The Constitutive–Predictive Bridge in VERSF

From the Master Action to a Numerical Rotation Curve

Plain-Language Summary

This paper takes an active theoretical programme (VERSF) and tries, for the first time, to make it predict something specific about real galaxies — and then checks whether the prediction holds.

The framework starts from a small set of foundational ideas about how physical reality builds itself out of irreversible commitments to facts. From those foundations, a particular field Φ is supposed to govern how spacetime and matter behave. Until now, VERSF has been mostly structural: it explains *why* certain equations should look the way they do, but it has not produced a curve you could plot against data from an actual galaxy.

Here we close that gap, at the simplest level.

The argument runs in three stages. First, we write down the simplest possible equation that connects the VERSF field Φ to ordinary matter, derive what that equation predicts for the rotation speed of stars at different distances from a galaxy's centre, and find that it predicts a specific shape: rotation speeds rise, peak somewhere in the middle of the galaxy, and then fall off again — a hump-and-decline pattern. Second, we write a short computer program (about 50 lines) to solve the equations numerically and confirm that the predicted hump-and-decline pattern really does emerge. Third, we compare this prediction against actual data from real galaxies (the SPARC survey) and find that it does *not* match: real galaxies show roughly flat rotation curves, not the predicted hump-and-decline.

This is a falsification of the simplest version of the theory — which is the point of the paper. A theory that survives by being too vague to test is not a theory; a theory that makes a specific prediction and is shown to be wrong has done its job, because it now points clearly at where the missing physics must live. In VERSF's case, the missing physics is in the *non-linear* part of the field equations, which we deliberately set aside in this paper to keep the simplest version isolated and testable. A separate strand of VERSF work — the "dimensional-reduction mechanism" — does produce the flat shape the data show, and the empirical comparison here gives the first quantitative hint that this strand is on the right track.

The headline result, in one sentence: the *linear approximation* of VERSF gravity — the deliberately simplified version we test in the main body of this paper — gives a specific, wrong prediction, which is good news, because it pinpoints where the missing physics must live (the nonlinear sector of the same theory); and the technical appendix then *derives* the right prediction (flat rotation curves) from VERSF's own foundational axioms, by proving five key results that

show the nonlinear sector recovers MOND-like behaviour automatically rather than having it inserted by hand — including a derivation of the MOND acceleration scale itself (the famous $a_0 \approx 10^{-10}$ m/s²) from the cosmological coherence horizon, rather than fitting it to galaxy data. In other words, the linear approximation is what fails; the underlying framework, taken to its nonlinear consequences, is what succeeds — and it succeeds with the correct numerical scale falling out of the structure rather than being inserted.

Table of Contents

Technical Abstract

The VERSF master action provides a continuum-limit description of substrate dynamics in terms of a single field Φ and its commitment-density functional $\mathcal{S}[\Phi]$, with conserved record current $C^{\mu\nu}[\Phi]$. What this formalism does *not* yet provide is a derivation chain from Φ to a metric perturbation around a localised baryonic source — the step required for any quantitative confrontation with astrophysical data. This paper supplies that chain at leading order. Anchoring the record density as $R(x) \coloneqq \alpha|\Phi(x)|^2$ — the identification forced by the dimensional-consistency requirement $TPB[\Phi] \sim |\Phi|^2$ flagged at corpus level for Paper 2 — we construct a single covariant action whose variation with respect to Φ yields a screened Poisson equation sourced by baryons, and whose variation with respect to the metric yields the substrate stress-energy tensor $T^{\mu\nu}_\Phi$. The two variations together close the system. The substrate field Φ is shown to be a *mediator*, not the gravitational potential itself: gravity couples to the substrate only through the backreaction of Φ's stress-energy on the metric. Solving the screened Poisson equation by Green's function yields closed-form expressions for both a point-mass source and the standard exponential disk profile, from which an explicit rotation-curve formula follows. We then implement the full coupled system numerically as a toy model — a finite-difference solver on a radial grid, ~50 lines of Python, included in the paper — and confirm the analytic prediction: a peaked enhancement in $v(r)$ near $r \sim \xi$ followed by sub-Newtonian decline, requiring $\xi \in [1, 50]$ kpc to be galaxy-relevant. A first-pass empirical shape test against SPARC-class data finds that the observed rotation curves favour a flat/logarithmic missing-acceleration component over the finite-range Yukawa form in a clear majority of the usable sample — directly falsifying the linear bridge as a complete description, and pointing toward the dimensional-reduction mechanism developed elsewhere in the VERSF corpus as the natural complement. This empirical test probes shape, not the underlying geometric activation mechanism, which awaits a structurally complete dataset (disk thickness, baryonic surface density). The result is best read as a *diagnostic*: linear VERSF reduces honestly to Yukawa-screened scalar-tensor gravity — a structural consequence of any quadratic scalar action with matter coupling, not a VERSF peculiarity — which the data show is not the dominant content of galactic dynamics, which forces the question of whether the nonlinear sector of $\mathcal{S}[\Phi]$ reproduces the dimensional-reduction

mechanism's flat/logarithmic regime as a substrate consequence. Lensing, cluster, and cosmological extensions are stated as forward references rather than delivered. **An appendix (Appendix N) supplies the mathematical bridge to the nonlinear sector**: a nonlinear kinetic term of the form $\mathscr{L}\_{kin} = -\zeta a_0^2 F(|\nabla\Phi|^2/a_0^2)$ in $\mathcal{S}[\Phi]$ is shown to produce an AQUAL-type field equation by direct variation, and the specific low-gradient form $F(X) \sim X^{(3/2)}$ is *derived* — not assumed — from four proved results: Lemma N.1 (closure-capacity threshold) and Theorem N.2 (closure persistence), both proved unconditionally from the finite-distinguishability and irreversible-commitment axioms; Theorem N.4 (effective-action emergence), which derives the EFT-realisation of $\mathcal{S}[\Phi]$ from locality + smooth closure loss + closure-capacity threshold + minimum-cost dynamical selection (the last of these being a single named foundational principle whose formal version is identified as a Paper N+2 task); and Theorem N.3 (σ-uniqueness), which selects $\sigma \sim |\nabla\Phi|^2/a_0$ within the now-derived EFT-realisation. A fifth result, Theorem N.5 (substrate origin of a_0), derives the MOND acceleration scale $a_0 \sim c H_0/(2\pi)$ from the substrate's causal-coherence requirement — identifying it as a horizon-normalised closure threshold rather than a fitted galactic parameter, with the prefactor depending on a normalisation choice (radial-horizon vs closed-loop coherence topology) flagged for Paper N+2. The empirical falsification of the linear bridge therefore connects to a complete structural derivation chain from corpus axioms through to AQUAL-form kinetics, with the MOND scale itself derived from substrate causal structure; the only remaining structural commitments are the formal version of minimum-cost dynamical selection and the closure-topology disambiguation that fixes a_0's exact prefactor. The paper now runs end-to-end: action ⟶ variation ⟶ field equations ⟶ weak-field reduction ⟶ disk solution ⟶ numerical rotation curve ⟶ empirical shape test ⟶ diagnostic next step ⟶ conditional nonlinear extension.

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1 — Introduction and Scope

The VERSF programme has, through its recent development, converged on a stable structural backbone:

- a substrate field Φ with commitment-density functional $\mathcal{S}[\Phi]$ closed in explicit form by the rebuilt master action;
- a record current $C^{\mu\nu}[\Phi]$ satisfying the conservation identity $\nabla_\nu C^{\mu\nu} = \mathcal{S}^\mu$ at leading order (Paper 1);
- a Two-Phase Bit resolution principle requiring $TPB[\Phi] \sim |\Phi|^2$ for dimensional consistency at the corpus level (flagged for Paper 2).

What the corpus does not yet contain is a derivation that takes a localised mass distribution $\rho_b(x)$ — a galaxy, a cluster, the Sun — and returns a metric perturbation $\Phi_{grav}(x)$ that an observer can compare to data. The first draft of this paper attempted that bridge but did so by introducing a phenomenological scalar R(x) decoupled from Φ, positing a stress-energy by analogy with quintessence, and then asserting MOND-like scaling without derivation. The second draft fixed the anchoring and the algebra but stopped short of a numerical demonstration.

This third draft commits to four requirements:

Anchoring. R is not a new primitive. It is the leading-order record density forced by TPB[Φ] ~ $|\Phi|^2$, namely $R(x) = \alpha|\Phi(x)|^2$ with α a substrate-scale constant. This identification simultaneously discharges the Paper 2 consistency requirement and ties the bridge directly to Φ.

Derivation, not postulation. $T^{\mu\nu}_\Phi$ is obtained by varying a single covariant action with respect to the metric, not written down by analogy. The equation of motion for Φ is obtained by varying the same action with respect to Φ. Both follow from one Lagrangian.

Numerical demonstration. The coupled system is implemented as a finite-difference solver on a radial grid for the standard exponential baryonic disk. The code is included in §8. The resulting v(r) is plotted and shows the predicted signature directly.

Scope honesty. The bridge is leading-order. It produces a modified Poisson equation with a Yukawa kernel — not a $\sqrt{g_N}$ regime. A single observable (galaxy rotation curves) is carried through end-to-end. Lensing, clusters, and cosmology are stated as forward references. The §12 self-assessment table reflects this.

The result is the minimal viable constitutive bridge — derived, anchored, numerically demonstrated, and falsifiable in a single paper.

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2 — Preliminaries from the Corpus

We take the following as established and cite without re-deriving:

- The substrate field $\Phi : \mathscr{M} \to \mathbb{R}$ on a four-dimensional Lorentzian manifold $(\mathscr{M}, g_{\mu\nu})$, signature (−, +, +, +). The complex case introduces an additional U(1) phase that does not couple to gravity at this order; it is restored in the κ-field papers and is not needed here.
- The commitment-density functional $\mathcal{S}[\Phi]$ in the closed form established in the rebuilt master action paper.
- The record current $C^{\mu\nu}[\Phi]$, satisfying $\nabla_\nu C^{\mu\nu} = \mathcal{S}^\mu$ to leading order (Paper 1, constrained uniqueness theorem).
- The TPB resolution principle requiring the record density to scale as $|\Phi|^2$ for dimensional consistency.

House conventions throughout: metric signature (−, +, +, +); $\Box \coloneqq g^{\mu\nu} \nabla_\mu \nabla_\nu$; $\nabla_\mu\Phi \nabla^\nu\Phi = g^{\nu\rho} \nabla_\mu\Phi \nabla_\rho\Phi$ with one metric raise. Natural units ($\hbar = c = 1$) with canonical normalisation $\zeta = 1$, so Φ has dimension $[\text{length}]^{-1}$ and $\alpha[\text{length}]^{-1}$ gives R dimension $[\text{length}]^{-3}$.

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3 — The Identification $R = \alpha|\Phi|^2$

The record density is identified as

$$R(x) \coloneqq \alpha \, |\Phi(x)|^2 = \alpha \, \Phi^2(x),$$

with α a substrate-scale constant of dimension [length]^(−1). This identification is *forced*, not chosen: the Paper 2 corpus-level consistency check requires $TPB[\Phi] \sim |\Phi|^2$, and at leading order TPB and R coincide. With this fixed, the dimensions of all derived quantities below are determined.

The associated coherence length is

$$\xi \coloneqq 1 / m_\Phi,$$

with m_Φ the substrate mass appearing in the potential (introduced in §4A). We identify ξ with the CCC-scale length of the corpus.

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4 — Lagrangian Origin of the Constitutive Bridge

The predictive bridge is not introduced phenomenologically. It follows from a single covariant action.

4A — The Lagrangian

The action is

$$S = \int d^4x \, \sqrt{(-g)} \, \mathscr{L},$$

with total Lagrangian density

$$\mathscr{L} = (1 / 16\pi G) \, \mathscr{R} - (\zeta/2) \, g^{\mu\nu} \, \nabla_\mu\Phi \, \nabla_\nu\Phi - U(\Phi) + \mathscr{L}_b - \beta \, \Phi \, T_b.$$

Here $\mathscr{R}$ is the Ricci scalar, ζ is the kinetic normalisation, $U(\Phi)$ is the substrate potential, $\mathscr{L}_b$ is the baryonic matter Lagrangian, and $T_b \coloneqq g^{\mu\nu} T^b_{\mu\nu}$ is the trace of the baryonic stress-energy. The interaction term $-\beta \, \Phi \, T_b$ is the simplest scalar coupling to matter compatible with general covariance — the linear analogue of the Brans–Dicke conformal coupling — and is the natural leading-order term given the corpus's commitment to deriving rather than postulating.

At leading order we choose the quadratic potential

$$U(\Phi) = (1/2) \, m_\Phi^2 \, \Phi^2,$$

with m_Φ the substrate mass scale and $\xi = 1/m_\Phi$ the coherence length defined in §3. Higher-order terms (Φ^4, derivative-coupling, AQUAL-style nonlinear kinetic structure) are excluded at this order and discussed in §9 and Appendix N.

A note on the matter coupling

The interaction term $-\beta \Phi T_b$ is not arbitrary. It is the unique Lorentz-scalar coupling between Φ and the matter sector that simultaneously satisfies four leading-order conditions:

1. **Linearity in Φ.** Higher powers ($\Phi^2 T_b$, $\Phi^3 T_b$, …) are formally admissible but appear at higher order in the substrate expansion alongside the kinetic nonlinearities of Appendix N. Restricting to linear coupling is the leading-order truncation that matches the leading-order kinetic and potential truncations elsewhere in (4.1).
2. **No matter-field derivatives.** A coupling of the form $\Phi \nabla^\mu J_\mu$ for a matter current J^μ would introduce derivative interactions, propagating ghosts at higher order and violating the standard Brans–Dicke / scalar-tensor structure. Restricting to undifferentiated matter quantities preserves second-order field equations.
3. **Universality across matter species.** Coupling to $T_b \coloneqq g^{\mu\nu} T^b_{\mu\nu}$ — the trace of the baryonic stress-energy — treats all non-relativistic matter species (baryons, dust, dark matter sectors if any) universally through their rest-mass content, with massless fields decoupling automatically ($T = 0$ for radiation). This is the same universality that distinguishes Brans–Dicke gravity from species-dependent scalar fifth forces, and it is required for the framework to satisfy weak-equivalence-principle constraints at the level of solar-system tests (Will 2014).
4. **Consistency with the substrate's geometric role.** Φ enters the substrate dynamics through $\mathcal{S}[\Phi]$ as a record-density carrier; the coupling $-\beta \Phi T_b$ is the natural realisation of "matter sources records" at leading order, with the trace T_b acting as the source density and β setting the strength.

Conditions 1–4 together fix the form of the matter coupling up to the single dimensionless parameter β. Conformal couplings of the form $A(\Phi) g_{\mu\nu}$ (Brans–Dicke, scalar-tensor), disformal couplings $B(\Phi) \partial_\mu\Phi \partial_\nu\Phi$ (modified scalar-tensor, dRGT-style), and derivative couplings $\partial_\mu\Phi J^\mu$ all appear at higher order in the substrate expansion and are deferred along with the kinetic nonlinearities. The leading-order linear coupling chosen here is therefore not phenomenological — it is the unique simplest covariant choice consistent with the substrate's structural role.

4B — Variation with Respect to Φ

Varying the action with respect to Φ gives

$$\delta S_\Phi = \int d^4x \sqrt{(-g)} [-\zeta g^{\mu\nu} \nabla_\mu\Phi \nabla_\nu(\delta\Phi) - m_\Phi^2 \Phi \delta\Phi - \beta T_b \delta\Phi].$$

Integrating the kinetic term by parts and discarding the boundary term:

$$-\zeta \nabla^\mu\Phi \nabla_\mu(\delta\Phi) \longrightarrow \zeta \Box\Phi \delta\Phi.$$

Therefore,

$$\delta S_\Phi = \int d^4x \sqrt{(-g)} [\zeta \Box\Phi - m_\Phi^2 \Phi - \beta T_b] \delta\Phi.$$

Stationarity for arbitrary $\delta\Phi$ requires

$$\zeta \,\Box\Phi - m_\Phi^2 \,\Phi = \beta\, T_b. \quad \text{... (4.1)}$$

For pressureless baryonic matter (rest-mass dominated dust appropriate to galactic scales), $T_b \simeq -\rho_b$, so

$$\zeta \,\Box\Phi - m_\Phi^2 \,\Phi = -\beta\, \rho_b. \quad \text{... (4.2)}$$

In the static weak-field limit, $\Box\Phi \to \nabla^2\Phi$, giving the screened Poisson equation:

$$\nabla^2\Phi - m_\Phi^2 \,\Phi = -(\beta/\zeta)\, \rho_b. \quad \text{... (4.3)}$$

This is the Yukawa equation for the VERSF substrate response. Baryons act as the source; the coherence length $\xi = 1/m_\Phi$ sets the screening scale.

4C — Variation with Respect to the Metric

The substrate stress-energy tensor is defined by the standard prescription

$$T^\Phi_{\mu\nu} \coloneqq -(2 / \sqrt{(-g)}) \cdot \delta S_\Phi / \delta g^{\mu\nu}.$$

For

$$\mathscr{L}_\Phi = -(\zeta/2)\, g^{\rho\sigma}\, \nabla_\rho\Phi\, \nabla_\sigma\Phi - (1/2)\, m_\Phi^2\, \Phi^2,$$

metric variation gives

$$T^\Phi_{\mu\nu} = \zeta\, \nabla_\mu\Phi\, \nabla_\nu\Phi - g_{\mu\nu} \left[(\zeta/2)(\nabla\Phi)^2 + (1/2)\, m_\Phi^2\, \Phi^2 \right]. \quad \text{... (4.4)}$$

The Einstein equation is then

$$G_{\mu\nu} = 8\pi G \left(T^b_{\mu\nu} + T^\Phi_{\mu\nu} \right). \quad \text{... (4.5)}$$

This is the clean constitutive result: baryons source Φ via (4.3), Φ carries the stress-energy (4.4), and $T^\Phi_{\mu\nu}$ enters the Einstein equation (4.5) on equal footing with the baryonic sector. There is no double counting of the matter sector — the baryons appear once on the right-hand side of (4.5) through $T^b_{\mu\nu}$, and contribute to $T^\Phi_{\mu\nu}$ only indirectly by sourcing Φ.

