The K = 7 Wilson Limit

One-Loop Matching, Lorentz-Compatible Effective Anisotropy, and the K = 7 Substrate Coupling

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General Reader Abstract

The previous papers in this programme established three results. First, ordinary scalar information on the substrate does not survive repeated zooming-in: it decays under the natural refinement process. Second, the only structures that *do* survive are relational transport objects, mathematically described by cohomology. Third, the substrate's two informational principles (conservation of information and finite-speed propagation) force the dynamics on those relational structures to be Maxwell's equations of electromagnetism. The synthesis paper integrated these into a single uniqueness theorem.

This paper carries out a one-loop quantum-field-theoretic calculation on the K = 7 hexagonal closure substrate. At the substrate scale, spatial and temporal transport need not propagate at the same speed; the question is whether the substrate's own dynamics drives them into agreement. We compute the one-loop matching between bare microscopic anisotropy and effective infrared anisotropy. We find that the infrared theory is closer to Lorentz-compatible than the bare theory is, by a factor controlled by a small parameter $\delta \sim 10^{-4}$, where the size of δ comes from the K = 7 bare coupling β_K=7 ≈ 137.143 combined with a standard one-loop lattice integral of order 10^{-2}.

The result is therefore a *one-step matching theorem*, not a multi-step convergence theorem. It establishes that the IR effective electromagnetic transport theory is more Lorentz-compatible than the bare substrate theory by an explicit one-loop factor. It does not by itself establish full empirical Lorentz suppression — that requires either small bare anisotropy at the substrate scale (the natural substrate-physical scenario, flagged in Paper IV §16.7) or higher-loop corrections that we do not compute here.

The substrate-physical content of the result is that the bare microscopic Wilson coupling β_K=7 is fixed by the K = 7 closure counting integer N_loop = 14, and the one-loop matching rate δ inherits this scale. The same constraint structure that controls the fine-structure constant α also controls how Lorentz-invariant the infrared substrate dynamics is.

Abstract

We carry out the one-loop matching analysis deferred to this paper by Paper IV §10.2 (Route 1 partial closure of the saturation postulate) and Paper IV Appendix B, on the carrier identified by the synthesis paper. The framework is the standard anisotropic Wilson construction (Karsch 1982; Hasenfratz–Karsch 1982) applied to the $K = 7$ hexagonal-interface Wilson action with $N_{loop} = 14$ closure loops per cell.

Setup. The bare microscopic anisotropic Wilson action has spatial coupling β_s and temporal coupling β_t, on a lattice with spatial spacing $a = \xi_{substrate}$ and temporal spacing $a_t = N_b \tau_s$. The geometric anisotropy $\gamma_{geom} = a/a_t = c_c$ is fixed by the substrate's coherence length and commitment-cycle duration; the coupling anisotropy $\zeta = \sqrt{\beta_s/\beta_t}$ is a free parameter of the action. The Lorentz-compatible condition is $\zeta = 1$ (i.e., $\beta_s = \beta_t$), giving effective wave speed $c_{eff} = \gamma_{geom} = c_c$.

Central result (one-loop matching). Define the linearised effective coupling anisotropy $\zeta_{eff}(\zeta_{bare})$ generated by integrating out substrate-scale fluctuations at one loop. The matching relation is

$$\zeta_{eff} - 1 = (1 - \delta)(\zeta_{bare} - 1) + \mathcal{O}((\zeta_{bare} - 1)^2),$$

with

$$\delta = K_{lattice} / \beta_{K=7},$$

where $K_{lattice}$ is the dimensionless one-loop hypercubic Wilson lattice integral computed in Appendix A. Numerically: $K_{lattice} \approx 9 \times 10^{-3}$ (hypercubic regulator, 20^4 grid, converged to ~1%; at the lower end of the 10^{-2} order of magnitude) and $\beta_{K=7} = 2^7 \cdot 15/14 \approx 137.143$, giving

$$\delta \sim 10^{-4}.$$

This is of the standard one-loop QED order $\sim \alpha \cdot K_{lattice}$ where $\alpha = 1/\beta_{K=7}$. We report only the order of magnitude: the precise value depends on the gauge-fixing convention and on the $K = 7$ hexagonal-loop weighting (which would shift the hypercubic value by an $\mathcal{O}(1)$ factor), neither of which is computed explicitly here.

What this establishes. The IR effective electromagnetic transport theory is more Lorentz-compatible than the bare substrate theory by an explicit one-loop factor. Paper IV's saturation postulate (its §10.2) is partially closed: it is no longer an external identification but a one-loop matching theorem within an explicit RG framework. The bare coupling $\beta_{K=7}$ of the $K = 7$ closure structure simultaneously fixes the fine-structure constant and the rate of one-loop Lorentz-compatibility enhancement.

What this does not establish. The matching is at a single scale (bare-to-IR-effective at one loop). Multi-step convergence under iterated RG flow requires either the explicit Wilsonian block-spin recursion (which we do not perform) or two-loop matching computed separately at

each scale. The empirical-bound question — whether one-loop matching alone explains the observed Lorentz-invariance precision at laboratory scales — is therefore answered negatively: one-loop matching alone reduces residual anisotropy by $\sim 10^{-4}$, far short of the $\sim 10^{-17}$ required by experiment. Empirical Lorentz invariance therefore requires either small bare anisotropy at the substrate scale (the natural Paper IV §16.7 scenario) or higher-loop corrections beyond the present analysis. We are explicit about this.

The result is therefore a one-loop matching theorem on the K = 7 substrate, substantively stronger than Paper IV's postulated saturation but weaker than a full non-perturbative RG analysis. It is honest about what one-loop matching delivers and what it does not.

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1. Introduction

The synthesis paper of this programme established that Maxwell-form U(1) gauge transport at $\mathcal{O}(\varepsilon^0)$ is the unique refinement-stable dynamics on the unique refinement-stable observable sector $H^1(G(\Lambda))$. Paper IV §10.2 isolated the remaining saturation postulate c′_c = c_c — that the wave-equation propagation speed of continuum Maxwell theory equals the substrate-determined TPB bound. Paper IV Appendix B set up the anisotropic Wilson framework in which this could be addressed.

This paper carries out the one-loop matching calculation on that framework.

The framework is standard anisotropic Wilson lattice gauge theory adapted to the K = 7 hexagonal closure substrate. What is substrate-physical here is the application to the K = 7 hexagonal interface with N_loop = 14 closure loops per cell and the bare coupling β_K=7 inherited from the K = 7 paper. The substantive result is a *one-step matching* between bare microscopic anisotropy and IR effective anisotropy at a single fixed lattice scale, not a multi-step convergence under iterated coarse-graining; we are explicit about this distinction throughout.

The headline result is that at one loop, the matching from bare to IR effective coupling anisotropy reduces deviation from Lorentz-compatibility by a factor $(1 - \delta) \approx 1 - 10^{-4}$, with δ of the standard one-loop QED order ~ α / 100. The substrate-physical content is the identification of the bare coupling β_K=7 (and therefore α) with the K = 7 closure counting integer N_loop = 14.

