

# The Persistent Sector Carries the Gauge Theory

## Maxwell Transport on Refinement-Stable Cohomology in VERSF

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### General Reader Abstract

Two earlier strands of the VERSF programme had remained compatible but not fully integrated. One strand, the *substrate refinement papers*, established that the only mathematical structure on a discrete substrate that survives the process of "zooming in" is a specific kind of relational object — cohomological transport, which assigns a value to closed loops in the substrate while ignoring the values at individual points. The other strand, the *Maxwell admissibility paper*, established that within a substrate satisfying two informational principles (conservation of information and finite-speed propagation), Maxwell's equations of electromagnetism are forced as the unique admissible local transport theory. The two papers spoke about closely related objects — relational transport, gauge equivalence, closed-loop quantities — but never made the identification explicit.

This paper makes the identification. It shows that the substrate papers' "refinement-persistent cohomology sector" is *the same object* as the Maxwell paper's "admissible transport sector." This isn't a coincidence; it is a structural necessity. The substrate refinement papers prove that the cohomology sector is the *only* relational structure that survives refinement. The Maxwell paper proves that Maxwell-form gauge transport is the *only* dynamics admissible on a sector of that type. Combined, the result is sharper than either alone: Maxwell-form gauge transport is the unique refinement-stable dynamics on the unique refinement-stable observable sector. Three further consequences follow. First, gauge redundancy is no longer "informational degeneracy" in a soft sense — it is exactly the equivalence relation forced by the substrate's identification of scalar gradients as refinement-trivial. Second, Wilson loops — the closed-loop quantities that physicists use to make gauge theory measurable — are mathematically justified as the unique observables that survive both refinement and gauge change. Third, the no-go results for scalar substrate observables now match the positive emergence of gauge theory: scalars trivialise, relational transport persists, and the persistent transport is electromagnetism.

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# Abstract

Four earlier VERSF papers had established complementary but unconnected results. The *coarse-graining* paper proved that scalar refinement on a single causal diamond trivialises bulk modes. The *refinement-stability* paper proved that scalar refinement on Level 2 substrates trivialises gluing and antichain modes, and derived the minimal non-trivial observable sector as

$$\mathcal{H}_{\text{R}}(\Lambda) = C^1(\Lambda) / \text{Im}(d^0) = H^1(G(\Lambda))$$

(edge transport modulo scalar gradients). The  $\Gamma^*$  paper proved that the natural lift  $L : H^1(\Lambda) \rightarrow H^1(\Lambda')$  is a canonical isomorphism and  $\Gamma^* \circ L = 2 \cdot \text{Id}_{\{H^1\}}$ , so cohomology classes are exactly refinement-persistent up to a structural scaling factor absorbed by renormalisation  $\Gamma^* := \frac{1}{2}\Gamma^*$ . The *Maxwell admissibility* paper proved that under substrate axioms BCB and TPB, within an admissibility class motivated by substrate locality, gauge redundancy, closure-geometry covariance, and a controlled  $\xi$ -EFT truncation, Maxwell-form U(1) gauge transport is the unique surviving theory at  $O(\varepsilon^0)$ .

These four results are compatible but were not fully integrated. The Maxwell paper introduced its potential  $A_c^\mu$ , gauge equivalence  $A_c \rightarrow A_c + \partial\chi$ , and Wilson holonomies as *postulated* substrate-physical objects with admissibility constraints (B1)–(B4). The substrate papers showed that exactly these objects — a 1-cochain potential, a gauge equivalence by gradients, and closed-loop holonomies — emerge as the forced refinement-stable structure on any admissible substrate, with the gauge equivalence motivated by scalar trivialisation rather than postulated as a symmetry.

This paper performs the integration. We establish four structural identifications:

**(I) The potential class is the cohomology class of the persistent sector.** The Maxwell paper's gauge-equivalence class  $[A_c]$  is identified at the cohomology-class level with  $[\omega] \in H^1(G(\Lambda))$ . The class-level identification follows from Paper III's canonical isomorphism  $L$ . A smooth-representative construction (an explicit smooth  $A_c$  discretised from a substrate cochain) requires the additional regularity machinery of Paper IV's emergent-Lorentz companion paper and is inherited from there as conditional; the class-level identification holds independently of those regularity questions.

**(II) Gauge redundancy is refinement equivalence.** The Maxwell paper's gauge transformation  $A_c \rightarrow A_c + \partial\chi$  is exactly the equivalence relation  $\omega \sim \omega + d^0\phi$  on cochains, with parameter  $\chi \equiv \phi$  drawn from the trivialised scalar sector  $C^0$ . This is not analogous to gauge redundancy; it *is* gauge redundancy, with the substrate paper's stronger justification (forced by scalar trivialisation) replacing the Maxwell paper's weaker one (informational degeneracy under an observable map).

**(III) Wilson loops are persistent observables.** The closed-walk integral  $\oint_\gamma A_c \cdot dl$  is the substrate-level pairing  $\langle [\gamma], [\omega] \rangle : H_{-1}(G(\Lambda)) \times H^1(G(\Lambda)) \rightarrow \mathbb{R}$  between a homology cycle class  $[\gamma]$  and a cohomology class  $[\omega]$ . Refinement persistence ( $\Gamma^* \circ L = 2 \cdot \text{Id}$ ) makes Wilson loops *exactly* refinement-stable up to the structural scaling factor.

#### **(IV) Maxwell is the dynamics on the persistent sector — the strong synthesis theorem.**

Combining the substrate papers' uniqueness of the persistent sector with the Maxwell paper's uniqueness of the dynamics within an admissibility class, we obtain:

**Theorem (Strong Synthesis).** Under substrate axioms (A1) BCB and (A2) TPB, within the admissibility class (B1)–(B4), Maxwell-form  $U(1)$  gauge transport at  $O(\varepsilon^0)$  is the unique refinement-stable dynamics on the unique refinement-stable observable sector  $H^1(G(\Lambda))$ .

This is substantially stronger than either input theorem read alone. The Maxwell paper proved uniqueness of the dynamics *given a carrier of the right type*; the substrate papers proved uniqueness of the carrier *given the scalar refutations*. The synthesis makes both qualifications mutually anchored: the carrier is *forced* by the scalar refutations, and the dynamics is *forced* on the carrier. There is no alternative refinement-stable transport structure, and there is no alternative admissible dynamics on it.

The paper does not introduce new substrate physics. It performs the structural identification that the four prior papers had jointly implied but never made explicit, and shows that the integration sharpens the headline result.

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## **1. Introduction and Inheritance**

The VERSF programme has produced four papers whose results are mutually compatible but were not until now fully integrated. Each paper has a precise scope and a precise list of open problems. The present paper does not add new results to any of them; it shows that their joint structural content is sharper than each individually.

## 1.1 The Four Papers

We refer throughout to:

- **Paper I** — *Admissible Coarse-Graining and Continuum Emergence in VERSF*. Establishes that the linearised refinement operator  $L_R$  on a single  $d$ -dimensional causal diamond has spectrum  $\{1, (d-1)/d, \dots, 0\}$  with binomial multiplicities; the eigenvalue-1 mode is one-dimensional (the constant counting invariant), and the corner-alternating mode is in the kernel.
- **Paper II** — *Refinement Stability on Diamond-Glued Substrates in VERSF*. Establishes that the lazy random walk  $W = \frac{1}{2}(I + D^{-1}A)$  on connected admissible glued substrates has a one-dimensional eigenvalue-1 scalar subspace (Proposition 1), bipartite covering-pair graphs (Theorem 1), and an exact closed-form near-marginal spectrum  $\lambda_2(k) = \cos^2(\pi/(4k))$  on chains (Proposition 2). The scalar refutations of Sections 5–8 force the minimal non-trivial observable sector to be  $H^1(G(\Lambda))$  (§11).
- **Paper III** — *Refinement Cohomology and the Spectrum of  $\Gamma^*$  on Admissible Substrates*. Establishes that the natural lift  $L : H^1(\Lambda) \rightarrow H^1(\Lambda')$  is a canonical isomorphism and  $\Gamma^* \circ L = 2 \cdot \text{Id}_{\{H^1(\Lambda)\}}$  under midpoint refinement (Theorem 1). The factor 2 is bookkeeping absorbed by  $\Gamma^* := \frac{1}{2}\Gamma^*$  giving  $\Gamma^* \circ L = \text{Id}$ .
- **Paper IV** — *Electricity and Magnetism from Bit Conservation & Balance and Ticks-Per-Bit*. Establishes that under substrate axioms (A1) BCB and (A2) TPB, within the admissibility class (B1) locality, (B2)  $\xi$ -EFT truncation, (B3) gauge redundancy, (B4) closure-geometry covariance, Maxwell-form  $U(1)$  gauge transport is the unique surviving theory at  $O(\epsilon^0)$ .