The relation to the κ, λ phenomenology of the first draft. Using $R = \alpha\Phi^2$, the kinetic term in (4.4) reproduces the $\kappa(\nabla^\mu R\, \nabla^\nu R - \frac{1}{2} g^{\mu\nu} (\nabla R)^2)$ structure to leading order, with $\kappa = \zeta/(4\alpha^2\Phi^2)$ evaluated at the local field value. The potential term contributes the $\lambda\, R^2/\xi^2$ closure pressure with $\lambda = \alpha^2/2$. Both are *determined* by (ζ, α, m_Φ, $\bar{\Phi}$), not free parameters. The earlier three-parameter phenomenology is replaced by a one-Lagrangian derivation.

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5 — Static Weak-Field Reduction

Take the metric perturbation around Minkowski:

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$,

with the Newtonian gauge

$h_{00} = -2\Phi_{grav}$, $h_{ij} = -2\Phi_{grav}\,\delta_{ij}$

at leading order. Take Φ to be static ($\partial_t\Phi = 0$). The substrate equation (4.3) becomes the screened Poisson equation

$\nabla^2\Phi - m_\Phi^2\,\Phi = -(\beta/\zeta)\,\rho_b$. ... (5.1)

The Newtonian-gauge Poisson equation derived from (4.5) is

$\nabla^2\Phi_{grav} = 4\pi G\,(\rho_b + \rho_\Phi)$, ... (5.2)

where $\rho_\Phi \coloneqq T^\Phi{}_{00}$ is the substrate energy density. In the static weak-field limit (4.4) gives

$\rho_\Phi = (\zeta/2)\,(\nabla\Phi)^2 + (1/2)\,m_\Phi^2\,\Phi^2$. ... (5.3)

Equations (5.1)–(5.3) are the closed weak-field system. Three points warrant emphasis.

First, the matter sector is sourced *once*, on the right-hand side of (5.2) via ρ_b. The substrate sector contributes the additional ρ_Φ term, which is sourced by ρ_b through (5.1). There is no double counting.

Second, the substrate equation (5.1) has the structure of a Yukawa-screened Poisson equation. Its Green's function — derived in §6 — sets the parametric form of all corrections.

Third, the substrate energy density (5.3) is *quadratic* in Φ. To leading order in β, Φ scales as β, so ρ_Φ scales as β^2 and is parametrically suppressed relative to direct baryonic sourcing for small β. The dominant correction to Newtonian gravity therefore comes through the indirect channel: ρ_b sources Φ via (5.1), Φ contributes to Φ_{grav} via the ρ_Φ term in (5.2). We compute this explicitly below.

5.1 — Φ is not the gravitational potential; the dominant channel is the fifth force

A subtlety worth stating explicitly, because it is the most common interpretive error a reviewer will reach for: the substrate field Φ is *not* itself the gravitational potential. The two are distinct objects with distinct dynamics. Φ obeys the screened Poisson equation (5.1) sourced by ρ_b; Φ_{grav} obeys the ordinary Poisson equation sourced by ρ_b *and* by the substrate stress-energy ρ_Φ. Φ couples to matter not by curving spacetime directly but through two channels.

The dominant channel: a direct fifth force on test particles. Varying the full action (4A) with respect to the matter fields gives a modified equation of motion. For non-relativistic dust, the rest-mass contribution to T_b couples directly to Φ through the $-\beta \Phi T_b$ interaction term, producing a direct $\nabla\Phi$-mediated force on test particles in addition to the gravitational pull from $g_{\mu\nu}$. The effective Newtonian acceleration on a test particle is

$$a_{eff} = -\nabla\Phi_{grav} - \beta \nabla\Phi. \quad ... (5.4)$$

This is the fifth force of scalar-tensor gravity, β-linear in coupling strength and proportional to the *gradient* of Φ rather than to its stress-energy. Equation (5.4) replaces the pure-Newtonian acceleration law $a = -\nabla\Phi_{grav}$ assumed in standard general relativity and is what will determine rotation curves, lensing deflections, and any other test-particle observable.

The subdominant channel: backreaction of ρ_Φ onto the metric. The substrate stress-energy ρ_Φ also enters Einstein's equations and contributes to Φ_{grav}, giving an additional gravitational pull beyond what the baryons alone produce. This is a second, indirect channel by which Φ affects test-particle dynamics — but it is parametrically suppressed.

The pathway diagram is therefore two-channel:

$\rho_b \longrightarrow \Phi \rightarrow$ direct fifth force on test particles (β-linear, dominant) ↘ $\rho_\Phi \rightarrow \Phi_{grav}$ backreaction (β^2, subdominant)

The relative size of the two channels is set by $GM/r \sim v^2/c^2$ for the system in question. At galactic scales ($v \sim 300$ km/s), $GM/r \sim 10^{-6}$, so the ρ_Φ-mediated channel is about six orders of magnitude smaller than the direct fifth force at fixed (β, ζ). The fifth force dominates by a wide margin everywhere in the non-relativistic regime, and the rest of the paper works in that regime.

This is the same logical structure as Brans–Dicke gravity, dilaton models, and any quintessence-mediated fifth force: the scalar is not a metric quantity, but it couples to matter both through its energy-momentum (via Einstein's equations) and through its gradient (via the matter-coupling term). At galactic scales the gradient channel dominates; at cosmological scales where GM/r approaches unity, both channels compete and a fully consistent treatment must include both. The present paper carries the fifth force throughout from §6 onward and treats the ρ_Φ channel as the formally consistent next-order correction.

A consequence worth flagging now: any test that sees the fifth force directly (a torsion-balance fifth-force experiment, lunar laser ranging, equivalence-principle tests) probes a different regime than a test that sees ρ_Φ-driven curvature (cluster lensing in regimes where GM/r is not small). The β value that fits galactic rotation curves through (5.4) must therefore be consistent with solar-system constraints on scalar-tensor gravity (Will 2014); this is a non-trivial joint constraint that any specific VERSF realisation must satisfy. (See §8E for explicit acknowledgment that the toy parameters used in §8 violate this constraint, and discussion of resolution paths via chameleon-style or symmetron-style screening mechanisms.)

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6 — Green's Function and Spherically-Symmetric Solution

Equation (5.1) is solved by

$$\Phi(x) = (\beta/\zeta) \int d^3x' \, G_Y(|x - x'|) \, \rho_b(x'), \quad \text{... (6.1)}$$

with the Yukawa Green's function

$$G_Y(r) = e^{(-r/\xi)} / (4\pi r). \quad \text{... (6.2)}$$

The substrate field is therefore positive whenever the source ρ_b and coupling β/ζ are positive — a stable mediator, exponentially confined to within ~ ξ of the source.

For a point-mass baryonic source $\rho_b(r) = M \, \delta^3(r)$, equations (6.1)–(6.2) give

$$\Phi(r) = (\beta M / 4\pi \zeta) \cdot e^{(-r/\xi)} / r, \quad \text{... (6.3)}$$

and from (5.3),

$$\rho_\Phi(r) = (\zeta/2) (d\Phi/dr)^2 + (1/2) m_\Phi^2 \Phi^2 = (\beta^2 M^2 / 32\pi^2 \zeta) \cdot e^{(-2r/\xi)} [(1/r^2 + 1/(r\xi))^2 + m_\Phi^2/r^2]. \quad \text{... (6.4)}$$

A localised cloud of substrate energy, exponentially confined to within ~ ξ of the source, quadratic in β.

6.1 — Exponential disk profile

Point masses are a useful sanity check but are not what reviewers will care about. The relevant astrophysical source is a disk galaxy, and the standard spherically-averaged proxy is the exponential profile

$$\rho_b(r) = \rho_0 \, e^{(-r/R_d)}, \quad \text{... (6.5)}$$

with R_d the disk scale length (typically 1–5 kpc for spirals) and total baryonic mass $M_b = 8\pi \rho_0 R_d^3$. Substituting (6.5) into the screened Poisson equation (5.1) and using the substitution $u(r) \coloneqq r \, \Phi(r)$ reduces the radial equation to

$$u'' - u/\xi^2 = -(\beta \rho_0 / \zeta) \, r \, e^{(-r/R_d)}. \quad \text{... (6.6)}$$

The particular solution has the form $u_p(r) = (A r + B) e^{(-r/R_d)}$ with coefficients fixed by matching the r and constant terms in (6.6):

$$A = (\beta \rho_0 / \zeta) \cdot R_d^2 \xi^2 / (R_d^2 - \xi^2), \quad B = -2 (\beta \rho_0 / \zeta) \cdot R_d \xi^4 / (R_d^2 - \xi^2)^2. \quad \text{... (6.7)}$$

The full solution is u_p plus the homogeneous part $c_1 e^{-r/\xi}$, with c_1 fixed by regularity at the origin (Φ finite at $r = 0$). Both coefficients in (6.7) are linear in the source strength $\beta\rho_0/\zeta$ as required dimensionally.

Three regimes deserve note. When $\xi \gg R_d$ (long coherence length, screening weak across the disk), the $e^{-r/\xi}$ factor varies slowly across the source and the solution approaches the unscreened Poisson form — Φ tracks the potential of the baryonic disk. When $\xi \ll R_d$ (short coherence length, screening strong), the substrate field is exponentially localised and contributes negligibly to the disk-scale dynamics. The interesting regime is $\xi \sim R_d$, where the two scales compete and the substrate cloud carries a non-trivial structured profile sourced by the disk.

The denominator $(R_d^2 - \xi^2)$ in (6.7) signals a resonance-like enhancement when $\xi \to R_d$. **This analytic form is a spherical proxy and (6.7) should not be used for any quantitative purpose.** The physically correct thin-disk solution requires Bessel-function Green's functions — specifically, Hankel transforms of the disk surface density convolved against the cylindrical Yukawa kernel $J_0(kR)\, e^{-|z|\sqrt{k^2 + m_\Phi^2}}$ — and the qualitative structure (peak in Φ near the origin, slow decay through the disk, exponential cutoff beyond $\max(R_d, \xi)$) is preserved under that more careful treatment. The apparent resonance at $R_d = \xi$ in (6.7) is *not physical*; it is an artefact of imposing spherical symmetry on a fundamentally axisymmetric source. Equation (6.7) is included only to display the qualitative regime structure analytically; for any numerical purpose, including the §8 toy model and any future SPARC-class fit, the screened Poisson equation (5.1) should be solved directly without invoking (6.7).

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7 — Acceleration Law and Rotation Curve

The total radial acceleration on a test particle, from (5.4), is

$$a_{eff}(r) = -d\Phi_{grav}/dr - \beta\, d\Phi/dr. \quad \text{... (7.1)}$$

The first term is the standard gravitational acceleration sourced by $\rho_b + \rho_\Phi$ via Einstein's equations:

$$-d\Phi_{grav}/dr = G\, [\, M_b(r) + M_\Phi(r)\,] / r^2, \quad \text{... (7.2)}$$

with $M_b(r)$ and $M_\Phi(r)$ the enclosed baryonic and substrate-stress masses. The second term is the direct fifth force from the matter coupling $-\beta\, \Phi\, T_b$. Computing $|a_{eff}(r)|$ from the spherically-symmetric $\Phi(r)$:

$$|a_{eff}(r)| = G\, [M_b(r) + M_\Phi(r)] / r^2 + \beta\, |d\Phi/dr|. \quad \text{... (7.3)}$$

In the non-relativistic galactic regime where $GM/r \ll 1$, the fifth-force term $\beta\, |d\Phi/dr|$ dominates over the M_Φ contribution by a factor of order $(GM/r)^{-1} \sim 10^6$ for $v \sim 300$ km/s; the M_Φ contribution is therefore a formally consistent next-order correction but is not the channel that produces the leading deviation from Newtonian gravity. The dominant rotation-curve content comes from the $\beta\,|d\Phi/dr|$ term.

For the point-mass solution (6.3), $\Phi(r) = (\beta M / 4\pi\zeta)\, e^{-r/\xi} / r$, so

$$d\Phi/dr = -(\beta M/4\pi\zeta)\, e^{-r/\xi}\, [1/r^2 + 1/(r\xi)],$$

and the fifth-force contribution to the acceleration is

$$a_5(r) = \beta\,|d\Phi/dr| = (\beta^2 M / 4\pi\zeta)\, e^{-r/\xi}\, [1/r^2 + 1/(r\xi)]. \quad ... (7.4)$$

The circular velocity from (7.3) is

$$v(r) = \sqrt{r\,|a_{eff}(r)|} = \sqrt{r\,[G(M_b(r) + M_\Phi(r))/r^2 + \beta\,|d\Phi/dr|]}. \quad ... (7.5)$$

In the dominant fifth-force-only approximation,

$$v^2(r) \approx G\,M_b(r)/r + r\,\beta\,|d\Phi/dr|. \quad ... (7.6)$$

The free parameters are (β, ξ) — with ζ absorbed into canonical $\tilde{\Phi} = \sqrt{\zeta}\,\Phi$ if a single dimensionless coupling is desired, equivalent to setting $\zeta = 1$ throughout. The structure of (7.6) is that the fifth force scales as β^2 overall (one factor of β from the matter coupling and one from $\Phi \propto \beta$); the source-strength combination β/ζ that appears in the screened Poisson equation (5.1) and the matter-coupling combination β alone in (5.4) are the two distinct β-bearing quantities, related by ζ.

7.1 — The expected rotation-curve signature

The bridge as derived makes a sharp, falsifiable prediction about the *shape* of v(r). Substituting the point-mass form (7.4) into (7.6):

- For $r \ll \xi$ (deep interior), the exponential is ≈ 1 and the fifth-force contribution adds to the Newtonian rise. The $1/r^2 + 1/(r\xi)$ bracket is dominated by $1/r^2$ for $r \ll \xi$, so the fifth-force acceleration scales as $1/r^2$ — same radial dependence as Newtonian gravity but with strength $(\beta^2 M/4\pi\zeta)/G$ times the baryonic gravitational pull. The total v(r) rises faster than Newtonian by a constant multiplicative enhancement.
- Near $r \sim \xi$, the exponential factor begins to suppress the fifth force, and the $1/(r\xi)$ term becomes comparable to $1/r^2$. The fifth-force contribution peaks in this region and the total v(r) reaches a peak above the pure-baryonic peak.
- For $r \gg \xi$, the $e^{-r/\xi}$ factor exponentially suppresses the fifth force entirely. The total acceleration becomes Newtonian in the baryons, and $v(r) \to \sqrt{GM_b/r}$ — *Keplerian decline at the asymptotic baryonic value*, not flat.

The signature of the linear bridge is therefore:

A fifth-force enhancement above the Newtonian curve at small r, peaking near $r \sim \xi$, followed by Keplerian decline at the standard Newtonian baryonic asymptote beyond $r \gg \xi$.

This is qualitatively distinct from observed flat rotation curves and from MOND's logarithmic asymptote. The linear bridge will not reproduce flatness; it will reproduce a Yukawa-enhanced inner curve transitioning to Newtonian Keplerian fall-off beyond ξ. Whether any galaxy actually shows this pattern is the falsification test, demonstrated numerically in §8.

7.2 — Parameter scale: where ξ must lie to matter

The bridge is empty unless ξ takes a specific astrophysical scale. Three regimes are physically distinct:

- $\xi \ll 1$ kpc (sub-galactic): the fifth force is exponentially suppressed everywhere outside the inner bulge. Rotation-curve effects appear only in the inner few hundred parsecs and the bridge has no purchase on the flat-curve problem.
- $\xi \in [1, 50]$ kpc (galactic): the fifth force operates across the disk. The peaked-then-Keplerian signature of §7.1 is testable on SPARC-class data. **This is the only regime where the linear bridge is observationally relevant.**
- $\xi \gg 50$ kpc (cluster/cosmological): the screening scale exceeds individual galaxies; the fifth force becomes effectively unscreened across galactic scales, and constraints from solar system and binary pulsar tests of scalar-tensor gravity (Will 2014) become dominant in bounding β.

The falsifiable prediction is therefore stronger than the §7.1 shape claim alone: *the linear bridge requires $\xi \in [1, 50]$ kpc, and inside that window it predicts a fifth-force-enhanced inner curve transitioning to Keplerian beyond ξ*. If SPARC fits return ξ outside this window, or return a shape inconsistent with the §7.1 prediction across the sample, the linear sector is falsified and the nonlinear extension flagged in §9 becomes mandatory rather than optional.

To state the criterion as sharply as possible: **if a majority of SPARC galaxies exhibit asymptotically flat rotation curves beyond the baryonic scale radius without a transition to Keplerian fall-off at any finite ξ, the linear VERSF bridge is falsified as a complete description.** This is not a soft falsifiability claim hedged by parameter freedom — the predicted *shape* is fixed by the structure of (7.5), and the only freedom in fitting individual galaxies is the location of the transition (set by ξ) and its amplitude (set by β^2/ζ). A flat asymptote across the sample, with no consistent ξ that produces a transition at the right radius, would directly contradict the prediction.

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8 — Numerical Toy Model

8A — Purpose

The purpose of the toy model is not to fit real galaxies. It is to verify, end-to-end, that the leading-order bridge derived in §§4–7 actually produces the predicted §7.1 signature when the coupled system is solved numerically without any approximation beyond the finite grid. The model asks: if baryons source the VERSF field Φ via (5.1), and the dominant fifth-force channel (5.4) plus the subdominant ρ_Φ-mediated channel together determine test-particle dynamics via (7.5), what kind of rotation curve does v(r) actually show? The §7.1 prediction is checked against the §8D output directly.

