigma_i \Sigma_j \nabla^i \mathscr{R}_{ij}(x)$, where ∇^i is the substrate-direction discrete derivative dual to the i-direction generator. This is linear in $\mathscr{R}_{ij}$ (matching the α-order of $\nabla^2\varepsilon_{gap}$) but acquires its second-derivative character from the derivative outside the curvature tensor — schematically, "the divergence of curvature produces a Laplacian of the field that sources it." The substrate-level question is whether the discrete contraction $\Sigma_i \nabla^i \mathscr{R}_{ij}(x)$, summed over j, reduces to $\nabla^2\varepsilon_{gap}(x)$ at leading order in α. Open.

(P2) Transport-action second variation. The candidate $\delta^2 S[\gamma]/\delta x^2|_x$ evaluated on the transport action $S[\gamma]$ of (6.1) at a point x, with γ varied as an infinitesimal loop encircling x. This is a direct construction: take the path-ordered transport functional, vary it twice with respect to the loop's spatial extent at x, and identify the resulting scalar with R(x). The conjecture is that the Hessian of the transport action at x equals (up to an explicit prefactor) the gap-field Laplacian. This route makes maximal use of the substrate's variational structure and is the most natural extension of the §6 geodesic-deviation argument from "deviation between two geodesics" to "second variation at a single point."

(P3) Holonomy-density projection. The candidate $\lim_{\varepsilon \to 0} \varepsilon^{-2} \cdot \hat{W}(\gamma_\varepsilon(x))$, where $\gamma_\varepsilon(x)$ is an ε-side-length loop centred at x and $\hat{W}(\gamma_\varepsilon(x))$ is the normalised loop functional of §7. This is the substrate analogue of identifying scalar curvature with the infinitesimal-area limit of loop-deviation density, in the spirit of the lattice gauge-theory plaquette-density continuum limit. The substrate-level question is whether this infinitesimal limit exists, whether it is direction-isotropic enough to define a scalar, and whether it reduces to $\nabla^2\varepsilon_{gap}(x)$.

These three candidates are not mutually exclusive — it is structurally possible that two or more of them yield the same scalar projection by different routes, in which case the agreement would itself be informative. Determining which (if any) of (P1)–(P3) succeeds is the principal technical target of the next-stage programme (cf. §11.9). The present paper's contribution is to identify the projection problem cleanly, to rule out the obvious-but-wrong Kretschmann candidate, and to lay out concrete candidate routes that the substrate's structure makes plausible.

5.5 The Stage V → X Nested Architecture

Even without the precise scalar-projection identification of §5.3, the broader architecture of the Stage V → X programme is clear. Stage IX Theorem 10.1 established that Stage VIII is the *fibrewise projection theory* of Stage IX coupled transport: the spatial fibre $P_x \cdot \mathbf{T} \cdot P_x$ of the Stage IX global operator equals the Stage VIII local operator $\hat{T}_x$ at each substrate position. The present paper establishes that Stage X is the *path-ordered tensorial extension* of Stage IX coupled transport: substrate parallel transport, holonomy, and transport-curvature tensors are built from path-ordered products of the same Stage IX local transport maps that Stage IX Theorem 10.1 projects onto Stage VIII fibres.

The picture across Stages V → X is therefore:

- **Stage V:** Lipschitz continuum from canonical wheel; scalar gap $\varepsilon_{gap} = ½$.
- **Stage VII:** Open universality class; scalar property of the wheel.
- **Stage VIII:** Localized defects produce scalar gap field $\varepsilon_{gap}(x)$, with four functionals including candidate curvature $R(x) = \nabla^2\varepsilon_{gap}(x)$.
- **Stage IX:** Global transport operator **T** with bulk bands, finite propagation, Birman–Schwinger trapped modes, Combes–Thomas localization; the fibrewise projection of **T** onto each spatial fibre is exactly the Stage VIII local operator $\hat{T}_x$ (Theorem 10.1), so Stage VIII is the fibrewise projection theory.
- **Stage X (present):** Substrate parallel transport, coherence holonomy, transport-curvature tensor $\mathscr{R}_{ij}$; the precise scalar projection $\mathscr{R} \to R$ recovering Stage VIII's R(x) =

$\nabla^2\varepsilon\_gap(x)$ is an open identification problem (Discussion 5.4), but the structural-correspondence content — both Stage VIII's R(x) and Stage X's $\mathscr{R}_{ij}$ are ε_gap-derived geometric objects with shared defect-localization — is established.

This nested structure (scalar → operator → tensor, with each stage containing the previous as a projection) is the architectural content of the geometry programme. It also suggests the natural next-stage targets: Stage X tensorial geometry should itself have a leading projection that, in the appropriate large-scale limit, recovers a metric / Lorentzian / gravitational structure — but that lift is the principal open problem (§11.6) and is not undertaken in the present paper.

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6. Geodesic Transport Distortion and the Deviation Equation

6.1 Effective Coherence Geometry near Defects

Stage IX §§5, 9 established that localized defects produce *coherence wells*: regions of suppressed ε_gap(x) where transport is heuristically slowed and where coherence trajectories (geodesics of the effective coherence metric g_coh of Stage IX Definition 5.3) bend toward the defect core (Stage IX Proposition 9.3). The transport-curvature tensor $\mathscr{R}_{ij}$ of §5 now allows us to refine this picture: the bending of geodesics near defects is governed by an equation of the substrate-level analogue of geodesic deviation.

Throughout this section we work with the heuristic register of Stage IX §9.1 ($v_c^{local}(x) \propto \varepsilon_{gap}(x)$), since the §6 geodesic-deviation equation depends on the same identification.

6.2 Transport Geodesics

Following Stage IX Definition 5.3 / Proposition 5.4, define a **coherence transport geodesic** to be a substrate path π that minimises the effective transport action

$S[\pi] := \Sigma_{u \in \pi} 1/\varepsilon_{gap}(u)$, (6.1)

equivalently a minimiser of the coherence-transport distance g_coh of Stage IX Definition 5.3. In the canonical vacuum $\varepsilon_{gap}(x) \equiv ½$, the action is uniform and geodesics are shortest substrate paths in the graph metric d_X. With defects, regions of suppressed ε_gap acquire larger action weight, and geodesics bend to avoid them (cf. Stage IX Proposition 9.3 — "coherence rays bend toward suppressed-gap regions" — which, despite its apparently paradoxical statement, follows from the Lagrangian structure: trajectories bend toward maxima of 1/ε_gap, equivalently minima of ε_gap; this is Fermat's principle, with bending toward suppressed-gap regions, not away).

Theorem 6.3 — Substrate Geodesic Deviation Equation (Heuristic)

Granting the heuristic identification $v_c^{local}(x) \propto \varepsilon_{gap}(x)$ of Stage IX §9.1 (whose epistemic register is recorded there), let π_1 and π_2 be two nearby coherence transport geodesics propagating through a region of varying gap field $\varepsilon_{gap}(x)$, with separation vector field $\xi(s) := \pi_2(s) - \pi_1(s)$ parametrised along arclength s. Then to leading order in the separation and in the defect-induced anisotropy, the separation vector satisfies the substrate analogue of the Riemannian geodesic-deviation equation:

$$\mathcal{D}^2\xi/ds^2 \sim \mathscr{R}(\xi, \dot{\pi})\cdot\dot{\pi}, \quad (6.2)$$

*where $\mathcal{D}/ds$ denotes the substrate-level **transport-derivative** along the geodesic (constructed from the perturbed transport generators $\tilde{\nabla}_i$ — see "Clarification on terminology" in §3.2 and the discussion in §11.8 below), $\dot{\pi}$ is the geodesic tangent, and $\mathscr{R}$ is the transport-curvature tensor of Definition 5.1 (contracted appropriately on its substrate-direction indices). Equivalently, in the discrete substrate setting:*

$$(\xi(s+1) - 2\xi(s) + \xi(s-1)) \sim \Sigma_{ij}\ \mathscr{R}_{ij}(\pi(s)) \cdot \dot{\pi}^i(s) \cdot \xi^j(s) + \mathcal{O}(\xi^2, \mathscr{R}^2), \quad (6.3)$$

the discrete-substrate analogue of the continuum equation. The notation $\mathcal{D}/ds$ is structural, not differential-geometric in the strict sense: we have no Levi-Civita connection, no metric tensor, and no tangent bundle in the present construction (cf. §3.2 Clarification on terminology, and §11.8 on the absence of a connection compatible with a substrate metric).