Papers I–III concern the refinement-stable observable sector at substrate level. Paper IV concerns admissible dynamics on a substrate satisfying BCB and TPB. They have an obvious meeting point — Paper IV's gauge structure looks remarkably like the cohomology structure Papers I–III derive — but the identification was previously left implicit.

## 1.2 What This Paper Does

We perform four structural identifications, in increasing order of substantive content:

- §3 identifies the gauge-equivalence class  $[A_c]$  of Paper IV's potential with the cohomology class  $[\omega] \in H^1(G(\Lambda))$  at the substrate level. The class-level identification is established by Paper III Theorem 1; the smooth-representative identification is acknowledged as conditional on Paper IV's emergent-Lorentz companion paper.
- §4 identifies Paper IV's gauge transformation  $A_c \rightarrow A_c + \partial\chi$  with the equivalence relation  $\omega \sim \omega + d^0\phi$  on edge cochains, with the substrate-physical justification (refinement trivality of scalars) inherited from Papers II–III rather than from Paper IV's §7.1.
- §5 identifies Paper IV's Wilson holonomy  $\oint_\gamma A_c \cdot dl$  with the canonical pairing  $H_1(G(\Lambda)) \times H^1(G(\Lambda)) \rightarrow \mathbb{R}$  between homology cycles and cohomology classes.
- §6 establishes refinement persistence of Wilson loops via Paper III's Theorem 1.

- §7 combines these into the strong synthesis theorem: Maxwell-form transport at  $O(\epsilon^0)$  is the unique refinement-stable dynamics on the unique refinement-stable observable sector.

We then catalogue (§8) what each paper contributed, (§9) what open problems are sharpened or partially closed by the synthesis, and (§10) the epistemic status of each claim.

### 1.3 Inheritance Discipline

Where this paper inherits a result from one of Papers I–IV, we cite the inherited claim with its original scope and conditionality. We do not strengthen inherited conditional results; we only combine them. In particular:

- Paper III's Theorem 1 ( $\Gamma^* \circ L = 2 \cdot \text{Id}$ ) is inherited as proved under midpoint refinement on connected admissible substrates. Its scope is not strengthened here.
- Paper IV's Maxwell admissibility theorem is inherited as proved under (A1)–(A2) and (B1)–(B4) at  $O(\epsilon^0)$ , with explicit external conditionalities on the saturation postulate (§10.2 of Paper IV) and current-identification (§12.2 of Paper IV). These conditionalities are *preserved* under the synthesis; the synthesis does not close them.

The synthesis adds structural identifications and a combined theorem. It does not add new substrate physics, new measurement theory, or new dynamics.

## 2. The Four Prior Results

Before performing the identifications, we restate the four prior results in compatible notation. We use the substrate-paper conventions throughout:  $G(\Lambda)$  is the covering-pair graph of an admissible substrate  $\Lambda$ ;  $C^k(\Lambda)$  are  $k$ -cochains;  $d^0$  is the coboundary;  $H^1(G(\Lambda)) = C^1/\text{Im}(d^0)$ .

### 2.1 Paper I — Bulk Scalar Triviality

On a  $d$ -dimensional causal diamond  $\Lambda^{(d)}$ , the linearised refinement operator  $L_R^{(d)}$  acts on vertex scalars with spectrum  $\{1, (d-1)/d, (d-2)/d, \dots, 0\}$  and binomial multiplicities. The eigenvalue-1 subspace is one-dimensional (the constant mode); all non-constant scalar modes are refinement-decaying.

Used in this paper as: scalar bulk modes on any single causal diamond are exhausted apart from the constant counting invariant.

### 2.2 Paper II — Forced Cohomology Sector

On any connected admissible glued substrate, the eigenvalue-1 subspace of  $W = \frac{1}{2}(I + D^{-1}A)$  is one-dimensional (Proposition 1). Antichain-supported scalar modes are not refinement-stable (§8). The minimal non-trivial observable sector compatible with the scalar refutations is

$$\mathcal{H}_R(\Lambda) := C^1(\Lambda) / \text{Im}(d^0) = H^1(G(\Lambda)),$$

with equivalence  $\omega \sim \omega + d^0\phi$  forced because the parameter space  $C^0$  is refinement-trivial (§11).

Used in this paper as:  $H^1(G(\Lambda))$  is the unique minimal refinement-stable observable sector within standard cochain geometry, and the gauge-type equivalence on it is forced rather than postulated.

### 2.3 Paper III — Refinement Persistence

The natural lift  $L : H^1(\Lambda) \rightarrow H^1(\Lambda')$  under midpoint refinement is a canonical isomorphism. The cochain pullback  $\Gamma^* : H^1(\Lambda') \rightarrow H^1(\Lambda)$  satisfies  $\Gamma^* \circ L = 2 \cdot \text{Id}_{\{H^1(\Lambda)\}}$ . Equivalently, the renormalised pullback  $\Gamma^* := \frac{1}{2}\Gamma^*$  satisfies  $\Gamma^* \circ L = \text{Id}$ .

Used in this paper as: cohomology classes are canonically identified across refinement scales by  $L$ ; closed-walk integrals (Wilson loops) inherit this identification with the structural factor 2 per refinement step, or exactly under renormalisation.

### 2.4 Paper IV — Maxwell Admissibility

Under (A1) BCB and (A2) TPB, within the admissibility class (B1) locality, (B2)  $\xi$ -EFT truncation, (B3) gauge redundancy, (B4) closure-geometry covariance, the surviving admissible transport theory at  $O(\varepsilon^0)$  is Maxwell-form  $U(1)$  gauge transport:

$$dF_c = 0, \delta F_c = \mu_c J_c,$$

with  $F_c = dA_c$ ,  $A_c^\mu$  the transport potential, and  $J_c^\mu$  the BCB-conserved current. Two external identifications (saturation §10.2, current-selection §12.2) are required for the empirical labelling "Maxwell electrodynamics."

Used in this paper as: Maxwell-form transport is the unique admissible dynamics within (B1)–(B4); the gauge redundancy (B3) and the potential  $A_c^\mu$  are the constructs in which the dynamics is naturally expressed.

The two external Paper-IV conditionalities (saturation, current-selection) are *not* addressed by the synthesis. They remain open in their original form.

## 3. The Identification of the Potential Class with the Cohomology Class

### 3.1 The Class Identification

Paper IV's transport potential  $A_c^\mu$  is a smooth one-form on the emergent-Lorentz substrate manifold (Paper IV §2.5 regularity assumptions). Papers II–III work at the discrete substrate

level, where the corresponding object is an edge cochain  $\omega \in C^1(G(\Lambda))$ . The natural identification proceeds at the level of cohomology *classes* rather than of specific representatives, which sidesteps the regularity questions involved in identifying a substrate cochain with a smooth form.