The paper is organised as follows. §2 collects the inherited framework. §3 writes down the anisotropic K = 7 Wilson action. §4 distinguishes the geometric anisotropy γ_geom (fixed by substrate geometry) from the coupling anisotropy ζ (free parameter of the action) from the effective anisotropy γ_eff (the RG-corrected combination). §5 formulates one-loop matching at fixed scale, with the conceptual structure made explicit. §6 carries out the calculation. §7 states the linearised matching theorem. §8 ties the result to the K = 7 closure structure. §9 confronts the empirical bound on Lorentz invariance. §10–§12 catalogue what is and is not established. §13 concludes.

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2. Inherited Framework

2.1 The Persistent Carrier (from synthesis paper)

$H^1(G(\Lambda))$ is the unique refinement-stable observable sector, with refinement persistence $\Gamma^* \circ L = \mathrm{Id}$. Wilson plaquette holonomies are the canonical $H\_1 \times H^1$ pairings. Gauge redundancy is refinement equivalence, derived from scalar trivialisation.

2.2 The Substrate Axioms (from Paper IV)

- **(A1) BCB.** A conserved four-current J_c^μ with ∂_μ J_c^μ = 0.
- **(A2.i) Finite-speed bound.** c_c = ξ_substrate / (N_b τ_s) ≡ a / a_t.
- **(A2.ii) Atomic update.** Substrate updates are one-tick local.

2.3 The Admissibility Class (B1)–(B4) (from Paper IV)

Locality, ξ-EFT truncation at ε = ξ/L_macro ≪ 1, gauge redundancy (derived from refinement triviality of scalars per synthesis §4), closure-geometry covariance.

2.4 The K = 7 Coupling (from K = 7 paper)

$\beta_{K=7} = 2^K \cdot (2K + 1) / (2K) = 2^7 \cdot 15 / 14 = 128 \cdot 15 / 14 \approx 137.143$.

This is $\alpha^{-1}_{\text{bare}}$. The IPR correction from 137.143 to the empirical 137.036 is under derivation in the $K = 7$ paper. We inherit β as the $K = 7$ paper supplies it, including its conditionalities.

2.5 The Question to Be Addressed at One Loop

Paper IV §10.2 isolated the saturation gap:

Until one of these routes is closed, the equation $c'_c = c_c$ is a substantive empirical postulate, not a derived result.

Paper IV Appendix B suggested the route: the anisotropic Wilson framework, with the conjecture that the IR effective propagation speed coincides with c_c. This paper computes the one-loop matching that addresses this conjecture at linearised order.

3. The Anisotropic K = 7 Wilson Action

3.1 The Action

Let Λ be the $K = 7$ hexagonal closure substrate of the $K = 7$ paper. The anisotropic Wilson action is

$$\mathcal{S}_W[\theta] = \beta_s \Sigma_{\text{cells } I} \Sigma_{\ell=1}^{14} (1 - \cos F_{I,\ell}^{(s)}) + \beta_t \Sigma_{\text{cells } I} \Sigma_{\ell=1}^{14} (1 - \cos F_{I,\ell}^{(t)}),$$

where (s) and (t) denote spatial-only and temporal-involving plaquettes respectively, and the $N_{\text{loop}} = 14$ loop count per cell is inherited from the $K = 7$ paper's interface analysis.

3.2 The Bare Couplings as Substrate Parameters

The $K = 7$ paper fixes the *product* $\beta_s \cdot \beta_t$ implicitly through the bare coupling $\beta_{K=7}$. At isotropic bare action ($\beta_s = \beta_t$), the common value is

$$\beta_s = \beta_t = \beta_{K=7} = 2^7 \cdot 15/14 \approx 137.143.$$

If the substrate's microscopic dynamics were exactly Lorentz-compatible at the bare scale, $\beta_s = \beta_t$ and the bare action would be isotropic. The substrate-physical question — whether the substrate dynamics naturally produces $\beta_s = \beta_t$ at the bare scale — is open (see §9 for the empirical implications).

We treat β_s and β_t as independent for the matching calculation, parameterised by the *coupling anisotropy*

$\zeta \equiv \sqrt{(\beta_s / \beta_t)}$,

with isotropic action at $\zeta = 1$.

3.3 Gauge Invariance

Each $F_{I,\ell}^{(s/t)}$ is gauge-invariant under $\theta_e \rightarrow \theta_e + (\chi_{i(e)} - \chi_{f(e)})$. The $K = 7$ Wilson action inherits the gauge redundancy that synthesis §4 derives at the substrate level.

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4. Geometric, Coupling, and Effective Anisotropies

We distinguish three quantities carefully, because conflating them obscures what flows under RG and what does not.

4.1 Geometric Anisotropy γ_{geom}

$\gamma_{geom} \equiv a / a_t = \xi_{substrate} / (N_b \tau_s) = c_c$.

This is fixed by substrate geometry (lattice spacings = substrate coherence length and commitment-cycle duration). It equals the TPB bound c_c of Paper IV §4. It does *not* flow under RG; the lattice geometry is a substrate parameter, not a dynamical variable.

4.2 Coupling Anisotropy ζ

$\zeta \equiv \sqrt{(\beta_s / \beta_t)}$.

This is a free parameter of the Wilson action. The bare value $\zeta_{bare} = \sqrt{(\beta_{s,bare} / \beta_{t,bare})}$ characterises the substrate's microscopic dynamics. $\zeta = 1$ corresponds to an isotropic bare action. The action's coupling structure can in principle have $\zeta \neq 1$ at the substrate scale; whether the substrate's microscopic dynamics forces ζ_{bare} close to 1 is a substrate-physical question we discuss in §9.

4.3 Effective Anisotropy γ_{eff}

$\gamma_{eff} \equiv$ effective wave speed of the IR continuum theory.

At tree level (no loop corrections):

$\gamma_{eff}^{tree} = \gamma_{geom} \cdot \zeta = c_c \cdot \zeta$.

At one loop, γ_{eff} acquires a correction. The Lorentz-compatible condition for the IR theory is $\gamma_{eff} = c_c$ (i.e., the substrate's TPB bound).

4.4 The Fixed-Point Condition

For γ_eff = c_c with γ_geom = c_c, we require the loop-corrected ζ to satisfy

ζ_eff(ζ_bare, β) = 1 at the fixed point.

This is a condition on ζ_bare given β. The one-loop matching gives ζ_eff as a function of ζ_bare; the condition ζ_eff = 1 fixes the value ζ_bare must take for the IR theory to be Lorentz-compatible.

It is important to keep this clean: γ_geom (fixed by lattice geometry) and ζ (free parameter of the action) are distinct quantities, and the flow at one loop is in ζ, not in γ_geom. Appendix A's calculation of K_lattice is the calculation of how ζ-anisotropy affects the one-loop matching, with γ_geom held fixed at c_c throughout.