Consequently:

- *Where $Tr(\mathscr{R}_{ij}) > 0$ in the direction of geodesic motion, the deviation $\mathcal{D}^2\xi/ds^2$ aligns with ξ, producing geodesic* **focusing** *(nearby trajectories accelerated toward each other).*
- *Where $Tr(\mathscr{R}_{ij}) < 0$ in the direction of geodesic motion, the deviation opposes ξ, producing geodesic* **defocusing** *(nearby trajectories accelerated apart).*
- *At trapped-mode defect cores (where transport curvature is positive and localized by §8), nearby coherence trajectories focus toward the defect, consistent with the §5 / §9 picture of defects as transport-attractors.*

Proof sketch. The derivation parallels the Riemannian geodesic-deviation argument, with the continuum Levi-Civita connection replaced by the substrate-level perturbed transport generators $\tilde{\nabla}_i$ and the Riemann tensor replaced by their commutator $\mathscr{R}_{ij}$. Take the second variation of the transport action (6.1) at a geodesic π_1 with infinitesimal deviation ξ to a nearby geodesic π_2:

$$\delta^2 S[\pi_1; \xi] = \Sigma_s\ [\langle\tilde{\nabla}\cdot\xi, \tilde{\nabla}\cdot\xi\rangle(s) - \langle\mathscr{R}(\xi, \dot{\pi})\cdot\dot{\pi}, \xi\rangle(s)] + \mathcal{O}(\xi^3).$$

The Euler–Lagrange equation for this second variation (with the substrate-level transport-derivative operator replacing the continuum Levi-Civita connection) gives the discrete deviation equation (6.3) as the condition that the variation is stationary, equivalently that ξ satisfies the substrate analogue of the Jacobi equation. The focusing / defocusing dichotomy follows from the sign of the right-hand side as in the standard Riemannian argument — with the same caveat that "Jacobi equation" here is structural-analogical, not a precise differential-geometric statement.

Epistemic register. Theorem 6.3 inherits the Stage IX §9.1 heuristic register: the geodesic-deviation form depends on the $v_c^{local}(x) \propto \varepsilon_{gap}(x)$ identification through the transport-action functional (6.1). The qualitative content — *nearby coherence trajectories focus / defocus near defect-induced transport curvature, with sign governed by $\mathscr{R}_{ij}$* — is robust at the structural level under any reasonable monotone $v_c^{local}(\varepsilon_{gap})$. The precise quantitative form (the factor of $\mathrm{Tr}(\mathscr{R}_{ij})$ on the right-hand side) is heuristic in its prefactor; a Born-series-rigorous derivation would either confirm the prefactor or correct it.

6.3 Coherence Wells Revisited

The §5 / §9 picture of localized defects as coherence wells acquires its tensorial completion through Theorem 6.3:

- **Slowing.** Stage IX §9.1: in the defect region, the heuristic local transport velocity $v_c^{local}(x) \propto \varepsilon_{gap}(x)$ is reduced. Coherence flowing into the well propagates more slowly.
- **Bending (refraction).** Stage IX Proposition 9.3: by Fermat's principle, coherence trajectories bend toward regions of suppressed ε_{gap}.
- **Focusing (curvature).** Theorem 6.3: nearby trajectories near a defect with positive $\mathrm{Tr}(\mathscr{R}_{ij})$ focus toward the defect, producing the substrate-level analogue of gravitational lensing — coherence trajectories converge on the defect.
- **Accumulation.** Stage IX Corollary 8.1 (entropy retention) + Stage IX §9.2 (coherence wells): with bent and focused trajectories converging on the defect and slowed propagation, coherence accumulates in the defect region.
- **Trapping.** Stage IX Theorem 7.1 + Corollary 8.2: in the strong-coupling / sufficient-defect-strength regime, the well becomes a genuine trap — some coherence is permanently localized as a Birman–Schwinger trapped mode, with Combes–Thomas-exponential localization length ξ.

The Stage VIII Defect-Coherence Principle therefore acquires its full tensorial-transport reading: roughening, candidate curvature, entropy retention, and trapped-mode persistence are all aspects of one underlying transport phenomenon — localized reduction of coherence-transport capacity, with the four functionals being scalar projections of the deeper transport-curvature structure and the trajectory-focusing behaviour governed by the transport-curvature tensor $\mathscr{R}_{ij}$.

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7. Wilson-Type Loop Functionals and Coherence Memory

7.1 Loop Transport Functionals

For a closed substrate loop γ, the holonomy operator $\mathscr{H}(\gamma)$ of §4 is an operator on $\mathbb{R}^{\mathcal{K}}$. Its *trace* extracts a scalar functional of the loop:

Definition 7.1 — Wilson-Type Loop Functional

The **Wilson-type loop functional** of a closed substrate loop γ is

$$W(\gamma) := \mathrm{Tr}\_\mathcal{K}(\mathscr{H}(\gamma)) = \Sigma\_\kappa \langle e\_\kappa, \mathscr{H}(\gamma)\, e\_\kappa\rangle\_\{L^2(\pi)\}, \quad (7.1)$$

where the trace is over the closure-catalogue indices in the $L^2(\pi)$ inner product on $\mathbb{R}$^$\mathcal{K}$.

7.2 Vacuum Value and Curvature Sensitivity

In the canonical vacuum, by Proposition 2.5:

$$W(\gamma) = \mathrm{Tr}\_\mathcal{K}(\gamma\text{^}n \cdot \hat{T}\text{^}n) = \gamma\text{^}n \cdot \mathrm{Tr}\_\mathcal{K}(\hat{T}\text{^}n) \text{ (canonical vacuum, n = loop length)}. \quad (7.2)$$

The vacuum value depends on n through both γ^n (spatial-coupling cumulative factor) and $\mathrm{Tr}(\hat{T}^n)$ (cumulative wheel trace). For large n, $\mathrm{Tr}(\hat{T}^n) \to 1$ (the Perron eigenvalue dominates and $\hat{T}^n$ converges in trace to the rank-1 projection onto the stationary state), so $W(\gamma) \to \gamma^n \cdot 1 = \gamma^n$ in the large-n limit. For small n, the wheel trace carries genuine information about the wheel structure: $\mathrm{Tr}(\hat{T}) = 1 + ½ + ½ - ¼ - 3/28 + 0 + 0 = 1 + 1 - ¼ - 3/28 = 47/28 \approx 1.679$, and so on.

As with the holonomy operator itself in §4, the relevant *curvature-sensitive* quantity is the deviation from the vacuum:

$$\hat{W}(\gamma) := W(\gamma) - \mathrm{Tr}\_\mathcal{K}(\gamma\text{^}n \cdot \hat{T}\text{^}n) = \mathrm{Tr}\_\mathcal{K}(\mathscr{H}(\gamma) - \gamma\text{^}n \cdot \hat{T}\text{^}n) = \mathrm{Tr}\_\mathcal{K}(\hat{H}(\gamma) \cdot \hat{T}\text{^}n) \cdot \gamma\text{^}n,$$
(Definition 7.2)

with $\hat{H}(\gamma)$ the normalised holonomy of Definition 4.2 (on the $\hat{T}$-invertible subspace). The normalised loop functional $\hat{W}(\gamma)$ strips off the vacuum cumulative-refinement trace and isolates the genuinely curvature-induced residual trace.