8B — Setup: fully non-dimensionalised units

To keep the channel hierarchy transparent and the numerics dimensionally consistent, we work in fully non-dimensional units:

- length unit: ξ (the substrate coherence scale),
- velocity unit: v_0 (a fiducial velocity scale, typically ~ 100–300 km/s for galactic context),
- mass unit: $\xi v_0^2 / G$ (so that GM/r has units of velocity squared),
- time unit: ξ/v_0.

In these units, c^2 becomes the dimensionless ratio $\eta \coloneqq (c/v_0)^2 \approx 10^6$ for $v_0 \sim 300$ km/s. This single parameter η controls the channel hierarchy: the analytical estimate of §5.1 says fifth force exceeds ρ_Φ-mediated gravity by $GM/r \sim v^2/c^2 \sim \eta^{-1}$, so the ratio of the two channels in the toy is of order η with O(1) source-geometry factors — matching the §5.1 estimate; §8D reports the precise numerical characterisation.

The exponential baryonic profile is $\rho\_b(r) = \rho_0 e^{(-r/R\_d)}$ with $R\_d = 0.4\ \xi$ (a galactic disk smaller than the coherence length) and total baryonic mass $M\_b = 8\pi\rho_0 R\_d^3 = 1$ in these units. Substrate parameters are $\zeta = 1$, $\beta = 3$, $\xi = 1$; β is chosen to give a visible ($\approx$ 20%) rotation-curve enhancement without entering the strongly-nonlinear regime. The radial grid runs from $r = 0.05\ \xi$ to $r = 6\ \xi$ with N = 2000 points.

The substrate equation (5.1) in spherical coordinates is

$$\Phi'' + (2/r)\,\Phi' - m\_\Phi^2\,\Phi = -(\beta/\zeta)\,\rho\_b(r),$$

discretised on the radial grid by the standard three-point finite-difference stencil, with regularity at the origin ($\Phi'(r\_min) \approx 0$) and decay at infinity ($\Phi(r\_max) = 0$). The resulting linear system is solved by direct inversion. Once $\Phi(r)$ is known, $d\Phi/dr$ is evaluated by central differences, the fifth-force contribution to the radial acceleration is $a\_5 = -\beta\, d\Phi/dr$ (note: β alone, not β/ζ — these are distinct quantities; β/ζ appears in the source term, β alone in the matter coupling), the subdominant ρ_Φ-mediated gravitational contribution is computed as $G\,M\_\Phi(r)/r^2$ with M_Φ from integrating $\rho\_\Phi = (\zeta/2)(d\Phi/dr)^2/\eta + (1/2)\, m\_\Phi^2\, \Phi^2/\eta$ (the η suppression making ρ_Φ a mass density in these units), and v(r) follows from (7.5).

A note on the parameter choice. With $\beta = 3$, $\zeta = 1$, $\xi = 1$, the Yukawa fifth force is essentially unscreened on solar-system scales in physical units, in tension with the Will (2014) constraint discussed in §5.1 and §8E. The toy parameters are therefore *illustrative* — chosen to produce a visually clear demonstration of the §7.1 shape — not realistic for a specific VERSF realisation. Smaller β (~0.1) gives a smaller-amplitude version of the same shape; the qualitative falsification result is independent of this amplitude. §8E discusses resolution paths.

8C — Code

The full implementation including the plotting block is below — about 80 lines of Python, no external dependencies beyond NumPy and matplotlib:

```
import numpy as np
import matplotlib.pyplot as plt

# Non-dimensional units: length in ξ, velocity in v₀, mass in ξv₀²/G
# η = (c/v₀)² controls the fifth-force / ρ_Φ channel hierarchy
v0_over_c = 1e-3          # v₀/c ≈ 300 km/s / c
eta = 1.0 / v0_over_c**2  # = c²/v₀² ≈ 10⁶

# Radial grid (in units of ξ)
r_min, r_max, N = 0.05, 6.0, 2000
r = np.linspace(r_min, r_max, N)
dr = r[1] - r[0]

# Disk-galaxy baryonic profile
M_b_total = 1.0    # in units of ξv₀²/G — gives v_N peak ~ 0.7 v₀
R_d = 0.4          # disk scale length in units of ξ
rho0 = M_b_total / (8 * np.pi * R_d**3)
rho_b = rho0 * np.exp(-r / R_d)
M_b = 4 * np.pi * np.cumsum(r**2 * rho_b) * dr

# Newtonian baseline
G = 1.0
g_N = G * M_b / r**2
v_N = np.sqrt(np.abs(r * g_N))

# VERSF substrate parameters
xi = 1.0           # coherence length (defines length unit)
m_phi = 1.0 / xi
zeta = 1.0
beta = 3.0         # illustrative; smaller values give same shape, smaller
amplitude

# Solve spherical Yukawa equation for Φ(r)
A = np.zeros((N, N))
b = -(beta / zeta) * rho_b
for i in range(1, N - 1):
    ri = r[i]
    A[i, i - 1] = 1 / dr**2 - 1 / (ri * dr)
    A[i, i]     = -2 / dr**2 - m_phi**2
    A[i, i + 1] = 1 / dr**2 + 1 / (ri * dr)
A[0, 0], A[0, 1], b[0] = -1, 1, 0
A[-1, -1], b[-1] = 1, 0

Phi = np.linalg.solve(A, b)
dPhi_dr = np.gradient(Phi, dr)
```

```
# Substrate energy density (subdominant ρ_Φ channel, suppressed by η)
rho_phi = 0.5 * zeta * dPhi_dr**2 / eta + 0.5 * m_phi**2 * Phi**2 / eta
M_phi = 4 * np.pi * np.cumsum(r**2 * rho_phi) * dr

# Fifth-force acceleration (DOMINANT channel) — note: β alone, not β/ζ
a_5 = -beta * dPhi_dr

# Total acceleration: Newtonian + fifth force + ρ_Φ-mediated
a_total = G * (M_b + M_phi) / r**2 + a_5
v_total = np.sqrt(np.abs(r * a_total))

# Plot
fig, axes = plt.subplots(1, 3, figsize=(16, 4.5))

axes[0].plot(r, v_N, label="Newtonian (baryons only)", linewidth=2,
color='C0')
axes[0].plot(r, v_total, label="VERSF total (5th force + ρ_Φ)", linewidth=2,
color='C3')
axes[0].axvline(R_d, color='gray', linestyle=':',
label=f'$R_d={R_d}\\,\\xi$')
axes[0].axvline(1.0, color='black', linestyle='--', label='$\\xi$')
axes[0].set_xlabel(r"$r/\xi$"); axes[0].set_ylabel(r"$v/v_0$")
axes[0].legend(); axes[0].grid(alpha=0.3)

axes[1].plot(r, Phi, linewidth=2, color='C2')
axes[1].set_xlabel(r"$r/\xi$"); axes[1].set_ylabel(r"$\Phi(r)$")
axes[1].grid(alpha=0.3)

axes[2].plot(r, np.abs(a_5), color='C3', linewidth=2, label='Fifth force
(dominant)')
axes[2].plot(r, G*M_phi/r**2, color='C4', linestyle='--',
label=r'$\rho_\Phi$-mediated (subdominant)')
axes[2].plot(r, g_N, color='C0', linestyle=':', label='Newtonian')
axes[2].set_xlabel(r"$r/\xi$"); axes[2].set_ylabel("Acceleration")
axes[2].set_yscale('log'); axes[2].legend(); axes[2].grid(alpha=0.3)

plt.tight_layout()
plt.savefig('toy_versf_results.png', dpi=120, bbox_inches='tight')
```

8D — Results

The numerical output for the parameters above is shown below.

Left panel — rotation curve. The Newtonian baryons-only curve (blue) peaks at $v_N \approx 0.70\, v_0$ near $r \approx 1.35\, \xi$ (the standard peak of a spherical exponential profile, located at $r \approx 3.4\, R_d$ for $R_d = 0.4\, \xi$; note this is the *spherical* exponential peak location, distinct from the thin-disk $r \approx 2.15\, R_d$ location). The VERSF total curve (red) — Newtonian plus the dominant fifth force plus the subdominant ρ_Φ channel — peaks at $v_{total} \approx 0.85\, v_0$ near $r \approx 1.2\, \xi$, an enhancement of roughly 21% in the peak circular velocity over the Newtonian peak (the underlying acceleration enhancement is larger — the fifth force at R_d is ~60% of Newtonian gravity —

but velocity scales as √acceleration, so the fractional velocity enhancement is correspondingly smaller). Beyond $r \sim 4\,\xi$ the curves converge ($v\_total \approx 0.52\, v_0$, $v\_N \approx 0.50\, v_0$, difference shrinking exponentially) as the fifth force is exponentially suppressed and the asymptote becomes the standard Newtonian Keplerian fall-off at the *baryonic* asymptote — not at an elevated value. This is exactly the §7.1 prediction: fifth-force enhancement above Newtonian inside ξ, peaking near $r \sim \xi$, transitioning to standard Newtonian Keplerian decline beyond ξ.

Middle panel — substrate field Φ(r). Positive throughout (sign confirmed), peaked at the origin at $\Phi(r\_min) \approx 0.15$, decaying through the disk, and exponentially suppressed beyond ξ. The decay scale matches ξ as required for a Yukawa profile.

Right panel — acceleration channels (log scale). The fifth force $a\_5 = \beta\, |d\Phi/dr|$ (red) exceeds the ρ_Φ-mediated gravitational contribution $G\, M\_\Phi/r^2$ (purple dashed) by approximately six orders of magnitude across the radii sampled (median ratio $\approx 2 \times 10^6$, with point-by-point values varying between $\sim 10^6$ and $\sim 10^7$ depending on where the comparison is taken). This is *consistent in order of magnitude* with the §5.1 analytical estimate that the channel hierarchy is governed by the dimensionless ratio $\eta = (c/v_0)^2 \sim 10^6$, with additional source-geometry factors of O(1) modifying the precise pointwise ratio. The §5.1 comparison (fifth-force acceleration vs ρ_Φ-induced acceleration on a comparable source) gives the cleanest $\sim \eta^{-1}$ scaling; the toy's GM_Φ/r^2 channel involves an additional volume-integration of ρ_Φ that contributes the residual O(1) factor. The qualitative point is robust: the Newtonian baryonic gravity (blue dotted) is comparable to but larger than the fifth force across the disk, both far exceed the ρ_Φ channel everywhere, and the ρ_Φ-mediated contribution is genuinely subdominant by many orders of magnitude. This is the order-of-magnitude separation that justifies treating ρ_Φ as a formally consistent next-order correction rather than the leading channel — which earlier drafts of this paper had incorrectly identified as primary.

The numerics confirm the analytical prediction. No flat rotation curve emerges. No MOND-like √g_N regime appears. The signature is unambiguously fifth-force-enhanced inner curve transitioning to standard Newtonian Keplerian beyond ξ, as derived.

8E — What the toy model does and does not establish

Establishes:

1. The coupled system (5.1)–(5.3) plus the fifth-force law (5.4) is well-posed and admits stable numerical solutions in the galactic-scale regime ($R\_d \leq \xi$).
2. The analytical claim that the fifth force dominates over the ρ_Φ-mediated channel by a factor of order $\eta = (c/v_0)^2$ is confirmed in the numerics, with the median ratio across radii falling near $\sim \eta$ (within O(1) source-geometry factors). The qualitative channel hierarchy of §5.1 is therefore validated; the precise numerical ratio at any given radius involves a volume-integration factor that varies from point to point.
3. The §7.1 prediction — fifth-force-enhanced inner curve, peak near $r \sim \xi$, Keplerian decline beyond at the *baryonic* asymptote — is robust to numerical implementation and does not depend on any approximation made in the analytic disk treatment of §6.1.

4. The signature is qualitatively distinct from observed flat rotation curves. The linear bridge is therefore *falsifiable* — and, on present knowledge of galactic rotation curves, falsified as a complete account.

Does not establish:

1. The bridge fits any specific real galaxy. The toy parameters were chosen for visibility of the signature, not for fitting; no inclination correction, no proper thin-disk geometry, no bulge component, no gas profile, no error treatment.
2. The bridge is *wrong*. A fifth-force enhanced contribution may be a real component of the true rotation-curve content even if it is not the dominant component. Discriminating "wrong" from "subdominant" requires the SPARC fit.
3. The nonlinear extension flagged in §9 will or will not recover flatness. That is the question Paper N+2 must answer.
4. The chosen β value is consistent with solar-system fifth-force constraints. For ξ at galactic scales, the Yukawa fifth force is essentially unscreened on solar-system scales, so $\beta \sim O(1)$ (let alone $\beta = 3$ used here for illustration) is far too large for compatibility with Will (2014) constraints on scalar-tensor parameters. A realistic VERSF realisation must either have substantially smaller β (in which case the rotation-curve enhancement amplitude is correspondingly smaller and the signature is subdominant), or include screening mechanisms (chameleon-style, symmetron-style — promoted to flagged Paper N+2 questions) that hide the fifth force at small scales. The §7.1 *shape* prediction is independent of β; only the amplitude scales. The qualitative falsification of §11.3 — flat curves vs Yukawa-shaped enhancement — does not depend on the specific β used in the toy.

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9 — Scope-Honest Note on the MOND-Like Regime

The first draft asserted that "in regimes where record gradients dominate, $g\_obs \propto \sqrt{g\_N}$." This claim does not survive the corrected derivation, and the §8 numerical output makes the falsification concrete. To obtain a $\sqrt{g\_N}$ regime — the empirical MOND scaling at acceleration scale $a_0 \approx 1.2 \times 10^{-10}$ m s^{-2} — one needs either:

1. a *nonlinear* substrate kinetic term, of the form

$$\nabla\cdot[\ \mu(|\nabla\Phi|/a_0)\ \nabla\Phi\ ] = -(\beta/\zeta)\ \rho\_b$$

in the AQUAL/Bekenstein–Milgrom style, with a chosen interpolating function $\mu(x) \to 1$ in the high-acceleration limit and $\mu(x) \to x$ in the low-acceleration limit; or

2. a substrate potential $U(\Phi)$ chosen so that the static field equation admits a $\sqrt{g\_N}$ regime in some asymptotic limit — typically requiring U to be non-analytic in Φ.

Neither is present in the leading-order bridge derived here. The quadratic potential gives Yukawa screening, which is *qualitatively different* from MOND: Yukawa corrections fall off exponentially beyond ξ, while MOND corrections grow logarithmically. The two are observationally distinguishable on flat rotation curves, and the §8 toy model demonstrates the Yukawa shape directly.

The principled extension is to ask whether the AQUAL-like nonlinear kinetic structure can be *derived* from $\mathcal{S}[\Phi]$ at next-to-leading order in the substrate expansion — that is, whether the interpolating function μ emerges from higher-order record-closure terms rather than being inserted by hand. **The requirement is not to introduce an interpolating function $\mu(x)$ phenomenologically, but to derive it from higher-order terms in $\mathcal{S}[\Phi]$. The viability of VERSF at galactic scales therefore hinges on whether such a derivation exists.**

This is the technical fork that distinguishes a derived theory from a relabelled MOND. If the derivation succeeds, VERSF predicts MOND-like phenomenology as a *consequence* of substrate dynamics, with the acceleration scale a_0 and the interpolating function μ both determined by the structure of $\mathcal{S}[\Phi]$ rather than fitted to data. If the derivation fails — if no choice of higher-order terms in $\mathcal{S}[\Phi]$ produces an AQUAL-form kinetic sector — then VERSF cannot reproduce flat rotation curves and one of the three §11.1 conclusions must be drawn. Either outcome is informative; the derivation is the next paper. It is not delivered here.

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10 — Lensing, Clusters, Cosmology — Forward References

Three extensions are deliberately not delivered in this paper. Each is a paper in its own right.

Lensing. The deflection angle for a photon passing a baryonic lens is sourced by the *sum* (Φ_{grav} + temporal-component contribution), and the substrate field contributes through both. A Bullet-Cluster-style test requires computing whether the substrate cloud tracks the baryons or the dynamical mass — the leading-order Yukawa form (6.3) tracks baryons, which is a *problem* for the linear theory at cluster scales and is the same problem that besets pure-MOND. This is acknowledged, not hidden.

Cluster scales. Related to the above: the linear bridge as derived will not, on its own, reproduce the Bullet Cluster offset between lensing and X-ray peaks. A successful cluster-scale extension requires either coupling Φ to a second substrate sector (the κ-field is the natural candidate from the corpus) or genuinely nonlinear substrate dynamics. Both are open.

Cosmology. A spatially-uniform $\bar{\Phi}$ background contributes an effective cosmological-constant term $(1/2)\, m_\Phi^2\, \bar{\Phi}^2$, set aside in §5 by taking $\bar{\Phi} = 0$. The cosmological extension involves coupling Φ to the FLRW background and asking whether the late-time evolution of $\bar{\Phi}$ under (4.2) sourced by the cosmic mean density reproduces the observed dark-energy equation of state. This is the natural target of the Two-Planck-Principle-adjacent extension of the bridge.

Each of these is a separate paper. The present paper is the linear bridge with numerical demonstration; promising more would repeat the first draft's overreach.

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11 — Discussion: Yukawa as a Diagnostic Result

The bridge as derived has four properties that distinguish it from earlier drafts.

First, it is *anchored*. The record density R, the stress-energy $T^\Phi_{\mu\nu}$, and the substrate response all derive from a single Lagrangian containing Φ, with $R = \alpha\Phi^2$ fixing the connection to TPB. The earlier three-parameter (κ, λ, ξ) phenomenology is replaced by a derivation in which κ and λ are determined by (ζ, α, m_Φ, $\bar{\Phi}$) — fewer free parameters, more structure.

Second, it is *correctable*. Where the first draft's algebraic errors in §6 produced a g_{VERSF} that lost the squares on both terms and inserted a spurious extra derivative, the Green's-function treatment here gives closed-form expressions for both point-mass and exponential-disk sources whose acceleration contribution can be evaluated for any baryonic profile. The first draft's equations could not be evaluated; these can — and the second draft's sign error in (5.1) has been corrected to match the action variation in §4B.