What Papers II–III establish is that the cohomology class  $[\omega] \in H^1(G(\Lambda_n))$  is canonically identified across refinement scales by the lift  $L : H^1(\Lambda_n) \rightarrow H^1(\Lambda_{n+1})$ , with  $\Gamma^*$  providing the inverse:  $\Gamma^* \circ L = \text{Id}$  (Paper III Theorem 1). The inverse limit

$$\lim_{\leftarrow} \{H^1(G(\Lambda_n))\}$$

with respect to the system  $(H^1(\Lambda_n), \Gamma^*)$  is therefore well-defined as a vector space, and  $L$  provides the consistent identification.

What Paper IV requires of  $A_c^\mu$  is its gauge-equivalence class  $[A_c]$ : the de Rham cohomology class to which  $A_c$  is a smooth representative. The Maxwell paper's substantive use of  $A_c$  is always through gauge-invariant constructions —  $F_c = dA_c$ ,  $\oint_\gamma A_c \cdot dl$ ,  $\partial_\mu F_c^{\mu\nu}$  — none of which depend on the choice of representative within the gauge-equivalence class.

The identification is therefore at the class level:

$$[\omega] \in H^1(G(\Lambda)) \text{ corresponds to } [A_c] \in H^1_{\{dR\}}(M),$$

where  $M$  is the emergent manifold of Paper IV §2.5 and the correspondence is mediated by whatever cochain-to-form construction (Whitney forms, simplicial cochains as currents, or similar) supplies the bridge between substrate cohomology and de Rham cohomology of the emergent manifold.

### 3.2 What Is Established and What Is Conjectured

This identification has two parts of distinct epistemic status, which we separate carefully.

**Established.** At the substrate-cohomology level, the inverse limit  $\lim_{\leftarrow} \{H^1(G(\Lambda_n))\}$  under  $\Gamma^*$  is a well-defined vector space, and  $L$  provides a canonical identification of  $H^1(\Lambda_n)$  with this limit for every  $n$ . Cohomology classes at every refinement depth correspond to a single inverse-limit class. This is what Paper III Theorem 1 directly delivers.

**Conjectured (inherited conditional).** That this inverse-limit vector space has the right additional structure (topology, smoothness, metric compatibility) to be identified with  $H^1_{\{dR\}}(M)$  for the emergent manifold  $M$  of Paper IV's emergent-Lorentz companion paper. This identification requires:

- (i) a topology and smooth structure on  $M$  (supplied by the emergent-Lorentz construction);

(ii) a cochain-to-form lift compatible with that smooth structure (e.g., Whitney forms; a standard construction in algebraic topology but not yet exhibited explicitly for the present refinement system);

(iii) a substrate analogue of the de Rham theorem identifying  $H^1_{\{dR\}}(M)$  with the inverse limit of substrate cohomology.

None of (i)–(iii) is established in this paper or in Papers II–III; they are background assumptions inherited from Paper IV's emergent-Lorentz companion paper. The synthesis identification at the *class* level (above) holds independently of these regularity questions; the identification at the *representative* level (specifying a smooth  $A_c$  that the substrate cochain  $\omega$  is a discretisation of) inherits them.

The body of the present paper uses the class-level identification throughout. Where Paper IV manipulates  $A_c$  (in §§7–9 of that paper, deriving  $F_c$ , gauge transformations, and the source equation), the constructions are all gauge-invariant, so the substantive content can be read at the class level without committing to a smooth-representative construction.

### 3.3 What the Maxwell Paper Implicitly Assumed

Paper IV §5 introduced  $A_c^\mu$  as a postulate, with the acknowledgement that "the explicit construction of  $A_c^\mu$  from substrate microstate is a separate problem." Papers II–III now supply that construction *at the class level*: the substrate microstate carries an edge cochain  $\omega \in C^1(\Lambda)$ , and the cohomology class  $[\omega] \in H^1(G(\Lambda))$  is the canonical object identified across refinement scales by  $L$ .

What Papers II–III do *not* supply is the smooth-form representative  $A_c$  on the emergent manifold; this requires the construction of Paper IV's emergent-Lorentz companion paper plus the additional regularity machinery of (i)–(iii) above.

The integration is therefore: Paper IV's  $[A_c]$  is identified at the cohomology-class level with the substrate-papers'  $[\omega] \in H^1(G(\Lambda))$ , with the smooth-representative construction inherited as conditional on Paper IV's emergent-Lorentz companion paper.

This is enough for the structural identifications of §§4–6 and the strong synthesis theorem of §7, all of which are gauge-invariant statements at the class level.

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## 4. Gauge Redundancy as Refinement Equivalence

### 4.1 The Two Independent Derivations of the Same Equivalence

Paper IV introduces gauge equivalence  $A_c^\mu \rightarrow A_c^\mu + \partial^\mu \chi$  as admissibility constraint (B3), and motivates it via §7.1 as "informational degeneracy" — equivalence classes of substrate

microstates under an observable map  $\mathcal{O}$ . The Maxwell paper acknowledges (§7.1, "honest acknowledgement"): "What §7.1 adds is substrate-physical interpretation. ... Genuine novelty would require exhibiting  $\mathcal{S}$  explicitly — a concrete substrate microstate space with a constructive observation map whose redundancy realises the gauge group — which is exactly what a toy substrate model (open problem 5) would supply."

Papers II–III derive the same equivalence by a stronger route. The equivalence  $\omega \sim \omega + d^0\varphi$  on  $C^1(G(\Lambda))$  is forced because:

- (R1, Paper II Proposition 1) The scalar sector  $C^0(G(\Lambda))$  is refinement-trivial except for the constant mode.
- (R2, Paper II §8) Scalar antichain modes are exhausted.
- (R3, Paper II §11) The parameter space  $\text{Im}(d^0) \subset C^1$  is therefore parametrised entirely by refinement-trivial data; any non-trivial observable structure on edge cochains must quotient by it.

The conclusion (Paper II §11.7, sharpened in Paper III §13) is that  $\omega \sim \omega + d^0\varphi$  is not a postulate but a substrate-level necessity: different choices of representative within a cohomology class are choices within the trivialised scalar sector.

These are not two analogous derivations of similar-looking equivalences. They are two independent derivations of the *same* equivalence:

Object	Paper IV name	Papers II–III name
Potential representative $A_c^\mu$		$\omega \in C^1(G(\Lambda))$
Equivalence	$A_c \rightarrow A_c + \partial\chi$	$\omega \rightarrow \omega + d^0\varphi$
Parameter	$\chi : V \rightarrow \mathbb{R}$ (scalar)	$\varphi : V \rightarrow \mathbb{R}$ (scalar)
Quotient	$A_c / (\text{gauge})$	$C^1 / \text{Im}(d^0) = H^1$
Justification	informational degeneracy	refinement triviality

The objects, the equivalence, the parameter, and the quotient are identical. Only the justifications differ — and the substrate-papers justification is strictly stronger, since it makes no appeal to a postulated observable map and instead derives the quotient from the substrate refinement structure itself.

## 4.2 What "Refinement Equivalence" Means

Gauge redundancy in this synthesis is therefore precisely:

**Definition (Refinement Equivalence).** Two cochain representatives  $\omega, \omega' \in C^1(G(\Lambda))$  are *refinement-equivalent* if they differ by an element of  $\text{Im}(d^0)$ , i.e.  $\omega' = \omega + d^0\varphi$  for some  $\varphi \in C^0(G(\Lambda))$ . Equivalently,  $\omega$  and  $\omega'$  are refinement-equivalent if they project to the same class in  $H^1(G(\Lambda))$ .

The substrate-physical content: refinement-equivalent representatives differ only in the trivialised scalar parameter sector, which the refinement flow does not distinguish. They produce identical refinement-stable observable content — the cohomology class — and identical pullbacks under the renormalised  $\Gamma^*$ .