7.3 Leading-Order Curvature Reading

For small loops surrounding weak defects, Taylor-expanding the holonomy in the perturbed-generator commutators gives the structural relation:

$$\hat{H}(\gamma) \approx \oint\_\gamma \tilde{\nabla} + ½ \oint\_\gamma [\tilde{\nabla}, \tilde{\nabla}] + \mathcal{O}(\tilde{\nabla}^3) \text{ (in the discrete-substrate, finite-difference sense)}$$

with the second term being the substrate analogue of the area integral of the curvature tensor over the surface bounded by γ. Taking the trace and using the antisymmetry of the commutator (so the first term contributes only via boundary effects that cancel on a closed loop):

$$\hat{W}(\gamma) \approx ½ \oint\gamma\ \mathit{Tr}(\mathscr{R}\{ij\}) \cdot dA\text{^}\{ij\} + \mathcal{O}(\mathscr{R}^2), \quad (7.3)$$

where dA^{ij} = −dA^{ji} is the antisymmetric discrete substrate-area element in the (i, j) plane bounded by γ — antisymmetric to match the antisymmetry of $\mathscr{R}$_{ij} = −$\mathscr{R}$_{ji}, so that the contraction Tr($\mathscr{R}$_{ij})·dA^{ij} is invariant under (i, j) relabelling. On $\mathbb{Z}^2$ with the canonical orientation, dA^{12} = +1 per anticlockwise-oriented unit plaquette and dA^{21} = −1; for general loops the integral $\oint$_γ … is computed via the discrete-Stokes plaquette decomposition of

Theorem 4.2 Step 2 with each plaquette contributing dA^{ij} according to its orientation. This is the substrate-level analogue of Stokes' theorem applied to the loop integral of a curvature tensor: the loop functional measures the *signed integrated curvature* enclosed by the loop, to leading order.

7.4 Wilson-Type Interpretation

Definition 7.1 and the leading-order curvature reading (7.3) make the operator-algebraic correspondence with lattice gauge theory explicit:

- In lattice gauge theory, the Wilson loop $W(\gamma) = \mathrm{Tr}(U_\gamma)$ for a closed lattice loop γ measures the trace of the gauge holonomy U_γ around γ, with vacuum value $\mathrm{Tr}(I) = \dim(\text{group})$ and deviation from vacuum measuring the enclosed lattice field strength (curvature) by a discrete Stokes-type identity.
- In the present substrate setting, $W(\gamma) = \mathrm{Tr}_{\mathcal{K}}(\mathscr{H}(\gamma))$ measures the trace of the coherence holonomy $\mathscr{H}(\gamma)$ around γ, with vacuum value $\gamma^n \cdot \mathrm{Tr}(\hat{T}^n)$ and deviation measuring the enclosed transport curvature by the discrete substrate Stokes-type identity (7.3).

The mathematical structure is *identical at the operator-algebraic level*. The structural difference is that no gauge group action is imposed on the substrate; the holonomy operator $\mathscr{H}(\gamma)$ is generated entirely by the substrate transport structure of Stage IX, not by an externally specified gauge field. The correspondence is *structural-analogical*, not derivational.

7.5 Coherence Memory

Repeated traversals of substrate loops accumulate substrate-level transport memory: the holonomy operator $\mathscr{H}(\gamma^k)$ for k traversals of γ is $\mathscr{H}(\gamma)^k$, which (for non-trivial holonomy) does not collapse to the identity even for large k — instead, the loop transport produces a substrate-level cyclic structure whose periodicity and spectrum are computable from the eigenvalues of $\mathscr{H}(\gamma)$. This is the operator-theoretic substrate analogue of the Aharonov–Bohm-style geometric phase accumulation: the substrate is no longer "memoryless" with respect to past transport, even though no clock or temporal record is built into the substrate dynamics.

This is one of the structurally deepest shifts in the VERSF programme. Through Stages V–IX, the substrate's geometry was *scalar*: distances, propagation speeds, and curvature precursors were all built from scalar fields on the substrate, and no observable encoded path information. With the present Stage X framework, the substrate acquires *path-sensitive structure*: loops carry information that distinguishes them from contractions to a point, and this information is operationally recoverable through the loop trace $W(\gamma)$ and the normalised holonomy $\hat{H}(\gamma)$. The substrate becomes, in operator-theoretic terms, a setting in which geometry is irreducibly tensorial — scalar fields no longer exhaust the observable content.

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8. Localization and Curvature Concentration at Trapped Modes

8.1 Holonomy and Curvature Localization

The Combes–Thomas localization of Stage IX (Theorem 8.1, Corollary 8.2) established that trapped eigenstates of the perturbed transport operator are exponentially localized near defects, with localization length $\xi = 1/\eta$ governed by the spectral distance δ from the bulk band edges. The transport-curvature tensor $\mathscr{R}_{ij}$ of §5, built from commutators of the perturbed transport generators, exhibits exponential localization with rate governed by a related but distinct mechanism — and the relationship between the two rates is the central structural content of this section.

Proposition 8.1 — Curvature Localization at Trapped-Mode Defects

Let V be a localized Stage VIII defect producing a trapped mode of the Stage IX coupled transport operator at spectral distance $\delta > 0$ from the bulk bands, with Combes–Thomas localization length $\xi_{trap} = 1/\eta_{trap}$ where $\eta_{trap} = \min(1, \delta/(2\gamma\cdot\rho(A_X)))$ (Stage IX Theorem 8.1). Then the associated transport-curvature tensor satisfies exponential localization at the trapped-mode rate:

$$\|\mathscr{R}_{ij}(x)\|_{op} \leq C \cdot e^{-d_X(x, x_0)/\xi_{trap}} \text{ for all } x \in X \text{ with } d_X(x, x_0) > r, \quad (8.1)$$

with C a constant depending on the defect strength α and on substrate-geometric prefactors, and ξ_{trap} the Combes–Thomas localization length of the underlying trapped mode.

Proof. By the construction of §3.3 (Definition 3.5), the perturbed directional generators $\tilde{\nabla}_i = \nabla i$ *$+ \Delta_i$ differ from the vacuum generators only by the defect-localized perturbations $\Delta_i = \gamma \cdot \Sigma\{$*$x \in B_r(x_0)\}$ $(\Delta T_{x_0,r}(x) \cdot |x + e_i\rangle\langle x|)$, supported on edges originating at defect-region vertices. The commutator $\mathscr{R}_{ij} = [\tilde{\nabla}_i, \tilde{\nabla}_j]$, expanded via Proposition 3.5 (equation (3.7)), is supported on the boundary shell of $B_r(x_0)$ and on its immediate neighbour shells where consecutive applications of perturbed generators sample fibres inside and outside the defect support.

For evaluation at points x with $d_X(x, x_0) > r$, the commutator contribution must propagate from the defect support $B_r(x_0)$ to x. The propagation is mediated by the resolvent of the perturbed transport operator $\mathbf{T} = \mathbf{T}_{bulk} + V$ evaluated at the trapped eigenvalue λ_{trap}. Specifically, the leading-order contribution is

$$\|\mathscr{R}_{ij}(x)\|_{op} \leq C \cdot \|R(\lambda_{trap})(x, x')\| \cdot \|\Delta_i \Delta_j\| \text{ for } x' \in B_r(x_0),$$

where $R(\lambda) := (\mathbf{T} - \lambda I)^{-1}$ is the perturbed resolvent. The matrix element $\|R(\lambda_{trap})(x, x')\|$ at the trapped eigenvalue diverges (λ_{trap} is in the discrete spectrum of $\mathbf{T}$), but the bulk resolvent $R_0(\lambda_{trap}) := (\mathbf{T}_{bulk} - \lambda I)^{-1}$ is well-defined since λ_{trap} lies in the gap region of the essential spectrum at spectral distance δ from the bulk band edges. The Combes–Thomas estimate of Stage IX Theorem 8.1 gives $\|R_0(\lambda_{trap})(x, x')\| \leq (2/\delta) \cdot e^{-\eta_{trap} \cdot d_X(x, x')}$, and the second-resolvent identity relates the perturbed resolvent's off-diagonal matrix elements to the bulk resolvent's. Using $d_X(x, x') \geq d_X(x, x_0) - r$ for $x' \in B_r(x_0)$ (triangle inequality, absorbing the $e^{\eta_{trap} \cdot r}$ factor into C) gives (8.1).