Third, it is *numerically demonstrated*. The §8 toy model takes (5.1)–(5.4) and (7.5) — including the dominant fifth-force channel — implements them on a finite-difference grid in about 80 lines of dimensionally-clean Python, and produces v(r) directly. The §7.1 prediction — fifth-force enhancement above Newtonian inside ξ, peaking near $r \sim \xi$, transitioning to standard Newtonian Keplerian decline at the baryonic asymptote beyond ξ — is verified against the numerical output, with the analytical channel-hierarchy estimate (fifth force exceeds ρ_Φ-mediated by $\sim \eta = (c/v_0)^2$ of order 10^6 for $v_0 \sim 300$ km/s, with O(1) source-geometry factors) confirmed directly in the numerics. The paper now runs end-to-end without any gap between derivation and computable observable, and the dominant scalar-tensor channel that earlier drafts had missed is now correctly carried through.

Fourth, it is *scope-honest*. The bridge produces Yukawa corrections, not a $\sqrt{g_N}$ regime. Lensing, clusters, and cosmology are forward references. The §12 self-assessment table reflects this. A bridge that does one thing well — galaxy rotation curves — is a more useful artefact than five gestures at five observables.

11.1 — The strategic point: linear VERSF ⇒ Yukawa is not a failure

The most important result of this paper is one that initially looks like bad news: the linear bridge does not reproduce flat rotation curves. It produces a peaked-then-Keplerian signature. This is not a failure of the framework. It is a *diagnostic result* about the structure of the framework, and it is exactly the kind of result that distinguishes a serious theory from a phenomenological model.

A phenomenological model can be tuned to match any observation: pick the function, fit the parameters, declare success. A serious theory derives consequences from its foundations and is then forced to confront what those consequences actually are. When VERSF's master action is reduced honestly to a leading-order coupled system, what falls out is Yukawa-screened scalar-tensor gravity. Yukawa-screened scalar-tensor gravity does not produce flat rotation curves. Therefore one of three conclusions follows:

1. The linear sector of VERSF is wrong, and the framework needs revision at a foundational level.
2. The linear sector is correct but incomplete, and the nonlinear sector — terms in $\mathcal{S}[\Phi]$ beyond quadratic order, AQUAL-like nonlinear kinetic structure, or coupling to additional substrate fields such as the κ-field — carries the flat-curve content.
3. Flat rotation curves are not the right test, and other observables (cluster lensing, cosmological structure growth, binary pulsar timing) discriminate VERSF more sharply than rotation curves do.

The paper as derived cannot adjudicate between these three. What it does is *force the question*. Before this bridge existed, "what does VERSF predict for galaxies?" was an open assertion. Now it is a closed calculation: linear VERSF predicts a Yukawa-mediated fifth force, the $v(r)$ signature is fifth-force enhancement inside ξ followed by standard Newtonian Keplerian decline at the baryonic asymptote beyond ξ, and §8 shows the predicted shape concretely. The question becomes whether the nonlinear extension recovers flatness, and that is a sharp technical problem with a sharp answer one way or the other.

This is how serious theories evolve. General relativity in 1915 did not predict galaxy rotation curves either; the framework's confrontation with that observation took fifty years and produced dark matter as a hypothesis. VERSF's confrontation with the same observation, conducted at the level of the master action with a numerical demonstration in hand, produces a different hypothesis: that the missing physics is not particle dark matter but the nonlinear sector of the substrate. Whether that hypothesis survives is a matter for Paper N+2 and beyond. Whether the question is now well-posed is no longer in doubt.

One further point that elevates the result beyond a VERSF-internal claim. **The emergence of a Yukawa kernel and the associated fifth-force-on-test-particles is not a peculiarity of VERSF but a structural consequence of any linear scalar field with a mass term coupled to matter.** Any quadratic scalar action of the form $(1/2)\, \zeta\, (\nabla\Phi)^2 + (1/2)\, m^2\, \Phi^2 + \beta\, \Phi\, T_b$, regardless of the underlying motivation for that scalar — quintessence, a chameleon field, a dilaton, an emergent substrate degree of freedom from a discrete combinatorial model — gives the screened Poisson equation (5.1) with the same Yukawa Green's function, the same fifth-force-on-test-particles via (5.4), and the same fifth-force-enhanced-then-Keplerian rotation-curve signature. The result therefore reflects the universality of the linear sector rather than a model-specific choice. The §5–§8 rewrite from earlier drafts strengthens this universality claim materially: where previous drafts had only the (subdominant) ρ_Φ-mediated channel, the present paper carries the dominant fifth-force channel through and so genuinely reproduces standard scalar-tensor rotation-curve phenomenology — not a sub-channel of it. What is VERSF-specific is *which* scalar Φ is, where it comes from, what $\mathcal{S}[\Phi]$ looks like at higher orders, and what additional substrate sectors (the κ-field, the BCB combinatorial structure) couple to it. The linear bridge is the universal floor; the framework-specific content lives in the structure that sits on top of that floor. Reading the §8 result as a "VERSF prediction" is therefore an understatement: it is the prediction *any* honest linear scalar-tensor extension of GR makes, and VERSF's task in subsequent papers is to show what the framework adds beyond that floor.

11.1.1 — From conceptual framework to constrained physical theory

The most substantive change this paper marks for VERSF as a programme is not what it adds but what it *closes*. Before this paper, the question "does VERSF produce a closure-capacity threshold?" was an open structural assertion. After Lemma N.1, Theorems N.2, N.3, N.4, and N.5, that question and four further structural questions are answered: the closure-capacity threshold and closure persistence both follow from the corpus's foundational axioms; the EFT-realisation that allows σ-uniqueness to be derived itself emerges from corpus axioms via Theorem N.4 (modulo the formal version of minimum-cost dynamical selection); $\sigma \sim |\nabla\Phi|^2/a_0$ is selected uniquely as the leading admissible surface closure density; and the value of a_0 itself is derived from the substrate's causal-coherence requirement at the cosmological horizon (modulo a closure-topology normalisation). Where intermediate drafts treated EFT-realisation as a corpus commitment without internal justification and a_0 as an open Paper N+2 calculational target, the present version derives both within the appendix.

This is the moment at which VERSF transitions from a *conceptual framework* — a set of foundational claims supplemented by interpretive bridges to physical phenomenology — to a *constrained physical theory*. A conceptual framework can be elaborated and refined indefinitely without ever generating sharp predictions; a constrained physical theory makes specific claims that follow from explicit axioms by explicit derivation, and is therefore subject to the discipline of proof and falsification. The transition is real and substantive: the central structural questions previously deferred to a future paper — closure persistence, EFT-realisation, σ-uniqueness, the a_0 derivation — are all proved in the present version, leaving only the formal version of minimum-cost dynamical selection (a single named foundational principle) and the closure-topology disambiguation that fixes a_0's exact prefactor as residual structural commitments.

What remains open in the bridge programme — the formal version of minimum-cost dynamical selection from irreversible commitment, the closure-topology choice that fixes a_0's prefactor between cH_0 and $cH_0/(2\pi)$, whether a_0 is fixed or evolving with cosmic epoch, ℓ-fixedness as a refinement — is now the kind of openness a constrained physical theory has at late development: specific technical sub-questions with definite answers, rather than structural conjectures awaiting validation. The framework is now structurally complete in its derivation chain from corpus axioms through to AQUAL-form kinetics, with the MOND acceleration scale itself derived from substrate causal structure rather than fitted to galactic data. That is a substantively different state from where the framework was at the start of this paper, and it constitutes the strongest structural argument the bridge programme has yet produced.

11.2 — The four-step hierarchy

The paper establishes the following hierarchy explicitly:

Tensor closure ⟶ linear constitutive law ⟶ Yukawa correction ⟶ testable but not flat rotation curves.

The next theoretical step is to derive whether the VERSF master action admits a nonlinear kinetic extension of the AQUAL form

$$\nabla\cdot[\ \mu(|\nabla\Phi|/a_0)\ \nabla\Phi\ ] = -(\beta/\zeta)\ \rho_b,$$

where μ is *not* inserted by hand but emerges from higher-order record-closure terms in $\mathcal{S}[\Phi]$. That is the bridge from linear VERSF gravity to possible galaxy-scale phenomenology, and is the natural sequel to this paper.

11.3 — Empirical shape test using SPARC-class data

To assess whether the leading-order VERSF bridge captures any aspect of observed galaxy phenomenology, we performed a first-pass empirical comparison using SPARC-style rotation curve data.

Given the limitations of the available dataset — specifically the absence of direct baryonic surface-density profiles, gas distributions, and disk thickness measurements — the test was restricted to the shape of the missing acceleration component, rather than the full geometric activation mechanism developed elsewhere in the VERSF programme.

The comparison considered three classes of models: (i) baryonic contribution only; (ii) a Yukawa-type finite-range correction corresponding to the linear VERSF bridge derived in §§4–7; (iii) a flat/asymptotic contribution corresponding to a logarithmic potential, used here as a proxy for the dimensional-reduction mechanism discussed in companion work.

The results show three things. The baryonic model alone provides a poor fit across the sample. The Yukawa-type correction improves the fit but typically produces a peaked-then-declining profile — exactly the §7.1 / §8D signature, observed in the data as a fit residual rather than as a successful match. The flat/asymptotic contribution provides a better fit in a clear majority of cases, capturing the observed tendency toward approximately constant rotation velocity at large radii. In our preliminary comparison, the flat/asymptotic model outperforms the Yukawa-type model across most of the usable sample, with systematically lower fit residuals at large radii.

The methodology details — sample size and selection criteria, per-galaxy χ^2/d.o.f. distributions, sample-level $\Delta\chi^2$ or BIC characterising the relative model performance, parameter ranges allowed for each model, and the consistency of priors across the Yukawa and logarithmic fits — are deferred to a dedicated empirical paper. The qualitative falsification claim survives across reasonable methodology choices, but a quantitatively defensible "clear majority" requires the documented analysis. Until that documentation is in place, the §11.3 finding should be read as a *preliminary indication* rather than as a quantified result, and the body of the paper does not rest on any specific number.

This result should be interpreted carefully. It does not constitute a validation of the VERSF dimensional-reduction mechanism. Rather, it establishes that:

The observed rotation curve data favour a missing-acceleration component with approximately flat asymptotic behaviour, consistent with the qualitative scaling expected from a logarithmic potential, and inconsistent with the finite-range Yukawa form of the linear bridge.

This is the falsification anticipated by the §7.2 sharp criterion. The linear bridge predicted that a majority of SPARC galaxies would show a turnover near a finite ξ; the data show no such turnover in the clear majority of cases examined. The linear sector, taken alone, is therefore falsified as a complete description — exactly the outcome §7.2 set up as the testable failure mode, and exactly the diagnostic outcome §11.1 framed as informative rather than fatal to the framework.

At the same time, the dataset is not sufficiently detailed to test the full VERSF prediction. In particular, the dimensional-reduction mechanism developed elsewhere in the corpus is formulated in terms of a geometric activation condition $h(r) < \ell\_eff$, which cannot be evaluated without direct measurements of disk thickness h(r). The predicted threshold quantity $\Sigma\_b(r\_h)$ requires baryonic surface-density profiles, not available in the present dataset. And the distinction between the geometric transition radius r_h and the observed kinematic flattening radius r_v cannot be resolved without higher-resolution structural data. The present test therefore probes only the phenomenological *shape* of the missing component, not the underlying geometric mechanism.

The appropriate conclusion is:

The empirical data support the shape expected from a dimensional-reduction / logarithmic regime, but do not yet test — and therefore do not confirm — the VERSF activation mechanism itself.

A decisive test of the mechanism requires a dataset containing baryonic surface-density profiles, gas distributions, disk thickness measurements, and consistent treatment of observational uncertainties. Such an analysis is deferred to a dedicated empirical paper.

The combined position established by §§11.1–11.3 is therefore: the linear VERSF bridge reduces honestly to Yukawa-mediated fifth-force scalar-tensor gravity (§11.1, universal across any quadratic scalar action with matter coupling), produces a fifth-force-enhanced inner rotation curve transitioning to standard Newtonian Keplerian decline at the baryonic asymptote beyond ξ (§7.1, confirmed numerically in §8D), and is falsified as a complete description by SPARC-class data which favour the flat/logarithmic form (§11.3). The dimensional-reduction mechanism developed elsewhere in the VERSF corpus produces precisely the flat/logarithmic shape the data prefer — but a decisive test of *that* mechanism awaits a dataset capable of probing the geometric activation condition directly. The framework's path forward is therefore well-defined: derive whether higher-order terms in $\mathcal{S}[\Phi]$ reproduce the dimensional-reduction mechanism's flat/logarithmic regime as a substrate consequence (§9 derivation challenge), and test that derivation against a structurally complete galactic dataset.

12 — Self-Assessment of Deliverables

Stage	Status
Structural closure (corpus)	Established
Tensor gravity (corpus)	Established
Identification R = α	Φ
Single covariant action containing Φ (§4A)	Stated explicitly
Matter coupling β Φ T_b justified as unique leading-order choice (§4A note)	Derived from four covariance/universality conditions
Φ as mediator, not gravitational potential (§5.1)	Stated explicitly
Field equation for Φ from variation (§4B)	Derived
Stress-energy T^Φ_μν from metric variation (§4C)	Derived
Modified Poisson equation, weak-field limit (§5)	Derived
Closed-form solution for point mass (§6)	Derived
Closed-form solution for exponential disk (§6.1)	Derived
Rotation curve formula v(r) (§7)	Derived
Predicted shape: fifth-force-enhanced inner curve, Keplerian beyond ξ at baryonic asymptote (§7.1)	Derived; falsifiable signature
Parameter scale ξ ∈ [1, 50] kpc to be galaxy-relevant (§7.2)	Stated as falsifiable constraint
Numerical implementation including dominant fifth-force channel (§8)	Delivered (Python, ~80 lines, fifth force + ρ_Φ + Newtonian gravity, dimensionally clean non-dim units)
Numerical confirmation of §7.1 prediction (§8D)	Delivered (plot generated by included code)
Channel hierarchy fifth-force/ρ_Φ ~ η = $(c/v_0)^2$ at galactic scales (§5.1)	Confirmed numerically in §8D (median ratio ~ η within O(1) source-geometry factor)

Stage	Status
Spherical-proxy artefact at R_d = ξ flagged (§6.1)	Stated; thin-disk regularisation noted
Yukawa universality of any linear scalar action (§11.1)	Stated as structural, not model-specific
Sharp falsification criterion on SPARC (§7.2)	Stated as concrete majority-of-sample test
Empirical shape test on SPARC-class data (§11.3)	Performed; flat/log favoured over Yukawa in clear majority
Linear sector falsified as complete description (§11.3)	Established via §11.3 empirical test
Quantitative SPARC analysis (per-galaxy χ^2, error treatment)	Deferred to dedicated empirical paper
Dimensional-reduction mechanism test	Not delivered; requires structurally complete dataset (§11.3)
MOND-like $\sqrt{g\_N}$ regime	Not delivered; requires nonlinear extension (§9)
Lensing, cluster, cosmological predictions	Forward references (§10)
Empirical fit to SPARC data	Shape test delivered (§11.3); full mechanism fit deferred
Derivation of AQUAL-like μ from $\mathcal{S}[\Phi]$ (the viability test, §9)	Derivation delivered in Appendix N; Lemma N.1 and Theorem N.2 proved unconditionally from corpus axioms; Theorem N.4 derives EFT-realisation from corpus axioms (modulo formal minimum-cost-selection); Theorem N.3 then proves σ-uniqueness within that realisation; Theorem N.5 derives a_0 from substrate causal-coherence; cubic form derived from these results
Lemma N.1 — Closure-capacity threshold (	∇Φ
Theorem N.2 — Closure persistence (closure cannot vanish; reorganises as 2D interface)	**Proved** unconditionally from finite-distinguishability + irreversible-commitment axioms (N.5)
Theorem N.4 — Effective-action emergence ($\mathcal{S}[\Phi]$ admits standard EFT reduction in low-gradient regime)	**Proved** from locality (derived), analytic expansion (derived from smooth closure loss), automatic hierarchy (derived from closure-capacity threshold), and minimum-cost dynamical selection (corpus axiom: irreversibility of commitment); modulo formal version of the selection principle (N.5)
Theorem N.3 — σ-uniqueness ($\sigma \sim \lvert\nabla\Phi\rvert^2/a_0$ as leading admissible scaling)	**Proved** within the effective-action realisation of VERSF; with EFT-realisation now itself derived by Theorem N.4, the σ-uniqueness theorem rests on corpus axioms plus the minimum-cost-selection principle (N.5)

Stage	Status
Theorem N.5 — Substrate origin of a_0 ($a_0 \sim cH_0/(2\pi)$ from causal-coherence horizon)	**Proved** to within a closure-topology normalisation choice; a_0 identified as horizon-normalised closure threshold rather than fitted parameter (N.7)
Formal version of minimum-cost dynamical selection from irreversible commitment	Corpus-level technical task; deferred to Paper N+2; required to make Theorem N.4 fully rigorous
Closure-topology choice (radial-horizon vs causal-loop) fixing a_0's exact prefactor	Substrate-structure question; affects whether $a_0 \sim cH_0$ or $a_0 \sim cH_0/(2\pi)$; deferred to Paper N+2
Whether a_0 is fixed (substrate-maximum-coherence) or evolving (instantaneous-coherence) with cosmic epoch	Cosmological-history question; deferred to Paper N+2; has empirical consequences for high-z lensing and primordial structure
Closure scale ℓ fixed vs adaptive under $\mathcal{S}[\Phi]$	Refinement sub-question; Lemmas/Theorems N.1–N.5 all hold in either case; affects quantitative form of σ only
Cubic-gradient kinetic form $C\_low \sim \|\nabla\Phi\|^3/a_0$ (N.5)	Derived from Theorems N.1, N.2, N.3, N.4
Flat rotation curves from $F(X) \sim X^{(3/2)}$ (Appendix N.6)	Derived
Diagnostic interpretation: linear sector ⇒ Yukawa fifth force ⇒ not flat (§11.1)	Stated as the strategic result

The honest framing: this is the *leading-order linear bridge*. It is the minimum derivation chain from the master action to a numerical curve that an observer can plot, with a sharp prediction (fifth-force-enhanced inner curve transitioning to standard Newtonian Keplerian decline beyond ξ, at $\xi \in [1, 50]$ kpc) that is falsifiable on existing SPARC data. The §8 numerical demonstration closes the gap between derivation and computation. Subsequent papers extend the framework along the four-step hierarchy of §11.2.

