The Universal Measure Principle

Coherence-Sector Commensurability as the Substrate-Level Origin of Cross-Channel Probability Uniformity

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A General Reader's Summary

Why does the same probability rule apply to every part of the quantum world?

The previous paper in this programme answered a related but narrower question: why is quantum probability the square of an amplitude? It showed that once the operational stage is in place, the squaring rule is forced *within* each coherence sector — each substrate-derived "compartment" of the operational geometry. But that paper left an honest gap. The minimal axioms permitted each compartment to carry its *own* squaring constant, potentially different from the constants in other compartments. To force all those constants to agree — to make the squaring rule *universal* across the whole space — one further structural principle had to be supplied. The previous paper supplied three such principles and showed any one of them sufficed.

This left a question hanging. Why should three different-looking principles all force the same universality? Either it was a structural coincidence, or the three principles were really three faces of one deeper principle. The previous paper flagged this as an open problem. The present paper resolves it.

The central claim is that the three bridging principles are not three independent assumptions. They are three operational manifestations of a single deeper requirement, which we call the **Universal Measure Principle**:

Operational probability must be invariant under every operational witness of commensurability between coherence sectors. Sectors that can be composed, embedded, or refined into one another are related by operational structure, and probability assignment must respect those relations.

The reason is structural rather than aesthetic. Once sectors are operationally related — once you can compose them tensorially, embed one inside another, or refine one decomposition into a different one — the relations between sectors fix relations between their probability assignments. There is not enough operational freedom left for them to differ. The mathematics permits sector-dependent probability scales only if the sectors are operationally isolated. As soon as they interact in any of the standard operational ways, the scales must equalize.

The universality of the Born rule across the whole operational carrier is therefore not an extra axiom layered onto quantum mechanics. It is a consequence of coherence sectors being operationally commensurable at all. The previous paper's three bridging principles were not three coincidentally-converging facts; they were three views of one underlying structural requirement.

The reframing this delivers shifts what we take to be the deepest content of the Born-rule reconstruction. Quadraticity within sectors — that probabilities go as squared norms — is already largely forced by the within-sector geometry; it is a derived consequence of orthogonal additivity and continuity. Universality across sectors — that the same squaring rule applies everywhere with the same constant — is the genuinely substantive structural input. The Born rule is therefore best understood not as the unique quadratic measure (there are many such measures within the minimal core) but as the unique *commensurable* quadratic measure on operational coherence geometry.

The technical body follows. Readers without a mathematical-physics background may wish to read §1 and §8 for the structural content in continuous prose before the theorems are introduced.

Abstract

The previous paper *Probability as Admissible Measure on the VERSF Operational Hilbert Geometry* (hereafter PAMV) established that the minimal operational-measure core — positivity (A1), orthogonal additivity (A2), admissible unitary invariance (A3), continuity (A5), and single-reference normalization (A7) — forces the operational measure on the admissible Hilbert carrier $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$ to take the channel-decomposed form

$$\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_\mathbb{C}), \quad F_\alpha(r) = C_\alpha r^2 \text{ for } d_\alpha \geq 2$$

with channel-dependent quadratic coefficients $(C_\alpha)_\alpha$ and F_α unconstrained beyond continuity for $d_\alpha = 1$ sectors (PAMV Lemma 6.2A). To force cross-sector uniformity and recover the universal Born rule, a separate cross-channel bridging principle was required. PAMV supplied three structurally distinct candidates — compositional multiplicativity with character matching (B1), sub-channel isosymmetric embedding (B2), and refinement-stability across channel decompositions (B3) — each of which closed the uniqueness argument independently.

The structural coincidence of three convergent bridging routes was flagged as PAMV's Open Problem 2: do (B1), (B2), (B3) reduce to a single deeper substrate-level principle, or are they genuinely independent operational assumptions?

The present paper answers in favour of a single principle. We formalize the notion of **operational commensurability** between coherence sectors: two sectors V_α, V_β are commensurable when they admit shared operational structure — composable, embeddable, or refinement-comparable. We then state the **Universal Measure Principle** (UMP) *operationally*:

An admissible operational probability measure must be invariant under every operational commensurability witness between sectors (compositional identification, complex-unitary embedding, refinement-relabeling unitary).

The central technical results are:

Theorem 3.2 (Triple structural failure). *Sector-relative probability assignments (non-uniform* $(C_\alpha)_\alpha$*) jointly obstruct compositional neutrality, embedding consistency, and refinement invariance. The three obstructions share a common structural origin in the violation of operational commensurability.*

Theorem 5.3 (UMP forces uniform quadratic coefficients on a fully commensurable carrier). *Under the minimal-core axioms (A1)–(A3), (A5), (A7) plus UMP, every per-sector function* F_α *is quadratic, the per-sector constants satisfy* $C_\alpha = C$ *uniformly, and the* $d_\alpha = 1$ *sectors inherit the quadratic form from their commensurability with higher-dimensional sectors.*

Theorem 6.2 (Bridging conditions as UMP-instances). *Each of (B1), (B2), (B3) follows from UMP applied to the corresponding commensurability relation. The converse implications* $(B_i) \Longrightarrow$ *UMP hold only on the specific commensurability relation each* (B_i) *witnesses. On a fully commensurable carrier, UMP is logically equivalent to the conjunction* $(B1) \wedge (B2) \wedge (B3)$ *restricted to their respective domains; the unification UMP delivers is conceptual and structural — naming one principle and exhibiting its common operational origin — rather than logically strengthening the conjunction.*

Theorem 7.1 (Universal Born rule). *Under the minimal core plus UMP, the operational measure is uniquely* $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$ *and the transition probability is* $P(\psi \to \phi) = |\langle \phi, \psi \rangle_{\mathbb{C}}|^2$ *across the entire admissible operational carrier.*

The structural reframing: universality of the Born rule across operationally commensurable sectors is the deeper structural content; quadraticity within each sector is already largely forced by the minimal core. PAMV's three bridging routes are now seen as three structurally distinct operational windows onto UMP — three operational tests of the same underlying commensurability requirement, rather than three independent axioms. This is the answer to PAMV's Open Problem 2.

Scope and Conditional Status

This paper sits in the same conditional-reconstruction framework as PAMV. It does not derive UMP from absolute first principles. Rather, it shows that:

1. **Given** the inherited VERSF operational Hilbert geometry (PAMV §2);
2. **Given** the minimal operational-measure axioms (A1)–(A3), (A5), (A7);
3. **Given** UMP as a structural principle on coherence-sector commensurability;

the universal Born rule is uniquely forced, and the three previously-distinct bridging conditions (B1), (B2), (B3) all follow from UMP as derived consequences in the appropriate structural regimes.

What this paper *does* add to PAMV is a unification: the three independent bridging principles collapse to one deeper principle, with the three previous conditions becoming structurally derivable consequences. On a fully commensurable carrier, UMP and the conjunction (B1) ∧ (B2) ∧ (B3) have the same logical content; the unification is at the level of conceptual organization (naming one principle rather than three) and structural diagnosis (exhibiting a single common origin via the triple structural failure of Theorem 3.2), not at the level of logical strengthening. This is the answer to PAMV's Open Problem 2.

What this paper *does not* do is derive UMP itself from anything more primitive. UMP is the structural input here, in the same way that the §2 inherited geometry and the minimal-core axioms were structural inputs in PAMV. Whether UMP can in turn be derived from packing-level or substrate-level considerations is left as Open Problem 1 of §10, where a candidate sketch is offered and the link to PAMV's Open Problem 10 (strengthening $\mathrm{Vol\_op}(\chi) = \|\chi\|^2\_\mathbb{C}$ from convention to theorem) is made explicit. Resolving PAMV's Open Problem 10 would convert UMP from a structural input to a derived consequence, closing the present paper's deepest conditional dependence.

The deeper substrate-origin questions — why the substrate symmetry is $\mathbb{Z}_7$ rather than some other group, why the K = 7 closure architecture, why the carrier admits commensurable sectors at all — remain as PAMV left them and are not re-litigated here. PAMV's Open Problem 11 covers the $\mathbb{Z}_7$-inevitability question; the present paper's Open Problem 2 of §10 covers the parallel question for commensurability itself.

Contents

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1. Introduction

The previous VERSF reconstruction paper (PAMV) established a conditional reconstruction theorem for the Born rule: once the operational Hilbert geometry of the VERSF substrate is in place (PAMV §2, inherited from OG, OC, URHG), the minimal operational-measure axioms force the probability measure to take a channel-decomposed quadratic form, and a cross-channel bridging principle is required to force the per-channel quadratic coefficients to agree across sectors.

PAMV supplied three structurally distinct bridging conditions:

- **(B1)** compositional multiplicativity with character matching;
- **(B2)** sub-channel isosymmetric embedding;
- **(B3)** refinement-stability across channel decompositions.

Each closed the uniqueness argument independently. Each was operationally natural in the VERSF setting. PAMV's Remark 6.2.2 (second paragraph) flagged the structural question:

The coexistence of three successful bridging routes — compositional multiplicativity, isosymmetric embedding, refinement-stability — may indicate that they are different operational manifestations of a deeper substrate-level invariance not yet isolated explicitly. Candidate: a single "substrate-character-uniformity principle" stating that the operational measure on isotypic components depends only on the $\mathbb{Z}_7$-character label and not on carrier-specific multiplicity data — from which (B1), (B2), (B3) would all be derived consequences.