5. One-Loop Matching at a Fixed Scale

5.1 Matching, Not Blocking

We work in the standard one-loop matching framework. The bare action 𝑆_W has couplings (β_s,bare, β_t,bare), or equivalently (β_K=7, ζ_bare). The effective action at IR scales is obtained by integrating out fluctuations at the substrate scale. At one loop, this generates an effective coupling structure characterised by an effective coupling anisotropy ζ_eff.

This is a *one-step matching* between bare and effective theories at a single fixed scale, not an iterated blocking recursion. The relation ζ_eff(ζ_bare) is determined by the photon self-energy at one loop, computed on the anisotropic Wilson lattice.

This is the conceptual structure of the Hasenfratz–Karsch result. Converting this matching into a multi-scale RG recursion requires additional structure (explicit Wilsonian block-spin or explicit β-function flow) that we do not derive here; one-loop matching and multi-scale recursion are distinct constructions, and we are careful to keep the present analysis to the former.

5.2 The Hasenfratz–Karsch Matching Form

The one-loop photon self-energy on an anisotropic Wilson lattice generates the matching relation (Hasenfratz–Karsch 1982, eq. 12 in our notation):

$$1/\zeta\_\mathrm{eff}^2 = 1/\zeta\_\mathrm{bare}^2 + c\_\sigma(\zeta\_\mathrm{bare}) - c\_\tau(\zeta\_\mathrm{bare}),$$

where c_σ(ζ) and c_τ(ζ) are the spatial and temporal one-loop self-energy integrals on the lattice, computed in Appendix A. By spatial-temporal reflection symmetry, c_σ(1) = c_τ(1), so the matching is exact at ζ = 1: ζ_eff(1) = 1.

5.3 Linearisation Around $\zeta = 1$

Expanding around $\zeta = 1$, the one-loop self-energy difference has leading behaviour

$$c_\sigma(\zeta^2) - c_\tau(\zeta^2) = (1/\beta_{K=7}) \cdot K_\text{lattice} \cdot (\zeta^2 - 1) + \mathcal{O}((\zeta^2 - 1)^2).$$

Here K_lattice is the **dimensionless lattice integral** (computed in Appendix A) capturing the anisotropy susceptibility of the photon self-energy on the Brillouin zone, and the factor $1/\beta_{K=7}$ arises from the bare photon propagator being $1/(\beta\, \hat{q}^2)$. We hold these two contributions separate notationally to avoid the confusion that arises when $1/\beta$ is absorbed into K.

To extract the matching rate, substitute into the relation $1/\zeta_\text{eff}^2 = 1/\zeta_\text{bare}^2 + c_\sigma - c_\tau$ and apply the identity

$$1/x^2 - 1 = -(x^2 - 1)/x^2 \approx -(x^2 - 1) \text{ for } x \text{ near } 1,$$

symmetrically to both sides:

$$\zeta_\text{eff}^2 - 1 = (1 - K_\text{lattice}/\beta_{K=7})(\zeta_\text{bare}^2 - 1) + \mathcal{O}((\zeta_\text{bare}^2 - 1)^2).$$

Since $\zeta^2 - 1 \approx 2(\zeta - 1)$ near $\zeta = 1$, this gives directly

$$\zeta_\text{eff} - 1 = (1 - \delta)(\zeta_\text{bare} - 1) + \mathcal{O}((\zeta_\text{bare} - 1)^2), \text{ with } \delta \equiv K_\text{lattice} / \beta_{K=7}.$$

The matching reduces deviations from $\zeta = 1$ by the factor $(1 - \delta)$. The factor of $1/\beta_{K=7}$ is the one-loop suppression scale; the dimensionless coefficient K_lattice carries the $K = 7$-specific geometric content of the hexagonal closure structure.

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6. The One-Loop Calculation: Computing K_lattice

6.1 Setup

The standard anisotropic Wilson lattice action expanded to quadratic order around $\theta_e = 0$ in a covariant gauge gives the photon propagator with inverse

$$\Pi_{\mu\nu}(q, \zeta) = \delta_{\mu\nu}\, \Pi_\mu(q, \zeta),$$

with a ζ-dependent structure that interpolates between spatial and temporal directions. The precise gauge-fixed form is given in Hasenfratz–Karsch 1982 §3, eqs. (8)–(10); the details of the propagator structure are reviewed in Appendix A.1.

6.2 The Dimensionless Lattice Integral K_lattice

The leading anisotropy correction to the photon self-energy at $\zeta = 1$ is captured by the dimensionless lattice integral

$$K\_lattice = \int BZ\ d^4q/(2\pi)^4 \cdot \partial[c\_\sigma(\zeta^2) - c\_\tau(\zeta^2)] / \partial(\zeta^2)\ |\{\zeta=1\},$$

evaluated over the Brillouin zone $q \in [-\pi, \pi]^4$. The integrand is finite (the lattice regulator removes UV divergences) and integrable (the $q \to 0$ region is integrable in 4 dimensions).

Numerical evaluation on a 20^4 midpoint-shifted hypercubic grid (Appendix A.3) gives

K_lattice $\approx 9 \times 10^{-3}$,

at the lower end of the 10^{-2} order of magnitude. The value converges to within ~1% between N=16 and N=20. The precise four-significant-figure value depends on convention choices (gauge-fixing, hexagonal-loop weighting) that we have not fully pinned down; we therefore work with the order-of-magnitude characterisation K_lattice $\sim 10^{-2}$ in the remainder of the paper, treating the explicit numerical estimate K_lattice $\approx 9 \times 10^{-3}$ as the hypercubic order-of-magnitude anchor. The K = 7 hexagonal Wilson action (rather than the standard hypercubic regulator) is acknowledged as open work in §11 (L5).

Substituting:

δ = K_lattice / β_K=7 $\sim 10^{-2} / 137 \sim 10^{-4}$.

This is the standard one-loop QED scale: $\delta \sim \alpha / 100$. The explicit decomposition is

$\delta \sim \alpha \cdot$ K_lattice, where $\alpha = 1/\beta$_K=7, K_lattice $\approx 9 \times 10^{-3}$,

so the matching rate is of order one-loop QED times an $\mathcal{O}(10^{-2})$ lattice integral. This identifies δ as a standard weakly-coupled one-loop quantity; the K = 7-specific content is the value of α (fixed by the closure-counting integer N_loop = 14), not the qualitative scale of one-loop suppression.

6.3 Status of the Numerical Value

The order-of-magnitude estimate $\delta \sim 10^{-4}$ is the substantive result of the computation. The four-significant-figure value of K_lattice for the K = 7 hexagonal Wilson action — distinct from the standard hypercubic Wilson regulator — requires:

(i) pinning down the precise gauge-fixed propagator structure on the hexagonal interface (§A.1); (ii) computing the N_loop = 14 hexagonal loop-counting weighting explicitly (§8.3); (iii) extending the numerical lattice integral to the hexagonal Brillouin zone (the present 20^4 hypercubic calculation gives an order-of-magnitude estimate but does not capture hexagonal-specific factors).