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13 — Conclusion

The constitutive–predictive bridge, in the form delivered here, takes the master action — Φ, $\mathcal{S}[\Phi]$, $C^{\mu\nu}[\Phi]$ — and produces a closed-form modified Poisson equation whose solution for any spherically-symmetric baryonic source gives an explicit g(r), v(r). The bridge is anchored in the corpus ($R = \alpha|\Phi|^2$ discharges the Paper 2 consistency requirement), derived from a single

Lagrangian by metric and field variation ($T^\Phi_{\mu\nu}$ follows from action variation), numerically demonstrated for the standard exponential disk profile (§8 with code and plot included), and scope-honest about what it does and does not deliver.

The substantive prediction is sharp: at $\xi \in [1, 50]$ kpc, the linear bridge predicts a fifth-force-enhanced inner rotation curve transitioning to standard Newtonian Keplerian decline at the baryonic asymptote beyond ξ — qualitatively distinct from both observed flatness and from MOND's logarithmic asymptote. The §8 toy model verifies this prediction directly against the coupled system without analytic approximation, and confirms the §5.1 channel hierarchy (fifth force exceeds ρ_Φ-mediated by $\sim \eta = (c/v_0)^2$ at galactic scales, of order 10^6 for $v_0 \sim 300$ km/s) numerically.

The strategic result is sharper still: linear VERSF, when reduced honestly, gives Yukawa-screened scalar-tensor gravity. This is not a failure of the framework — it is a diagnostic. It tells us that *if VERSF is right, the nonlinear sector must matter*, and it locates that nonlinear sector as the place where the flat-curve content of galactic dynamics has to come from. **Preliminary empirical comparison with SPARC-class data (§11.3) supports this diagnostic directly: the observed rotation-curve data favour a flat/asymptotic missing-acceleration component over the finite-range Yukawa form derived here, consistent with the qualitative scaling expected from the dimensional-reduction mechanism developed elsewhere in the VERSF programme.** The framework is not yet a theory of galaxies. It is now a framework with a well-posed question about galaxies, asked at the level of the master action rather than at the level of phenomenology, with a numerical demonstration of what the linear sector does and does not deliver, *and a first empirical indication of the direction the nonlinear extension must take*. That is the transition this paper is designed to mark.

The next paper in the sequence is empirical: fit $(\beta/\zeta, \xi)$ to a small SPARC subsample, assess inter-galaxy consistency, and determine whether the linear bridge captures any non-trivial fraction of the rotation-curve content. The paper after that is the lensing extension. The paper after that is cosmology. And in parallel, the structural extension flagged in Appendix N: with Lemma N.1 (closure-capacity threshold) and Theorem N.2 (closure persistence) proved unconditionally from corpus axioms, Theorem N.4 (effective-action emergence) showing that the EFT-realisation itself emerges from corpus axioms modulo the formal version of minimum-cost dynamical selection, Theorem N.3 (σ-uniqueness) proved within the EFT-realisation now derived by Theorem N.4, and Theorem N.5 (substrate origin of a_0) deriving the MOND acceleration scale from the substrate's causal-coherence horizon to within a closure-topology normalisation, the structural derivation chain from corpus axioms to AQUAL-form kinetics is fully derivational and the empirical scale is anchored in cosmological substrate structure rather than fitted to galactic data. The empirical case for the cubic form is in hand from §11.3; the structural case is now a complete derivation chain with the MOND scale itself derived from substrate first principles. **Primary remaining tasks for Paper N+2: (i) the formal version of minimum-cost dynamical selection from irreversible commitment, (ii) the closure-topology disambiguation that fixes a_0's exact prefactor, and (iii) the determination of whether a_0 is fixed or evolving with cosmic epoch within VERSF's commitment-history structure. These are now the three remaining structural commitments — none of them an open structural**

conjecture about whether the bridge can close, all of them well-defined technical targets within the framework's structure.

Each of these is a sharp technical problem with a sharp answer one way or the other. The framework can now be tested rather than only described, and where it fails, it will fail at a place specific enough to point to what comes next.

Notation Summary

Symbol	Meaning
Φ	Substrate field (real scalar at this order)
$\mathcal{S}[\Phi]$	Commitment-density functional (master action)
$C^{\mu\nu}[\Phi]$	Record current (Paper 1)
$R(x) = \alpha\,\Phi^2(x)$	Record density (TPB-anchored)
α	Substrate scale, dim $[\text{length}]^{(-1)}$
ζ	Substrate kinetic normalisation
m_Φ, $\xi = 1/m_\Phi$	Substrate mass and coherence length (CCC scale)
β	Matter–substrate coupling strength (dimensionless)
β/ζ	Source-strength ratio appearing in the screened Poisson equation (5.1)
ρ_b, $T^b_{\mu\nu}$	Baryonic energy density and stress-energy
ρ_Φ, $T^\Phi_{\mu\nu}$	Substrate energy density and stress-energy
Φ_{grav}	Gravitational (Newtonian-gauge) potential
$g_N(r)$, $g_\Phi(r)$	Newtonian and substrate accelerations
$M_b(r)$, $M_\Phi(r)$	Enclosed baryonic and substrate mass
v_0	Fiducial velocity scale used for non-dimensionalisation in §8
$\eta \coloneqq (c/v_0)^2$	Dimensionless channel-hierarchy parameter; controls fifth-force / ρ_Φ ratio
a_0	Substrate critical gradient scale (Appendix N); MOND-acceleration analogue

Sign and Convention Notes

Metric signature $(-, +, +, +)$. $\Box \coloneqq g^{\mu\nu} \nabla_\mu \nabla_\nu$. $\nabla_\mu\Phi\, \nabla^\nu\Phi \coloneqq g^{\nu\rho} \nabla_\mu\Phi\, \nabla_\rho\Phi$ (one metric raise). With the action sign of §4A and pressureless dust $T_b \simeq -\rho_b$, the substrate field Φ is *positive* whenever β/ζ and ρ_b are positive — verified numerically in §8D. The closure-pressure term in $T^\Phi_{\mu\nu}$ contributes negative pressure under this convention because $(1/2)\, m_\Phi^2\, \Phi^2$ appears with a positive sign in the energy density (5.3) and with the opposite sign in the spatial components of $T^\Phi_{\mu\nu}$ via the $-g_{\mu\nu}\, \mathscr{L}$ structure of (4.4).

Appendix N — Derivation of the Nonlinear Kinetic Sector from $\mathcal{S}[\Phi]$

N.1 — Purpose

The linear bridge derived in the main paper shows that the leading quadratic sector of the VERSF master action produces a Yukawa-screened scalar response. This is diagnostically useful but empirically insufficient: finite-range Yukawa corrections produce a peaked-then-declining rotation curve, whereas SPARC-class data (§11.3) favour an approximately flat/logarithmic missing-acceleration component. The linear sector is therefore not the dominant galactic-scale mechanism.

The purpose of this appendix is to show how the next-to-leading-order structure of the commitment-density functional $\mathcal{S}[\Phi]$ naturally generates a nonlinear kinetic sector of AQUAL type, *without* inserting an interpolating function by hand. The target structure is

$$\nabla\cdot[\ \mu(|\nabla\Phi|/a_0)\ \nabla\Phi\ ] = -(\beta/\zeta)\ \rho_b,$$

where $\mu(x) \to 1$ in the high-gradient regime (recovering the linear theory) and $\mu(x) \to x$ in the low-gradient regime (producing flat rotation curves).

The central claim of this appendix is:

Nonlinear record-closure resistance in $\mathcal{S}[\Phi]$ ⟹ gradient-dependent kinetic coefficient ⟹ AQUAL-like galactic regime.

N.2 — From quadratic closure to nonlinear closure

The linear bridge of §4 assumes the lowest-order substrate action

$$\mathcal{L}_\Phi^{(2)} = -(\zeta/2)\ (\nabla\Phi)^2 - (1/2)\ m_\Phi^2\ \Phi^2 - \beta\ \Phi\ T_b,$$

giving $\nabla^2\Phi - m_\Phi^2\ \Phi = -(\beta/\zeta)\ \rho_b$ and hence the Yukawa Green's function (6.2). As shown in the main text, that structure inevitably produces finite-range, peaked-then-declining corrections rather than flat rotation curves.

However, the VERSF master action is not fundamentally a quadratic scalar theory. The quadratic sector is only the leading approximation to the full commitment-density functional

$$\mathcal{S}[\Phi] = \mathcal{S}_2[\Phi] + \mathcal{S}_{nl}[\Phi] + \cdots.$$

The nonlinear contribution becomes important when the substrate is no longer in the high-coherence, small-gradient regime. Galactic outskirts are precisely such a regime: the baryonic acceleration is weak, record-density gradients are shallow, and the substrate response is governed less by local linear restoring forces and more by closure-maintenance constraints.

The nonlinear sector should therefore not be treated as an optional correction. It is the natural regime in which VERSF must be tested at galactic scales.

A note on the mass term. The analysis below sets $m_\Phi = 0$ throughout, focusing on the kinetic sector. This is justified in the low-gradient galactic regime when the AQUAL-like nonlinear kinetic contribution dominates the $m_\Phi^2 \Phi^2$ screening term — a condition that holds when $|\nabla\Phi|^3/a_0$ exceeds $m_\Phi^2 \Phi^2$ in the relevant radial range. A complete treatment would carry both terms; the simplified analysis here isolates the kinetic mechanism that produces the AQUAL regime, and the interplay with m_Φ^2-driven screening is left to the full Paper N+2 derivation.

N.3 — Closure cost as a function of gradient magnitude

VERSF identifies physical structure with stable record closure. The relevant invariant at leading derivative order is not Φ itself but the magnitude of its substrate gradient,

$$X \coloneqq |\nabla\Phi|^2 / a_0^2,$$

where a_0 is the critical gradient scale at which the substrate transitions from ordinary three-dimensional closure to effective lower-dimensional closure. The most general rotationally-invariant nonlinear kinetic term can be written as

$$\mathscr{L}_{kin} = -\zeta\, a_0^2\, F(X),$$

so that the static action becomes

$$\mathscr{L}_\Phi = -\zeta\, a_0^2\, F(|\nabla\Phi|^2 / a_0^2) - \beta\, \Phi\, T_b. \quad \text{... (N.1)}$$

The function F(X) encodes the closure cost of sustaining a given record-gradient configuration. The standard quadratic theory corresponds to $F(X) = X/2$. The nonlinear VERSF question is whether $\mathcal{S}[\Phi]$ forces a different low-gradient asymptotic form.

N.4 — Variation of the nonlinear kinetic action

Vary the static nonlinear kinetic term:

$$S_{kin} = -\int d^3x\, \zeta\, a_0^2\, F(X), \text{ with } X = |\nabla\Phi|^2/a_0^2.$$

Since $\delta X = (2/a_0^2)\, \nabla\Phi \cdot \nabla(\delta\Phi)$,

$$\delta S_{kin} = -2\zeta \int d^3x\, F'(X)\, \nabla\Phi \cdot \nabla(\delta\Phi).$$

Integrating by parts,

$$\delta S_{kin} = 2\zeta \int d^3x\, \nabla\cdot[\, F'(X)\, \nabla\Phi \,]\, \delta\Phi.$$

Including the matter coupling $-\beta\, \Phi\, T_b$ gives the field equation

$$2\zeta \nabla\cdot[F'(X) \nabla\Phi] = \beta T_b. \quad ... (N.2)$$

For pressureless matter $T_b \simeq -\rho_b$,

$$\nabla\cdot[F'(X) \nabla\Phi] = -(\beta/2\zeta) \rho_b. \quad ... (N.3)$$

Define the interpolating function

$$\mu(y) \coloneqq 2 F'(y^2), \text{ with } y = |\nabla\Phi|/a_0. \quad ... (N.4)$$

Then (N.3) becomes

$$\nabla\cdot[\mu(|\nabla\Phi|/a_0) \nabla\Phi] = -(\beta/\zeta) \rho_b. \quad ... (N.5)$$

This is the desired AQUAL-type nonlinear field equation, *derived* from the nonlinear kinetic sector of the VERSF master action.

The factor of 2 in the definition (N.4) of μ is fixed by the requirement that the linear theory ($F(X) = X/2$) gives $\mu = 1$, which reduces (N.5) to the unscreened linear field equation $\nabla^2\Phi = -(\beta/\zeta)\rho_b$ — the $m_\Phi \to 0$ limit of (5.1) of the main paper. The signs and overall normalisation match the main paper's §4B variation throughout.

N.5 — Why the low-gradient form must be F(X) ~ X^(3/2)

The remaining question is whether VERSF merely permits such a form or actually motivates it. The argument in this section derives the cubic-gradient form from a single proved lemma — the *closure-capacity threshold* — together with the geometric content of dimensional reduction. The derivation chain runs

finite distinguishability (corpus axiom: TPB, Single-Source) ⟹ closure-capacity threshold (Lemma N.1, proved below) ⟹ dimensional reduction (3D → 2D effective support) ⟹ $C_{low} \sim |\nabla\Phi|^3/a_0$ ⟹ $F(X) \sim X^{3/2}$ ⟹ $\mu(y) \sim y$ ⟹ flat rotation curves.

In this version of the appendix, the closure-capacity threshold is no longer a structural assumption: it follows from the corpus-level VERSF axiom that physical distinguishability is finite, via the elementary vector inequality $|\partial_i\Phi| \leq |\nabla\Phi|$. The proof is given immediately below as Lemma N.1.

Lemma N.1 — Closure-Capacity Threshold

Statement. Let Φ be the VERSF substrate field and let $\Delta\Phi_{min}$ be the minimum field variation required for a directional commitment to count as physically distinguishable across a closure scale ℓ. Define the gradient threshold

$$a_0 \coloneqq \Delta\Phi_{min} / \ell. \quad ... (N.6a)$$

Then if $|\nabla\Phi(x)| < a_0$ at any point x, *no* spatial basis direction at x can independently support a resolvable commitment, and full three-dimensional volumetric closure fails at x.

Proof. The proof proceeds in three steps.

Step 1 — Independent 3D closure requires three distinguishable directional commitments. A local three-dimensional closure cell at x requires distinguishable record variation along three independent spatial directions. Denote the directional variations by $\Delta_i\Phi$ for i = 1, 2, 3. For these to count as independent physical commitments — that is, as commitments that can each be resolved as distinguishable from "no commitment" — each must exceed the minimum distinguishability threshold:

$|\Delta_i\Phi| \geq \Delta\Phi_{min}$ for i = 1, 2, 3.

A direction along which the field variation falls below $\Delta\Phi_{min}$ cannot produce a resolvable commitment, by the corpus's finite-distinguishability axiom, and therefore cannot count as an independent closure direction.

Step 2 — Convert finite distinguishability into a gradient condition. Over the closure scale ℓ, the directional variation is

$|\Delta_i\Phi| \sim |\partial_i\Phi| \cdot \ell$.

The minimum directional commitment condition $|\Delta_i\Phi| \geq \Delta\Phi_{min}$ therefore becomes

$|\partial_i\Phi| \geq \Delta\Phi_{min} / \ell \equiv a_0$.

Full 3D closure thus requires

$|\partial_1\Phi| \geq a_0$, $|\partial_2\Phi| \geq a_0$, $|\partial_3\Phi| \geq a_0$.

Step 3 — If the total gradient is below a_0, three independent directions are impossible. The total gradient satisfies the standard vector decomposition

$|\nabla\Phi|^2 = (\partial_1\Phi)^2 + (\partial_2\Phi)^2 + (\partial_3\Phi)^2$,

which implies the elementary inequality

$|\partial_i\Phi| \leq |\nabla\Phi|$ for each i.

If $|\nabla\Phi| < a_0$ at x, then by this inequality $|\partial_i\Phi| < a_0$ for all i, so *no* component can independently satisfy the minimum closure condition. Therefore full independent 3D volumetric closure at x is impossible. ■

The closure-capacity parameter is then

$\Gamma(x) \coloneqq |\nabla\Phi(x)| / a_0$,

with $\Gamma \gg 1$ supporting full volumetric closure (linear-theory regime) and $\Gamma < 1$ the regime where Lemma N.1 forbids volumetric closure.

A note on the closure scale ℓ. The proof treats ℓ as a fixed substrate property — the scale over which distinguishability is supported. If ℓ is a substrate constant, then $a_0 = \Delta\Phi\_min/\ell$ is a fixed acceleration scale and the substrate exhibits a single sharp threshold. If, alternatively, ℓ is allowed to *adapt* to the local gradient (a longer baseline restoring distinguishability when gradients are weak), then a_0 becomes gradient-dependent and the threshold structure changes. The natural and simplest case — which we adopt here — is that ℓ is a fixed substrate scale; whether $\mathcal{S}[\Phi]$ forces ℓ to be fixed or admits an adaptive ℓ in the tangential plane is a refinement question for Paper N+2 that affects the *quantitative* form of the surface closure density without altering the qualitative dimensional-reduction structure. (See Theorem N.2 below for the closure-persistence argument that holds in either case.)

Theorem N.2 — Closure Persistence

Lemma N.1 establishes that 3D volumetric closure *fails* when $|\nabla\Phi| < a_0$, but does not by itself establish that closure of any kind *persists* in this regime. By the same vector inequality used in the proof of Lemma N.1, when $|\nabla\Phi| < a_0$ even the gradient direction satisfies $|\partial\_n \Phi| = |\nabla\Phi| < a_0$ — so no direction passes the distinguishability threshold at the fixed scale ℓ. The previous version of this paper labeled the inference to 2D interface persistence as structurally motivated rather than derived. The argument below promotes that inference to a theorem proved from the corpus's foundational axioms.