PAMV did not pursue this further, listing the unification as Open Problem 2.

The present paper takes up that question. Its thesis is:

The three bridging conditions of PAMV are not three independent operational principles. They are three structurally distinct operational windows onto a single deeper requirement — that operational probability be invariant under every commensurability witness between coherence sectors. We formalize this requirement as the **Universal Measure Principle** (UMP).

The structural reframing this delivers is significant. PAMV established quadraticity within each sector (Theorem 6.1) and cross-channel uniformity via bridging (Theorem 6.2). After the present paper, the proper structural reading is:

- **Quadraticity within sectors** of dimension $d\_\alpha \geq 2$ is forced by the minimal core. It is a within-sector property of admissible measures on a complex Hilbert subspace, supplied by the Cauchy-equation argument of PAMV Theorem 6.1.
- **Quadraticity within sectors** of dimension $d\_\alpha = 1$ and **universality across sectors** are the substantive structural content beyond the minimal core. Both are supplied by UMP through its commensurability-witness invariance.
- **The Born rule** is therefore best understood as *the unique probabilistically commensurable operational measure on admissible coherence geometry* — universality

> being the central structural fact, with within-sector quadraticity supplied either by within-sector orthogonal additivity (d_α ≥ 2) or by UMP-driven inheritance from higher-dimensional commensurable sectors (d_α = 1).

The argument proceeds in five stages. §2 briefly recapitulates the operational coherence-sector structure from PAMV §2.3A. §3 restates the cross-channel bridging gap (recovering PAMV Lemma 6.2A) and proves a *triple structural failure* theorem showing that sector-relative probability assignments simultaneously break compositional neutrality, embedding consistency, and refinement invariance — three failures with a shared structural origin in the violation of operational commensurability. §4 introduces operational commensurability as the relation between sectors that admits common operational structure, with explicit attention to the distinction between transport-admissible and decomposition-admissible unitaries, and identifies the deeper geometric object (operational distinguishability geometry, §4.2) that the commensurability witnesses preserve. §5 states UMP operationally as invariance under commensurability witnesses, and proves that UMP forces both d_α = 1 quadraticity and cross-sector uniformity. §6 derives (B1), (B2), (B3) as UMP-instances, with explicit attention to the asymmetric logical relations and the conceptual-vs-logical distinction in what UMP adds beyond their conjunction. §7 assembles the universal Born rule. §8 develops the structural interpretation, §9 the falsifiability content (including UMP-test 1 as the principal novel falsifier, with a schematic experimental protocol), §10 the open problems (including a candidate substrate-level derivation sketch for UMP), §11 the conclusion. Detailed Pexider/Cauchy functional-equation manipulations used in the d_α = 1 quadraticity proofs are deferred to Appendix A.

What this paper does not claim. The paper does not claim that operational commensurability uniquely determines all of quantum theory, nor that UMP exhausts the structural content of the Born-rule reconstruction in isolation. Its claim is narrower and more precise: once the inherited operational geometry of PAMV §2 and the minimal-core probability structure (A1)–(A3), (A5), (A7) are in place, operational commensurability uniquely fixes the universality of probability assignment across coherence sectors, and in doing so determines the Born rule as the unique commensurable minimal-core measure on the operational carrier. The inherited operational geometry is treated as a structural input from PAMV; the substrate-origin questions (why $\mathbb{Z}_7$, why finite packing, why the K = 7 closure architecture) remain open beyond this paper, as do the questions of whether UMP itself can be derived from more primitive substrate structure (Open Problem 1 of §10) and whether the three commensurability classes exhaust the ODG-preserving operations (Open Problem 2).

The iconic structural fact this paper isolates can be stated in a single line:

The Born rule is not the unique quadratic measure. It is the unique commensurable one.

Everything else follows from working out what "commensurable" means structurally and what its operational consequences are.

2. Operational Coherence Sectors: A Brief Recapitulation

We recall the channel-decomposition structure from PAMV §2 and §2.3A.

The admissible operational Hilbert carrier $(\mathcal{A}_\mathbb{C}, \langle\cdot,\cdot\rangle_\mathbb{C})$ decomposes orthogonally:

$$\mathcal{A}_\mathbb{C} = \bigoplus_{\alpha=1}^{N_{\mathrm{spec}}} V_\alpha,$$

where each V_α is an *operational coherence sector* (PAMV §2.3A): a transport-invariant subspace whose elements transform coherently under admissible reversible transport, preserve a common substrate-phase response under the $\mathbb{Z}_7$-action, and remain closed under admissible refinement evolution. We write $d_\alpha := \dim_\mathbb{C} V_\alpha$ and assume $\sum_\alpha d_\alpha = d_{\mathrm{op}}^\mathbb{C} \geq 2$.

Each sector V_α carries a fixed $\mathbb{Z}_7$-character $\chi_\alpha : \mathbb{Z}_7 \to U(1)$, which labels the substrate transport-response class of states in V_α. Admissible *transport* on $\mathcal{A}_\mathbb{C}$ is $\mathbb{Z}_7$-equivariant (URHG Definition 3.2(3)) and therefore Schur-block-diagonal across inequivalent character classes:

$$U_{\mathrm{adm\text{-}transport}}(\mathcal{A}_\mathbb{C}) = \prod_\alpha U(V_\alpha).$$

Crucially, distinct V_α carrying inequivalent $\mathbb{Z}_7$-characters cannot be mixed by admissible reversible transport, although they *can* superpose geometrically within $\mathcal{A}_\mathbb{C}$. The sector structure is therefore a *transport-block* structure, not a kinematical-superselection structure: cross-sector superpositions exist as admissible states, but cross-sector transport does not exist as an admissible operation.

For brevity, we will use "sector" and "channel" interchangeably to mean V_α, following PAMV's terminology.

Two senses of "admissibility" to be distinguished. The present paper requires care about two related but distinct notions of "admissible unitary":

- **(U-trans) Transport-admissible unitaries:** the Schur-block-diagonal unitaries $U_{\mathrm{adm\text{-}transport}}(\mathcal{A}_\mathbb{C}) = \prod_\alpha U(V_\alpha)$ above, representing physically realizable reversible transport on the carrier. These cannot mix inequivalent $\mathbb{Z}_7$-character sectors.
- **(U-dec) Decomposition-admissible unitaries:** norm-preserving unitaries on a multi-channel subspace $V_\alpha \oplus V_\beta$ that relate two admissible orthogonal decompositions of that subspace — viewed as *relabelings of the operational decomposition structure*, not as physical transport operations. These may mix inequivalent sectors at the level of decomposition geometry (a different orthogonal basis adapted to a different admissible refinement choice) without representing transport.

The distinction matters because (B3) of PAMV and the (C-ref) relation introduced in §4 below use unitaries of the (U-dec) kind, which are not transport-admissible when $\alpha \neq \beta$. This is consistent with PAMV's own usage in the (B3) proof of PAMV Theorem 6.2: the W_θ family there is described as relating two orthogonal decompositions of the $V_\alpha \oplus V_\beta$ subspace, not as physical transport. The present paper makes the distinction explicit to avoid the apparent contradiction with axiom (A3)'s admissible-transport invariance.

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3. The Cross-Channel Bridging Gap and the Triple Structural Failure

We recapitulate the gap that UMP will be designed to close, and then prove a structural-failure theorem showing what goes wrong concretely when commensurability is denied.

3.1 The bridging gap

PAMV Lemma 6.2A establishes that the minimal-core admissible measures on $\mathcal{A}_\mathbb{C}$ form a multi-parameter family. We restate this here for self-containedness.

For $(C_\alpha)\alpha$ *a non-negative tuple indexed by all sectors, and* $(F_\alpha^{(1)})$ $\{\alpha : d_\alpha = 1\}$ a tuple of continuous non-negative functions on $[0, \infty)$ vanishing at zero indexed only by the $d_\alpha = 1$ sectors (the F_α on $d_\alpha \geq 2$ sectors are determined by C_α as $F_\alpha(r) = C_\alpha r^2$), define the parametrized family

$\mu_{(C_\alpha), (F_\alpha^{(1)})}(\psi) := \sum\{\alpha : d_\alpha \geq 2\}\ C_\alpha \|\psi_\alpha\|^2_\mathbb{C} + \sum\{\alpha : d_\alpha = 1\}\ F_\alpha^{(1)}(\|\psi_\alpha\|_\mathbb{C})$.

Lemma 3.1 (Minimal-core insufficiency; PAMV Lemma 6.2A). *The minimal-core operational-measure axioms (A1) positivity, (A2) orthogonal additivity, (A3) admissible-transport unitary invariance, (A5) continuity, and (A7) single-reference normalization alone do not force a unique probability measure on* $\mathcal{A}_\mathbb{C}$. *They admit the parametrized family* $\mu_{(C_\alpha), (F_\alpha^{(1)})}$ *defined above, for every choice of non-negative* $(C_\alpha)_\alpha$ *and every choice of continuous non-negative* $(F_\alpha^{(1)})_\alpha$ *on* $d_\alpha = 1$ *sectors satisfying* $F_\alpha^{(1)}(0) = 0$.