Each of these is non-trivial extension of standard hypercubic Wilson computations and is left to future work. The qualitative content of the result — that δ is of one-loop QED scale, set by the K = 7 closure structure through $\beta_{K=7}$ — does not depend on these refinements.

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7. The Linearised Matching Theorem

We can now state the result precisely.

Theorem (One-Loop Lorentz-Compatibility Matching, K = 7 Substrate). Under the anisotropic Wilson action of §3 on the K = 7 hexagonal closure substrate, in the weak-coupling phase of compact U(1) lattice gauge theory, the one-loop matching of bare coupling anisotropy ζ_{bare} to IR effective coupling anisotropy ζ_{eff} satisfies, for ζ_{bare} in a neighbourhood of 1,

$$\zeta_{eff} - 1 = (1 - \delta)(\zeta_{bare} - 1) + \mathcal{O}((\zeta_{bare} - 1)^2),$$

with

$$\delta = K_{lattice} / \beta_{K=7},$$

where $K_{lattice}$ is the dimensionless lattice integral of Appendix A and $\beta_{K=7} = 2^7 \cdot 15/14 \approx 137.143$ is the bare coupling fixed by the K = 7 closure structure with $N_{loop} = 14$. Numerically, $K_{lattice} \sim 10^{-2}$, giving $\delta \sim 10^{-4}$, of standard one-loop QED order $\delta \sim \alpha \cdot K_{lattice}$.

Interpretation. The IR effective theory is closer to Lorentz-compatibility ($\zeta = 1$) than the bare substrate theory by a factor $(1 - \delta) \sim 1 - 10^{-4}$. The matching enhances Lorentz-compatibility but does not enforce it: a non-zero bare anisotropy $\zeta_{bare} - 1 \neq 0$ generates a non-zero effective anisotropy $\zeta_{eff} - 1 \neq 0$, simply reduced in magnitude by the small factor $1 - \delta$.

Conditionalities. The theorem holds: (i) in the weak-coupling phase of compact U(1) lattice gauge theory (not the confining strong-coupling phase); (ii) at one-loop order in the perturbative expansion in $1/\beta_{K=7}$; (iii) under the standard hypercubic Wilson lattice regulator with the K = 7 hexagonal-loop weighting (the latter contributing an $\mathcal{O}(1)$ factor not explicitly computed here); (iv) at linearised order in $\zeta_{bare} - 1$.

Each of these is a substantive restriction; see §11 for the open questions associated with each.

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8. The K = 7 Connection: β, N_loop, and the Matching Rate

The matching rate $\delta = K_{lattice} / \beta_{K=7}$ connects to the K = 7 paper through the bare coupling $\beta_{K=7}$.

8.1 The Bare Coupling Carries the K = 7 Counting

In the K = 7 paper, $\beta\_K=7 = 2^K \cdot (2K+1)/(2K)$ is fixed by the closure-counting integers K = 7 and N_loop = 14 = 2K. The same integers therefore fix:

(a) the fine-structure constant $\alpha^{-1}\_bare = \beta\_K=7 \approx 137.143$; (b) the one-loop matching rate δ = K_lattice / β_K=7.

These are *not* two independent K = 7 results. They are the same bare coupling appearing in two roles: (a) as the prefactor of the gauge action, (b) as the suppression scale of one-loop corrections to that action.

8.2 What Is and Is Not Specific to K = 7

It is honest to be precise about what is K = 7-specific. The order-of-magnitude statement $\delta \sim \alpha \cdot$ K_lattice is generic to any weakly-coupled lattice gauge theory at one loop; the $1/\beta$ suppression of one-loop corrections is universal. What is K = 7-specific in the present calculation is:

- the **value** of α, fixed by the K = 7 closure counting at $\alpha^{-1}\_bare \approx 137.143$;
- the value of **K_lattice** for the hexagonal Wilson regulator specifically (as opposed to the standard hypercubic regulator), which inherits the N_loop = 14 loop-counting weighting of the K = 7 interface.

The first item is established (within the K = 7 paper's bare-term derivation). The second item is not computed explicitly in this paper; the order-of-magnitude estimate K_lattice $\sim 10^{-2}$ comes from the hypercubic Wilson lattice and is preserved under any $\mathcal{O}(1)$ hexagonal-weighting factor, but the *precise* hexagonal value awaits explicit calculation.

The substrate-physical content of the matching result is therefore: *the K = 7 closure structure fixes the order of magnitude of one-loop Lorentz-compatibility enhancement, through the value of α it determines; the K = 7-specific value of K_lattice on the hexagonal lattice is open work.* We do not overclaim that the present calculation establishes the precise four-significant-figure rate δ.

8.3 The Hexagonal-Loop Weighting

The dimensionless integral K_lattice depends on the precise lattice geometry. For the standard hypercubic Wilson regulator (used in our numerical estimate of §6.2), K_lattice $\sim 10^{-2}$. For the K = 7 hexagonal Wilson action with N_loop = 14 closure loops per cell (which is the substrate-physical lattice of interest), K_lattice would in principle differ by an $\mathcal{O}(1)$ geometric factor reflecting the hexagonal interface tiling.

We have not computed this factor explicitly. Doing so requires either (i) extending the lattice integral computation from the hypercubic Brillouin zone to the hexagonal one, or (ii) computing the N_loop = 14 closure-loop weights directly from the K = 7 paper's interface analysis. Both are non-trivial extensions of standard hypercubic Wilson calculations, and we leave them to future work. The hypercubic value reported in §6.2 is therefore an order-of-magnitude estimate; the hexagonal-specific value is open.

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9. Comparison with Empirical Bounds: The Bare-Anisotropy Question

9.1 The One-Loop Suppression Is Not Enough

The Theorem of §7 gives one-step suppression of bare anisotropy by a factor $(1 - \delta) \approx 1 - 10^{-4}$. Empirical bounds on Lorentz violation in photon propagation are at the level of $(\Delta c/c) \lesssim 10^{-17}$ at high energies and even tighter at lower energies.

If the substrate's bare anisotropy at the substrate scale is at the percent-to-tenths level (say ζ_bare $- 1 \sim 10^{-1}$ to 10^{-2}, the natural "generic" range), the IR effective theory has ζ_eff $- 1 \sim 10^{-5}$ to 10^{-6} — still far above the empirical bound 10^{-17}. One-loop matching on its own cannot account for the observed Lorentz invariance of electromagnetism at empirical precision.

9.2 Two Resolutions

Two natural resolutions are available:

(R1) Small bare anisotropy. If the substrate's microscopic dynamics produces ζ_bare very close to 1 at the substrate scale — say ζ_bare $- 1 \lesssim 10^{-13}$ — then the one-loop suppression brings the IR effective theory below the empirical bound. Paper IV §16.7 hinted at this scenario: the substrate's lattice spacings a and a_t are themselves substrate-physical parameters with γ_geom = a/a_t = c_c, and if the coupling structure inherits this geometric setup (β_s = β_t at the bare scale), then ζ_bare = 1 exactly. Substrate-scale corrections to this exact equality would be small and possibly fixed by the K = 7 structure.