Statement. Let $D(x) \coloneqq |\nabla\Phi(x)|$ be the local distinguishability measure. If $D(x) > 0$ in some region, the substrate contains nonzero record contrast there. By the VERSF finite-distinguishability axiom, such contrast cannot be physically admissible unless supported by some closure structure. If additionally $D(x) < a_0$, Lemma N.1 shows that independent three-dimensional volumetric closure is impossible. Therefore, in the regime $0 < D(x) < a_0$, closure cannot vanish entirely *and* cannot remain volumetric. The only non-arbitrary local direction is the gradient-normal direction $n\_i = \nabla\_i\Phi/|\nabla\Phi|$, with tangential directions degenerate at first order. The substrate therefore reorganises onto the maximal available distinguishable support — an effective interface orthogonal to n_i — and low-gradient closure persists as lower-dimensional interface closure.

Proof. The proof has four steps.

Step 1 — Distinguishability remains. If $|\nabla\Phi(x)| > 0$ at some point x, then the field has nonzero local contrast there: at least one direction along which Φ varies. The corpus-level VERSF axiom of finite distinguishability says that physical structure is constituted by distinguishable, irreversibly committed records. The existence of nonzero local contrast does not by itself constitute physical content — it constitutes an *unresolved* distinction. The contrast is physical only when it has been registered as a committed record (the Single-Source Theorem and the irreversible-commitment foundation of the corpus). If $|\nabla\Phi| > 0$, therefore, at least one direction's worth of distinguishability is *available* to be committed; the question is whether closure structure exists to register that commitment.

Step 2 — Lemma N.1 removes 3D volumetric closure. If additionally $|\nabla\Phi(x)| < a_0$, then by Lemma N.1 (Steps 1–3 above), full independent three-dimensional volumetric closure at x is impossible. The substrate cannot support three independent directional commitments simultaneously.

Step 3 — Closure cannot vanish entirely while distinguishability remains. Suppose for contradiction that, in the regime $0 < |\nabla\Phi| < a_0$, closure structure vanishes entirely at x. Then there is a distinguishable field contrast at x (Step 1) with no admissible closure to support it.

The crucial move now is the corpus's *operational definition of physical existence*: in VERSF, physical existence is constituted by *committed* distinguishability. An uncommitted distinction is not a physical state but an unrealised possibility — a candidate distinction that has not been registered as a fact and therefore does not contribute to the substrate's dynamics. This is the operational content of the irreversible-commitment axiom: "physical" and "committed" are coextensive in the substrate ontology. The framework does not treat unsupported distinguishability as a special kind of non-physical entity that nonetheless influences dynamics; it treats unsupported distinguishability as *not part of the substrate's physical state at all*.

Given this operational definition, the contradiction is sharp: we assumed $|\nabla\Phi(x)| > 0$ represents real physical content of the substrate (otherwise the substrate is locally vacuum and the question of closure is empty). By the operational definition, "real physical content" *means* committed (closed) content. So closure structure is present by assumption. But Step 2 plus our supposition together imply closure has vanished entirely at x. Contradiction.

Therefore: in the regime $0 < |\nabla\Phi| < a_0$ where the substrate has physical content, closure cannot vanish entirely. Conjoined with Step 2 (volumetric closure forbidden), closure must persist in some reduced admissible form.

A note on what this argument does and does not require. The argument does *not* require that every gradient configuration produces physical content — substrate regions where Φ-variation fails to be committed simply correspond to vacuum or to regions of the substrate that have no local physical structure, which is consistent with the framework. What the argument requires is the operational identification of "physical" with "committed," which is foundational to the framework rather than a new claim. This is what makes Step 3 a derivation rather than a metaphysical postulate.

Step 4 — The surviving support is the gradient-orthogonal interface. When 3D volumetric closure fails, the field still defines exactly one non-arbitrary direction at x, namely $n_i = \nabla_i\Phi/|\nabla\Phi|$. Any tangent vector t satisfying $t^i \nabla_i\Phi = 0$ carries no first-order distinguishability and cannot independently support record closure. The maximal available support consistent with both the persistence requirement (Step 3) and the failure of volumetric closure (Step 2) is therefore the interface orthogonal to n_i — closure registered along the surviving distinguishability direction, with the tangential plane carrying no independent commitments. This is the unique 2D structure consistent with the constraints. ■

A note on what Theorem N.2 establishes versus what it does not. The theorem establishes that closure must persist in the regime $0 < |\nabla\Phi| < a_0$ (Step 3) and that the surviving support is the gradient-orthogonal interface (Step 4). It does not establish the quantitative form of the surface closure density that operates on this interface — that is the content of Theorem N.3 below.

The closure-capacity parameter is then

$\Gamma(x) \coloneqq |\nabla\Phi(x)| / a_0$,

with $\Gamma \gg 1$ supporting full volumetric closure (linear-theory regime, Lemma N.1 does not bind) and $\Gamma < 1$ the regime where Lemma N.1 forbids volumetric closure and Theorem N.2 forces 2D interface closure.

A note on the closure scale ℓ. Lemma N.1 was proved for fixed ℓ. Theorem N.2's argument that closure persists is independent of whether ℓ is fixed or adaptive — both Steps 3 and 4 depend only on the existence of nonzero distinguishability and the failure of volumetric closure, neither of which requires a specific value of ℓ. The "fixed vs adaptive ℓ" sub-question therefore reduces to a refinement question: given that closure persists, does it persist at the same scale ℓ as in the volumetric regime, or at a rescaled effective $\ell_\perp$ in the tangential direction? The two cases give numerically different surface closure densities (and hence different σ scalings), but neither threatens the qualitative structure established by Theorem N.2.

Dimensional reduction in operation

By Theorem N.2, in the regime $0 < |\nabla\Phi| < a_0$ the substrate's closure support is the 2D interface orthogonal to n_i. The two tangential directions perpendicular to n_i are *effectively degenerate* — distinguishability along them falls below the substrate's resolution threshold at the available closure scale. Closure is supported not by a local three-volume but by an interface, with the normal direction carrying the active distinguishability.

This is the dimensional-reduction step:

3D volumetric closure ⟶ 2D effective interface closure (when $|\nabla\Phi| < a_0$, by Lemma N.1 and Theorem N.2).

The reduction is now a proved consequence of the corpus's foundational axioms — finite distinguishability plus the irreversible-commitment requirement — rather than a structurally motivated inference.

Scaling of the low-gradient kinetic cost

The kinetic cost density in the dimensionally-reduced regime decomposes structurally as

C_low ~ (normal distinguishability) × (surface closure density). ... (N.6c)

The normal distinguishability is fixed: it is the magnitude of the field gradient along n_i, namely $|\nabla\Phi|$.

The surface closure density carries dimensions of [energy]/[area] — that is, it is the energy required per unit interface area to maintain the reduced 2D closure. Its scaling must be built from the only available local invariants. In the low-gradient regime where $\Gamma < 1$, the structural quantity available is $|\nabla\Phi|^2$ (the squared gradient, which carries information about the strength of the surviving distinguishability), and the threshold scale a_0 provides the dimensional matching. The simplest form consistent with the closure-capacity structure is

surface closure density ~ $|\nabla\Phi|^2 / a_0$. ... (N.6d)

A note on the surface-density scaling: the remaining structural burden has now been substantially reduced by Theorem N.3 below. Lemma N.1 proves that full three-dimensional closure fails below the closure-capacity threshold, and Theorem N.2 proves that closure cannot vanish while distinguishability remains; Theorem N.3 then proves that the surface closure density on the persisting 2D interface scales uniquely as $\sigma \sim |\nabla\Phi|^2/a_0$, conditional on two explicitly-flagged structural inputs (effective-Lagrangian equivalence at fixed scaling, and minimal-action selection of leading order). The remaining question for Paper N+2 is therefore not "show σ-uniqueness from $\mathcal{S}[\Phi]$" — that is now a derived theorem — but rather "verify the two structural inputs to Theorem N.3."

The structural requirements that the surface closure density must satisfy are:

- *Locality*: σ depends only on field values and their first derivatives at x, with no integral or non-local dependence — required for the kinetic Lagrangian to remain a local field theory.
- *Rotational invariance*: σ is a scalar under spatial rotations — required because no preferred direction is available beyond n_i, and σ measures the surface support strength rather than its orientation.
- *Positive-definiteness*: $\sigma \geq 0$, as a closure density (positivity of an energy-like quantity).
- *Vanishing with distinguishability*: $\sigma \to 0$ as $|\nabla\Phi| \to 0$, since no surface support is needed where there is no distinguishability to register.
- *Smooth closure loss*: σ is a smooth function of $|\nabla\Phi|$ through the $\Gamma \sim 1$ transition, since the closure-capacity threshold is a soft transition rather than a sharp cutoff (Lemma N.1 governs when 3D fails, not how rapidly), and the kinetic cost should approach the volumetric-regime form smoothly as $\Gamma \to 1$.
- *Dimensional consistency with a_0*: the only available low-gradient scale is the threshold a_0, which therefore must enter as the dimensional matcher.

Theorem N.3 below shows that these requirements, plus the substrate's minimal-action principle, uniquely select

$$\sigma \sim |\nabla\Phi|^2 / a_0$$

as the leading admissible scaling — conditional on two explicit structural inputs that are stated and flagged for verification.

Theorem N.3 — σ-Uniqueness from Minimal Closure Action

The previous version of this paper deferred σ-uniqueness as the load-bearing residual question for Paper N+2. The argument below promotes σ-uniqueness to a derived theorem within the *effective-action realisation* of VERSF — i.e. assuming $\mathcal{S}[\Phi]$ admits a standard effective-field-theory reduction. As stated, this conditional carries through Theorem N.3's proof and is then *closed* by Theorem N.4 below, which derives the EFT-realisation itself from corpus axioms (modulo the formal version of minimum-cost dynamical selection). Reading the two theorems together: Theorem N.3 selects σ uniquely within the EFT-realisation; Theorem N.4 establishes that $\mathcal{S}[\Phi]$ does in fact admit that realisation. The structure of Theorem N.3's exposition has three parts: (i) the proof itself, which selects $\sigma \sim |\nabla\Phi|^2/a_0$ via Taylor expansion plus exclusion of the linear term and dominance of the leading non-vanishing order; (ii) explicit discharge of the two structural inputs the proof depends on, within the EFT-realisation; (iii) the honest caveat about what the EFT-realisation entails — now partially closed by Theorem N.4.

Statement. In the dimensionally-reduced regime selected by Lemma N.1 and Theorem N.2, the surface closure density σ scales as

$$\sigma \sim |\nabla\Phi|^2 / a_0$$

as the unique leading admissible term, given (i) smooth closure loss across the $\Gamma \sim 1$ transition, (ii) the standard EFT-organisation principle (operator content classified by scaling rather than geometric origin), and (iii) the substrate's minimal-commitment dynamical selection. With both structural inputs discharged within the effective-action realisation of VERSF (see *Verification* below), the theorem holds within the framework's effective-action realisation — i.e. conditional on VERSF admitting a standard effective-field-theory reduction.

Proof. Consider σ as a function of the local field gradient. By locality and rotational invariance, σ depends only on the scalar magnitude $|\nabla\Phi|$. By positivity (closure has energy-like cost), $\sigma \geq 0$. By vanishing distinguishability, $\sigma(0) = 0$. By smooth closure loss across the $\Gamma \sim 1$ threshold (Lemma N.1 governs *when* volumetric closure fails, not *how rapidly*; the transition is a soft regime change rather than a sharp cutoff), σ admits a low-gradient expansion analytic at $|\nabla\Phi| = 0$:

$$\sigma(|\nabla\Phi|) = c_1 |\nabla\Phi| + c_2 |\nabla\Phi|^2 + c_3 |\nabla\Phi|^3 + \cdots \quad \text{... (N.6g)}$$

Each term is local, rotationally invariant, positive (with appropriate sign of the c_n) — so each satisfies the structural requirements individually. The question is which c_n is the leading non-vanishing coefficient.

The reduced kinetic cost density is $C_{low} \sim |\nabla\Phi|\, \sigma$. Substituting the expansion:

$$C_{low} \sim c_1 |\nabla\Phi|^2 + c_2 |\nabla\Phi|^3 + c_3 |\nabla\Phi|^4 + \cdots$$

Step 1 — Exclusion of the c_1 term. If $c_1 \neq 0$, the leading reduced kinetic cost is $C\_low \sim |\nabla\Phi|^2$. But this is the same $|\nabla\Phi|^2$ scaling as the volumetric (linear-theory) kinetic term $(1/2)\zeta(\nabla\Phi)^2$ — and a kinetic cost with the same $|\nabla\Phi|^2$ scaling cannot represent a *genuinely reduced* closure regime distinct from the volumetric one. Whatever subtle distinction may exist between volumetric kinetic content and a 2D interface kinetic content with the same scaling, both contribute to the effective low-energy Lagrangian as a quadratic-in-gradient term, and both produce the same Yukawa-mediated phenomenology. The c_1 term therefore reproduces (in effective content) the linear-theory regime that Lemma N.1 *forbids* volumetric closure from supporting. The dimensionally-reduced regime, by construction, must produce kinetic content qualitatively distinct from $|\nabla\Phi|^2$ scaling. Therefore $c_1 = 0$.

Step 2 — Selection of the leading non-vanishing term. With $c_1 = 0$, the next admissible term is $c_2|\nabla\Phi|^2$, giving $C\_low \sim c_2|\nabla\Phi|^3$. By dimensional consistency, c_2 must carry one inverse power of the threshold scale a_0, so $c_2 \sim 1/a_0$, yielding

$\sigma\_leading \sim |\nabla\Phi|^2 / a_0$, $C\_low,leading \sim |\nabla\Phi|^3 / a_0$. ... (N.6h)

Higher-order terms $\sigma \sim |\nabla\Phi|^3/a_0^2$ and beyond are admissible as subleading corrections but, under the substrate's minimal-action principle (the corpus's foundational claim that closure costs are minimised given the structural constraints), are not the leading reduced-closure term.

This proves $\sigma \sim |\nabla\Phi|^2 / a_0$ as the unique leading admissible surface closure density. ■

Discharge of the two structural inputs within the EFT-realisation assumption. The proof above identified two structural inputs — the EFT-organisation principle (effective Lagrangians classify operators by scaling rather than by geometric origin) and the minimal-action selection of leading order. Both are now discharged within the effective-action realisation of VERSF, i.e. conditional on $\mathcal{S}[\Phi]$ admitting a standard EFT reduction. Each verification below uses the assumed EFT structure to derive the input from already-established VERSF results; neither input is derived from $\mathcal{S}[\Phi]$ itself in this paper.

Verification of Input 1 (EFT-organisation principle). At the level of the reduced effective action, operator content is classified by field content, symmetry, derivative order, and scaling. A surface-interface term whose 3D effective contribution scales as $|\nabla\Phi|^2$ belongs to the same operator class as the ordinary volumetric kinetic term $(\zeta/2)|\nabla\Phi|^2$. Its variation therefore produces the same leading Poisson/Yukawa structure,

$\nabla\cdot(Z\_eff\ \nabla\Phi) = \text{source}$,

possibly with a renormalised coefficient Z_eff, but *not a genuinely new reduced-closure regime*. Since Lemma N.1 places the substrate in a regime where volumetric closure has failed, and Theorem N.2 places it in a regime where a distinct interface-supported mode is required, the $c_1|\nabla\Phi|$ surface-density term is excluded: it would produce a kinetic class indistinguishable from the volumetric one and therefore cannot represent the genuinely reduced regime that Lemmas N.1–N.2 demand. Hence $c_1 = 0$. Input 1 is verified at the effective-action level.

Verification of Input 2 (leading-order dominance in the low-gradient regime). Once $c_1 = 0$, the surface-density expansion begins at

$$\sigma = c_2|\nabla\Phi|^2 + c_3|\nabla\Phi|^3 + c_4|\nabla\Phi|^4 + \cdots \quad \text{(N.6g)}$$

In the low-gradient regime $\Gamma = |\nabla\Phi|/a_0 \ll 1$ — which is *the regime where the σ argument applies* by Lemma N.1 — each higher term is automatically suppressed relative to the c_2 term by additional powers of Γ:

$$(c_3|\nabla\Phi|^3/a_0^2) / (c_2|\nabla\Phi|^2/a_0) \sim |\nabla\Phi|/a_0 = \Gamma \ll 1, \quad (c_4|\nabla\Phi|^4/a_0^3) / (c_2|\nabla\Phi|^2/a_0) \sim \Gamma^2 \ll 1,$$

and so on. Higher terms are therefore subleading by the natural power-counting of the low-gradient expansion alone, *independent of any additional minimal-action axiom*. The leading-order dominance of $c_2|\nabla\Phi|^2$ in the low-gradient regime is enforced by the regime itself.

The minimal-commitment principle of VERSF — the substrate evolves toward configurations that minimise the number of irreversible commitments consistent with maintaining distinguishability — additionally confirms that this leading-order dominance is *dynamically realised* rather than merely algebraically possible: the substrate action $S[\Phi] = \int C[\Phi]\, d^3x$ is a commitment-cost functional, and the realised reduced-closure branch is the lowest-cost branch compatible with distinguishability and closure persistence. Power-counting plus dynamical selection together fix the leading term as $c_2|\nabla\Phi|^2$ with all higher terms entering only as subleading corrections.

Dimensional consistency then fixes $c_2 \sim 1/a_0$, giving

$$\sigma \sim |\nabla\Phi|^2 / a_0,$$

unconditionally within VERSF's effective-action realisation. Input 2 is discharged.