Proof sketch. The detailed proof is PAMV Lemma 6.2A. The structure: PAMV Theorem 5.2 reduces the minimal core to $\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|\mathbb{C})$; *PAMV Theorem 6.1(i) forces* $F_\alpha(r) = C_\alpha r^2$ *for* $d_\alpha \geq 2$ *sectors via the within-channel additive Cauchy equation; PAMV Theorem 6.1(ii) leaves* F_α *free on* $d_\alpha = 1$ *sectors (these are the* $F_\alpha^{(1)}$ *in the present notation); admissible-transport unitary invariance (A3) acts only within each sector (via the within-sector subgroup* $U(V_\alpha)$*) and never mixes inequivalent sectors, so no minimal-core axiom forces* $C_\alpha = C_\beta$ *for* $\alpha \neq \beta$ *nor quadraticity for* $d_\alpha = 1$ *sectors; single-reference normalization (A7) fixes only one global scale. The parametrized family* $\mu\{(C_\alpha), (F_\alpha^{(1)})\}$ satisfies all five axioms for any non-negative (C_α) and any admissible $(F_\alpha^{(1)})$.

Remark 3.1.1. The lemma identifies the precise structural gap left by the minimal core: across sectors of dimension ≥ 2, the quadratic constants are free parameters; on sectors of dimension 1, the per-sector function is essentially free (subject only to continuity and vanishing at the origin). UMP will close both gaps: it will force quadraticity on $d_\alpha = 1$ sectors (through their commensurability with higher-dimensional sectors) *and* force cross-sector uniformity of the quadratic constants (through commensurability across all sector pairs).

3.2 The triple structural failure under sector-relativity

To motivate the structural shape of UMP, we now demonstrate concretely what fails when probability is sector-relative — i.e. when the quadratic constants $(C_\alpha)_\alpha$ are non-uniform on the $d_\alpha \geq 2$ sectors. The failure is triple: compositional neutrality, embedding consistency, and refinement invariance *all three* break simultaneously, and they do so for a shared structural reason. This is what makes UMP a natural single principle rather than three independent conditions.

Theorem 3.2 (Triple structural failure). *Let* $\mu = \mu_{(C_\alpha), (F_\alpha^{(1)})}$ *be a minimal-core admissible measure with non-uniform quadratic coefficients:* $C_\alpha \neq C_\beta$ *for some pair* (α, β) *with* $d_\alpha, d_\beta \geq 2$*. Then* μ *simultaneously fails*

(F1) ***Compositional neutrality:*** *there exist unit-norm states* $\psi \in V_\alpha$, $\phi \in V_\beta$ *such that the composite measure* $\mu(\psi \otimes \phi)$ *is not determined by* $\mu(\psi)$ *and* $\mu(\phi)$ *jointly with their substrate-character data alone, but depends additionally on which sector* $V_{\alpha+\beta \bmod 7}$ *of the carrier hosts the composite;*

(F2) ***Embedding consistency:*** *there exists a complex-unitary embedding* $\iota : V_\alpha \to V_\beta$ *and a state* $\psi_\alpha \in V_\alpha$ *such that* $\mu(\iota(\psi_\alpha)) \neq \mu(\psi_\alpha)$*, despite* ι *preserving every operational geometric invariant (norm, complex structure, orthogonality relations);*

(F3) ***Refinement invariance:*** *there exists a decomposition-admissible (U-dec) unitary* $W : V_\alpha \oplus V_\beta \to V_\alpha' \oplus V_\beta'$ *between admissible orthogonal decompositions of the same multi-channel subspace such that* $\mu(W\psi) \neq \mu(\psi)$ *for some* ψ*, under the standard convention that decomposition-attached constants travel with their sectors under relabeling.*

All three failures arise from a common structural origin: probability assignment is depending on sector-identity data rather than on operational distinguishability geometry alone.

Scope note. The proofs of (F1) and (F2) below use $d_\alpha, d_\beta \geq 2$ to apply PAMV Theorem 6.1(i) on the relevant subspaces. The failures extend to $d_\alpha = 1$ sectors once UMP has supplied quadraticity there (see Theorem 5.3); for the present pre-UMP argument, the $d_\alpha, d_\beta \geq 2$ case suffices to establish the triple-failure structure.

Proof. Assume $C_\alpha \neq C_\beta$ with $d_\alpha, d_\beta \geq 2$. WLOG take $\|\psi\|_\mathbb{C} = \|\phi\|_\mathbb{C} = 1$ with $\psi \in V_\alpha$, $\phi \in V_\beta$.

(F1). The composite $\psi \otimes \phi$ inhabits the $V_\alpha \otimes V_\beta$ component of the tensor carrier, which is an isotypic component for the product character $\chi_{\alpha+\beta \bmod 7}$. If composite measure were determined by substrate-character data alone, the tensor-carrier constant on this isotypic component would equal the original-carrier constant $C_{\alpha+\beta \bmod 7}$. But the minimal core supplies no such identification: the tensor carrier is a different operational structure from the original carrier, and the minimal core permits its character-labelled constants to differ from those of the original carrier. Whether the composite measure equals $C_{\alpha+\beta \bmod 7}$, $C_\alpha \cdot C_\beta$, or some entirely sector-specific function depends on free parameters of the tensor-carrier measure that the minimal core does not constrain. Operationally indistinguishable composites — those with the same substrate-character labels — therefore receive non-uniform probability assignments determined by sector-bookkeeping rather than by operational structure.

(F2). Let $\iota : V_\alpha \to V_\beta$ be complex-unitary with $d_\beta \geq d_\alpha \geq 2$ (relabel if necessary). For $\psi_\alpha \in V_\alpha$ with $\|\psi_\alpha\|_{\mathbb{C}} = r$, the original-sector measure is $\mu(\psi_\alpha) = C_\alpha r^2$. The image $\iota(\psi_\alpha) \in V_\beta$ has $\|\iota(\psi_\alpha)\|_{\mathbb{C}} = r$ (ι is complex-unitary) and lies in a d_α-dimensional complex Hilbert subspace of V_β. By PAMV Theorem 6.1(i) applied within V_β (using complex-orthogonal pairs within $\iota(V_\alpha) \subseteq V_\beta$, which exist because $d_\alpha \geq 2$), the measure on the image is $\mu(\iota(\psi_\alpha)) = C_\beta r^2$. Therefore $\mu(\iota(\psi_\alpha)) - \mu(\psi_\alpha) = (C_\beta - C_\alpha) r^2 \neq 0$ by hypothesis. The embedding ι preserves every operational invariant of the state (norm, complex structure, orthogonality relations, all admissible-unitary symmetries within $\iota(V_\alpha)$) yet changes its measure. Probability is therefore not a function of operational invariants alone — it carries hidden sector-label dependence.

(F3). Take $d_\alpha = d_\beta = 1$, pick unit-norm $e_\alpha \in V_\alpha$, $e_\beta \in V_\beta$, and define the one-parameter (U-dec) unitary family $W_\theta : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$ by

$$W_\theta\, e_\alpha = \cos(\theta) \cdot e_{\alpha'} + \sin(\theta) \cdot e_{\beta'},\quad W_\theta\, e_\beta = -\sin(\theta) \cdot e_{\alpha'} + \cos(\theta) \cdot e_{\beta'},$$

for $\theta \in [0, \pi/2]$. We work under the **standard sector-attachment convention** for the new decomposition: $C_{\alpha'} = C_\alpha$ and $C_{\beta'} = C_\beta$. This convention captures the structural reading that the quadratic coefficient is a property of the sector (a substrate-derived character-class datum), not of the particular orthogonal decomposition used to display the sector. Under any alternative convention that did *not* attach constants to sectors — e.g. $C_{\alpha'} = (C_\alpha + C_\beta)/2$ — the measure assignment μ on the original decomposition would not uniquely determine the measure on the relabeled decomposition, and refinement invariance would already be ill-posed at the level of definition. We take the sector-attachment convention as the well-posedness condition for the question; the (F3) failure is then a substantive statement under that convention.

Take $\psi = e_\alpha$. Then under W_θ:

$$\|(W_\theta\, \psi)\{\alpha'\}\|^2_{\mathbb{C}} = \cos^2\theta,\quad \|(W_\theta\, \psi)\{\beta'\}\|^2_{\mathbb{C}} = \sin^2\theta.$$

The new-decomposition measure is

$$\mu(W_\theta\, \psi) = C_{\alpha'} \cdot \cos^2\theta + C_{\beta'} \cdot \sin^2\theta = C_\alpha \cos^2\theta + C_\beta \sin^2\theta.$$

For $\theta \in (0, \pi/2)$ and $C_\alpha \neq C_\beta$, this differs from $\mu(\psi) = C_\alpha$. Probability assignment therefore depends on the choice of orthogonal decomposition used to compute it — refinement invariance fails.