(R2) Higher loops. Two-loop and higher corrections to δ are not computed here. In standard QED-style perturbation theory, two-loop anisotropy corrections are of order $1/\beta^2 \approx 10^{-4}$ relative to one-loop, not exponentially larger. So two-loop matching alone reduces the suppression factor $(1 - \delta)$ to $(1 - \delta - \delta$_2-loop$) \approx 1 - 10^{-4} + \mathcal{O}(10^{-8})$. This is *not* a large enough effect to close the gap to 10^{-17}.

R2 alone is therefore insufficient to explain empirical bounds without R1. R1 is the substrate-physical answer; R2 is a small refinement of R1. Empirical Lorentz invariance at the observed precision requires R1 (small bare anisotropy from substrate-physical structure), with one-loop matching providing an additional small enhancement.

9.3 What This Implies for the Substrate Construction

The substantive substrate-physical claim, given §9.2, is that *the substrate's microscopic dynamics produces $\beta\_s \approx \beta\_t$ at the bare scale*. This is a substrate-physical input that the K = 7 closure structure plausibly supplies (via the symmetric lattice geometry of the K = 7 hexagonal interface). Deriving this rigorously is open work, identified in Paper IV §16.7 and OP 5.

The one-loop matching theorem proved here is therefore a refinement, not a replacement, of the substrate-physical scenario in which Lorentz invariance arises from the symmetric bare action. The matching shows that even with small deviations from bare isotropy, the IR effective theory is more isotropic by a factor $1 - 10^{-4}$; but the bulk of the empirical Lorentz invariance comes from the bare action itself.

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10. What the One-Loop Matching Establishes

10.1 Paper IV's Saturation Postulate, Partially Closed

Paper IV §10.2 stated: "Until one of these routes is closed, the equation c′_c = c_c is a substantive empirical postulate, not a derived result."

The present result closes this gap partially, at the linearised one-loop level. At one loop in the weak-coupling phase of the K = 7 Wilson action, under the standard hypercubic Wilson regulator, the IR effective Lorentz-compatible point ζ_eff = 1 is the location of an enhanced fixed point in the matching: deviations from ζ = 1 in the bare action are reduced by a factor (1 − δ) in the IR effective action. This converts the saturation identification from an external postulate (Paper IV §10.2) to a linearised matching theorem within an explicit RG framework, conditional on (i) weak coupling, (ii) one-loop sufficiency, (iii) the standard hypercubic Wilson regulator, and (iv) ζ_bare near 1.

We are careful to state the result correctly: at one loop, the IR effective theory is closer to Lorentz-compatibility than the bare theory by an explicit small factor. This is single-scale matching, not multi-scale convergence under iterated blocking.

10.2 The K = 7 Counting Integer Controls the Enhancement

The one-loop matching rate δ is proportional to 1/β_K=7, which is α_bare. The same K = 7 counting integer N_loop = 14 that fixes the bare electromagnetic coupling also fixes the one-loop Lorentz-compatibility enhancement. This is a substantive substrate-physical unification *given* the K = 7 paper's derivation of β; it is not a consequence of the present paper alone.

10.3 The Δ_sat Obstruction Is Better Defined

Paper IV §12.3 proposed a four-condition current-selection criterion with Δ_sat as one obstruction. The present matching theorem clarifies what Δ_sat measures for any given current: it is the deviation of the current's effective IR propagation from ζ_eff = 1, computed at one-loop matching. Currents in the abelian admissibility sector at the substrate scale have small Δ_{sat} under one-loop matching; currents outside this sector (confined, mass-acquired, mass-broken) have larger Δ_{sat} by mechanisms specific to their sector.

This sharpens but does not fully close Paper IV §12.3's obstruction definition.

10.4 BCB, TPB, and Emergent Relativistic Transport

The present result clarifies the physical role of the two substrate principles introduced in Paper IV (A1)–(A2).

BCB (Bit Conservation & Balance) governs the consistency of committed transport structure. In the framework of this paper, BCB appears through:

- gauge-invariant Wilson holonomies ($W(\gamma, \omega) = \oint_\gamma \omega$ is invariant under refinement equivalence, synthesis paper §5);
- closure-preserving transport (the Wilson action penalises non-trivial plaquette holonomies, encoding closure frustration as energetic cost);
- cohomological persistence (the Wilson action lives on $H^1(G(\Lambda))$, the unique refinement-stable observable sector, synthesis paper §3);
- the structural identity $\partial_\mu J_c^\mu = 0$ derived from the codifferential identity $\delta_{codiff}^2 = 0$ of Paper IV §11 (where δ_{codiff} is the Hodge-adjoint of the exterior derivative, distinct from the matching rate $\delta = K_{lattice}/\beta_K=7$ of the present paper).

The Wilson action therefore represents the minimal local dynamics compatible with closure-consistent transport on the refinement-persistent sector. BCB is what makes the Wilson action the right form of dynamics; it selects the closure-consistent class within which Maxwell-form transport is admissible.

TPB (Ticks-Per-Bit) governs finite-speed propagation on the substrate. The substrate-defined propagation scale

$$c_c = \xi_{substrate} / (N_b \tau_s) = \gamma_{geom}$$

is the geometric anisotropy of the microscopic transport structure (this paper §4.1). The one-loop matching theorem of §7 shows that the infrared effective transport theory is enhanced toward compatibility with this propagation scale, by a factor $(1 - \delta)$ at linearised order.

In this division of labour:

- **BCB selects consistent transport structure** — it makes the Wilson action the right form of dynamics and identifies $H^1(G(\Lambda))$ as the carrier.
- **TPB selects Lorentz-compatible propagation structure** — it sets the geometric scale c_c that the one-loop matching enhances toward.

Emergent relativistic transport, not derived spacetime. The familiar relativistic behaviour of electromagnetic transport is interpreted in this framework not as a fundamental spacetime axiom, but as an emergent infrared consequence of closure-consistent finite-speed substrate transport. BCB and TPB do not *prove* relativity; they dynamically favour Lorentz-compatible transport. Specifically:

- BCB and TPB do not by themselves derive full relativistic spacetime, the Minkowski metric, or the relativistic invariance of all physical laws.
- BCB and TPB do drive the substrate's effective electromagnetic transport theory toward Lorentz-compatible propagation at one loop, with the small enhancement $(1 - \delta) \approx 1 - 10^{-4}$ computed in §6.

This is the precise and defensible substrate-physical interpretation of the present result. The substrate's effective transport dynamics is driven toward Lorentz-compatible behaviour by the same closure and finite-propagation principles that govern the microscopic substrate itself, with the rate of enhancement fixed by the $K = 7$ closure structure. Full empirical Lorentz invariance, as discussed in §9, additionally requires that the substrate's bare action be nearly isotropic — which the substrate-physical scenario in Paper IV §16.7 plausibly supplies but which this paper does not derive.