Remark 8.2 — Subcritical Defects: Curvature Localization Without Trapping

For defects that *do not* produce a trapped mode (subcritical defects, with α below the Birman–Schwinger threshold α_{trap} of Stage IX §7.4), the spectral distance δ as defined above does not exist. The transport-curvature tensor $\mathscr{R}_{ij}(x)$ is still localized — by Definition 3.5, the perturbations Δ_i remain supported on $B_r(x_0)$, and $\mathscr{R}_{ij}(x)$ at points x outside the defect support still requires propagation from the defect — but the decay rate is no longer governed by the trapped-mode Combes–Thomas formula $\eta_{trap} = \delta/(2\gamma \cdot \rho(A_X))$.

Instead, for subcritical defects, the decay rate is governed by Helffer–Sjöstrand-type resolvent estimates at points λ in the *resolvent set* of $\mathbf{T}_{bulk}$. The relevant decay rate is

$\eta_{sub} := \min(1, \mathrm{dist}(\lambda_{ref}, \mathrm{spec}(\mathbf{T}_{bulk}))/(2\gamma \cdot \rho(A_X)))$, (Remark 8.2)

where λ_{ref} is the natural reference point for the curvature propagation — for the leading-order curvature contribution from the cross-commutator $[\nabla_i, \Delta_j]$, the relevant λ_{ref} is the nearest point of $\mathrm{spec}(\mathbf{T}_{bulk})$ to the wheel eigenvalue most affected by the defect (typically the ½-band edge for Stage VIII §9.1-type boundary defects, giving $\mathrm{dist}(\lambda_{ref}, \mathrm{spec})$ of order $\gamma \cdot \rho(A_X)$ − (small Birman–Schwinger residual)).

The structural point is: subcritical defects have curvature localization rate η_{sub} set by *distance to the nearest bulk-spectrum point* of $\mathbf{T}_{bulk}$, while critical / supercritical defects have curvature localization rate η_{trap} set by *distance from the trapped eigenvalue to the bulk-band edge*. Generically $\eta_{sub} > \eta_{trap}$ for defects close to but below threshold, since the trapped eigenvalue at the threshold has $\delta \to 0$ and $\eta_{trap} \to 0$ while η_{sub} remains bounded away from zero. As α crosses the Birman–Schwinger threshold from below, the curvature localization rate transitions from η_{sub}-governed to η_{trap}-governed.

8.2 Structural Reading: Trapped Modes as Rate-Coincidence Configurations

Proposition 8.1 and Remark 8.2 together supply a structural reading of trapped modes, with a careful caveat on the scope of the claim:

Within the present construction, the trapped-mode configurations are *structurally distinguished* by the property that the transport-curvature localization rate matches the eigenvector-localization rate (the rate-coincidence property of Proposition 8.1).

This is a *correlation* established under the present operator-theoretic construction — coexistence, shared scaling, shared localization, and shared defect-dependence — and not a proof of strict equivalence in any deeper sense. Specifically, we have not shown that trapped-mode existence and curvature concentration are *necessarily* equivalent properties; only that under the present construction they appear together and share their characteristic scales.

For any defect (subcritical or supercritical), the substrate's response to localized perturbation has two associated decay scales: a curvature-decay scale $\xi_{curv} = 1/\eta$ (with $\eta \in \{\eta_{sub}, \eta_{trap}\}$ depending on whether trapping occurs) and an eigenvector-decay scale ξ_{eig}. For subcritical defects there *is no* trapped eigenvector, so ξ_{eig} is undefined and the curvature simply decays at the η_{sub} rate without an eigenvector partner. For supercritical defects, both scales exist, and Proposition 8.1 establishes $\xi_{curv} = \xi_{eig} = \xi_{trap} = 1/\eta_{trap}$ — the two rates coincide.

The structural unification at the level of *coexistence* is therefore:

- **Subcritical defects** produce localized transport curvature with rate η_{sub} but no localized trapped state.
- **Critical defects** (at the Birman–Schwinger threshold) have $\eta_{trap} \to 0$ and produce delocalized incipient trapped states with marginal-rate curvature.
- **Supercritical defects** produce localized transport curvature with rate η_{trap} and a localized trapped eigenstate at the same rate.

This is the substrate-level operator-theoretic content underneath the structural picture of trapped modes as *defects in the geometric vacuum*: under the present construction they appear as operator-theoretic objects whose trapped-eigenstate localization and transport-curvature concentration share the same characteristic scale. Whether this rate-coincidence reflects a deeper underlying equivalence — i.e., whether trapped-mode existence *necessarily* implies curvature concentration in some construction-independent sense, or whether the present correlation is an artefact of the specific operator-theoretic apparatus we have built — is open.

The structural correspondence is the substrate-level analogue of how, in general relativity, localized matter / energy concentrations are accompanied by localized Schwarzschild-style curvature and by localized bound states of test fields in the resulting metric, with the bound-state localization scale and the curvature concentration scale set by the same Schwarzschild radius. The substrate version is *non-metric* (no Lorentzian or Riemannian structure is constructed) and *operator-theoretic* (the trapped state and the curvature concentration are both operators on $\mathscr{H}$); the analogy is at the level of structural correlation under a chosen construction, not at the level of strict mathematical equivalence.

8.3 Curvature Clustering and Multi-Defect Hybridization

For multiple defects within one localization length of each other (Stage IX §12.2 "regime of strong hybridization"), the individual trapped-mode-localized curvature concentrations interact. The two-defect splitting analysis of Stage IX Proposition 12.1 (bonding / antibonding modes with exponentially-decaying interaction energy $J(d) \propto e^{-\eta_{trap} \cdot d}$) lifts directly to the present setting: the transport-curvature tensors at the two defects hybridize with coupling strength inheriting the same exponential distance-dependence, with rate η_{trap} matched to the hybridized trapped-mode rate.

For N defects in a superlattice arrangement, Stage IX §12.4 derived an impurity-band of trapped modes with bandwidth $\propto e^{-\eta_{trap} \cdot d_{min}}$. At the curvature level, this produces a corresponding *curvature band*: a continuous spectrum of localized transport-curvature configurations whose spatial structure interpolates between the single-defect form (sharply localized) and a quasi-uniform background curvature (in the dense-defect limit). The structural correspondence with impurity-band physics — both for the trapped-mode spectrum and for the curvature distribution — holds at the operator-algebraic level under the present construction; whether the trapped-mode band and the curvature band are strictly equivalent objects, or only structurally correlated, is a refinement-stability question of the kind flagged in §11.11.

8.4 Emergent Geometric Sources

The structural content of §8 is that trapped modes act as *emergent geometric sources*:

- They are operator-theoretically distinguished (Birman–Schwinger isolated eigenvalues outside the bulk bands).
- They produce localized transport curvature with matched-rate decay (Proposition 8.1).
- They interact via exponentially-decaying matrix elements (Stage IX Proposition 12.1) at the trapped-mode level, with structurally correlated decay at the transport-curvature level under the present construction.
- They focus coherence trajectories toward themselves (Theorem 6.3 with localized positive $\mathscr{R}$).
- They accumulate coherence (Stage IX §§9.2, 12).

This is, structurally, the strongest bridge yet developed between substrate transport, localization theory, and gravity-like geometry. The matter / particle-content reading remains conjectural in the strict Stage VIII sense — no specific particle phenomenology is derived, no mass / charge / spin assignments are made, and the trapped-mode spectrum at this stage is purely operator-theoretic without quantum-mechanical interpretation. What *is* established under the present construction is that the substrate's operator-theoretic localized states appear *in coexistence with* localized geometric sources — with their characteristic scale (the Combes–Thomas length ξ_{trap}) shared between the eigenstate and the curvature concentration. Whether this coexistence reflects a deeper underlying mechanism that makes the two phenomena strictly equivalent rather than merely correlated under the present construction is left open. The link between the two roles, at the level established here, is the rate-coincidence structure embodied in Proposition 8.1 and Remark 8.2.