### 4.3 Stronger Than "Informational Degeneracy"

The substrate-papers framing is stronger than Paper IV's §7.1 informational-degeneracy framing in three precise senses:

**(i) No appeal to a postulated observable map.** Paper IV §7.1 defines the equivalence via an observable map  $\mathcal{O} : \mathcal{S} \rightarrow \mathcal{O}$  and an equivalence relation  $s_1 \sim s_2 \Leftrightarrow \mathcal{O}(s_1) = \mathcal{O}(s_2)$ . The substrate-physical content of "what  $\mathcal{O}$  returns" was left for an explicit substrate model (Paper IV OP 5). Papers II–III need no such postulated map: the equivalence emerges from the spectral structure of  $W$  on  $C^0$  and the cochain pullback  $\Gamma^*$  on  $C^1$ , both of which are intrinsic to the substrate refinement structure.

**(ii) The parameter sector is identified, not asserted.** Paper IV §7.1 asserted that "the substrate carries more degrees of freedom than the observable map distinguishes; gauge symmetry is the redundant representational structure that tracks this disparity." Papers II–III identify the redundant degrees of freedom explicitly: they are vertex scalars  $\varphi \in C^0$ , and they are refinement-trivial by Proposition 1 (Paper II).

**(iii) The substrate framing commits to additive  $\mathbb{R}$ -equivalence and makes the compactification question precise.** The substrate-papers framing commits to scalar-parameter equivalence with additive  $\mathbb{R}$ -group structure, which is the cochain analogue of *non-compact*  $U(1)$  (i.e., the gauge group  $\mathbb{R}$  rather than the circle). Compactification to  $U(1)$  proper — periodic identification of cochain values modulo  $2\pi$  — is not addressed by either framing. It remains Paper IV OP 2 and the  $K = 7$  companion paper's question of whether the substrate also enforces a periodic identification. The substrate framing's strength on this axis is therefore to make the compactification question *precise* (is there an additional substrate structure forcing periodic identification of edge-cochain values?), not to answer it. The synthesis preserves Paper IV's gap on gauge-group compactification; it does not narrow it.

Paper IV's §7.1 framing is preserved as compatible; the synthesis observes that it is strengthened by the substrate-papers' derivation.

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## 5. Wilson Loops as the Canonical Pairing

### 5.1 The Substrate Definition

Paper IV introduces Wilson holonomies as gauge-invariant closed-loop integrals:

$$W(\gamma) := \oint_{\gamma} A_c \cdot d\ell,$$

where  $\gamma$  is a closed loop in the substrate manifold. Paper IV remarks (Aharonov–Bohm reference, §14) that these are gauge-invariant despite  $A_c$  not being gauge-invariant.

At the substrate level, the analogue is direct. Let  $\gamma \subset G(\Lambda)$  be a closed walk in the covering-pair graph — equivalently, a 1-cycle in the simplicial structure of  $G(\Lambda)$ . Let  $\omega \in C^1(G(\Lambda))$  be an edge cochain. The substrate Wilson loop is

$$W(\gamma, \omega) := \sum_{\{(e_i \rightarrow e_{i+1}) \in \gamma\}} \omega(e_i \rightarrow e_{i+1}),$$

with the convention that reverse-traversal edges contribute negative signs (Paper III §5.2 convention). This is the closed-walk sum.

## 5.2 $W(\gamma, \omega)$ Is the Canonical Pairing $H_1 \times H^1 \rightarrow \mathbb{R}$

The first homology  $H_1(G(\Lambda))$  is the space of closed walks modulo boundaries of 2-cells. On a graph (no 2-cells specified),  $H_1(G(\Lambda)) = \ker(\partial) \subseteq C_1(G(\Lambda))$  where  $C_1$  is the space of 1-chains and  $\partial$  is the boundary operator. By duality,  $\dim H_1(G(\Lambda)) = \dim H^1(G(\Lambda)) = |E| - |V| + 1$  on a connected graph.

The pairing

$$\langle \cdot, \cdot \rangle : H_1(G(\Lambda)) \times H^1(G(\Lambda)) \rightarrow \mathbb{R}, ([\gamma], [\omega]) \mapsto W(\gamma, \omega)$$

is well-defined on classes:

**(a) Independence of cocycle representative.** If  $\omega' = \omega + d^0\varphi$  for some scalar  $\varphi$ , then

$$W(\gamma, \omega') = W(\gamma, \omega) + \sum_{\{(e_i \rightarrow e_{i+1}) \in \gamma\}} (\varphi(e_{i+1}) - \varphi(e_i)).$$

The sum telescopes around the closed walk  $\gamma$ : each vertex enters once with positive sign and once with negative sign, so the sum vanishes. Hence  $W(\gamma, \omega') = W(\gamma, \omega)$ .

**(b) Independence of cycle representative.** If  $\gamma' = \gamma + \partial\beta$  for some 2-chain  $\beta$ , then  $\gamma$  and  $\gamma'$  differ by the boundary of a 2-cell. On a graph without specified 2-cells, this is vacuous; with 2-cells specified, the resulting condition is the standard "Wilson loop is well-defined on homology" statement.

The pairing  $W(\gamma, \omega)$  is therefore the canonical pairing of 1-homology with 1-cohomology — the substrate-level analogue of integrating a 1-form over a 1-cycle in de Rham cohomology.

## 5.3 The Plaquette Holonomy on $\Lambda^{(2)}$

The simplest example, recalling Paper III §5: on the single  $d = 2$  causal diamond  $\Lambda^{(2)}$ ,  $\dim H^1 = 1$  and the unique cohomology generator is represented by the plaquette walk  $a \rightarrow b \rightarrow d \leftarrow c \leftarrow a$

with  $\omega_{\square} = (+1, -1, +1, -1)$  in the basis  $(a \rightarrow b, a \rightarrow c, b \rightarrow d, c \rightarrow d)$ . The corresponding Wilson loop is

$$W(\square, \omega_{\square}) = \omega_{\square}(a \rightarrow b) - \omega_{\square}(a \rightarrow c) + \omega_{\square}(b \rightarrow d) - \omega_{\square}(c \rightarrow d) = 1 + 1 + 1 + 1 = 4$$

(with the convention that  $\leftarrow$  contributes negation of the basis-edge value). This is the substrate-level Wilson loop on the smallest possible plaquette and the simplest substrate-level analogue of the electromagnetic field flux through a plaquette in lattice gauge theory.

The value 4 is the *un-renormalised* substrate pairing at refinement depth 0 with our specific cycle and cochain normalisations. Under the natural  $\Gamma^* = \frac{1}{2}\Gamma^*$  renormalisation (Paper III §11.3), the corresponding refinement-stable pairing is  $4/2^n = 4$  at every depth, where the factors  $2^n$  from lifting are exactly cancelled by the  $1/2^n$  from renormalising. So the value is convention-dependent (it depends on the choice of basis cochain normalisation and traversal sign convention) but the refinement-stability of the value is intrinsic — §6 establishes that the pairing is exactly preserved across refinement scales after renormalisation.

## 5.4 What Paper IV Implicitly Used

Paper IV §7.1 acknowledged the Aharonov–Bohm holonomies as part of the observable map  $\mathcal{O}$  but did not specify their substrate-level form. The synthesis identifies them as the  $H_{-1} \times H^1$  pairing. Paper IV §14 cites the Aharonov–Bohm effect as evidence that gauge-invariant holonomies (and not the potential itself) are physical; Papers II–III now derive this from the substrate refinement structure, with Wilson loops emerging as the canonical pairing of the persistent cycle and cohomology classes.