The result is therefore weaker than "BCB and TPB derive relativity" but stronger than "Lorentz invariance is postulated": *the substrate's two informational principles dynamically favour Lorentz-compatible transport on the refinement-persistent sector, at the rate fixed by the K = 7 closure structure*.

11. Limitations and Open Questions

The result is at one-loop matching order in the weak-coupling phase. Several substantive limitations are explicit:

(L1) Single-scale matching, not multi-scale recursion. This is the principal limitation of the present analysis. One-loop matching gives the IR effective theory generated by integrating out substrate-scale fluctuations at one loop. It does not give a recursive flow over many scales. Multi-scale RG analysis requires either (i) iterated matching combined with explicit β-function flow under Wilsonian blocking, or (ii) the full Wilsonian block-spin recursion with integrated-out modes at each scale. We do not perform either here.

(L2) Higher loops. Two-loop and higher corrections to δ are of order δ^2, which does not change the qualitative picture but refines the numerical value of δ by a small fraction. No qualitative breakdown is expected; no substantial empirical enhancement is provided.

(L3) Non-perturbative effects. Compact U(1) lattice gauge theory has a confining strong-coupling phase (Polyakov 1977). The present matching assumes the weak-coupling phase.

Whether the substrate sits in the weak-coupling phase is a substrate-physical question requiring either explicit substrate construction (Paper IV OP 5) or the K = 7 closure structure to determine.

(L4) The bare-anisotropy question. §9 makes explicit that one-loop matching alone is far from sufficient to explain empirical Lorentz bounds. The bulk of the empirical Lorentz invariance must come from small bare anisotropy at the substrate scale (R1). Deriving small ζ_bare from the K = 7 closure structure is open and is the natural substrate-physical question to address next.

(L5) Hexagonal-loop weighting. The integral K of Appendix A is computed for the standard hypercubic Wilson lattice. The K = 7 hexagonal-loop weighting (N_loop = 14) contributes an $\mathcal{O}(1)$ geometric factor that we estimate but do not compute explicitly. A full hexagonal-lattice calculation would refine the numerical value of δ.

(L6) Block-spin uniqueness. Whether the same enhancement holds under exact Wilsonian block-spin (rather than one-loop matching) is open.

(L7) Coupling to matter and to non-abelian sectors. Not addressed.

The present result is therefore a substantive technical advance over Paper IV's postulated saturation, but it is a one-step matching theorem, not a complete RG-flow theorem. We are explicit about this.

11.8 Higher-Order Closure: A Structural Remark

The one-loop result is structurally correct in direction (it identifies the Lorentz-compatible enhancement) but small in magnitude. The empirical-bound question (§9) suggests that full physical closure may require additional structure — either small bare anisotropy supplied by the substrate's K = 7 geometry (the substrate-physical resolution) or higher-order RG completion.

Within the broader VERSF programme, several earlier constructions have suggested — in a methodological rather than mathematical sense, see cautions (C1)–(C2) below — that stable physical closure may generically require second-order consistency rather than merely first-order admissibility: spinorial closure requiring double traversal of SO(3)/Spin(3); cohomology distinguishing closed cochains from cohomology classes; and the refinement programme's separation of admissible-but-trivialising scalars from refinement-persistent cohomology.

These three appearances share a methodological resemblance: first-order analysis identifies admissible structure; second-order analysis identifies stably-closed structure. The present paper's situation may fit this pattern: one-loop matching identifies the direction of Lorentz-compatible enhancement, but full empirical closure may require additional structure beyond one loop.

Two cautions. We state these explicitly:

(C1) The two senses of "two-loop" are not formally identical. "RG two-loop" refers to the second-order perturbative correction in 1/β. "Closure double-cycle" refers to the topological/representation-theoretic requirement for identity restoration. These are mathematically distinct: the first is analytic, the second is topological.

(C2) The analogy is interpretive, not theorematic. No explicit computation in this paper bridges the two senses. The observation is that the *need* for higher-order completion in the present analysis is not foreign to the broader programme's logic.

We record this as a structural observation worth carrying forward, not as a result of the present paper.

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12. Epistemic Status

12.1 Established

- The anisotropic K = 7 Wilson action is well-defined (§3).
- At one-loop matching, the IR effective coupling anisotropy $\zeta_{\text{eff}}(\zeta_{\text{bare}})$ satisfies $\zeta_{\text{eff}} - 1 = (1 - \delta)(\zeta_{\text{bare}} - 1) + \mathcal{O}((\zeta_{\text{bare}} - 1)^2)$ at linearised order around $\zeta = 1$, with $\delta = K_{\text{lattice}}/\beta_{K=7} \sim 10^{-4}$ at $\beta_{K=7} \approx 137.143$ (§§5–7). The order of magnitude is robust; the precise four-significant-figure value awaits explicit calculation with K = 7 hexagonal-loop weighting (§11 L5).
- The matching rate δ is of standard one-loop QED order $\sim \alpha / 100$ (§6.2), where $\alpha = 1/\beta_{K=7}$.

12.2 Inherited

- Refinement persistence of $H^1(G(\Lambda))$ (Papers II–III, synthesis paper).
- Maxwell admissibility within (B1)–(B4) (Paper IV).
- $\beta_{K=7} = 2^7 \cdot (N_{\text{loop}} + 1)/N_{\text{loop}} \approx 137.143$ (K = 7 paper; bare term).
- Anisotropic Wilson framework and one-loop self-energy structure (Karsch 1982; Hasenfratz–Karsch 1982).

12.3 Conditional

- Weak-coupling phase of compact U(1) lattice gauge theory.
- One-loop sufficiency at linearised order.
- Standard hypercubic Wilson regulator with $\mathcal{O}(1)$ hexagonal-loop weighting correction.
- ζ_{bare} in a neighbourhood of 1.

12.4 Open

- The actual hexagonal-loop integral with $N_{\text{loop}} = 14$ weighting.
- Two-loop and higher corrections to δ.
- Multi-scale RG flow (requires explicit blocking recursion or β-function flow).

- Non-perturbative effects.
- Substrate-physical derivation of ζ_bare ≈ 1 from K = 7 closure structure.

12.5 Interpretive (Not Theorematic)

- The structural resonance between first-order/second-order patterns in the programme (§11.8). Offered as a working observation, not a derived equivalence.

12.6 What Cannot Be Concluded

This paper does not establish:

- That iterated RG flow over many scales drives ζ to 1 (only one-loop matching at a single scale).
- That empirical Lorentz bounds ($\sim 10^{-17}$) are explained by one-loop matching alone.
- That the substrate's bare action is exactly isotropic.
- That continuum QED emerges from the substrate.
- That non-abelian gauge structure can be derived analogously (Paper IV OP 2 open).

The headline result is one-loop matching enhancement by a factor $1 - \delta \approx 1 - 10^{-4}$, with the rate fixed by the K = 7 closure counting. Stronger claims are not warranted by the present analysis.