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9. The Extended Coherence Transport Scaling Identity

9.1 The Four Fundamental Transport Scales

The Stage IX Coherence Transport Scaling Identity (Stage IX Theorem 13.1) identified three primary scales of the transport framework:

- **Coherence velocity** $v_c := \gamma \cdot \rho(A_X)$ (rate of bulk propagation).
- **Spectral distance** $\delta := \mathrm{dist}(\lambda, \mathrm{spec\_ess}(\mathbf{T}\_\mathrm{bulk}))$ (binding strength of a trapped eigenvalue).
- **Localization length** $\xi := 1/\eta$ where $\eta = \min(1, \delta/(2\gamma\cdot\rho(A_X)))$ (Combes–Thomas decay rate).

These were related by the basic identity $\xi \cdot \delta \sim v_c$ (the substrate-level uncertainty-style bound). The present paper introduces a fourth scale:

- **Transport-curvature magnitude** $\|\mathscr{R}\| := \|\mathscr{R}_{ij}\|_{op}$ (strength of the local transport-curvature tensor at the defect core).

We now establish that this fourth scale is governed by the same underlying scaling structure.

9.2 The Curvature Scale

From the §3.3 construction of perturbed generators and the §3.4 worked example, the leading-order transport-curvature magnitude at the defect core is

$\|\mathscr{R}\| \sim \alpha \cdot \gamma^2 \cdot (\text{substrate-geometric factor}) \sim \gamma \cdot |\Delta_i| \cdot |\Delta_j|$. (9.1)

In the heuristic register of Stage IX §9.1, the defect-induced generator anisotropy $|\Delta_i|$ can be related to the local-gap depression: $\|\Delta_i\| \sim \gamma \cdot \|\nabla_i \varepsilon\_gap\|$ (the i-direction perturbation of the transport generator scales with the i-direction gradient of the gap field). Substituting:

$\|\mathscr{R}\| \sim \gamma^3 \cdot |\nabla\varepsilon\_gap|^2 \sim (1/\xi) \cdot |\nabla\varepsilon\_gap|$, (9.2)

using the localization-length / spectral-distance relation $\xi \cdot \delta \sim v_c = \gamma\cdot\rho(A_X)$ to convert $\gamma^3 \cdot |\nabla\varepsilon\_gap|^2$ into a form involving ξ. The second form (9.2) is the **curvature scaling**: transport-curvature magnitude scales as gap gradient divided by localization length.

Proposition 9.2 (Heuristic) — Extended Coherence Transport Scaling Identity

Granting the heuristic identification $v_c^{local}(x) \propto \varepsilon\_gap(x)$ of Stage IX §9.1 (whose epistemic register is recorded there), the four fundamental transport scales — coherence velocity v_c, spectral distance δ, localization length ξ, transport-curvature magnitude $\|\mathscr{R}\|$ — together with the underlying gap-gradient field $|\nabla\varepsilon\_gap|$ are linked by the extended scaling structure:

$\xi \cdot \delta \sim v_c$ (basic localization identity, Stage IX Theorem 13.1) $\|\mathscr{R}\| \sim |\nabla\varepsilon\_gap| / \xi \sim \delta \cdot |\nabla\varepsilon\_gap| / v_c$. (curvature extension, present paper) (9.3)

Consequently, defining the **dimensionless coherence-curvature scaling number**

$\mathcal{N}\_curv := \|\mathscr{R}\| \cdot \xi / |\nabla\varepsilon\_gap| = \|\mathscr{R}\| \cdot v_c / (\delta \cdot |\nabla\varepsilon\_gap|)$, (9.4) *we have $\mathcal{N}$_curv ~ 1 throughout the substrate, with deviations only in the strong-coupling regime, at band edges, or for defects violating the small-Δ regime.*

All four scales are functionals of the single underlying field ε_gap(x), with explicit proportionality constants determined by γ, ρ(A_X), and substrate-geometric factors.

Proof. The basic identity $\xi \cdot \delta \sim v_c$ is Stage IX Theorem 13.1. The curvature scaling $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}| / \xi$ is established by (9.2) using the §3.3 construction of perturbed generators in the heuristic register. The dimensionless $\mathcal{N}_{curv} \sim 1$ follows from substituting the basic identity into the curvature scaling. The functional dependence on ε_gap(x) is inherited from Stage IX Theorem 13.1's reduction of $v_c^{local}(x)$ to ε_gap(x) plus the present-paper identification of $|\Delta_i|$ with $|\nabla_i \varepsilon_{gap}|$.

9.3 Structural Climax

Proposition 9.2 extends the Stage IX Coherence Transport Scaling Identity into a unified scaling structure linking localization, curvature, transport velocity, and gap gradients. The Stage VIII Defect-Coherence Principle, which Stage IX upgraded from "four functionals of one field" to "four functionals of one field under one scaling identity", now upgrades again:

One field (ε_gap), four functionals (K_∞, R, entropy retention, trapped-mode persistence), one transport geometry (substrate parallel transport, holonomy, curvature tensor), one extended scaling identity ($\xi\cdot\delta \sim v_c$, $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}|/\xi$).

This is the structural climax of the Stage V → X arc. The four Stage VIII functionals are specific extractions from one underlying transport / curvature structure governed by one extended scaling identity, with each Stage VIII functional being a specific mathematical order or contraction of the deeper tensorial object:

- **Continuum roughening** $K_\infty \propto 1/\varepsilon_{gap}$ is τ_relax-type (inverse scale).
- **Candidate curvature** $R = \nabla^2\varepsilon_{gap}$ is closely related to $\mathscr{R}_{ij}$, with the precise scalar projection an open identification problem (Discussion 5.4).
- **Entropy retention** $(1 - \varepsilon_{gap})^{2n}$ is the $L^2(\pi)$ contraction-rate factor.
- **Trapped-mode persistence** $\xi \propto \varepsilon_{gap} / \delta$ is the localization scale.
- **Transport curvature** $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}| / \xi$ is the tensorial-geometric magnitude.

All five are scalar projections or magnitudes of the underlying transport-curvature structure, and all are linked by the extended scaling identity (9.3).

9.4 Epistemic Register

Proposition 9.2 inherits the Stage IX §9.1 heuristic register through the $v_c^{local}(x) \propto \varepsilon_{gap}(x)$ identification and through the §3 / §5 relation between Δ_i and $\nabla_i \varepsilon_{gap}$. The basic identity $\xi \cdot \delta \sim v_c$ is rigorous at the bulk level (Stage IX Theorem 13.1); the curvature extension $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}|/\xi$ is heuristic in its leading-order proportionality constant. A future paper carrying out the explicit Born-series expansion of the perturbed transport generators would either confirm the

(1/ξ) scaling or correct it to a related form (e.g., (1/ξ)^p for some p), preserving the qualitative structural unification while sharpening the quantitative law.

The structural-unification content — *one underlying scaling structure governs all five transport scales* — is robust under the operator-theoretic framework and is the central content of the result.

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10. Relation to Earlier VERSF Stages

10.1 Relation to Stage VIII

Stage VIII established the local spectral-gap field ε_gap(x) and identified its candidate scalar curvature $R(x) = \nabla^2 \varepsilon_{gap}(x)$ along with three other ε_gap-functionals: continuum roughening, entropy retention, trapped-mode persistence. The Stage VIII structural unification was *scalar*: all four functionals depend on the same underlying scalar field ε_gap(x).

The present paper supplies the tensorial completion: the Stage VIII candidate scalar curvature R(x) and the substrate's transport-curvature tensor $\mathscr{R}_{ij}$ are both ε_gap-derived geometric objects from the same defect structure with shared exponential localization (Discussion 5.4), the trapped-mode persistence functional acquires the structural reading as *localized concentration of transport curvature* with matched-rate decay (Proposition 8.1), and the Stage VIII Defect-Coherence Principle upgrades to a tensorial-transport principle with the extended scaling identity of Proposition 9.2. The precise scalar projection $\mathscr{R} \to R$ that recovers Stage VIII's $R(x) = \nabla^2 \varepsilon_{gap}(x)$ is, however, an open identification problem (see Discussion 5.4 for the analysis of why the naive Kretschmann-style contraction does not suffice).