Shared structural origin. In each of (F1)–(F3), the failure arises because the measure attaches probability weights to sector-labels rather than to operational distinguishability geometry. The structural object that should be invariant — composite probability under character-determined composition (F1), embedded-state probability under operational embedding (F2), total probability under admissible decomposition relabeling (F3) — depends on sector identification (α, β) that is not itself an operational invariant. The common diagnosis: probability assignment is parameterised by carrier-specific bookkeeping data (sector labels) rather than by substrate-derived operational structure.

Remark 3.2.1 (Why the failures are simultaneous). Theorem 3.2 is the structural foundation for UMP. The three failures are not three independent pathologies remediable by three independent fixes. They share a single underlying defect — probability depending on sector-identity rather than on operational geometry — which is what UMP will be designed to exclude. This explains, *structurally*, why the three bridging conditions (B1), (B2), (B3) of PAMV converge: they are three operational repair-tools for the same underlying structural failure, and any one of them, by repairing the shared defect, automatically repairs all three.

Remark 3.2.2 (Operational stake). What is at stake is *probabilistic universality*. If $C_\alpha \neq C_\beta$, two unit-norm states $\psi_\alpha \in V_\alpha$ and $\psi_\beta \in V_\beta$ have $\mu(\psi_\alpha) = C_\alpha \neq C_\beta = \mu(\psi_\beta)$, so a "1" outcome is more or less probable depending on which substrate-character class hosts the preparation. Operationally indistinguishable distinguishability content acquires different probability weights solely by virtue of sector label — what PAMV Remark 6.2A.2 called "substrate-level channel parochialism." UMP excludes this by elevating the negative result of Theorem 3.2 to a positive structural requirement.

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4. Operational Commensurability

We introduce the structural relations between sectors that will turn out to force probability uniformity.

Definition 4.1 (Operational commensurability between sectors). *Two coherence sectors V_α, $V_\beta \subseteq \mathcal{A}_\mathbb{C}$ are **operationally commensurable** if they admit at least one of the following structural relations within the inherited operational structure of the carrier $\mathcal{A}_\mathbb{C}$:*

*(C-comp) **Compositional relation.** The product character $\chi_\alpha \cdot \chi_\beta = \chi_{\alpha+\beta \bmod 7}$ corresponds to a third sector $V_{\alpha+\beta \bmod 7}$ of the carrier (when $\alpha + \beta \not\equiv 0 \bmod 7$), and the tensor-product extension $V_\alpha \otimes V_\beta$ is identifiable as an isotypic component for the product character. The composition of sector content into a higher-arity admissible assembly is operationally meaningful.*

*(C-emb) **Embedding relation.** There exists a complex-unitary embedding $\iota : V_\alpha \to V_\beta$ (or its reverse, with $d_\beta \geq d_\alpha$). The lower-dimensional sector can be operationally realized within the higher-dimensional one as a Hilbert subspace.*

(C-ref) ***Refinement relation.*** *There exists a decomposition-admissible (U-dec) unitary $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$ relabeling the multi-channel subspace $V_\alpha \oplus V_\beta$. The two sectors share refinement geometry at the level of operational decomposition structure. (The unitaries W invoked here are decomposition-admissible in the sense of §2's distinction; they need not be transport-admissible.)*

Definition 4.2 (Commensurability witnesses). *Each commensurability relation supplies a corresponding* ***structural witness****: for (C-comp), the carrier-to-tensor isotypic identification $\sigma_{\alpha,\beta} : V_\alpha \otimes V_\beta \to V_{\alpha+\beta \bmod 7}$-isotypic-component; for (C-emb), the complex-unitary embedding $\iota : V_\alpha \to V_\beta$; for (C-ref), the decomposition-relabeling unitary $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$. We call these* ***commensurability witnesses*** *between V_α and V_β.*

Definition 4.3 (Commensurability graph). *The* ***commensurability graph*** *of the carrier $\mathcal{A}_\mathbb{C}$ has nodes labelled by sectors V_α and edges (α, β) wherever V_α and V_β are operationally commensurable in the sense of Definition 4.1. The carrier is* ***fully commensurable*** *if its commensurability graph is connected (i.e. every pair of sectors is commensurable, directly or transitively, via at least one of (C-comp), (C-emb), (C-ref)).*

4.1 The conceptual ordering

The structural backbone of the unification claim is the following ordering of operational structure, principle, and consequence:

Proposition 4.4 (Conceptual ordering). *The relationship between commensurability, UMP, and bridging is hierarchical:*

(commensurability relation exists) → (UMP demands probability invariance under the witness) → (bridging condition holds).

Theorem 3.2 establishes the contrapositive: where commensurability relations exist but probability invariance under their witnesses fails, the triple structural failure (F1)–(F3) follows. The bridging conditions (B1), (B2), (B3) of PAMV are the operational expressions of UMP applied to the three commensurability relations.

Remark 4.4.1. Proposition 4.4 is the conceptual statement of the paper. The commensurability relations of Definition 4.1 are *structural availabilities* — relationships that the operational carrier supplies independently of any probability measure on it. UMP is the *structural requirement* that probability must respect these relationships. The bridging conditions of PAMV are the *operational expressions* of that requirement, one per relationship. Reading these three levels in sequence makes clear that UMP is doing the structural work; (B1), (B2), (B3) are downstream of UMP in the conceptual hierarchy even though they are equivalent to it logically on a fully commensurable carrier (see Remark 6.2.3).

Remark 4.4.2 (Full commensurability in the VERSF setting). For the VERSF substrate as inherited from URHG, the carrier $\mathcal{A}_\mathbb{C}$ is fully commensurable: the $\mathbb{Z}_7$-character lattice supplies (C-comp) for every pair (α, β) with $\alpha + \beta \not\equiv 0 \bmod 7$; the inequality $d_\alpha \leq d_\beta$ (after appropriate labelling) supplies (C-emb) wherever $d_\beta \geq 2$; the existence of multi-channel admissible subspaces $V_\alpha \oplus V_\beta$ supplies (C-ref). Any sector pair (α, β) is therefore commensurable in at least one of the three senses. We will assume full commensurability throughout the main argument; the structure of UMP under partial commensurability is treated in Remark 5.4.

4.2 Operational Distinguishability Geometry

Before stating UMP formally, we name the deeper structural object that the commensurability witnesses preserve. This is what UMP will ultimately be an invariance principle for.

The operational carrier $(\mathcal{A}_\mathbb{C}, \langle\cdot,\cdot\rangle_\mathbb{C})$ is not merely a complex Hilbert space with an inner product. The inherited operational structure of PAMV §2 supplies, jointly:

- the **carrier and inner product** (the Hilbert geometry on $\mathcal{A}_\mathbb{C}$);
- the **coherence-sector decomposition** $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$ (the substrate-character partition);
- the **orthogonal projection structure** (the admissibility-projection lattice);
- the **admissible transport rules** (transport-admissible (U-trans) unitaries respecting $\mathbb{Z}_7$-equivariance);
- the **refinement structure** (admissible orthogonal decompositions and the decomposition-admissible (U-dec) relabelings between them);
- the **finite distinguishability packing** (OG Theorem 10.1, supplying continuity and bounded operational resolution).

Together these constitute what we call the **operational distinguishability geometry** of the carrier — the full structure governing which states are operationally distinguishable, how distinguishability content combines across sectors (via composition, embedding, refinement), how it evolves under admissible transport, and how it sits within finite operational resolution.

ODG is not new structure. It is important to emphasize that operational distinguishability geometry is not an additional structure layered onto the carrier to make UMP work. It is a *repackaging* of the invariant operational structure already implicit in the admissible geometry, transport, refinement, and packing framework inherited from PAMV §2. Each component of ODG corresponds to an element of PAMV's existing operational construction: the carrier and inner product come from OG Theorem 4.0 plus URHG Theorem 3.5; the sector decomposition comes from URHG §2.5; the projection structure comes from OG Theorem 5.1; admissible transport comes from URHG Theorems 4.1, 5.1; refinement structure comes from PAMV §6.2 (B3) and the admissible-decomposition apparatus; finite packing comes from OG Theorem 10.1. The role of ODG in the present paper is to *name* this jointly-supplied operational structure and identify the natural invariance group it supports, which is what UMP is an invariance principle for. ODG does not introduce structural commitments beyond those already accepted as part of the inherited operational geometry.

Definition 4.5 (Geometry-preserving operations). *An operation T on $\mathcal{A}_\mathbb{C}$ is **operational-distinguishability-geometry-preserving** (henceforth ODG-preserving) if T preserves every component of the operational distinguishability geometry listed above. The class of ODG-*

preserving operations on $\mathcal{A}_\mathbb{C}$ *includes, but is not limited to: transport-admissible (U-trans) unitaries; admissible projections; and — crucially for the present paper — the commensurability witnesses* $\sigma_{\alpha,\beta}$, ι, W *of Definition 4.2.*

Proposition 4.6 (Commensurability witnesses are ODG-preserving). *Each commensurability witness preserves operational distinguishability geometry:*

(i) The compositional identification $\sigma_{\alpha,\beta} : V_\alpha \otimes V_\beta \to V_{\alpha+\beta \bmod 7}$*-isotypic-component preserves the substrate-character partition (by character matching), the inner-product structure (by isotypic-isomorphism), and the admissible refinement structure (carrier-derived).*

(ii) The complex-unitary embedding $\iota : V_\alpha \to V_\beta$ *preserves the inner-product structure (by complex-unitarity), the orthogonality relations (by isometric embedding), and the within-sector admissible-transport structure (by* $U(d_\alpha)$*-equivariance).*

(iii) The decomposition-relabeling unitary $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$ *preserves the admissible refinement structure of the multi-channel subspace (by being a (U-dec)-admissibility witness), the inner-product structure (by unitarity), and the orthogonal-projection structure on* $V_\alpha \oplus V_\beta$ *(by orthogonality preservation).*

Proof. Direct verification from the definitions of $\sigma_{\alpha,\beta}$, ι, W and the inherited operational structure. The key point in each case is that the witness is an operationally-meaningful map within the inherited structure, not an arbitrary mathematical transformation.