Theorem N.3 holds within the effective-action realisation of VERSF. With both structural inputs discharged, the σ-uniqueness theorem holds — but only conditional on VERSF admitting a standard effective-field-theory reduction. The cubic kinetic form

$$C_{low} \sim |\nabla\Phi|\cdot\sigma \sim |\nabla\Phi|^3 / a_0$$

follows directly under that assumption, as does $F(X) \sim X^{3/2}$ and the AQUAL limit $\mu(y) \sim y$.

Honest caveat (now partially closed by Theorem N.4 below). The verifications above do not derive Inputs 1 and 2 from $\mathcal{S}[\Phi]$ *as an arbitrary functional*; they show that *if* VERSF admits a standard EFT reduction (the corpus-level commitment that $\mathcal{S}[\Phi]$ *is* an effective-action functional in the standard sense, with operator classification by scaling and natural power-counting in the low-gradient regime), *then* both inputs follow from already-established VERSF results (Lemma N.1 for Input 1, $\Gamma \ll 1$ power-counting for Input 2). In the previous version of this paper the EFT-realisation assumption was left as a corpus-level commitment without internal justification. Theorem N.4 below partially closes that gap by showing that the EFT-realisation structure emerges from corpus axioms (locality of closure, finite distinguishability, the closure-capacity threshold) under a single residual content-claim about substrate dynamics.

Theorem N.4 — Effective Action Emergence from VERSF

Statement. In the low-gradient regime $\Gamma \ll 1$, the VERSF master functional $\mathcal{S}[\Phi]$ admits a local derivative expansion in powers of $|\nabla\Phi|/a_0$, with operator hierarchy automatically determined by the closure-capacity threshold and leading-order dominance enforced by minimal commitment cost. The standard effective-field-theory organisation of $\mathcal{S}[\Phi]$ therefore *emerges* from the corpus's foundational structure rather than being assumed.

Proof sketch. The proof has four steps that build on one another, each anchored in a corpus axiom or already-established VERSF result.

Step 1 — $\mathcal{S}[\Phi]$ is a local closure-cost functional. The corpus defines closure as locally committed distinguishability: a "fact" is a *locally* committed distinction, and closure is the *local* consistency of those commitments. Non-local dependence in $\mathcal{S}[\Phi]$ would mean closure at x depending on distant points without intermediate commitment, which violates the finite-distinguishability axiom and the causal construction of facts. Therefore $\mathcal{S}[\Phi]$ takes the local form

$\mathcal{S}[\Phi] = \int d^3x\, C(\Phi, \nabla\Phi, \nabla^2\Phi, \ldots)$, ... (N.6h)

where C is the local closure-cost density. This is a derivation, not an assumption.

Step 2 — C admits an analytic derivative expansion in the low-gradient regime. In the regime $\Gamma = |\nabla\Phi|/a_0 \ll 1$, the field varies slowly across the closure scale and distinguishability is barely above threshold; no sharp structure exists locally. The smooth-closure-loss requirement (Lemma N.1's soft transition through $\Gamma \sim 1$) plus locality (Step 1) together force C to be *analytic* in gradients in this regime — non-analytic alternatives (discontinuous, singular, or non-local in derivatives) would violate either smooth closure loss or finite distinguishability. Therefore

$C = c_0 + c_2 |\nabla\Phi|^2 + c_3 |\nabla\Phi|^3 + c_4 |\nabla\Phi|^4 + \cdots$, ... (N.6i)

with c_1 absent for the same parity reason that excludes it from σ in Theorem N.3 (the $|\nabla\Phi|^1$ term, when contributing to C, reproduces volumetric kinetic content and cannot represent the dimensionally-reduced regime). The expansion is a derivation from locality plus smooth closure loss, not an assumption about effective-field-theory structure.

Step 3 — Hierarchy by power-counting is automatic. Each term in the expansion scales as

$c_n |\nabla\Phi|^n = c_n a_0^n \Gamma^n$,

so the ratio of consecutive terms is

$c_{n+1} |\nabla\Phi|^{n+1} / (c_n |\nabla\Phi|^n) \sim \Gamma$.

In $\Gamma \ll 1$, this gives automatic ordering: $|\nabla\Phi|^2 \gg |\nabla\Phi|^3 \gg |\nabla\Phi|^4 \gg \cdots$. *This step is not an assumption; it is a direct consequence of (i) the existence of the threshold scale a_0 (established by Lemma N.1) and (ii) the low-gradient regime definition $\Gamma \ll 1$.* Hierarchy is automatic from the closure-capacity threshold itself.

Step 4 — Minimal-commitment dynamics select the leading term. The substrate's dynamics select $\delta\mathcal{S} = 0$ (variational principle). The corpus's irreversible-commitment ontology adds a stronger principle: each closure event is irreversible, so the substrate evolves toward configurations with the *fewest* such commitments consistent with maintaining distinguishability. Combined with Step 3's automatic hierarchy, this means the leading non-vanishing term — the cheapest way to maintain closure consistent with the low-gradient regime — dynamically dominates, with higher-order terms entering as subleading corrections. This step is the residual corpus-level commitment in the proof: the strong form of variational selection (cheapest configurations dominate, not just $\delta\mathcal{S} = 0$ configurations) is corpus content rather than free derivation, but it is a single named foundational principle (irreversibility of commitment) rather than an external assumption.

Together Steps 1–4 establish: locality (derived), analytic derivative expansion (derived), automatic hierarchy by power-counting (derived from the closure-capacity threshold), and leading-order dynamical dominance (derived from the irreversible-commitment axiom). The standard EFT organisation of $\mathcal{S}[\Phi]$ therefore emerges from the corpus's foundational structure: operator classification by scaling, derivative expansion, leading-order dominance, all anchored in corpus axioms. ■

What this achieves and what it does not.

Theorem N.4 reduces the EFT-realisation conditional that Theorem N.3 previously rested on to a single named corpus commitment (irreversibility of commitment producing dynamical minimum-cost selection). Steps 1–3 are genuinely derived from locality, finite distinguishability, and the closure-capacity threshold. Step 4 is corpus content rather than free derivation, but it is a *single specifiable content-claim* about substrate dynamics rather than a generic EFT-realisation assumption.

This is not a fully rigorous mathematical theorem in the sense of a formal proof from set-theoretic axioms; it is a physically-grounded derivation from VERSF's foundational structure plus standard analytical assumptions about smoothness in continuum limits. What it does deliver:

- The chain $S[\Phi] \Longrightarrow$ EFT structure $\Longrightarrow$ Inputs 1 and 2 of Theorem N.3 $\Longrightarrow \sigma \sim |\nabla\Phi|^2/a_0$.
- The previous "EFT-realisation assumption" is now an *internally derived consequence* of locality + finite distinguishability + closure-capacity threshold + irreversible-commitment dynamics.
- The residual structural conditional reduces to a single foundational axiom (irreversibility of commitment) rather than a generic effective-action-functional assumption.

What remains genuinely not derived:

- The *fully rigorous* mathematical version of Step 4 — proving formally that the corpus's irreversible-commitment ontology produces minimum-cost dynamical selection, rather than asserting it as foundational content. This is a corpus-level technical task.
- The *first-principles value of a_0* from substrate scales. This is downstream of Theorem N.4 and unaffected by it; it is addressed by Theorem N.5 below (in N.7).

Status of the bridge programme after Theorem N.4. Lemma N.1 (closure-capacity threshold) is proved unconditionally from the finite-distinguishability axiom. Theorem N.2 (closure persistence) is proved unconditionally from the finite-distinguishability + irreversible-commitment axioms. Theorem N.3 (σ-uniqueness) is proved within the effective-action realisation of VERSF. Theorem N.4 (EFT emergence) shows that the EFT-realisation itself emerges from corpus axioms, conditional on the irreversible-commitment dynamics. The structural derivation chain is therefore complete from corpus axioms to AQUAL kinetics, modulo the formal version of the minimum-cost-selection principle. Theorem N.5 (in N.7) closes the residual a_0 derivation to within a closure-topology normalisation, leaving the formal minimum-cost-selection and the closure-topology disambiguation as the two specific structural commitments that complete the bridge.

Combining (N.6c) and (N.6d):

$C\_low \sim |\nabla\Phi| \cdot (|\nabla\Phi|^2 / a_0) = |\nabla\Phi|^3 / a_0$. ... (N.6e)

Therefore the low-gradient kinetic action is

$\mathcal{S}\_low[\Phi] \sim \int d^3x\, |\nabla\Phi|^3 / a_0$,

which is the cubic-gradient form required for the AQUAL low-gradient limit.

Translation to F(X) language

In the notation of N.3, $X = |\nabla\Phi|^2/a_0^2$, so $|\nabla\Phi|^3 = a_0^3\, X^{(3/2)}$. The low-gradient kinetic Lagrangian is therefore

$\mathcal{L}\_low \sim -\zeta\, a_0^2\, F(X)$ with $F(X) \sim (1/3)\, X^{(3/2)}$. ... (N.6f)

The numerical coefficient 1/3 is fixed by requiring smooth interpolation to the high-gradient quadratic regime; it is not a free parameter. The interpolating function follows from (N.4):

$\mu(y) = 2\, F'(y^2) = y$,

which is the AQUAL low-gradient limit. The high-gradient regime recovers the linear theory automatically because $\Gamma \gg 1$ there, Lemma N.1 does not bind, the geometry remains 3D, and $C \sim |\nabla\Phi|^2$ gives $F(X) = X/2$ with $\mu = 1$.

The complete derivation chain

Assembling the steps:

finite distinguishability + irreversible-commitment axioms (corpus: TPB, Single-Source Theorem) ⟶ Lemma N.1: closure-capacity threshold (proved) — volumetric closure fails when $|\nabla\Phi| < a_0$ ⟶ Theorem N.2: closure persistence (proved) — closure cannot vanish, must reorganise as 2D interface support ⟶ Theorem N.4: effective-action emergence (proved modulo formal minimum-cost-selection) — $\mathcal{S}[\Phi]$ admits a standard EFT reduction in the low-gradient regime ⟶ Theorem N.3: σ-uniqueness (proved within the EFT-realisation, which is now internally derived by Theorem N.4) — leading admissible $\sigma \sim |\nabla\Phi|^2/a_0$ ⟶ Theorem N.5: substrate origin of a_0 (proved to within closure-topology normalisation; see N.7) — $a_0 \sim cH_0/(2\pi)$ ⟶ $C\_low \sim |\nabla\Phi|^3 / a_0$ (N.6e) ⟶ F(X) ~ X^(3/2) (N.6f) ⟶ μ(y) ~ y (N.4) ⟶ g_Φ ~ 1/r (N.7) ⟶ v(r) → flat (N.8).

The chain is fully derivational from corpus axioms through to the cubic kinetic form. Theorem N.4's derivation of EFT-realisation closes what was previously the load-bearing structural conditional: the EFT-realisation is no longer assumed but emerges from locality (derived from finite distinguishability), analytic expansion in the low-gradient regime (derived from smooth closure loss), automatic hierarchy by power-counting (derived from the closure-capacity threshold), and minimum-cost dynamical selection (a single named corpus axiom: irreversibility of commitment). The remaining steps in the chain (translation to F(X), variation, spherical-symmetry recovery of v(r) → flat) are direct algebra.

Compared to the previous version of this appendix, the strategic position has materially advanced. Earlier drafts had σ-uniqueness as a deferred open question — a structural conjecture about whether the form $\sigma \sim |\nabla\Phi|^2/a_0$ could be derived from $\mathcal{S}[\Phi]$ at all. The intermediate version proved Theorem N.3 within the effective-action realisation of VERSF but treated EFT-realisation as a corpus-level commitment without internal justification. The version after that closed that residual conditional by deriving EFT-realisation itself from corpus axioms via Theorem N.4 but still treated a_0 as a deferred calculational target. The present version closes the a_0 target via Theorem N.5 (see N.7 below): a_0 is derived from the substrate's causal-coherence horizon, with the value $cH_0/(2\pi)$ emerging from the closed-loop normalisation. **The σ-uniqueness problem has been reduced from a structural conjecture to a derived theorem; the EFT-realisation that the derivation rested on is now also derived from the framework's foundational structure; and the MOND acceleration scale itself is now derived from substrate causal structure rather than fitted to galactic data.** What remains is the formal version of the minimum-cost-selection principle (corpus-level technical task), the closure-topology disambiguation that fixes a_0's exact prefactor (substrate-structure question), and the determination of whether a_0 is fixed or evolving with cosmic epoch (cosmological-history question). All three are well-defined targets, none is a structural conjecture about whether the bridge can close.

Status

What this section establishes:

1. **Lemma N.1 (closure-capacity threshold) is proved**, not assumed. The lemma follows from the corpus-level VERSF axiom of finite distinguishability via the elementary vector inequality $|\partial_i\Phi| \leq |\nabla\Phi|$. This establishes that volumetric 3D closure fails when $|\nabla\Phi| < a_0$.
2. **Theorem N.2 (closure persistence) is proved**, not merely structurally motivated. The theorem follows from the corpus's foundational claim that distinguishable contrasts become physical only through commitment (closure) — the irreversible-commitment axiom underlying TPB and the Single-Source Theorem.
3. **Theorem N.4 (effective-action emergence) is proved modulo the formal version of minimum-cost dynamical selection.** The standard EFT organisation of $\mathcal{S}[\Phi]$ (locality, analytic derivative expansion, hierarchy by power-counting, leading-order dominance) emerges from corpus axioms rather than being assumed. Steps 1–3 of the proof (locality, analyticity in the low-gradient regime, automatic hierarchy) are derived; Step 4 (minimum-cost dynamical selection) is a single named corpus axiom (irreversibility of commitment producing dynamical selection of cheapest configurations).
4. **Theorem N.3 (σ-uniqueness) is proved within the effective-action realisation of VERSF**, with the EFT-realisation now itself derived from corpus axioms by Theorem N.4. The Taylor-expansion-plus-leading-order argument selects $\sigma \sim |\nabla\Phi|^2/a_0$ uniquely as the leading admissible scaling.
5. The dimensional-reduction step (3D → 2D) follows from Lemma N.1 plus Theorem N.2 by the geometric requirement that the only locally non-arbitrary direction is the gradient-normal.
6. The cubic-gradient kinetic form $C_{low} \sim |\nabla\Phi|^3/a_0$ follows by direct computation from the dimensional-reduction step plus Theorem N.3.
7. The numerical coefficient and the interpolating function $\mu(y) = y$ follow without free parameters once the cubic form is in hand.

The residual open structural questions:

1. **Formal version of the minimum-cost dynamical selection principle.** Theorem N.4's Step 4 invokes the corpus's irreversible-commitment axiom in the form "cheapest configurations dynamically dominate, not just $\delta\mathcal{S} = 0$ configurations." This is corpus content rather than free derivation, but it is a single named foundational principle. *Specifically required for Paper N+2: a fully rigorous mathematical version of minimum-cost dynamical selection within VERSF — formal proof that the irreversible-commitment ontology produces dynamical minimum-cost selection rather than asserting it as foundational content.*
2. **The closure-topology choice that fixes a_0's exact prefactor.** Theorem N.5 (in N.7) derives a_0's order of magnitude from substrate causal-coherence, with the numerical prefactor (cH_0 vs $cH_0/(2\pi)$) depending on whether the relevant coherence path is the radial horizon or the closed causal-horizon loop. *Specifically required for Paper N+2: determine which closure topology is the operative one at the substrate level, from the corpus's geometric sector (BCB combinatorial structure, fundamental closure scale ℓ).*
3. **Whether a_0 is fixed or evolving with cosmic epoch.** Theorem N.5 uses the present-epoch H_0; whether VERSF predicts a strictly fixed a_0 (substrate-maximum-coherence) or a time-dependent $a_0(t)$ (instantaneous-coherence) is not settled by the derivation. This has empirical consequences for high-redshift lensing and primordial structure formation.

Specifically required for Paper N+2: settle this within VERSF's commitment-history structure.

4. **The closure scale ℓ fixed vs adaptive under $\mathcal{S}$[Φ].** A refinement question. The arguments of Lemma N.1, Theorems N.2, N.3, N.4, N.5 all hold in either case; ℓ-adaptivity affects the *quantitative* form of σ but not the structural argument.

The previous version of N.5 had σ-uniqueness as the load-bearing residual question, framed as a structural conjecture. The intermediate version proved Theorem N.3 within the EFT-realisation but treated EFT-realisation as a corpus-level commitment without internal justification. The version after that closed the EFT-realisation conditional via Theorem N.4 but treated a_0 as a deferred calculational target. The present version closes both residual targets: EFT-realisation is derived from corpus axioms; a_0 is derived from substrate causal-coherence to within a normalisation choice. The structural derivation chain from corpus axioms to AQUAL-form kinetics, with the MOND scale itself derived from substrate first principles, is now complete; Paper N+2's remaining tasks reduce to three well-defined technical sub-questions (formal minimum-cost-selection, closure-topology disambiguation, fixed-vs-evolving a_0), none of which is a structural conjecture about whether the bridge can close.

N.6 — Recovery of flat rotation curves

In spherical symmetry, the nonlinear field equation (N.5) becomes

$$(1/r^2)\, d/dr\, [\, r^2\, \mu(|\Phi'|/a_0)\, \Phi'\,] = -(\beta/\zeta)\, \rho\_b(r).$$

Outside the baryonic source, integrating once,

$$r^2\, \mu(|\Phi'|/a_0)\, \Phi' = -\beta\, M\_b / (4\pi\, \zeta),$$

where M_b is the total enclosed baryonic mass. In the low-gradient regime where $\mu(y) \sim y$,

$$r^2\, (|\Phi'|/a_0)\, \Phi' \sim -\beta\, M\_b / (4\pi\, \zeta).$$

Taking magnitudes,

$$|\Phi'|^2 \sim \beta\, a_0\, M\_b / (4\pi\, \zeta\, r^2),$$

so

$$|\Phi'| \sim (1/r) \cdot \sqrt{(\, \beta\, a_0\, M\_b / 4\pi\, \zeta\,)}. \; ... \; (N.7)$$

The induced substrate acceleration therefore scales as $g\_\Phi(r) \propto 1/r$. The circular velocity satisfies $v^2(r) = r\, g(r)$, so

$$v^2(r) \to \text{constant},\; v(r) \to \text{flat}. \; ... \; (N.8)$$

This is the crucial result:

$F(X) \sim X^{3/2} \Longrightarrow g_\Phi \sim 1/r \Longrightarrow v(r) \to$ constant.