Stage VIII is therefore retrospectively interpretable as *carrying* the scalar projection theory of the present Stage X tensorial transport geometry, in the same way that Stage IX Theorem 10.1 identified Stage VIII as the *fibrewise projection theory* of Stage IX — with the caveat that the precise projection map in the Stage VIII ↔ Stage X direction remains to be derived. The structural-correspondence chain is:

Stage X (tensorial transport curvature $\mathscr{R}_{ij}$) ↓ scalar projection $\mathscr{R} \to R$: open identification problem (Discussion 5.4) Stage VIII candidate curvature $R(x) = \nabla^2 \varepsilon_{gap}(x)$ ↑ derived from Stage IX local spatial fibre (Theorem 10.1) ↑ derived from Stage IX coupled transport operator **T**

with each stage containing the previous as a projection / contraction / restriction. The architectural unity of the geometry programme is now fully visible across all four stages V–X.

10.2 Relation to Stage IX

Stage IX established the coupled transport operator **T**, the bulk transport-band structure, finite coherence propagation, Birman–Schwinger trapped-mode criteria, Combes–Thomas eigenvector localization, and the Coherence Transport Scaling Identity $\xi \cdot \delta \sim v_c$. All of these were spectral

/ propagation / localization results — the operator-theoretic infrastructure for studying localized perturbations of the substrate. None of them were *tensorial*: they were all built from scalar properties of the transport operator (spectra, propagation speeds, decay rates, gap fields).

The present paper supplies the tensorial layer on top of the Stage IX scalar infrastructure: parallel transport operators $\mathcal{P}_\pi$ are built from products of local transport maps which are themselves built from blocks of **T**; the transport-curvature tensor $\mathscr{R}_{ij} = [\tilde{\nabla}_i, \tilde{\nabla}_j]$ is built from commutators of directional decompositions of $\gamma \cdot A_X$; holonomy $\mathscr{H}(\gamma)$ is the loop integral of the parallel transport; trapped-mode curvature localization (Proposition 8.1) inherits the Combes–Thomas decay rate from Stage IX Theorem 8.1; and the extended scaling identity $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}|/\xi$ extends the Stage IX basic identity.

Stage IX therefore supplied the operator-theoretic infrastructure required for the present tensorial transport framework; the present paper supplies the tensorial geometric content that the Stage IX infrastructure naturally supports but did not itself develop.

10.3 Programme Trajectory

The geometry programme trajectory across Stages V–X is now:

- **Stage V** — Coherent continuum emergence. The canonical wheel produces a Lipschitz continuum at large scales, with regularity controlled by the spectral gap $\varepsilon_{gap} = ½$.
- **Stage VII** — Universality and robustness. The canonical wheel sits inside an open universality class $\mathcal{C}_{K=7}$, robust to admissibility-preserving perturbations.
- **Stage VIII** — Localized coherence defects. Defects produce a spatially-varying local gap field $\varepsilon_{gap}(x)$ with four functionals (continuum roughening, candidate scalar curvature $R(x) = \nabla^2\varepsilon_{gap}(x)$, entropy retention, trapped-mode persistence) sharing a common ε_{gap}-dependence.
- **Stage IX** — Global coherence transport. The substrate is a coupled dynamical system with transport bands, finite propagation speed $v_c = \gamma \cdot \rho(A_X)$, Birman–Schwinger trapped modes, Combes–Thomas localization length ξ, and the scaling identity $\xi \cdot \delta \sim v_c$. The Stage VIII analysis is identified as the fibrewise projection theory.
- **Stage X (present)** — Tensorial transport geometry. The substrate carries parallel transport $\mathcal{P}_\pi$, coherence holonomy $\mathscr{H}(\gamma)$, transport-curvature tensor $\mathscr{R}_{ij}$, geodesic deviation $\mathcal{D}^2\xi/ds^2 \sim \mathscr{R}(\xi)$, Wilson-type loop functionals $W(\gamma)$, and the extended scaling identity $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}|/\xi$. The Stage VIII scalar candidate curvature is structurally related to $\mathscr{R}_{ij}$ via shared defect-localization and ε_{gap}-dependence, with the precise scalar projection $\mathscr{R} \to R$ an open identification problem (Discussion 5.4); trapped modes are identified as localized curvature concentrations with matched-rate decay (Proposition 8.1).

The shift from Stage IX to Stage X is the transition from *scalar coherence geometry* (Stages V–IX, where all geometric observables are scalars) to *genuinely tensorial transport structure* (Stage X, where parallel transport, holonomy, and curvature tensors are the relevant objects, with scalars appearing as contractions). This is the central structural advance of the present paper.

11. Limitations and Open Problems

The present paper closes the Stage VIII §12.1 / Stage IX §14.6 tensorial-curvature open problem at the substrate-level operator-theoretic / combinatorial level, while leaving the broader set of Stage VIII–IX open problems and introducing some new ones specific to the present framework.

11.1 No Einstein Equations. The present framework develops transport curvature but does not derive Einstein field equations. No gravitational dynamics is proposed, no stress-energy tensor is identified on the substrate side, and the gravitational-coupling question is not addressed. Whether the substrate transport-curvature tensor $\mathscr{R}_{ij}$ satisfies any analogue of an Einstein equation — perhaps in a large-scale continuum limit, perhaps with a substrate stress-energy built from coherence flux through ε_{gap}-gradients — is the most natural next-stage target and remains the principal open question.

11.2 No Lorentzian Geometry. The transport cones of Stage IX (strict cone $v_{strict} = 1$, effective cone $v_c = \gamma \cdot \rho(A_X)$) are propagation bounds on the substrate, *not* light cones in any Minkowski sense. The temporal direction in the present framework is the refinement-step index n, which is a *discrete* substrate parameter rather than a continuum time coordinate. Whether the substrate possesses an emergent Lorentzian structure — perhaps via a continuum limit in which substrate refinement steps coalesce into a continuum time and the strict cone becomes a Minkowski light cone — remains an open question. We have no commitment on this at present.

11.3 No Gauge Theory. The Wilson-loop analogy of §7 is structural: the operator algebra of substrate parallel transport and loop trace functionals is mathematically isomorphic to lattice gauge-theory Wilson loops. But no gauge group is imposed on the substrate, no gauge-covariant derivative structure is derived, and no gauge field is identified. Whether the substrate carries an emergent gauge structure — with the closure-catalogue $\mathbb{R}^{\mathcal{K}}$ playing the role of an internal gauge space and the perturbed transport generators carrying gauge-covariant derivatives — is an open question. The Stage VIII Standard-Model-gauge-derivation work suggests that gauge structure does emerge at large scales; whether and how this connects to the present substrate-level transport structure remains to be established.

11.4 No Quantization. The present framework is classical / operator-theoretic, as Stage VIII and Stage IX. The trapped-mode spectrum is discrete and the transport curvature is operator-valued, both structurally suggestive of quantum-mechanical apparatus, but the relevant quantization (canonical, geometric, holonomy-loop, ...) is not derived. Whether substrate transport admits a natural quantization — perhaps via a path-integral over substrate loops weighted by $\mathrm{Tr}(\mathscr{H}(\gamma))$ — is an open question that connects to the Stage IX §14.8 quantization problem.

11.5 Strong-Coupling Regime. The present analysis assumes the Stage IX weak-coupling regime (W): $\gamma \cdot \rho(A_X) < \frac{1}{2}$. Outside this regime — overlapping transport bands, vanishing trap gaps, non-isolated Perron eigenvalues — the operator-theoretic structure becomes substantially more delicate. Whether the tensorial transport framework survives into the strong-coupling regime (with appropriately modified scaling identities, possibly non-perturbative transport-curvature contributions, and possibly topological transport transitions akin to integer-quantum-Hall-style invariants) is an open question. The Stage IX §14.1 caveat applies in full to the present paper.

11.6 Non-Commuting Bulk Directional Decompositions. The standing hypothesis (S_comm) of §3.1 — $[A_i, A_j] = 0$ in the bulk — is satisfied for all canonical regular substrates ($\mathbb{Z}^d$, hexagonal, finite torus) of Stage IX §3.4. Generalisations to substrates with *non-commuting bulk directional decompositions* (which can arise for irregular substrates or non-translation-invariant coupling) would produce *bulk* transport curvature even without defects — a structurally distinct scenario with potential physical interest (e.g., as substrate-level analogues of curved-background lattices). We have not pursued this generalisation; it remains open.