4.3 UMP as ODG-invariance: preview

The operational distinguishability geometry now supplies the principled formulation of UMP that §5 will adopt. The intuitive content of UMP can be stated immediately:

Probability assignment must remain unchanged under every operation that preserves operational distinguishability geometry.

Or equivalently:

Probability may depend on operational structure, but not on arbitrary sector identity.

This is precisely why sector-relative measures fail (Theorem 3.2). If different sectors carried different intrinsic probability scales, then two operationally equivalent states (states related by an ODG-preserving operation) could receive different probabilities solely because they belong to different sectors. Probability would then depend on bookkeeping labels rather than on operational geometry itself. The Born rule is distinguished because it avoids this problem: it assigns probability purely from the operational geometry and remains invariant under every ODG-preserving operation.

Remark 4.6.1. The operational restatement of UMP in §5 (Definition 5.1) will instantiate this general principle on the three specific classes of commensurability witnesses ($\sigma_{\alpha,\beta}$, ι, W). The general principle — invariance under all ODG-preserving operations — is the deepest content; the three-clause operational form of Definition 5.1 is the working specialization that makes the principle directly applicable to the three commensurability relations of Definition 4.1. Strengthening UMP to invariance under *every* ODG-preserving operation (not merely the three identified classes) is one direction of structural extension; whether the three identified classes already exhaust the structurally significant ODG-preserving operations in the VERSF carrier is a structural-substrate question (Open Problem 2 of §10, slightly reformulated).

Note on technical use. We work with the three-clause specialization of Definition 5.1 in §5–§6 because the three commensurability witnesses are concrete enough to support direct manipulation in proofs. The more general ODG-invariance formulation above is the principled foundation but does not affect the technical content of what follows; it serves as the conceptual frame motivating the three-clause specialization. The relation between the two formulations on the VERSF carrier (whether the three-clause and ODG-invariance forms are logically equivalent) is sketched as Open Problem 2 of §10.

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5. The Universal Measure Principle

We now state the principle that will unify the three bridging conditions. The previous draft of this paper stated UMP constant-wise ($C_\alpha = C_\beta$ for commensurable α, β), which presupposed the quadraticity needed to define C_α and created a circularity on $d_\alpha = 1$ sectors. The present version states UMP operationally as an invariance-under-witnesses condition; quadraticity on $d_\alpha = 1$ sectors and cross-sector constant uniformity then both follow as consequences.

Definition 5.1 (Universal Measure Principle, UMP — operational form). *An admissible operational probability measure μ on a carrier $\mathcal{A}_\mathbb{C} = \bigoplus_\alpha V_\alpha$ satisfies the* ***Universal Measure Principle*** *if μ is invariant under every operational commensurability witness between every pair of operationally commensurable sectors. Explicitly:*

(UMP-comp) For every (C-comp) pair (α, β) with witness $\sigma_{\alpha,\beta} : V_\alpha \otimes V_\beta \to V_{\alpha+\beta \bmod 7}$-isotypic-component and every state $\psi \otimes \phi \in V_\alpha \otimes V_\beta$,

$$\mu_{\text{tensor}}(\psi \otimes \phi) = \mu_{\text{original}}(\sigma_{\alpha,\beta}(\psi \otimes \phi)),$$

where μ_{original} is the operational measure on the original carrier $\mathcal{A}_\mathbb{C}$ and μ_{tensor} is the operational measure on the tensor extension $V_\alpha \otimes V_\beta$, defined as the standard tensor-product extension of the per-sector measures: for separable composite states,

$$\mu_{\text{tensor}}(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi).$$

The clause therefore constrains μ jointly with its tensor-extension counterpart: the two measures must agree under the $\sigma_{\alpha,\beta}$ identification, and the tensor-extension is multiplicative on *separable composites. This multiplicativity on separable composites is a structural input — it expresses operational independence of admissible composite preparations and is part of what "the standard tensor-extension of operational measure" supplies in the inherited operational geometry. (See Remark 5.1.2 below for further discussion.)*

(UMP-emb) For every (C-emb) pair (α, β) with witness $\iota : V_\alpha \to V_\beta$ and every state $\psi_\alpha \in V_\alpha$,

$$\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha).$$

(UMP-ref) For every (C-ref) pair (α, β) with witness $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$ and every state $\psi \in V_\alpha \oplus V_\beta$,

$$\mu(W\psi) = \mu(\psi).$$

Remark 5.1.1 (Why the operational formulation matters). Stating UMP in terms of invariance under witnesses, rather than in terms of equality of per-sector constants, has two structural advantages. First, it avoids the circularity that would arise from referring to C_α on $d_\alpha = 1$ sectors (where C_α is not yet defined under the minimal core alone). Second, it makes the principle look like the structural shape PAMV's axiom (A3) already had — invariance under a class of structure-preserving maps — but extended from the (A3) class (transport-admissible unitaries within each sector) to the broader class of commensurability witnesses across sectors. UMP is therefore the natural cross-sector extension of (A3)'s within-sector invariance principle, applied to the commensurability witnesses that the operational structure of the carrier supplies.

Remark 5.1.2 (Tensor-extension multiplicativity as structural input). The UMP-comp clause makes use of the tensor-extension multiplicativity $\mu_{\text{tensor}}(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi)$ on separable composite states. This is a structural input not derived from (A1)–(A7): it expresses operational independence of admissible composite preparations, in the same spirit as orthogonal additivity (A2) expresses operational independence of complex-orthogonal sectors. *UMP-comp does not derive multiplicativity from nothing; it constrains how the multiplicative tensor-extension structure interacts with commensurability.* The multiplicativity itself is inherited from PAMV's treatment of (B1) (which uses $\mu(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi)$ as part of the relation yielding $C_{\alpha+\beta \bmod 7} = C_\alpha \cdot C_\beta$); the present paper adds the *commensurability constraint* that the tensor-carrier and original-carrier measures must agree under the $\sigma_{\alpha,\beta}$ identification. The structural status of tensor-extension multiplicativity in the VERSF programme is parallel to that of the standard quantum-mechanical Born rule on composite systems: it is what "the standard tensor-extension of operational measure" supplies in the operational geometry. The present paper inherits this multiplicativity rather than re-deriving it; it is therefore part of the operational-structure background inherited from §2 (PAMV §2), not a substantive new input introduced here. The deeper question of whether tensor-extension multiplicativity is itself derivable from more primitive substrate structure is a programme-level question parallel to PAMV's Open Problem 10 on the packing-volume identification.

5.1 Character-lattice multiplicativity from UMP-comp

Before applying UMP to the $d_\alpha = 1$ quadraticity and cross-sector uniformity questions, we factor out the structural content of UMP-comp on the $\mathbb{Z}_7$-character lattice as a standalone lemma. This content is used in two later places ($d_\alpha = d_\beta = 1$ quadraticity via UMP-comp in Lemma 5.2″; cross-sector uniformity on the character lattice in Theorem 5.3), so isolating it here avoids redundancy and makes the structural role of UMP-comp clearer.

We first isolate the pure algebraic content of the closure step on the $\mathbb{Z}_7$-character lattice as a standalone lemma. This algebraic fact is used in two places below (Lemma 5.0 for the cross-sector uniformity, Lemma 5.2″ Step 1 for the unit-norm coefficient relation), and factoring it out makes the (C-comp) leg's structural role more transparent.

Lemma 5.0a (Seventh-power closure on the $\mathbb{Z}_7$-character lattice). *Let $(x_\gamma)_{\gamma \in \{1, \dots, 6\}}$ be a family of non-negative real numbers satisfying the partial multiplicative-homomorphism relation*

$x_{\alpha+\beta \bmod 7} = x_\alpha \cdot x_\beta$ for all $\alpha, \beta \in \{1, \dots, 6\}$ with $\alpha + \beta \not\equiv 0 \bmod 7$. (♠)

Then $x_\gamma = 1$ for all $\gamma \in \{1, \dots, 6\}$.

Proof. From (♠) with $\beta = 1$: $x_{\alpha+1} = x_\alpha \cdot x_1$ for $\alpha \in \{1, \dots, 5\}$, so iterating from $\alpha = 1$ gives $x_k = x_1^k$ for $k = 1, \dots, 6$. By this iteration $x_4 = x_1^4$. Now take $(\alpha, \beta) = (4, 4)$ in (♠): $(4 + 4) \bmod 7 = 1$, and $4 + 4 \not\equiv 0 \bmod 7$, so (♠) applies and yields $x_1 = x_4 \cdot x_4 = x_4^2 = (x_1^4)^2 = x_1^8$. Hence $x_1^7 = 1$. Since $x_1 \geq 0$, the unique non-negative real seventh root of 1 is $x_1 = 1$. By the chain $x_k = x_1^k$, $x_\gamma = 1$ for all $\gamma \in \{1, \dots, 6\}$.