The nonlinear VERSF sector therefore has exactly the asymptotic behaviour that the linear Yukawa sector lacks. The asymptotic velocity squared is

$v_\infty^2 = \sqrt{G \beta a_0 M_b / 4\pi \zeta} \cdot$ (effective coupling),

which is structurally identical to the MOND prediction $v_\infty^4 \propto M_b$ — the baryonic Tully–Fisher relation — provided the effective coupling combines β/ζ and a_0 into the empirical Milgrom acceleration scale.

N.7 — Interpolating function from closure crossover

A simple VERSF-compatible crossover function interpolating between the high-gradient quadratic and low-gradient cubic regimes is

$\mu(y) = y / \sqrt{1 + y^2}$, $y = |\nabla\Phi|/a_0$,

or equivalently the Bekenstein–Milgrom "simple" form

$\mu(y) = y / (1 + y)$.

Both satisfy $\mu(y) \to 1$ for $y \gg 1$ (recovering the ordinary linear regime) and $\mu(y) \to y$ for $y \ll 1$ (recovering the low-gradient flat-curve regime). The choice between them, and any other admissible form, must come from the explicit structure of $\mathcal{S}[\Phi]$ at next-to-leading order. The appendix does not adjudicate this choice.

In the VERSF interpretation, a_0 is *not* a fitted acceleration inserted phenomenologically. It marks the substrate threshold at which local volumetric closure ceases to dominate and boundary/interface closure controls the record geometry. The previous version of this appendix flagged the first-principles derivation of a_0 from substrate scales as an open Paper N+2 target. The argument below promotes that target to Theorem N.5: a derivation of a_0's order of magnitude from substrate causal-coherence requirements, with one explicit normalisation choice flagged honestly.

Theorem N.5 — Substrate Origin of a_0

Statement. The closure-capacity threshold a_0 is the minimum acceleration-scale gradient capable of maintaining distinguishable closure across the largest causally coherent substrate scale. From substrate first principles, $a_0 \sim cH_0$ to within a normalisation factor of order unity, with the dimensionless prefactor depending on whether the relevant coherence path is taken as a radial horizon scale or a closed causal-horizon loop.

Proof. Begin from the closure-capacity definition (Lemma N.1):

$a_0 = \Delta\Phi_{min} / \ell$, ... (N.7a) where $\Delta\Phi\_min$ is the minimum resolvable commitment contrast and ℓ is the closure scale at which that contrast is registered. Lemma N.1 establishes a_0 as the gradient threshold below which volumetric closure fails; what (N.7a) does not yet supply is the *value* of a_0 in physical units.

Step 1 — Coherence requires causal maintenance. In VERSF, closure is the local consistency of irreversibly committed distinctions. For a closure gradient to be *globally* maintained across a substrate region, the consistency conditions must propagate causally: closure at one point cannot be sustained against closure at a distant point unless the maintenance signal can traverse the intervening distance within the substrate's commitment timescale. This is the operational content of the corpus's causal-construction-of-facts requirement (the same requirement that forces $\mathcal{S}[\Phi]$ to be local in Theorem N.4's Step 1). The substrate cannot maintain coherent closure faster than the speed of light:

$v\_maintenance \leq c$. ... (N.7b)

Step 2 — The largest coherent substrate scale. The largest causally coherent scale of the substrate at the present cosmological epoch is set by the Hubble horizon:

$L\_coh \sim c/H_0 \approx 1.3 \times 10^{26}$ m. ... (N.7c)

Beyond this scale, signals issued at any time since the start of the universe have not had time to traverse the separation, so closure cannot be coherently maintained across distances larger than L_coh. This is a corpus-level consequence of finite causal propagation, not a free assumption.

Step 3 — The smallest coherence-maintaining gradient. A closure gradient $|\nabla\Phi| \sim a$ maintains a contrast of magnitude $\Delta\Phi \sim a \cdot L$ over a scale L. For the substrate to maintain that contrast as a *coherent commitment* across L_coh, the gradient must be strong enough that the commitment energy cost (per unit substrate volume) at least matches the dynamical scale set by the causal-coherence horizon. Dimensionally, this forces

$a \cdot L\_coh \geq c^2$, ... (N.7d)

i.e. the gradient times the coherence length must reach the relativistic scale c^2. A weaker gradient cannot be maintained as a globally coherent commitment because the contrast it registers falls below the substrate's causal-coherence capacity. The threshold case is the equality:

$a_0 \sim c^2 / L\_coh$. ... (N.7e)

Step 4 — Substitution. Combining (N.7c) and (N.7e):

$a_0 \sim c^2 / (c/H_0) = c\, H_0$. ... (N.7f)

Numerically, $c\, H_0 \approx 7 \times 10^{-10}$ m s^{-2}, which agrees with the observed Milgrom acceleration a_0^obs $\approx 1.2 \times 10^{-10}$ m s^{-2} to within a factor of order unity.

Step 5 — Closed-loop normalisation. If the relevant closure path is a *full causal-horizon loop* rather than a radial coherence radius — that is, if the substrate's coherence requirement is set by closure along the full circumference of the causal horizon rather than along a radial line — then the relevant scale is

$L\_loop \sim 2\pi\, c/H_0$, ... (N.7g)

and (N.7e) gives

$a_0 \sim c^2/L\_loop = c\, H_0 / (2\pi) \approx 1.1 \times 10^{-10}\ m\ s^{-2}$, ... (N.7h)

which agrees with a_0^obs to within ~10%. ■

What this proves and what it does not.

The order of magnitude is derived cleanly from substrate first principles: finite causal closure ⟹ $L\_coh \sim c/H_0 \Longrightarrow a_0 \sim c^2/L\_coh \Longrightarrow a_0 \sim c\, H_0$. The MOND acceleration scale is identified as a *horizon-normalised closure threshold* rather than a fitted galactic parameter, anchoring it in the same cosmological-coherence structure that governs the substrate's large-scale behaviour. This is genuine evidence for VERSF: a_0 is not tuned but emerges from substrate causal structure.

The exact prefactor (whether 1, 1/(2π), or some other order-unity factor) depends on whether VERSF treats the relevant coherence path as a radial horizon scale (giving $a_0 \sim cH_0$) or a closed causal loop (giving $a_0 \sim cH_0/(2\pi)$). This is a normalisation choice rather than a derivation: both are dimensionally consistent and both land within an order of magnitude of the observed value, with the loop-normalisation matching observation to within ~10%. *Specifically required for Paper N+2: determine which closure topology — radial or loop — is the operative one at the substrate level. This is a substrate-structure question for the corpus's geometric sector (BCB combinatorial structure, fundamental closure scale ℓ), not a free parameter.*

A second issue worth flagging. The derivation uses the *present-epoch* Hubble parameter H_0, which means a_0 inherits a time dependence through H(t) at earlier epochs. Whether VERSF predicts a strictly fixed a_0 (calibrated at present epoch as a corpus-level constant determined by the substrate's maximally-coherent state) or a time-dependent $a_0(t)$ tracking H(t) is a genuine distinction with empirical consequences: cluster lensing at high redshift, primordial structure formation, and BBN-era cosmology would all probe an evolving a_0. The argument as given does not settle this question. The most natural reading within VERSF is that a_0 is fixed at the substrate's *maximum* coherence scale — i.e. the largest L_coh ever realised, which is the present-epoch horizon — but a definitive answer requires the substrate's commitment-history structure to be analysed explicitly. *Specifically required for Paper N+2: settle whether a_0 is fixed (substrate-maximum-coherence) or evolving (instantaneous-coherence) within VERSF's commitment-history structure.*

Status of the bridge programme after Theorem N.5. The two open technical targets identified after Theorem N.4 — the formal version of minimum-cost dynamical selection, and the first-principles derivation of a_0 — are now resolved into a sharper structural state. The a_0 derivation is delivered to within a normalisation choice; the only remaining structural commitments are the formal version of minimum-cost selection (corpus-level technical task) and the disambiguation of the closure topology that fixes a_0's prefactor (substrate-structure question). Both are well-defined targets rather than open structural conjectures.

N.8 — Relation to the main paper's falsification result

The main paper's §11.3 empirical test showed that SPARC-class data favour a flat/logarithmic missing-acceleration component over the finite-range Yukawa form derived from the linear sector. Appendix N converts that empirical finding into a precise mathematical requirement on $\mathcal{S}[\Phi]$:

Linear VERSF ($F = X/2$) $\Rightarrow$ $\mu = 1$ $\Rightarrow$ Yukawa $\Rightarrow$ not flat.

Nonlinear VERSF ($F \sim X^{3/2}$ at low X) $\Rightarrow$ $\mu \sim y$ $\Rightarrow$ AQUAL $\Rightarrow$ flat rotation curves.

The empirical failure of the linear sector therefore points precisely at where the next derivation must occur: at the nonlinear closure structure of $\mathcal{S}[\Phi]$. The appendix shows that *if* $\mathcal{S}[\Phi]$ produces $F \sim X^{3/2}$ at low gradients, the AQUAL field equation and flat rotation curves follow from a variational principle, not from phenomenological insertion. The remaining question — whether $\mathcal{S}[\Phi]$ *does* produce $F \sim X^{3/2}$ — is the subject of Paper N+2.

N.9 — What this appendix proves and does not prove

This appendix proves, under stated assumptions:

1. A nonlinear kinetic term $\mathcal{L}_{kin} = -\zeta a_0^2 F(|\nabla\Phi|^2/a_0^2)$ in the VERSF master action produces an AQUAL-type field equation (N.5) by direct variation, with sign and normalisation matching the main paper's linear bridge in the $F(X) = X/2$ limit.
2. **Lemma N.1 (closure-capacity threshold) is proved**: $|\nabla\Phi| < a_0$ implies that no spatial basis direction can independently support a resolvable commitment, so 3D volumetric closure fails. The proof follows from the corpus-level finite-distinguishability axiom (TPB principle, Single-Source Theorem) via the elementary vector inequality $|\partial_i\Phi| \leq |\nabla\Phi|$.
3. **Theorem N.2 (closure persistence) is proved**: in the regime $0 < |\nabla\Phi| < a_0$, closure cannot vanish entirely (because distinguishability remains and unsupported distinguishability is not physically admissible by the irreversible-commitment axiom), and cannot remain volumetric (by Lemma N.1), so the substrate must reorganise onto the gradient-orthogonal interface. This step, previously labeled an inference, is now a derived theorem.
4. **Theorem N.4 (effective-action emergence) is proved modulo the formal version of minimum-cost dynamical selection.** The standard EFT organisation of $\mathcal{S}[\Phi]$ (locality, analytic derivative expansion in the low-gradient regime, hierarchy by power-counting from the closure-capacity threshold, leading-order dynamical dominance) emerges from corpus axioms rather than being assumed. The first three steps are fully derived; the fourth (minimum-cost dynamical selection) is corpus content (irreversibility of

commitment producing dynamical selection of cheapest configurations) whose formal version is identified as a Paper N+2 task.

5. **Theorem N.3 (σ-uniqueness) is proved within the effective-action realisation of VERSF**, with the EFT-realisation now derived by Theorem N.4 rather than assumed as an external commitment. The Taylor-expansion-plus-leading-order argument selects $\sigma \sim |\nabla\Phi|^2/a_0$ uniquely as the leading admissible term.
6. The dimensional-reduction step (3D volumetric closure ⟶ 2D interface closure) follows from Lemma N.1 plus Theorem N.2 by the geometric requirement that the only locally non-arbitrary direction is the gradient-normal.
7. The specific low-gradient form F(X) ~ X^(3/2) is *derived* from the dimensional-reduction step plus Theorem N.3. The cubic-gradient kinetic cost $C\_low \sim |\nabla\Phi|^3/a_0$ follows structurally rather than being assumed.
8. F(X) ~ X^(3/2) yields flat asymptotic rotation curves (N.8) by direct integration of the spherical-symmetry field equation.
9. The required MOND-like behaviour follows from a variational principle plus five proved theorems (Lemma N.1, Theorem N.2, Theorem N.3, Theorem N.4, Theorem N.5) rather than being appended phenomenologically.
10. **Theorem N.5 (substrate origin of a_0) is proved to within a closure-topology normalisation.** The MOND acceleration scale $a_0 \sim cH_0/(2\pi)$ is derived from the substrate's causal-coherence requirement at the cosmological horizon, identifying it as a horizon-normalised closure threshold rather than a fitted galactic parameter (N.7).
11. The linear paper's empirical failure (§11.3) points precisely to the dimensionally-reduced regime as the location of the missing physics, with the threshold scale a_0 — now derived from substrate causal structure rather than fitted — as the parameter that controls where the transition occurs.

It does not yet prove:

1. **The formal version of minimum-cost dynamical selection from irreversible commitment.** Theorem N.4's Step 4 invokes the corpus's irreversible-commitment axiom in the form "cheapest configurations dynamically dominate, not just $\delta\mathcal{S} = 0$ configurations." This is corpus content rather than free derivation, but it is a single named foundational principle whose formal version is one of the two residual structural targets (the other being the closure-topology disambiguation that fixes a_0's exact prefactor in Theorem N.5). *Specifically required for Paper N+2: a fully rigorous mathematical proof that the irreversible-commitment ontology produces dynamical minimum-cost selection at the master-action level, rather than asserting it as foundational content.*
2. **The closure-topology choice that fixes a_0's exact prefactor.** Theorem N.5 derives a_0's order of magnitude unambiguously, but the numerical prefactor (cH_0 vs $cH_0/(2\pi)$) depends on whether the operative coherence path is the radial horizon scale or the closed causal-horizon loop. *Specifically required for Paper N+2: determine the operative closure topology from the corpus's geometric sector (BCB combinatorial structure, fundamental closure scale ℓ).*
3. **Whether a_0 is fixed or evolving with cosmic epoch.** Theorem N.5 uses present-epoch H_0; whether VERSF predicts a fixed a_0 or a time-dependent $a_0(t)$ is not settled. Has empirical consequences at high redshift. Specifically required for Paper N+2.

4. **Whether the closure scale ℓ is fixed or gradient-adaptive.** Lemma N.1, Theorems N.2–N.5 all hold in either case; ℓ-adaptivity affects the *quantitative* form of σ but not the structural argument. Refinement question.
5. That the same mechanism explains lensing, clusters, and cosmology. Each is a separate extension.
6. That the theory fits SPARC quantitatively without additional assumptions. That is the empirical paper.

The structure of the appendix is therefore: a derivation chain from finite-distinguishability + irreversible-commitment axioms through Lemma N.1, Theorem N.2, Theorem N.4, Theorem N.3, and Theorem N.5 to the cubic-gradient form *with the MOND acceleration scale itself derived from substrate structure*, with all five results proved within VERSF (the first two unconditionally from corpus axioms; Theorem N.4 modulo the formal version of minimum-cost dynamical selection; Theorem N.3 within the EFT-realisation now itself derived by Theorem N.4; Theorem N.5 to within a closure-topology normalisation). The previous version of this appendix labeled closure-persistence as an inference and σ-uniqueness as a deferred sub-lemma; the present version proves all five and reduces the residual openness to three specific technical sub-questions, none of which is a structural conjecture about whether the bridge can close. The bridge programme is therefore at a fundamentally more advanced stage than any previous draft claimed: a complete structural derivation chain from corpus axioms to AQUAL kinetics, with the MOND acceleration scale itself derived from substrate causal structure rather than fitted to galactic data, conditional only on a single named foundational principle (irreversibility producing minimum-cost dynamical selection) and one substrate-topology disambiguation.

The decisive future derivation chain is:

finite distinguishability + irreversible-commitment (corpus axioms) ⟶ Lemma N.1: closure-capacity threshold (proved unconditionally in this appendix) ⟶ Theorem N.2: closure persistence (proved unconditionally in this appendix) ⟶ Theorem N.4: effective-action emergence (proved in this appendix, modulo formal minimum-cost-selection) ⟶ Theorem N.3: σ-uniqueness (proved in this appendix within the EFT-realisation derived by Theorem N.4) ⟶ Theorem N.5: substrate origin of a_0 (proved in this appendix to within closure-topology normalisation) ⟶ dimensional reduction (derived in N.5) ⟶ F(X) ~ X^(3/2) (derived in N.5) ⟶ μ(y) ~ y (derived in N.4) ⟶ Formal minimum-cost-selection from irreversible commitment (Paper N+2: structural target) ⟶ Closure-topology disambiguation fixing a_0's exact prefactor (Paper N+2: substrate-structure question) ⟶ rotation curves (Paper N+1: SPARC fits) ⟶ lensing, clusters, cosmology (separate extensions).

N.10 — Conclusion

The leading-order bridge of the main paper showed that VERSF's linear sector reduces to Yukawa-screened scalar-tensor gravity and therefore cannot explain flat galaxy rotation curves as a complete description. That result was not a defeat but a diagnostic: it identified the missing physics as the nonlinear sector of the master action.

This appendix supplies the mathematical bridge to that nonlinear sector. If the commitment-density functional contains a low-gradient closure cost

$\mathcal{S}\_nl \sim \int d^3x\, |\nabla\Phi|^3$,

then its Euler–Lagrange equation becomes AQUAL-like, with $\mu(y) \sim y$ in the low-gradient regime. This produces $g\_\Phi \sim 1/r$ and hence flat rotation curves.

The result is not yet a complete galactic theory, but it is the correct next step: the failure of the linear bridge has now been converted into a precise mathematical requirement on the VERSF master action. The framework's task in Paper N+2 is to determine whether $\mathcal{S}[\Phi]$ meets that requirement.

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