# 6. Refinement Persistence of Wilson Loops

## 6.1 Lifted Cycle, Lifted Cocycle

Let  $\gamma \subset G(\Lambda_n)$  be a coarse cycle, with natural lift  $\gamma' = L(\gamma) \subset G(\Lambda_{\{n+1\}})$  under midpoint refinement (each edge in  $\gamma$  is replaced by its two-edge subdivision). Let  $\omega \in C^1(\Lambda_n)$  be a coarse cocycle, with natural lift  $\omega' = L(\omega) \in C^1(\Lambda_{\{n+1\}})$  (each fine edge inherits the value of the coarse edge it subdivides).

The Wilson loop on the lifted data is

$$W(\gamma', \omega') = \sum_{\{\text{fine edges in } \gamma'\}} \omega'(\text{fine edge}).$$

Each coarse edge of  $\gamma$  contributes two fine edges to  $\gamma'$ , each carrying the same cochain value, so

$$W(\gamma', \omega') = 2 \cdot W(\gamma, \omega).$$

## 6.2 Persistence of the Class Pairing

This is the Wilson-loop manifestation of Paper III's Theorem 1:  $\Gamma^* \circ L = 2 \cdot \text{Id}_{\{H^1\}}$ . Under the renormalised pullback  $\Gamma^* := \frac{1}{2}\Gamma^*$ , the relation becomes

$$W(\gamma', \omega') / 2 = W(\gamma, \omega).$$

The renormalised pairing  $2^{-1} W(L \gamma, L \omega)$  equals  $W(\gamma, \omega)$  at every refinement step, not merely in an asymptotic limit. Iterating,

$$2^{-n} \cdot W(L^{\wedge n} \gamma_0, L^{\wedge n} \omega_0) = W(\gamma_0, \omega_0) \text{ for every } n \geq 0.$$

The renormalised refinement-stable Wilson pairing is therefore *exactly* the coarse pairing at every refinement depth —  $n$ -independent by structural consequence, not by limit-taking. This is the strong form of refinement persistence: the Wilson loop is not approximately stable, not stable only in the IR limit, but exactly equal across all refinement scales after the natural  $\frac{1}{2}$ -per-step renormalisation.

## 6.3 Wilson Loops as the Operational Content of $[\omega]$

Combining §5.2 and §6.2: Wilson loops are the canonical pairing of the persistent homology class with the persistent cohomology class, and the pairing value is exactly refinement-stable (after renormalisation). The substrate's refinement-stable observables are precisely the Wilson loops — equivalently, the values of  $[\omega]$  on  $[\gamma]$  for any cycle  $[\gamma] \in H_1$ .

The operational content of "[ $\omega$ ] is a cohomology class" is: it produces refinement-stable real-number outputs when paired with refinement-stable cycle classes. Wilson loops are not auxiliary gauge-invariant constructs; they *are* the observable content of the cohomology sector.

This is what Paper IV §7.1 attempted to formalise abstractly via the observation map  $\mathcal{O}$  returning "all directed-loop holonomies." The synthesis makes this concrete:  $\mathcal{O}([\omega])$  is the function  $[\gamma] \mapsto W(\gamma, \omega)$  from  $H_1(G(\Lambda))$  to  $\mathbb{R}$ . The substrate microstate space modulo the equivalence relation is precisely the function space  $\text{Hom}(H_1(G(\Lambda)), \mathbb{R}) = H^1(G(\Lambda))$ , where the last equality is the substrate-level Poincaré duality.

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# 7. The Strong Synthesis Theorem

We now combine the four prior results.

## 7.1 Statement

**Theorem (Strong Synthesis).** Let  $\Lambda$  be an admissible substrate satisfying:

- **(A1) BCB.** There exists a conserved four-current  $J_c^\mu$  with  $\partial_\mu J_c^\mu = 0$ .
- **(A2) TPB.** Commitment propagation has finite invariant speed  $c_c = \xi/(N_b \tau_s)$  under atomic substrate update.

Within the admissibility class:

- **(B1)** Locality.
- **(B2)**  $\xi$ -controlled EFT truncation at parameter  $\varepsilon = \xi/L_{\text{macro}} \ll 1$ .
- **(B3)** Gauge redundancy (now read as refinement equivalence, §4).
- **(B4)** Closure-geometry covariance.

The substrate observable sector (within the standard cochain framework, per Paper II §11.6) is uniquely fixed by Papers II–III as  $H^1(G(\Lambda))$  — the refinement-stable cohomology sector — and the surviving admissible dynamics at  $O(\varepsilon^0)$  on this sector is Maxwell-form  $U(1)$  gauge transport:

$$dF_c = 0 \text{ (homogeneous, from } F_c = dA_c \text{ structural identity)}$$

$$\delta F_c = \mu_c J_c \text{ (inhomogeneous, unique source equation at } O(\varepsilon^0))$$

with  $[A_c]$  the gauge-equivalence class corresponding to  $[\omega] \in H^1(G(\Lambda))$  at the substrate level (§3),  $F_c = dA_c$  the curvature 2-form, and  $J_c^\mu$  the BCB-conserved current.

The uniqueness has two relativisations, in parallel: the carrier uniqueness is relative to the standard cochain framework (Paper II §11.6), and the dynamics uniqueness is relative to the admissibility class (B1)–(B4) (Paper IV §2).

## 7.2 The Strengthening Over Paper IV

Paper IV alone proved: *given the admissibility class (B1)–(B4), Maxwell-form transport is the unique dynamics at  $O(\varepsilon^0)$ .* The class (B1)–(B4) was *motivated* by substrate informational structure but not *derived* from it — it was a stated admissibility class within which uniqueness was proved.

The synthesis strengthens this in two ways:

**(i) (B3) is now derived.** The gauge redundancy  $A_c \rightarrow A_c + \partial\chi$  is no longer a postulated admissibility constraint; it is the refinement equivalence  $\omega \sim \omega + d^0\phi$  forced by scalar trivialisation (§4). This converts (B3) from "motivated" to "derived from (A1) BCB and the spectral structure of  $W$  on  $C^0$ ."

**(ii) The carrier is forced.** Paper IV's transport potential  $A_c^\mu$  was introduced as an admissibility postulate (§5). The synthesis identifies its gauge-equivalence class  $[A_c]$  at the cohomology-class level with  $[\omega] \in H^1(G(\Lambda))$ , which is the unique minimal refinement-stable observable sector within the standard cochain framework (Paper II §11). The carrier is no longer assumed; it is forced. (The smooth-representative refinement of the  $[A_c] \leftrightarrow [\omega]$  identification inherits the regularity conditionality of Paper IV's emergent-Lorentz companion paper; see §3.2.)

Combined: the synthesis converts Paper IV's "uniqueness within an admissibility class" to "uniqueness on the unique refinement-stable observable sector with derived gauge equivalence." The qualifier "within an admissibility class" weakens to "with substrate axioms (A1)–(A2), with derived locality conjecturally tied to atomic update (B1, Paper IV OP 4), with  $\xi$ -EFT truncation as the operational expansion parameter (B2), and with closure-geometry covariance inherited from emergent Lorentz (B4)."

The remaining unforced admissibility constraints are (B1) and (B2). (B3) and (B4) are now derived from the substrate refinement structure and the emergent-Lorentz companion paper respectively. (B1) and (B2) remain as in Paper IV.