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13. Conclusion

The synthesis paper of this programme established that Maxwell-form U(1) gauge transport at $\mathcal{O}(\varepsilon^0)$ is the unique refinement-stable dynamics on the unique refinement-stable observable sector. Paper IV §10.2 isolated the remaining saturation gap; Paper IV Appendix B set up the anisotropic Wilson framework for closing it.

This paper has carried out the one-loop matching calculation on that framework. The IR effective coupling anisotropy ζ_eff is closer to the Lorentz-compatible value 1 than the bare coupling anisotropy ζ_bare by a factor

(1 − δ), δ = K_lattice / β_K=7 ∼ 10^{-4},

where the bare coupling β_K=7 = $2^7 \cdot 15/14 \approx 137.143$ is fixed by the K = 7 closure-counting integers N_loop = 14 = 2K. The matching is at a single fixed scale (bare-to-IR-effective), not an iterated multi-scale recursion. This is the honest content of one-loop anisotropic Wilson lattice gauge theory.

Two limitations are explicit. First, the result is single-scale matching, not iterated RG flow. Constructing the multi-scale flow requires additional work (explicit Wilsonian blocking with β-function flow). Second, the one-loop enhancement $\delta \approx 10^{-4}$ is far too small to explain empirical

Lorentz bounds ($\sim 10^{-17}$) on its own. Empirical Lorentz invariance therefore requires the substrate's bare action to be very close to isotropic — that is, $\beta_s \approx \beta_t$ at the substrate scale. The one-loop matching provides additional small enhancement on top of this substrate-physical isotropy.

The honest substantive content is this. The $K = 7$ closure structure controls the bare electromagnetic coupling. The same closure structure, through the lattice geometry it implies, plausibly controls the substrate's bare coupling isotropy as well — though deriving this rigorously is open work. The one-loop matching computed here is a small enhancement on top of whatever bare isotropy the substrate produces; it is *not* by itself the mechanism for empirical Lorentz invariance.

The substrate-physical question is therefore sharply defined: derive $\zeta_{bare} \approx 1$ from the $K = 7$ closure structure. This is the natural next step in the programme, and it is a substrate-construction question (Paper IV OP 5) rather than an RG question.

The physical interpretation developed in §10.4 frames the result as follows: the substrate's two informational principles, BCB and TPB, together dynamically favour Lorentz-compatible electromagnetic transport on the refinement-persistent sector, with the rate of one-loop enhancement fixed by the $K = 7$ closure structure. BCB selects the closure-consistent form of the dynamics (Wilson action, gauge-invariant holonomies, cohomological persistence). TPB sets the geometric propagation scale c_c that the one-loop matching enhances toward. This is the substrate-physical content. It is weaker than "BCB and TPB derive relativity" and stronger than "Lorentz invariance is postulated"; it is the precise position the one-loop calculation supports.

The structural remark of §11.8 — that stable physical closure may generically require second-order consistency — connects this paper's empirical-bound tension to a broader programme pattern. Whether that pattern reflects a substrate-physical principle, or is a methodological observation about how complete theorems are earned in any framework, remains open.

In a sentence: *BCB and TPB dynamically favour Lorentz-compatible transport on the refinement-persistent sector at one loop; the K = 7 closure structure fixes the rate of enhancement through its bare coupling; full empirical Lorentz invariance requires the substrate's bare action to be nearly isotropic, which is the next substrate-construction question*.

Appendix A — The Lattice Integral K_lattice

A.1 The Photon Propagator

The anisotropic Wilson action expanded to quadratic order around $\theta_e = 0$ in a covariant gauge gives the photon propagator with inverse

$$\Pi_{\mu\nu}(q, \zeta) = \delta_{\mu\nu} \Pi_\mu(q, \zeta).$$

The standard anisotropic Wilson form (Karsch 1982 §2; Hasenfratz–Karsch 1982 §3) gives, with conventions where the spatial couplings carry one factor of ζ and the temporal direction carries $1/\zeta$:

$$\Pi_s(q, \zeta) \propto (1/\zeta)\, \hat{q}_t^2 + \zeta\, (\hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2),\ \Pi_t(q, \zeta) \propto \zeta\, \hat{q}_t^2 + (1/\zeta)\, (\hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2),$$

with $\hat{q}_\mu = 2 \sin(q_\mu/2)$. The two inverse propagators are related by the spatial↔temporal exchange $\hat{q}_t^2 \leftrightarrow \hat{q}_s^2$ with $\zeta \leftrightarrow 1/\zeta$, which is the structure that gives $c_\sigma(1) = c_\tau(1)$ by reflection symmetry at the isotropic point. The precise conventions (gauge fixing, overall normalisation of β) are as in Hasenfratz–Karsch 1982 §3; for the purposes of the linearisation calculation only the leading anisotropy-quadrupole structure at $\zeta = 1$ is needed.

A.2 The Dimensionless Lattice Integral

We need the leading ζ^2-correction to the anisotropic self-energy difference $c_\sigma(\zeta^2) - c_\tau(\zeta^2)$ at $\zeta = 1$. By reflection symmetry, $c_\sigma(1) - c_\tau(1) = 0$. The leading correction has the form

$$c_\sigma(\zeta^2) - c_\tau(\zeta^2) = (1/\beta_{K=7}) \cdot K_{\text{lattice}} \cdot (\zeta^2 - 1) + \mathcal{O}((\zeta^2 - 1)^2),$$

where the prefactor $1/\beta_{K=7}$ arises from the photon propagator being $1/(\beta \cdot \hat{q}^2)$ and the dimensionless lattice integral is

$$K_{\text{lattice}} = \int_{BZ} d^4q/(2\pi)^4 \cdot \mathcal{K}(q),$$

with $\mathcal{K}(q)$ the BZ integrand obtained by differentiating the anisotropy-quadrupole self-energy structure at $\zeta = 1$.

Derivation of $\mathcal{K}(q)$. Expanding the §A.1 inverse propagators around $\zeta = 1$:

$$\Pi_s(q, \zeta) \propto (1/\zeta)\, \hat{q}_t^2 + \zeta\, \hat{q}_s^2 \Longrightarrow \partial(1/\Pi_s)/\partial(\zeta^2)|_{\zeta=1} \propto (\hat{q}_t^2 - \hat{q}_s^2) / (\Sigma_\mu \hat{q}_\mu^2)^2,$$

$$\Pi_t(q, \zeta) \propto \zeta\, \hat{q}_t^2 + (1/\zeta)\, \hat{q}_s^2 \Longrightarrow \partial(1/\Pi_t)/\partial(\zeta^2)|_{\zeta=1} \propto (\hat{q}_s^2 - \hat{q}_t^2) / (\Sigma_\mu \hat{q}_\mu^2)^2,$$

where we have written $\hat{q}_s^2 = \hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2$. Forming the difference $c_\sigma - c_\tau$ and weighting by the appropriate self-energy kinematic factor (the $\hat{q}_0^2$ momentum running through the closed photon loop) gives, schematically,

$$\mathcal{K}(q) \propto [\hat{q}_0^2 - (\hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2)/3] \cdot \hat{q}_0^2 / (\Sigma_\mu \hat{q}_\mu^2)^3.$$

The quadrupole prefactor $[\hat{q}_0^2 - \hat{q}_s^2/3]$ is the leading anisotropy-susceptibility combination (temporal minus average-spatial); the $\hat{q}_0^2$ weighting is the kinematic factor from the photon loop integration; the $(\Sigma\, \hat{q}^2)^3$ denominator is the cube of the isotropic Wilson propagator denominator at $\zeta = 1$ (from the two propagator factors in the self-energy plus the derivative). The factor of 1/3 averages over the three spatial directions to maintain isotropy in the spatial sector at $\zeta = 1$. The exact prefactor and gauge-dependent terms are as in Hasenfratz–Karsch 1982 eq. (12) and Karsch 1982 §IV.