11.7 The C ≠ 0 Closure-Mixing Coupling. Throughout Part I we work in the minimal-coupling regime $C = 0$. The qualitative structure of the present framework (parallel transport, holonomy, transport-curvature tensors, geodesic deviation, curvature localization at trapped modes) extends to $C \neq 0$ in principle, but the explicit form of the perturbed transport generators acquires additional Duhamel-style corrections (cf. Stage IX §4 "Note on $C \neq 0$"), and the directional decomposition (3.1) requires careful treatment of the C-coupling contribution to each direction. A systematic treatment is deferred. The principal qualitative content survives: defects break directional symmetry, generators acquire non-commuting corrections, holonomy is non-trivial, and curvature localizes at trapped modes.

11.8 Tensorial Completion / Metric Emergence. The present transport curvature tensor $\mathscr{R}_{ij}$ is substrate-level and operator-theoretic. Whether it can be promoted into a full Riemannian metric geometry (with a metric tensor g_{ij}, connection coefficients Γ^k_{ij} from a Levi-Civita compatibility condition, and a Riemann tensor R^a_{bcd} from the metric in the usual way) is open. The natural candidate metric would be the *Hessian of the coherence-transport distance* $g_{ij}(x) := \partial^2 g_{coh}(x, y)/\partial y^i \partial y^j|_{y=x}$ (with g_{coh} the Stage IX Definition 5.3 effective coherence metric), but whether this Hessian satisfies the algebraic / differential properties of a Riemannian metric, and whether the associated Levi-Civita connection coincides with the substrate-level perturbed transport generators $\tilde{\nabla}_i$, remains to be established. This is the most concrete next-stage target.

11.9 Born-Series-Rigorous Heuristic Identifications and the Scalar Projection Problem. The heuristic identification $v_c^{local}(x) \propto \varepsilon_{gap}(x)$ of Stage IX §9.1 propagates throughout the present paper, affecting Theorem 6.3 and Proposition 9.2 in their precise quantitative forms. Additionally, the open scalar projection problem (Discussion 5.4) — the question of how Stage VIII's $R(x) = \nabla^2\varepsilon_{gap}(x)$ recovers from a scalar projection of $\mathscr{R}_{ij}$ — is the most concrete next-stage technical target. Both questions are most naturally addressed by a Born-series expansion of the perturbed transport generators around the canonical vacuum, with explicit tracking of (i) how local-fibre defects propagate through the spatial coupling to produce the heuristic $v_c^{local}(x)$ relation, and (ii) how the second variation of the path-ordered transport product generates $\nabla^2\varepsilon_{gap}$ as a scalar projection of the curvature tensor. A successful Born-series derivation would either confirm the heuristic forms with explicit prefactors, identify the correct scalar projection $\mathscr{R} \to R$, or correct the framework in concrete ways. The structural-correspondence content of the results survives either way, but the quantitative form and the precise projection map depend on the derivation.

11.10 Connection to Cosmological / Matter Content. The Stage VIII §12.5 matter-interpretation question and the Stage IX §14.9 cosmological-structure question are not addressed by the present paper. Trapped modes appear here as localized concentrations of transport curvature with no specific particle-physics or cosmological reading. Whether the substrate transport-curvature framework supports a substrate-level energy / matter content (with the transport curvature playing a stress-energy-like role and trapped modes playing a localized-matter role) is the natural next-stage target — and depends on §§11.1 (Einstein equations) and §11.4 (quantization) being addressed first.

11.11 Refinement Stability of Holonomy. The present paper establishes non-trivial loop transport at a *fixed* substrate scale: given the Stage IX coupled transport operator **T** on a fixed substrate graph, the parallel transport $\mathcal{P}_\pi$, the holonomy $\mathscr{H}(\gamma)$, and the transport-curvature tensor $\mathscr{R}_{ij}$ are well-defined objects with the properties developed in §§2–8. What is *not* established is whether these objects are stable under the substrate-refinement / coarse-graining operations that distinguish a genuine emergent geometric observable from a substrate-scale artefact.

The precise question is: for a sequence of substrate refinements $X_0 \subset X_1 \subset X_2 \subset \ldots$ converging to a continuum in the Stage V Lipschitz sense, with each level n carrying its own transport operator $\mathbf{T}_n$ and induced curvature tensor $\mathscr{R}_{ij}^{(n)}$, does the loop-trace functional $W_n(\gamma)$ — evaluated on substrate loops γ that converge to a fixed continuum loop in an appropriate sense — converge to a well-defined limit? Equivalently, does the holonomy operator $\mathscr{H}(\gamma)$ extracted at each refinement level converge (in operator norm, weak operator topology, or some appropriate sense) to a refinement-stable limit?

A positive answer would substantially strengthen the interpretation of $\mathscr{R}_{ij}$ as a *genuine emergent geometric object* of the VERSF programme — refinement-stable, scale-invariant in the appropriate sense, and a candidate for connection to continuum geometric structures (a metric tensor §11.8, an Einstein-equation analogue §11.1). A negative answer would localize the curvature concept to the substrate scale and force a structurally different interpretation — perhaps as a *renormalisable* observable in the sense of statistical mechanics, with explicit substrate-scale dependence.

The Stage V continuum-emergence machinery (Lipschitz convergence of substrate averages, spectral robustness of the canonical wheel under refinement) and the Stage VII open-universality-class result are the natural starting points for addressing this question. The technical target is to lift the Stage V Lipschitz convergence from scalar substrate observables (e.g., the gap field $\varepsilon_{gap}(x)$) to operator-valued path-ordered substrate observables (the transport functionals $W(\gamma)$, $\mathscr{H}(\gamma)$, $\mathscr{R}_{ij}$). This is open.

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12. Conclusion

Programme map. The VERSF geometry programme has now developed in six structural stages: Stage V (coherent continuum emergence), Stage VII (open universality class), Stage VIII (localized coherence defects with four scalar ε_{gap}-functionals), Stage IX (global coupled transport operator with bands, finite propagation, Birman–Schwinger trapped modes, Combes–Thomas localization, and the basic scaling identity $\xi \cdot \delta \sim v_c$), and now Stage X (tensorial transport geometry: parallel transport, coherence holonomy, transport-curvature tensors, geodesic deviation, Wilson-type loops, curvature concentration at trapped modes, and the extended scaling identity $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}|/\xi$). The shift from Stage IX to Stage X is the transition from *scalar coherence geometry* to *genuinely tensorial transport structure* — the central structural advance of the present paper.

Principal results. The paper establishes:

- **Substrate parallel transport.** Composite local transport maps $\mathcal{T}_{x\to y} = \gamma\cdot\hat{T}_x$ (Definition 2.1, including one fibre step and one spatial step) assemble into directional parallel transport operators $\mathcal{P}_\pi$ along directed substrate paths π. In the canonical vacuum, $\mathcal{P}_\pi = \gamma^n\cdot\hat{T}^n$ is path-independent (Proposition 2.5) — the substrate is *flat*. The earlier-paper spatial-coupling-only form $P_y\cdot\mathbf{T}\cdot P_x$ failed to see Stage VIII diagonal-in-X defects through the transport machinery and is superseded by the composite form here.
- **Directional transport generators and curvature tensor.** A directional decomposition $A_X = \Sigma_i A_i$ induces transfer-operator generators $\nabla_i := \gamma \cdot \Sigma_x (\hat{T} \cdot |x + e_i\rangle\langle x|)$ (Definition 3.1), combining one fibre step with one i-direction position shift. For commuting bulk decompositions ((S_comm) of §3.1), vacuum generators commute: $[\nabla_i, \nabla_j] = 0$ *(Proposition 3.3). Defects breaking i ↔ j directional symmetry, or with non-zero fibre commutator* $[\hat{T}, \Delta T_{x_0,r}(x)]$*, produce perturbed generators* $\tilde{\nabla}_i$ *with* $[\tilde{\nabla}_i, \tilde{\nabla}_j] \neq 0$ *(Proposition 3.5), defining the transport-curvature tensor* $\mathscr{R}_{ij} := [\tilde{\nabla}_i, \tilde{\nabla}_j]$ (Definition 5.1) which vanishes in the vacuum and is non-zero of order $\alpha\gamma^2$ on the boundary shell of the defect support.
- **Coherence holonomy.** Loop transport $\mathscr{H}(\gamma) := \mathcal{P}_\gamma$ produces non-trivial normalised holonomy $\hat{H}(\gamma) := \gamma^{-n}\cdot\hat{T}^{-n}\cdot\mathscr{H}(\gamma) - I_{\mathcal{K}}$ (on the $\hat{T}$-invertible subspace; the loop-trace functional $W(\gamma)$ of §7 carries the same information without invertibility caveats) for loops surrounding defects (Theorem 4.2), with holonomy magnitude bounded by an integrated commutator estimate and decaying exponentially with the loop's distance from the defect at the Combes–Thomas rate η_{trap} of Stage IX for defects producing trapped modes (Proposition 8.1), or the subcritical rate η_{sub} of Remark 8.2 otherwise.
- **Stage VIII scalar curvature — open identification (Discussion 5.4).** The Stage VIII candidate scalar curvature $R(x) = \nabla^2\varepsilon_{gap}(x)$ and the present transport-curvature tensor $\mathscr{R}_{ij}(x)$ are both ε_{gap}-derived geometric objects from the same defect structure, with shared exponential localization near defects. The precise scalar projection $\mathscr{R} \to R$ recovering Stage VIII's $R(x)$ is, however, *not* the naive Kretschmann-style contraction (which has wrong α-order and sign behaviour) and has not yet been established; the diagonal-sum vanishes by antisymmetry, and divergence-type contractions remain unexplored. The structural correspondence between $R(x)$ and $\mathscr{R}_{ij}(x)$ is established; the precise operator-theoretic projection map is the most concrete next-stage technical target (§11.9).

- **Geodesic deviation equation.** Nearby coherence transport geodesics satisfy a substrate analogue of the Riemannian geodesic-deviation equation, $\mathcal{D}^2\xi/ds^2 \sim \mathscr{R}(\xi)\cdot\pi$ (Theorem 6.3), with focusing for positive transport curvature and defocusing for negative — the substrate-level operator-theoretic content underneath the §5 / §9 picture of defects as transport-attractors.
- **Wilson-type loop functionals.** The trace functional $W(\gamma) := \mathrm{Tr}_{\mathcal{K}}(\mathscr{H}(\gamma))$ measures integrated transport curvature, with vacuum value $\gamma^n\cdot\mathrm{Tr}(\hat{T}^n)$ (where $\mathrm{Tr}(\hat{T}) = 47/28 \approx 1.679$ for the canonical wheel; cf. §7.2) and deviation $\hat{W}(\gamma) \approx \gamma^n\cdot\oint_\gamma \mathrm{Tr}(\mathscr{R}_{ij}\cdot\hat{T}^n)\cdot dA^{ij}$ to leading order — the substrate Stokes-type identity. The structural correspondence with lattice-gauge Wilson loops is exact at the operator-algebraic level; the absence of gauge content is explicit.
- **Curvature concentration at trapped modes.** Stage IX trapped modes appear in coexistence with localized transport-curvature concentrations satisfying $\|\mathscr{R}_{ij}(x)\| \leq C\cdot e^{-d_X(x, x_0)/\xi}$ (Proposition 8.1), inheriting the Combes–Thomas decay rate of Stage IX Theorem 8.1. Under the present construction, trapped substrate excitations are localized eigenstates *and* are accompanied by localized curvature sources at the same characteristic scale — the substrate-level operator-theoretic correlate of matter / energy concentrations producing localized curvature in classical gravity. Whether this coexistence reflects a deeper equivalence rather than a correlation under the chosen construction is open (§8.2 caveat).
- **Extended Coherence Transport Scaling Identity.** The Stage IX basic identity $\xi \cdot \delta \sim v_c$ extends to include curvature: $\|\mathscr{R}\| \sim |\nabla\varepsilon_{gap}| / \xi \sim \delta \cdot |\nabla\varepsilon_{gap}| / v_c$ (Proposition 9.2), with the dimensionless scaling number $\mathcal{N}_{curv} \sim 1$ throughout the substrate. The Stage VIII "one field, four functionals" upgrade through Stage IX "one scaling identity" now reaches Stage X "one transport geometry, one extended scaling structure".
- **Symmetry-protected vanishing.** Defects respecting the substrate's directional symmetry (e.g., the Stage VIII §9.6 hub-coupling defect, globally spectrally invisible by Stage IX §11.3) produce vanishing transport-curvature: they are *both* spectrally invisible and transport-curvature invisible, a structurally satisfying check that the local Stage VIII symmetry protection lifts cleanly to the present tensorial level.

Scope clarification. The present framework establishes *tensorial transport structure*, not gravitational dynamics. Parallel transport, holonomy, and curvature-like commutators emerge directly from substrate coherence propagation through the operator-theoretic machinery of Stages IX–X — but no metric tensor, no Lorentzian structure, no field equations of Einstein type, and no Einsteinian dynamics have been derived. The transport-curvature tensor $\mathscr{R}_{ij}$ is a substrate-direction-indexed operator-valued tensor; it is *not* the Riemann tensor of any metric, and the section identifying the "three levels of curvature structure" (§5.2) is explicit that the present paper operates entirely at level 2 (tensorial transport) and not at level 3 (metric curvature). The correct reading of the present paper is therefore: *emergent transport geometry first; possible gravitational interpretation later*. The structural correspondences with general relativity that appear throughout — geodesic-deviation form, focusing of nearby trajectories, curvature concentration at trapped sources — are operator-algebraic structural analogies, and their physical interpretation (if any) as gravitational content remains a programme target rather than an established result.

What this paper establishes and what remains. The paper closes the Stage VIII §12.1 / Stage IX §14.6 tensorial-curvature open problem at the substrate-level operator-theoretic / combinatorial level. The framework now possesses substrate parallel transport, holonomy, curvature tensors, geodesic deviation, loop functionals, and an extended scaling identity — the full tensorial geometric backbone. What remains is the lift from substrate-level tensorial geometry to *continuum-level* geometric structure: a Riemannian metric (§11.8), Lorentzian signature (§11.2), gauge structure (§11.3), Einstein-type field equations (§11.1), quantization (§11.4), refinement stability of the holonomy and curvature observables (§11.11), and matter / cosmological content (§11.10). Each of these is a substantial subsequent step; the present paper provides the operator-theoretic foundation on which any such step would have to be built.

The honest summary: this paper does not derive gravity. It derives the substrate-level operator-theoretic bridge toward gravity-like geometry. The framework now possesses finite propagation structure (Stage IX), localization theory (Stage IX), transport geodesics (Stage IX), parallel transport (present), holonomy (present), curvature tensors (present), geodesic deviation (present), Wilson-type loops (present), and scaling identities (Stage IX extended in the present) — all emerging directly from substrate coherence transport, with no manifold, metric, connection, or gauge field assumed. The remaining programme targets are the tensorial completion in §§11.1, 11.2, 11.3, 11.4, 11.8, 11.10; the present paper supplies the operator-theoretic geometric backbone upon which those targets would naturally build.

Final position. Through Stages V–X, the VERSF geometry programme has progressed from *coherence* (V) to *robust coherence* (VII) to *defected coherence* (VIII) to *globally transported coherence* (IX) to *tensorially structured transported coherence* (X). The Stage VIII Defect-Coherence Principle's "one field, four functionals" has, through Stage IX's "one scaling identity" and the present paper's "one tensorial transport geometry, one extended scaling identity", reached its mature structural form. The substrate is not merely a coherent dynamical system but a coherent dynamical system with intrinsic transport-tensorial structure — a setting in which classical-geometric apparatus (parallel transport, holonomy, curvature, geodesic deviation) emerges from substrate-level operator algebra without being imposed by hand. The transition from this tensorial-transport setting to a full continuum-geometric / gravitational / gauge / quantum setting remains the principal target of the programme's subsequent stages.