### 7.3 What the Synthesis Does Not Strengthen

The synthesis does not address:

- **The cochain-framework relativisation.** Paper II §11.6 carefully relativised the forcing of  $H^1(G(\Lambda))$  to the standard cochain framework (cochains  $C^k$ , coboundary  $d^k$ , gradient quotient). Non-cochain observable structures — coefficient sheaves, higher categories, derived structures — are not refuted. The synthesis preserves this relativisation; the carrier-uniqueness claim is parallel in strength to the dynamics-uniqueness claim, which is similarly relativised to the admissibility class.
- **(B1) locality** as a theorem rather than admissibility constraint. This remains Paper IV OP 4.
- **(B2)  $\xi$ -EFT truncation** as a derivation rather than power-counting analogy. This remains Paper IV OP 5.
- **The saturation postulate**  $c'_c = c_c$ . This remains Paper IV §10.2.
- **The current-selection problem.** This remains Paper IV §12.2.
- **Non-Abelian extension.** This remains Paper IV OP 2.
- **Higher-order corrections** at  $O(\epsilon)$ ,  $O(\epsilon^2)$ . These remain Paper IV OP 3.
- **The class-to-representative refinement.** The §3 identification holds at the cohomology-class level; the smooth-representative construction inherits the regularity conditionality of Paper IV's emergent-Lorentz companion paper.

The synthesis sharpens the structural foundation of Maxwell transport. It does not close any of the originally open dynamical or empirical conditionalities.

### 7.4 Proof Outline

The proof is a composition of the four prior theorems, with the structural identifications of §§3–6 supplying the bridges.

**Step 1.** By Paper II Proposition 1, the scalar sector  $C^0(G(\Lambda))$  on any connected admissible  $\Lambda$  is refinement-trivial except for the constant mode. By Paper II §8, antichain-supported scalar modes are exhausted.

**Step 2.** By Paper II §11, the minimal non-trivial observable sector consistent with Step 1 within the standard cochain framework is  $H^1(G(\Lambda)) = C^1(\Lambda) / \text{Im}(d^0)$ , with equivalence  $\omega \sim \omega + d^0\phi$  forced because the parameter space  $C^0$  is refinement-trivial.

**Step 3.** By Paper III Theorem 1, the natural lift  $L : H^1(\Lambda_n) \rightarrow H^1(\Lambda_{n+1})$  is canonical and  $\Gamma^* \circ L = 2 \cdot \text{Id}_{\{H^1\}}$ . So the cohomology sector is preserved exactly across refinement scales under the renormalised  $\Gamma^* = \frac{1}{2}\Gamma^*$ .

**Step 4.** By §§3–6, Paper IV's gauge-equivalence class  $[A_c]$  corresponds at the cohomology-class level to  $[\omega] \in H^1(G(\Lambda))$  (with the smooth-representative refinement of this identification inherited as conditional on Paper IV's emergent-Lorentz companion paper; see §3.2 and §10.3); Paper IV's gauge transformation is the refinement equivalence forced by Step 2; Paper IV's Wilson loops are the canonical pairing of  $H_{-1}$  with  $H^1$ .

**Step 5.** By Paper IV's Maxwell admissibility theorem, under (A1)–(A2) and (B1)–(B4) restricted to the sector identified in Step 4, the unique surviving dynamics at  $O(\epsilon^0)$  is Maxwell-form  $U(1)$  gauge transport.

The composition: under (A1)–(A2), the only refinement-stable observable sector is  $H^1(G(\Lambda))$  (Steps 1–3), and the only admissible dynamics on it at  $O(\epsilon^0)$  is Maxwell-form  $U(1)$  gauge transport (Step 5 with carrier identification from Step 4). No other refinement-stable observable sector exists; no other admissible dynamics exists on it. The composition is therefore a strong uniqueness claim on both sides.

## 8. What Each Paper Contributed and What the Synthesis Adds

For clarity, we tabulate the contribution of each prior paper and what the present synthesis adds.

Component	Paper providing it	Status before synthesis	Status after synthesis
Scalar bulk triviality on single diamonds	Paper I	Proved	Inherited; used as Step 1
Scalar gluing/antichain triviality on Level 2	Paper II	Proved	Inherited; used as Step 1
Forced cohomology sector $H^1(G(\Lambda))$	Paper II §11	Proved	Inherited; used as Step 2
Lift $L$ canonical, $\Gamma^* \circ L = 2 \cdot \text{Id}$	Paper III	Proved	Inherited; used as Step 3
Maxwell admissibility within (B1)–(B4)	Paper IV	Proved	Inherited; used as Step 5

Component	Paper providing it	Status before synthesis	Status after synthesis
(B3) gauge redundancy as substrate-physical	Paper IV §7.1	Motivated via $\mathcal{S}/\sim$	<b>Derived</b> from Step 2 (§4)
[A_c] (gauge class) as substrate-physical	Paper IV §5	Postulated	<b>Derived at class level</b> (§3); smooth representative inherits Paper IV emergent-Lorentz companion paper conditionality
Wilson loops as substrate observables	Paper IV §7.1, §14	Gauge-invariant by construction	<b>Derived</b> as $H_{-1} \times H^1$ pairing (§5)
Refinement persistence of Wilson loops	Not in Paper IV	Not addressed	<b>New</b> (§6)
Maxwell on $H^1$ as unique on unique sector	Combination	Not stated	<b>New theorem</b> (§7)

The synthesis adds three structural identifications (§§3–5), one refinement-persistence result for Wilson loops that follows directly from Paper III's Theorem 1 (§6), and the strong synthesis theorem (§7) that combines the four prior results into a single uniqueness statement on both the carrier and the dynamics.

No new substrate dynamics, measurement theory, or empirical predictions are added. The synthesis is a structural integration of existing results.

## 9. Consequences for Open Problems

The synthesis partially clarifies some of the open problems flagged in the prior papers, without closing any.

### 9.1 Paper IV OP 5 (Substrate Construction)

Paper IV OP 5 sought "a concrete substrate model whose dynamics generate the admissibility constraints from below" with the goal of converting (B1)–(B4) from substrate-motivated postulates to substrate theorems. The synthesis closes one of these — (B3) gauge redundancy — by deriving it from the spectral structure of  $W$  on  $C^0$  (Papers I–II). It also converts the potential  $A_c^\mu$  from a substrate postulate to a substrate-derived inverse-limit object (Papers II–III).

The remaining (B1), (B2), and the saturation postulate continue to require substrate construction. The synthesis does not change their status; it does narrow the scope of "what needs to be

substrate-constructed" from "(B1)–(B4) plus  $A_c$ " to "(B1)–(B2) plus saturation," with (B3) and the carrier identification now derived.

## 9.2 Paper IV §12.2 (Current Selection)

Paper IV §12.2 noted that BCB delivers a conserved current  $J_c^\mu$  but does not internally identify it as the electromagnetic current; the substrate may carry multiple conserved vector currents (baryon number, lepton number, etc.) each independently satisfying  $\partial_\mu J^\mu = 0$ . Paper IV §12.3 proposed a four-condition minimisation programme as a candidate current-selection criterion.

The synthesis does not close current-selection. It does observe that the cohomology sector  $H^1(G(\Lambda))$  is intrinsic to the substrate refinement structure — it is determined by  $G(\Lambda)$  alone, not by the choice of which conserved current Maxwell transport governs. So the question becomes: which conserved current's dynamics is supported on  $H^1(G(\Lambda))$ , as opposed to on some richer cochain structure (e.g.,  $H^1$  with a different coefficient group, or higher cohomology)? Paper IV's four-condition criterion (massless, unconfined, unbroken, TPB-saturating) remains the candidate answer.

## 9.3 Paper IV OP 2 (Non-Abelian Extension)

The substrate papers used  $\mathbb{R}$ -valued cochains throughout. The non-Abelian extension would replace  $C^1$  with a richer cochain structure (matrix-valued or Lie-algebra-valued cochains), and the equivalence  $\omega \sim \omega + d^0\phi$  with a non-Abelian gauge transformation.

The synthesis observes that the underlying spectral argument ( $W$  trivialises  $C^0$ ; the parameter space of  $\text{Im}(d^0)$  is correspondingly trivialised) generalises to richer coefficient structures only when the  $W$ -spectrum behaves correspondingly. The non-Abelian extension is therefore a question about non-Abelian-valued substrate refinement: does the analogue of  $W$  on Lie-algebra-valued vertex functions still trivialise except for a "constant" mode? If yes, the cohomology forcing argument extends; if no, the carrier is not forced and the non-Abelian extension requires a different structural foundation.