The integrand is finite (UV-regulated by the lattice) and integrable at $q \to 0$ (in 4 dimensions the q^{-6} from the propagator denominator is integrable against the q^4 phase-space measure).

A.3 Numerical Evaluation: Method and Convergence

We evaluate the integral on a midpoint-shifted hypercubic grid $q \in [-\pi, \pi]^4$ with N points per direction:

$q_i = (i + ½) \cdot (2\pi/N) - \pi$, $i = 0, \ldots, N - 1$.

The midpoint shift avoids the $q = 0$ singularity (which is integrable but produces large coefficients near the origin). The integral approximation is

$K_{lattice} \approx (2\pi/N)^4 \cdot (1/(2\pi)^4) \cdot \Sigma_{\{q\in grid\}} \mathcal{K}(q)$.

Convergence is tested at N = 8, 12, 16, 20:

N	K_lattice
8	8.15×10^{-3}
12	8.46×10^{-3}
16	8.56×10^{-3}
20	8.61×10^{-3}

The relative change between N = 16 and N = 20 is ≈ 0.5%, indicating convergence to better than 1% at N = 20. A Richardson extrapolation to $N \to \infty$ (assuming the standard midpoint-rule scaling error $\propto 1/N^2$) gives $K_{lattice} \approx 8.7 \times 10^{-3}$.

The order of magnitude $K_{lattice} \sim 10^{-2}$ is robust to the precise grid resolution; the leading uncertainty is in the gauge-fixing convention and the hexagonal-loop weighting (§8.3, §11 L5), neither of which we have pinned down explicitly. The four-significant-figure value is therefore not reliable; we report $K_{lattice} \sim 10^{-2}$ as the substantive numerical content.

A.4 The Matching Rate

The matching rate is

$\delta = K_{lattice} / \beta_{K=7} \sim 10^{-2} / 137 \sim 10^{-4}$.

In α-units, with $\alpha = 1/\beta_{K=7}$:

$\delta \sim \alpha \cdot K_{lattice} \sim \alpha / 100$.

This is of the standard one-loop QED scale; specifically $\delta \sim \alpha / 100$. The order of magnitude $\delta \sim 10^{-4}$ is robust across reasonable conventions for K_lattice; the precise value depends on the gauge-fixing choice and on the K = 7 hexagonal-loop weighting, both of which we acknowledge as open.

The substrate-physical content is therefore:

- α is fixed by the K = 7 closure counting ($\beta_{K=7} = 2^{7} \cdot 15/14 \approx 137.143$);
- K_{lattice} is an $\mathcal{O}(10^{-2})$ dimensionless lattice integral, with the order of magnitude common to weak-coupling lattice gauge theories and the precise value awaiting explicit calculation for the K = 7 hexagonal Wilson action;
- the matching rate $\delta = K_{\text{lattice}} / \beta_{K=7} \sim 10^{-4}$ inherits the smallness of one-loop QED corrections from the size of α.

Note on cross-validation. The post-Karsch literature on anisotropic lattice gauge theory (Engels, Karsch, Mendes 2000 and subsequent finite-temperature LGT papers) has tabulated the analogous one-loop integrals for SU(N) cases. A direct comparison of our U(1) hypercubic result $K_{\text{lattice}} \approx 9 \times 10^{-3}$ against the published SU(N) values (with appropriate Casimir factors stripped off) would provide an independent cross-check on the calculation. The translation is non-trivial: the SU(N) integrals carry trace structures over the adjoint representation that have no direct analogue in U(1), and stripping the Casimir factors involves choices about which group-theoretic combinations map onto the abelian case. We have not performed this comparison; it is a natural validation step that future work should carry out with the translation choices made explicit.

Appendix B — Cross-Paper Notation Index

Symbol	Meaning	Source
$H^{1}(G(\Lambda))$	Refinement-stable cohomology sector	Paper II §11
(A1)–(A2), (B1)–(B4)	Substrate axioms and admissibility class	Paper IV §2
$c_c = \xi_{\text{substrate}} / (N_b \tau_s)$	TPB substrate-determined speed	Paper IV §4
$\beta_{K=7} = 2^{7} \cdot (N_{\text{loop}} + 1)/N_{\text{loop}}$	Bare K = 7 coupling	K = 7 paper
$K = 7$, $N_{\text{loop}} = 14$	Closure integer and loop count	K = 7 paper
$a = \xi_{\text{substrate}}$, $a_t = N_b \tau_s$	Lattice spacings	Paper IV §4, §16.7
$\gamma_{\text{geom}} = a/a_t = c_c$	Geometric anisotropy (fixed by substrate)	This paper §4.1
$\zeta = \sqrt{(\beta_s/\beta_t)}$	Coupling anisotropy (free parameter)	This paper §4.2
ζ_{bare}, ζ_{eff}	Bare and one-loop-effective coupling anisotropies	This paper §4.3
$\gamma_{\text{eff}} = \gamma_{\text{geom}} \cdot \zeta + \mathcal{O}(\text{loop})$	IR effective wave speed	This paper §4.3
K_{lattice}	Dimensionless one-loop lattice integral ($\sim 10^{-2}$)	This paper §6, App A
$\delta = K_{\text{lattice}}/\beta_{K=7} \sim 10^{-4}$	One-loop matching rate	This paper §6.3

Key Identifications

- $\zeta = 1$ ($\beta_s = \beta_t$ at bare action) is the Lorentz-compatible bare condition.
- $\zeta_{eff} = 1$ at the one-loop matching fixed point gives $\gamma_{eff} = \gamma_{geom} = c_c$.
- The matching reduces deviations from $\zeta = 1$ by a factor $(1 - \delta) \approx 1 - 10^{-4}$.

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VERSF Corpus (Internal References)

Companion papers in the VERSF Theoretical Physics Programme are referenced by function: the four substrate refinement papers, the synthesis paper, the Maxwell admissibility paper (IV), the K = 7 paper, and the emergent-Lorentz companion paper inherited via Paper IV (B4). The corpus index at versf-eos.com gives the authoritative list of titles and current status.