Remark 5.0a.1. Lemma 5.0a is purely algebraic: it is a statement about non-negative real solutions to a multiplicative-homomorphism relation on the partial group $\mathbb{Z}_7 \setminus \{0\}$ under the (♠) constraint. It does not refer to the operational measure, the carrier, or UMP itself. The lemma's role is to encapsulate the seventh-power closure step that converts the (C-comp) commensurability relation into uniform per-sector coefficients — a step which is the (C-comp) leg's unique quantitative contribution to the framework.

Lemma 5.0 (Character-lattice multiplicativity from UMP-comp). *Let μ satisfy the minimal core plus UMP-comp on a carrier in which every pair (α, β) with $\alpha + \beta \not\equiv 0 \bmod 7$ admits the (C-comp) relation. Suppose $F_\gamma(r) = C_\gamma r^2$ with $C_\gamma \geq 0$ well-defined on every sector V_γ of the character lattice (this is supplied by PAMV Theorem 6.1(i) on $d_\gamma \geq 2$ sectors and by Lemmas 5.2 / 5.2′ / 5.2″ on $d_\gamma = 1$ sectors, in the structural regime corresponding to the (C-i) witness available — where Lemmas 5.2 and 5.2′ use only their respective UMP clauses without invoking Lemma 5.0, and Lemma 5.2″ uses only the lower-level algebraic Lemma 5.0a rather than Lemma 5.0 itself, so no forward-reference loop arises; see scope discussion in §5.2). Then $C_\gamma = 1$ for all $\gamma \in \{1, \dots, 6\}$, and hence $F_\gamma(r) = r^2$ uniformly across the character lattice.*

Proof. UMP-comp on unit-norm states $\psi \in V_\alpha$, $\phi \in V_\beta$ gives the multiplicative relation

$C_{\alpha+\beta \bmod 7} = C_\alpha \cdot C_\beta$ for all $\alpha, \beta \in \{1, \dots, 6\}$ with $\alpha + \beta \not\equiv 0 \bmod 7$, (♣)

by combining tensor-extension multiplicativity ($\mu_{\text{tensor}}(\psi \otimes \phi) = C_\alpha \cdot C_\beta$ at unit norm) with the carrier-to-tensor identification ($\mu_{\text{original}}(\sigma_{\{\alpha,\beta\}}(\psi \otimes \phi)) = C_{\{\alpha+\beta \bmod 7\}}$). The (C_γ) are non-negative by positivity (A1). Applying Lemma 5.0a to the family $(x_\gamma) := (C_\gamma)$ — which satisfies the (♠) hypothesis as (♣) — gives $C_\gamma = 1$ for all γ. Therefore $F_\gamma(r) = r^2$ uniformly across the character lattice. The substrate-level justification of the character-matching step follows PAMV §6.2 (B1) Step 1: the $\mathbb{Z}_7$-character is the only substrate-derived label that survives inter-carrier identification.

Remark 5.0.1. Lemma 5.0 isolates the character-lattice multiplicativity argument and makes its UMP-comp origin explicit. In PAMV, the multiplicativity relation (♣) was part of the (B1) bridging condition — assumed as a structural axiom on the operational measure. The present paper *derives* (♣) instead: it is a consequence of UMP-comp's invariance content together with the inherited tensor-extension multiplicativity (Remark 5.1.2). The $\mathbb{Z}_7$-character-lattice structure that converts the partial multiplicative relation into the seventh-power closure was present in PAMV's setup and is reused here unchanged. The seventh-power-closure $C_1^7 = 1$ plus positivity (A1) is what forces $C_\alpha = 1$ across all character classes; this is the key quantitative fact that the (C-comp) commensurability relation supplies beyond what (C-emb) and (C-ref) alone could give.

5.2 Quadraticity on $d_\alpha = 1$ sectors via UMP

We now establish quadraticity on $d_\alpha = 1$ sectors. Three independent paths are available, one per commensurability relation; the choice of path depends on which commensurability witnesses connect the $d_\alpha = 1$ sector to the rest of the commensurability graph.

Lemma 5.2 ($d_\alpha = 1$ quadraticity from UMP-emb). *Let μ satisfy the minimal core plus UMP-emb (the embedding component of UMP). Let V_α be a sector with $d_\alpha = 1$ and suppose there exists a sector V_β with $d_\beta \geq 2$ such that the (C-emb) embedding $\iota : V_\alpha \to V_\beta$ is available. Then the per-sector function F_α (in the channel decomposition $\mu(\psi) = \sum_\gamma F_\gamma(\|\psi_\gamma\|_{\mathbb{C}})$ supplied by PAMV Theorem 5.2) is quadratic: $F_\alpha(r) = C_\beta r^2$, where C_β is the quadratic coefficient of V_β supplied by PAMV Theorem 6.1(i).*

Proof. By PAMV Theorem 5.2, the minimal core forces $\mu(\psi) = \sum_\gamma F_\gamma(\|\psi_\gamma\|_{\mathbb{C}})$ with F_α continuous, non-negative, $F_\alpha(0) = 0$ (and otherwise unconstrained for $d_\alpha = 1$ by PAMV Theorem 6.1(ii)). For $\psi_\alpha \in V_\alpha$ with $\|\psi_\alpha\|_{\mathbb{C}} = r$, $\mu(\psi_\alpha) = F_\alpha(r)$. The image $\iota(\psi_\alpha)$ lies in a 1-dimensional complex Hilbert subspace of V_β (since $d_\alpha = 1$) with $\|\iota(\psi_\alpha)\|_{\mathbb{C}} = r$. By UMP-emb, $\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha) = F_\alpha(r)$.

Now compute $\mu(\iota(\psi_\alpha))$ within V_β. The image $\iota(\psi_\alpha)$ is a vector in V_β supported on the one-dimensional subspace $\iota(V_\alpha)$. The measure on V_β satisfies $F_\beta(s) = C_\beta s^2$ for $s \geq 0$ (PAMV Theorem 6.1(i), using $d_\beta \geq 2$ to apply the within-channel Cauchy-equation argument). For $\xi \in V_\beta$ supported on any one-dimensional subspace of V_β with $\|\xi\|\mathbb{C}$ = *r, the within-sector decomposition of ξ against any orthonormal basis of V_β has all mass on a single coordinate (the one aligned with the support subspace), so* $\|\xi\|\{V_\beta\} = r$ and $F_\beta(\|\xi\|_{\{V_\beta\}}) = C_\beta r^2$. Therefore $\mu(\iota(\psi_\alpha)) = F_\beta(\|\iota(\psi_\alpha)\|_{\mathbb{C}}) = C_\beta r^2$.

Combining: $F_\alpha(r) = C_\beta r^2$. So F_α is quadratic on V_α with coefficient C_β.

Remark 5.2.1. The lemma shows that UMP-emb does real work even for $d_\alpha = 1$ sectors: it imports the quadratic form from a higher-dimensional commensurable sector. The minimal core leaves F_α essentially free on $d_\alpha = 1$ sectors (PAMV Theorem 6.1(ii)); UMP-emb plus the existence of a (C-emb) witness to a $d_\beta \geq 2$ sector forces quadraticity on the $d_\alpha = 1$ sector and ties its coefficient to C_β. This dissolves the circularity that a constant-wise statement of UMP would otherwise face.

Lemma 5.2′ ($d_\alpha = 1$ quadraticity from UMP-ref). *Let μ satisfy the minimal core plus UMP-ref. Let V_α, V_β be sectors with $d_\alpha = d_\beta = 1$ and suppose the (C-ref) refinement-relabeling family $W_\theta : V_\alpha \oplus V_\beta \to V_\alpha' \oplus V_\beta'$ ($\theta \in [0, \pi/2]$) is available. Then F_α and F_β are both quadratic with a common coefficient: $F_\alpha(r) = F_\beta(r) = C r^2$ for some $C \geq 0$.*

Structural sketch. UMP-ref applied to the family W_θ at unit-norm $\psi = e_\alpha$ gives the constraint $F_\alpha(\cos \theta) + F_\beta(\sin \theta) = F_\alpha(1)$ for all $\theta \in [0, \pi/2]$. Extending this constraint to rescaled inputs $\lambda \cdot e_\alpha$ ($\lambda \in (0, 1]$) and to the symmetric $\psi = e_\beta$ instance gives a pair of Pexider-type functional equations on the squared-argument variables. The pair reduces, by standard manipulation of Pexider systems with shared boundary conditions (Aczél, *Lectures on Functional Equations*, Ch. 3–4), to a single additive Cauchy equation; continuous solutions are linear (Kuczma's theorem, the same theorem invoked in PAMV Theorem 6.1(i)); positivity (A1) selects the $C \geq 0$ branch; and the boundary condition $F_\alpha(1) = F_\beta(1) = C$ extends to $F_\alpha(r) = F_\beta(r) = Cr^2$ for all $r \geq 0$. The full proof is given in Appendix A.1.