This is not closure — it is a sharpened formulation of OP 2.

## 9.4 Wilson Loops and the Aharonov–Bohm Effect

Paper IV §14 cited the Aharonov–Bohm effect as a refutation-test for (B3): if a measurable quantity depended on  $A_c$  at a point rather than on  $F_c$ , (B3) would be refuted. The synthesis converts this from a structural observation to a substrate prediction: the Aharonov–Bohm phase  $\oint_\gamma A_c \cdot d\ell$  is the substrate Wilson loop  $W(\gamma, \omega)$ , which by §6 is exactly refinement-stable under the substrate refinement structure. The non-locality of the Aharonov–Bohm effect (the phase depends on global topology of  $\gamma$  rather than on local field values) is therefore a substrate-level consequence of the  $H^1$  structure, not an accidental feature of continuum electrodynamics.

This sharpens Paper IV §14's refutation framing: AB phases are predicted to be refinement-stable substrate observables, with the substrate-level mechanism being the  $H_{-1} \times H^1$  pairing.

## 9.5 Paper II's Programmatic Conjecture on Coherence Bands

Paper II §6 introduced the near-marginal scalar mode  $\lambda_{-2}(k) = \cos^2(\pi/(4k))$  on chains, with the programmatic conjecture that these long-lived modes correspond to "coherence band" structures in the wider VERSF programme. The synthesis paper's spectral discipline forces a sharpening of this conjecture.

The substrate exhibits near-marginal scalar modes alongside the exactly-persistent cohomological transport sector, but these are not parallel structures. By their spectral nature, near-marginal scalars are *transient* — they decay to zero under iterated refinement, with mixing time  $\tau_{\text{mix}} \sim k^2$ , leaving only the constant counting mode in the long-refinement limit. They cannot be carriers of refinement-stable physics. The synthesis theorem singles out the cohomology sector precisely because it is the *only* exactly-persistent observable structure; the near-marginal scalars do not share this status.

Whether the near-marginal scalars correspond to anything physically meaningful at finite scales is a separate programmatic question — they are long-lived but ultimately decaying. Paper II's "coherence band" framing was speculative at the time of writing; the synthesis paper's spectral discipline now makes the conjecture more precise: it must be a claim about finite-scale phenomena (where mixing times are long enough to be physically operational), not about the refinement limit (where the modes are trivialised). Whether this finite-scale framing can be made rigorous is a separate question not addressed here.

The Maxwell sector and the coherence-band programme are therefore *not* parallel substrate phenomena. The Maxwell sector lives on the unique exactly-persistent observable structure; the coherence-band programme, if it is to be made precise, must live at finite scales where near-marginal decay has not yet completed. The synthesis does not unify them; it sharpens the distinction.

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## 10. Epistemic Status

We catalogue the status of each claim made in this paper. The discipline mirrors that of the prior papers: distinguish proved from inherited from conditional from interpretive.

### 10.1 Proved (within this paper)

- **(§3) Class-level identification of  $[A_c]$  with  $[\omega]$ .** At the cohomology-class level, Paper IV's gauge-equivalence class  $[A_c]$  corresponds to  $[\omega] \in H^1(G(\Lambda))$ , with the canonical isomorphism  $L$  (Paper III Theorem 1) supplying the cross-scale identification. The substrate-cohomology inverse limit  $\lim_{\leftarrow n} H^1(G(\Lambda_n))$  under  $\Gamma^*$  is well-defined as a vector space. The smooth-representative construction (an explicit smooth  $A_c$  discretised

from  $\omega$ ) is conditional on Paper IV's emergent-Lorentz companion paper; this conditional is listed in §10.3.

- **(§4) Refinement-equivalence = gauge-equivalence.** The equivalence  $A_c \rightarrow A_c + \partial\chi$  on continuum potentials is precisely the cochain-level equivalence  $\omega \sim \omega + d^0\varphi$  forced by Paper II §11.
- **(§5) Wilson loop =  $H_1 \times H^1$  pairing.** The closed-walk integral  $W(\gamma, \omega)$  descends to a well-defined pairing of homology classes with cohomology classes on  $G(\Lambda)$ , via the standard telescoping argument for closed walks.
- **(§6) Refinement persistence of Wilson loops.**  $W(L\gamma, L\omega) = 2 \cdot W(\gamma, \omega)$  follows directly from Paper III Theorem 1 applied to the closed-walk sum; renormalisation by  $\frac{1}{2}$  per refinement step gives an exact invariant.

## 10.2 Inherited (from Papers I–IV)

- **Spectral results.** Paper I's d-cube spectrum, Paper II's Proposition 1, 2, Theorem 1, and Corollary 1, Paper III's Theorem 1. Inherited at the conditionalities and scopes of their original papers.
- **Maxwell admissibility theorem.** Paper IV's main theorem inherited at its original scope (under (A1)–(A2) and (B1)–(B4) at  $O(\epsilon^0)$ , with explicit external conditionalities on saturation and current-identification).
- **The structural identification of  $H^1(G(\Lambda))$  as forced.** Paper II §11.6, with the relativisation to standard cochain geometry preserved.

## 10.3 Conditional

- **(§3.2) Smooth-representative identification.** The class-level identification of §3.1 (substrate cohomology  $H^1(G(\Lambda_n)) \leftrightarrow$  inverse-limit vector space, via Paper III's  $L$  and  $\Gamma^*$ ) is established. The further identification of the inverse-limit vector space with  $H^1_{\{dR\}}(M)$  of the emergent manifold  $M$  — and the construction of a smooth representative  $A_c$  from a substrate cochain  $\omega$  via Whitney forms or similar — requires Paper IV's emergent-Lorentz companion paper. This identification is inherited as conditional. The three sub-conditionalities are listed in §3.2 (i)–(iii): smooth structure on  $M$ , a cochain-to-form lift compatible with that structure, and a substrate analogue of the de Rham theorem identifying  $H^1_{\{dR\}}(M)$  with the inverse limit of substrate cohomology.
- **(§9.3) Non-Abelian extension.** The synthesis sharpens but does not address Paper IV OP 2. Whether the substrate refinement structure generalises to Lie-algebra-valued cochains is open.

## 10.4 What the Synthesis Does Not Prove

We are explicit about what is *not* added by this paper:

- The synthesis does not derive (B1) or (B2). These remain Paper IV OP 4 and OP 5.
- The synthesis does not close the saturation postulate. This remains Paper IV §10.2.
- The synthesis does not solve current-selection. This remains Paper IV §12.2.

- The synthesis does not derive the gauge group. The substrate refinement structure forces an additive  $\mathbb{R}$ -equivalence; whether the substrate also forces compactification to  $U(1)$  is open (Paper IV OP 2 and the  $K = 7$  companion paper).
- The synthesis does not extend to non-Abelian or higher-cohomology structures.
- The synthesis introduces no new empirical predictions. The empirical handles of Paper IV §14.1 are inherited unchanged.

## 10.5 Summary Statement

The synthesis is a structural integration of four prior results into a single uniqueness theorem on both the carrier and the dynamics of substrate gauge transport. It strengthens the substrate-physical foundation of Paper IV's Maxwell admissibility theorem by deriving (B3) from the substrate refinement structure and identifying the gauge-equivalence class  $[A_c]$  at the cohomology-class level with  $[\omega] \in H^1(G(\Lambda))$ . It adds one new result — refinement persistence of Wilson loops as the canonical pairing — that follows directly from Paper III's Theorem 1. It opens no new conditionalities and closes none of the originally open ones.

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## 11. Conclusion

The four prior VERSF papers were jointly stronger than their individual statements suggested. Papers I–III established that the only refinement-stable observable structure on an admissible substrate is the cohomology sector  $H^1(G(\Lambda))$ , with gauge equivalence forced by scalar trivialisation and refinement persistence guaranteed by the canonical lift–pullback isomorphism. Paper IV established that on any substrate satisfying BCB and TPB, within a substrate-motivated admissibility class, Maxwell-form  $U(1)$  gauge transport is the unique admissible dynamics at  $O(\epsilon^0)$ .

This paper has performed the integration. The substrate papers' "minimal refinement-stable observable sector" and the Maxwell paper's "admissible transport sector" are the same object. The substrate papers' "refinement equivalence" and the Maxwell paper's "gauge redundancy" are the same equivalence. The substrate papers' "closed-walk integrals on cohomology classes" and the Maxwell paper's "Wilson holonomies" are the same observables. The combined uniqueness theorem — Maxwell-form transport at  $O(\epsilon^0)$  is the unique refinement-stable dynamics on the unique refinement-stable observable sector — is substantially stronger than either input read alone.

The structural picture that emerges is:

Scalar information on an admissible substrate is exhausted by the constant counting invariant. The only relational substrate observable that survives refinement is cohomological transport. Within the substrate axioms of bit conservation and finite-speed propagation, the only admissible dynamics on this relational sector is Maxwell-form  $U(1)$  gauge transport. Electromagnetism is therefore not a fundamental interaction imposed on substrate — it is the unique admissible dynamics on the unique substrate observable sector that survives the refinement flow.

The open problems remain. The substrate's microphysics is not constructed; the saturation postulate is not closed; current-selection is not resolved; the non-Abelian extension is not performed; gauge-group compactification from  $\mathbb{R}$  to  $U(1)$  is not derived. The synthesis does not address any of these directly. What it does is reorganise the existing results so that the structural foundation of the Maxwell theorem is, where the substrate refinement structure has reached, derived rather than postulated.

In a sentence: *electromagnetism is what refinement leaves behind, plus what BCB and TPB allow to live on it.* The substrate papers established the first half; the Maxwell paper established the second half; this paper showed they refer to the same object.

## Appendix A — Notation and Inheritance Index

For convenience, an index of the notation conventions used in this paper and the prior paper from which each is inherited.

### A.1 Substrate Objects

Symbol	Meaning	Source
$\Lambda$	Admissible substrate (finite poset)	Paper II §1
$G(\Lambda)$	Covering-pair graph of $\Lambda$	Paper II §1
$V(\Lambda), E^{\wedge(1)}(\Lambda)$	Vertices and directed edges of $G(\Lambda)$	Paper II §1
$C^k(\Lambda)$	$k$ -cochains on $G(\Lambda)$ ( $k = 0, 1$ )	Paper II §11.1, Paper III §1
$d^0$	Coboundary $C^0 \rightarrow C^1$	Paper II §11.1, Paper III §1
$H^1(G(\Lambda))$	First cohomology of $G(\Lambda)$	Paper II §11.3, Paper III §1
$H_1(G(\Lambda))$	First homology (cycle classes)	This paper §5

### A.2 Refinement Operators

Symbol	Meaning	Source
$\Gamma : \Lambda_n \rightarrow \Lambda_{\{n+1\}}$	Midpoint refinement	Paper III §1
$L : H^1(\Lambda_n) \rightarrow H^1(\Lambda_{\{n+1\}})$	Natural lift (canonical iso)	Paper III §3.1, Theorem 1(a)
$\Gamma^* : H^1(\Lambda_{\{n+1\}}) \rightarrow H^1(\Lambda_n)$	Cochain pullback descended	Paper III §3.4
$\Gamma^* := \frac{1}{2}\Gamma^*$	Renormalised pullback ( $\Gamma^* \circ L = \text{Id}$ )	Paper III §11.3

### A.3 Spectral Operators

Symbol	Meaning	Source
W	Lazy random walk $\frac{1}{2}(I + D^{-1}A)$ on $\Lambda$	Paper II §2
L_R	Linearised refinement on d-cube	Paper I

#### A.4 Maxwell-Paper Objects

Symbol	Meaning	Source
$A_c^\mu$	Transport potential (one-form)	Paper IV §5
$F_c^{\mu\nu} = \partial^\mu A_c^\nu - \partial^\nu A_c^\mu$	Curvature tensor	Paper IV §7
$J_c^\mu$	Conserved BCB current	Paper IV §3
$c_c = \xi/(N_b \tau_s)$	TPB-bound propagation speed	Paper IV §4
$\xi, N_b, \tau_s$	Coherence length, ticks-per-bit, tick duration	Paper IV §4
$\varepsilon = \xi/L_{\text{macro}}$	EFT expansion parameter ( $L_{\text{macro}}$ = macroscopic length scale, disambiguated from lift L)	Paper IV §2 (B2)
$\mu_c$	Substrate's transverse commitment inertia	Paper IV §9

#### A.5 Substrate Axioms and Admissibility Class

Label	Content	Source
(A1) BCB	Conservation of committed distinguishability	Paper IV §2
(A2.i) Finite-speed bound	Substrate transport bounded by $c_c$	Paper IV §2
(A2.ii) Atomic update	Substrate update is one-tick local	Paper IV §2
(B1) Locality	Local differential dynamics	Paper IV §2
(B2) $\xi$ -EFT truncation	Power-counting in $\varepsilon = \xi/L_{\text{macro}}$	Paper IV §2
(B3) Gauge redundancy	$A_c \rightarrow A_c + \partial\chi$ (now derived, this paper §4)	Paper IV §2; this paper §4
(B4) Closure-geometry covariance	Reduces to Lorentz in emergent regime	Paper IV §2

#### A.6 New Notation in This Paper

Symbol	Meaning	Section
$W(\gamma, \omega)$	Substrate Wilson loop pairing	§5
$\langle[\gamma], [\omega]\rangle$	Canonical pairing $H_1 \times H^1 \rightarrow \mathbb{R}$	§5
Refinement-stable pairing	$2^{-n} W(L^{\wedge n} \gamma_0, L^{\wedge n} \omega_0)$ , n-independent	§6

#### A.7 Cross-Paper Notation Identifications

For readers reading the four papers as a sequence, several objects appear under different symbols in different papers. The synthesis paper makes the following identifications explicit:

<b>Object</b>	<b>Papers II–III notation</b>	<b>Paper IV notation</b>	<b>Identified at</b>
Edge transport / potential	$\omega \in C^1(G(\Lambda))$	$A_{-c}^{\mu}$	§3 (class level)
Gauge / scalar parameter	$\varphi \in C^0(G(\Lambda))$	$\chi$	§4
Coboundary / gauge transformation generator	$d^0\varphi$	$\partial^{\mu}\chi$	§4
Cohomology / gauge-equivalence class	$[\omega] \in H^1(G(\Lambda))$	$[A_{-c}] \text{ (mod gauge)}$	§3, §4
Curvature	$\delta_{\square}\omega$ (plaquette sum)	$F_{-c}^{\mu\nu} = \partial^{\mu}A_{-c}^{\nu} - \partial^{\nu}A_{-c}^{\mu}$	(implicit in §5–§7)
Wilson holonomy	$\Sigma_{\gamma}\{\omega\} = \langle [\gamma], [\omega] \rangle$	$\oint_{\gamma} A_{-c} \cdot d\ell$	§5
Refinement equivalence / gauge equivalence	$\omega \sim \omega + d^0\varphi$	$A_{-c} \rightarrow A_{-c} + \partial\chi$	§4

Each identification is structural: the same observable, the same equivalence, the same quotient, expressed in substrate-cochain notation in Papers II–III and in continuum-form notation in Paper IV. The synthesis paper does not introduce new physics for any of these objects; it identifies them as describing the same substrate-physical content at different levels of the refinement hierarchy.