Lemma 5.2″ ($d_\alpha = 1$ quadraticity from UMP-comp). *Let μ satisfy the minimal core plus UMP-comp on a carrier where every pair (α, β) with $\alpha + \beta \not\equiv 0 \bmod 7$ admits the (C-comp) relation. Let V_α, V_β be sectors with $d_\alpha = d_\beta = 1$, and suppose at least one sector V_δ on the character lattice $\{V_1, ..., V_6\}$ has $d_\delta \geq 2$. Then F_α and F_β are both quadratic with $F_\alpha(r) = F_\beta(r) = r^2$.*

(The scope condition $d_\delta \geq 2$ for some δ is automatically satisfied in the VERSF setting per Remark 4.4.2, where the inherited operational coherence-sector decomposition supplies at least some $d_\gamma \geq 2$ sectors. In the degenerate edge case where every sector on the character lattice has $d_\gamma = 1$, Lemma 5.2″ establishes the weaker uniform-power form $F_\gamma(r) = r^k$ for some k and an additional structural input is required to fix $k = 2$; see Appendix A.2 Case (ii) for the honest scope discussion.)

Structural sketch. For each sector V_γ on the character lattice, the value $\kappa_\gamma := F_\gamma(1)$ is well-defined regardless of quadraticity. UMP-comp's tensor-extension multiplicativity (Definition 5.1, Remark 5.1.2) applied at unit norm yields the partial multiplicative relation $\kappa_{\alpha+\beta \bmod 7} = \kappa_\alpha \cdot \kappa_\beta$; iteration around the $\mathbb{Z}_7$-character lattice plus seventh-power closure plus positivity (A1) forces $\kappa_\gamma = 1$ for all γ — fixing F_γ at unit norm but not yet at $r \neq 1$.

Extending UMP-comp to non-unit-norm composites then gives the multiplicative functional equation $F_\alpha(r) \cdot F_\beta(s) = F_{\alpha+\beta \bmod 7}(rs)$ across the character lattice. Logarithmic transformation reduces this to the Cauchy Pexider multiplicative-additive equation, whose continuous positive solutions establish $F_\gamma(r) = r^k$ *uniformly across the character lattice* for some single exponent k. The exponent $k = 2$ is fixed by applying PAMV Theorem 6.1(i) on any $d_\delta \geq 2$ sector V_δ of the character lattice (where the within-sector Cauchy-equation argument forces $F_\delta(r) = r^2$), combined with the uniformity of k. The argument is self-contained on the character lattice and does not depend on Lemma 5.0 (it operates on κ_γ at unit norm in the first step, where quadraticity of F_γ on $r \neq 1$ is not yet assumed). The full proof is given in Appendix A.2, including the explicit scope discussion of the edge case in which no $d_\delta \geq 2$ sector is available on the character lattice.

Theorem 5.3 (UMP forces uniform quadratic operational measure). *Under the minimal-core axioms (A1)–(A3), (A5), (A7) plus UMP on a fully commensurable carrier, every per-sector function F_α is quadratic, $F_\alpha(r) = C r^2$ for a single constant $C \geq 0$ uniform across all α, and (A7) fixes $C = 1$.*

Proof. By PAMV Theorem 5.2, the minimal core forces $\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_\mathbb{C})$. For $d_\alpha \geq 2$ sectors, $F_\alpha(r) = C_\alpha r^2$ by PAMV Theorem 6.1(i). For $d_\alpha = 1$ sectors, F_α is quadratic by Lemma 5.2 (when a $d_\beta \geq 2$ sector is available via (C-emb)), Lemma 5.2′ (when (C-ref) is available), or Lemma 5.2″ (when (C-comp) is available). Full commensurability (Remark 4.4.2) supplies at least one of these witnesses for every $d_\alpha = 1$ sector via its connection to other sectors in the commensurability graph. Therefore F_α is quadratic on every sector: $F_\alpha(r) = C_\alpha r^2$ with $C_\alpha \geq 0$ well-defined.

For cross-sector uniformity, UMP applied to each commensurable pair (α, β) forces $C_\alpha = C_\beta$:

- **(C-comp) pairs:** Lemma 5.0 (character-lattice multiplicativity from UMP-comp) forces $C_\alpha = 1$ uniformly across all character classes via the seventh-power-closure plus positivity. This handles all (C-comp)-commensurable pairs simultaneously in a single argument.
- **(C-emb) pairs:** UMP-emb applied to $\iota : V_\alpha \to V_\beta$ with $d_\alpha, d_\beta \geq 2$ gives $\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha)$, i.e. $C_\beta r^2 = C_\alpha r^2$, hence $C_\alpha = C_\beta$. (For the $d_\alpha = 1$ case, Lemma 5.2 already established $C_\alpha = C_\beta$ as part of the quadraticity step.)
- **(C-ref) pairs:** UMP-ref applied to W_θ as in PAMV Theorem 6.2 (B3) proof gives $C_\alpha \cos^2\theta + C_\beta \sin^2\theta = C_\alpha$ for all $\theta \in [0, \pi/2]$, hence $C_\alpha = C_\beta$. (For the $d_\alpha = d_\beta = 1$ case, Lemma 5.2′ already established $C_\alpha = C_\beta = C$ as part of the quadraticity step.)

Full commensurability connects every sector pair via at least one of these three witnesses, so $C_\alpha = C$ uniformly across α. On the VERSF carrier specifically, the (C-comp) leg via Lemma 5.0 already determines $C = 1$ directly through the character-lattice seventh-power closure; the (C-emb) and (C-ref) legs are then consistency checks confirming the uniform $C = 1$ result. Single-reference normalization (A7) is satisfied automatically by the $C = 1$ solution (PAMV Theorem 6.3).

Remark 5.3.1 (Where the work happens). Theorem 5.3 does genuine structural work, distinguishing it from a thin graph-connectivity observation: it both *establishes quadraticity on $d_\alpha = 1$ sectors* (via Lemmas 5.2, 5.2′, 5.2″) and *forces cross-sector uniformity*. The $d_\alpha = 1$ quadraticity is genuinely new content beyond PAMV Theorem 6.1, and could not be obtained from the minimal core alone — it requires UMP. The cross-sector uniformity then follows from UMP applied to each commensurable pair, with Lemma 5.0 doing the heaviest lifting (the (C-comp) leg uniformly determines C = 1 on the entire character lattice in a single character-multiplicativity argument), and the (C-emb) and (C-ref) legs supplying consistency checks. The graph-connectivity step is the final assembly rather than the main content.

Remark 5.3.2 (Why UMP is non-trivial). A reader might object that UMP is a thinly-disguised restatement of "C_α = C" or that it amounts to imposing the result. Neither is correct. UMP asserts something more specific than uniformity: that probability is invariant under operational commensurability *witnesses*. Incommensurable sectors — sectors with no operational composition, embedding, or refinement relation — are not constrained by UMP. If the carrier happened to contain operationally isolated sectors, UMP would permit them to carry independent per-sector functions. The substantive structural content of UMP is the implication

(operational commensurability witness exists) ⟹ (probability invariant under that witness),

not the bare statement of cross-sector uniformity. UMP is therefore falsifiable in a structurally specific way: a theory in which sectors were operationally related (via composition, embedding, or refinement) but probabilistically independent would violate UMP; a theory with operationally isolated sectors carrying different scales would not. UMP picks out a specific structural relation between operational geometry and probability, rather than imposing global uniformity by fiat.

Remark 5.3.3 (Relation to the substrate-only principle). UMP can be read as the operational formulation of the substrate-only principle invoked in PAMV §6.2, Step 1: probability assignment depends only on substrate-derived operational structure. The ℤ_7-character is the only substrate-derived label that survives inter-carrier identification, and the sector-identification (α-labels) is *not* itself substrate-derived data — it is a carrier-specific bookkeeping convention. Under UMP, probability cannot depend on α directly; it can depend only on the substrate-derived operational structure that the sectors collectively realize, which is the operational distinguishability geometry. Operationally indistinguishable preparations therefore receive equal probability, regardless of sector label.

Remark 5.4 (Partial commensurability). Suppose the carrier is not fully commensurable: the commensurability graph splits into disconnected components. UMP forces equal probability scales *within* each component, but admits potentially distinct scales *across* disconnected components. This is consistent with the structural content of UMP: incommensurable sectors are not required to be probabilistically commensurable. For the VERSF substrate as inherited from URHG, Remark 4.4.2 shows full commensurability holds and this case does not arise. But the structural framework is robust enough to handle partial commensurability if a substrate were to supply it. The resulting "multi-component" admissible measures would be a structural generalization of standard superselection sectors — see Open Problem 3 of §10.

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6. The Three Bridging Conditions as Consequences of UMP

We now show that the three bridging conditions (B1), (B2), (B3) of PAMV are each operationally entailed by UMP — i.e. that UMP implies each of them in the appropriate structural regime. This is the unification claim of the paper.

Mapping to PAMV. For readers coming from PAMV, the following table maps PAMV's structural results to their UMP-framework counterparts in the present paper. The mapping makes explicit how each PAMV bridging condition decomposes into a UMP-derivation plus the necessary supporting lemma in the present framework.

PAMV result	Role in PAMV	UMP-framework counterpart
Theorem 6.1(i)	Within-sector quadraticity on $d_\alpha \geq 2$ sectors from minimal core	Reused unchanged; supplies $F_\delta(r) = r^2$ on $d_\delta \geq 2$ sectors of the character lattice
Theorem 5.2	Channel decomposition $\mu(\psi) = \sum_\gamma F_\gamma(\|\psi_\gamma\|_{\mathbb{C}})$	Reused unchanged; supplies the per-sector structure used throughout
Lemma 6.2A	Minimal-core insufficiency for cross-channel uniformity	Reused as Lemma 3.1 of the present paper
Theorem 6.2 (B1) bridging	Compositional multiplicativity assumed; supplies C_α uniformity via character-lattice multiplicativity	Theorem 6.1.1 derives (B1) from UMP-comp via Lemmas 5.0a and 5.0 (which supply the character-lattice multiplicativity and seventh-power closure) and Theorem 5.3 (which lifts uniformity to all sectors)
Theorem 6.2 (B2) bridging	Isosymmetric-embedding consistency assumed; supplies cross-sector C_α equality on (C-emb) pairs	Theorem 6.1.2 derives (B2) directly from UMP-emb; Lemma 5.2 supplies $d_\alpha = 1$ quadraticity from UMP-emb on the same witnesses
Theorem 6.2 (B3) bridging	Refinement-stability consistency assumed; supplies cross-sector C_α equality on (C-ref) pairs	Theorem 6.1.3 derives (B3) directly from UMP-ref; Lemma 5.2′ supplies $d_\alpha = d_\beta = 1$ quadraticity from UMP-ref on the same witnesses
Theorem 6.3	Normalization (A7) fixes $C = 1$	Reused unchanged in Theorem 5.3 of the present paper
Theorem 7.1	Born rule on the carrier under three bridging conditions	Replaced by Theorem 7.1 (Universal Born rule) of the present paper, under UMP rather than the conjunction of three bridging conditions
Open Problem 2	Whether (B1), (B2), (B3) are independent or three faces of one principle	Resolved by the present paper: they are three operational windows onto UMP (Theorem 6.2)

The structural moves the present paper makes relative to PAMV are visible in the right column: (i) the three bridging conditions are *derived* from UMP rather than assumed; (ii) the multiplicative-homomorphism structure on the $\mathbb{Z}_7$-character lattice (which was implicit in PAMV's (B1) bridging condition) is factored out into Lemma 5.0 plus the algebraic Lemma 5.0a;

(iii) the $d_\alpha = 1$ quadraticity (which PAMV's Theorem 6.1(ii) left unconstrained beyond continuity) is supplied by UMP through three structurally distinct paths (Lemmas 5.2 / 5.2′ / 5.2″); and (iv) the conjunction-equivalence of UMP with the three (B_i) on a fully commensurable carrier (Theorem 6.2) shows that the unification UMP delivers is conceptual and structural rather than logically strengthening — the answer to PAMV's Open Problem 2.

A note on what to expect from the proofs that follow. Under the operational restatement of UMP (Definition 5.1), the embedding clause UMP-emb *is* the (B2)-type invariance and the refinement clause UMP-ref *is* the (B3)-type invariance, restricted to the corresponding commensurability witnesses. Theorems 6.1.2 and 6.1.3 below are therefore one-line invocations of the corresponding clauses of Definition 5.1, and their structural content is correspondingly direct. The (B1) leg requires more work — Theorem 6.1.1 — because the (C-comp) commensurability witness $\sigma_{\alpha,\beta}$ is a carrier-to-tensor identification, and the bridging condition (B1) couples this identification with the multiplicativity of μ on tensor products. The asymmetry in proof length reflects this structural asymmetry: (B2) and (B3) are immediate UMP-axiom-instances, while (B1) requires combining UMP-comp with the multiplicative-homomorphism structure on the character lattice (much of which was already factored out in Lemma 5.0).

6.1 (B1) as a compositional UMP-instance

Theorem 6.1.1 (UMP ⟹ B1 under (C-comp)). *Suppose* μ *satisfies the minimal core, UMP holds on the carrier* $\mathcal{A}_\mathbb{C}$*, and the compositional commensurability relation (C-comp) connects sectors* V_α *and* V_β *through their product character* $\chi_{\alpha+\beta \bmod 7}$*. Then the multiplicativity relation*

$$\mu(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi)$$

and the character-matching relation

$$C_{\alpha+\beta \bmod 7}(\text{tensor carrier}) = C_{\alpha+\beta \bmod 7}(\text{original carrier})$$

both follow from UMP applied to the commensurability between $(\alpha, \beta, \alpha+\beta \bmod 7)$*.*

Proof. By Theorem 5.3, UMP plus the minimal core forces $F_\gamma(r) = r^2$ uniformly across γ (with $C = 1$ by normalization). On unit-norm states $\psi \in V_\alpha$, $\phi \in V_\beta$:

$$\mu(\psi) = 1, \ \mu(\phi) = 1.$$

The composite $\psi \otimes \phi$ lies in the $V_\alpha \otimes V_\beta$ component of the tensor carrier, which (by (C-comp)) is operationally identified with an isotypic component for character $\chi_{\alpha+\beta \bmod 7}$. UMP-comp applied to this identification (Definition 5.1, UMP-comp clause) gives:

$$\mu_{\text{tensor}}(\psi \otimes \phi) = \mu_{\text{original}}(\sigma_{\alpha,\beta}(\psi \otimes \phi)).$$

The image $\sigma_{\alpha,\beta}(\psi \otimes \phi)$ is a unit-norm element of the $V_{\alpha+\beta \bmod 7}$ isotypic component in the original carrier, so μ_{original} of it equals 1 by Theorem 5.3. Therefore $\mu_{\text{tensor}}(\psi \otimes \phi) = 1$

$= 1 \cdot 1 = \mu(\psi) \cdot \mu(\phi)$. The character-matching relation is the same UMP-instantiation, restricted to the equality of the two quadratic constants on isotypic components for character $\chi_{\alpha+\beta \bmod 7}$.

Remark 6.1.1. (B1) of PAMV is recovered as a UMP-instance: the compositional multiplicativity is the operational consequence of UMP-comp applied to the (C-comp) witness $\sigma_{\alpha,\beta}$, and the character-matching consistency is the substrate-only specialization of UMP to isotypic components. The seventh-power closure argument of PAMV Theorem 6.2 (Step 3) emerges in the present framework as the consistency check that the partial multiplicative homomorphism on the $\mathbb{Z}_7$-character lattice factors through the trivial homomorphism, which UMP-comp already forces directly.

6.2 (B2) as an embedding UMP-instance

Theorem 6.1.2 (UMP ⟹ B2 under (C-emb)). *Suppose μ satisfies the minimal core, UMP holds on the carrier $\mathcal{A}_\mathbb{C}$, and the embedding commensurability relation (C-emb) connects sectors V_α and V_β via a complex-unitary embedding $\iota : V_\alpha \to V_\beta$ with $d_\beta \geq d_\alpha$. Then the isosymmetric-embedding consistency*

$$\mu(\iota(\psi_\alpha)) = \mu(\psi_\alpha)$$

follows from UMP.

Proof. This is exactly UMP-emb (Definition 5.1, UMP-emb clause) applied to the (C-emb) witness ι. The result is immediate from the definition of UMP.

Remark 6.1.2. Under the operational formulation of UMP, (B2) is a direct UMP-axiom-instance: UMP-emb *is* the (B2)-type invariance, restricted to the (C-emb) commensurability witnesses. The structural unification at this leg is therefore particularly transparent.

6.3 (B3) as a refinement UMP-instance

Theorem 6.1.3 (UMP ⟹ B3 under (C-ref)). *Suppose μ satisfies the minimal core, UMP holds on the carrier $\mathcal{A}_\mathbb{C}$, and the refinement commensurability relation (C-ref) connects sectors V_α and V_β via the existence of a decomposition-admissible (U-dec) unitary $W : V_\alpha \oplus V_\beta \to V_{\alpha'} \oplus V_{\beta'}$. Then the refinement-stability consistency*

$$\mu(W\psi) = \mu(\psi)$$

for the (U-dec) unitary W follows from UMP.

Proof. This is exactly UMP-ref (Definition 5.1, UMP-ref clause) applied to the (C-ref) witness W. The result is immediate from the definition of UMP.

Remark 6.1.3. Like (B2), (B3) is recovered as a direct UMP-axiom-instance under the operational formulation: UMP-ref *is* the (B3)-type invariance restricted to the (C-ref) commensurability witnesses. The (B3) and (B2) legs are particularly clean under the operational UMP; the (B1) leg requires the additional carrier-to-tensor identification step (the σ_{α,β} witness) before the UMP-comp invariance takes effect.

6.4 The unification statement

Theorem 6.2 (Unification: UMP unifies B1, B2, B3). *Under the minimal core plus UMP on a fully commensurable carrier, the three bridging conditions (B1), (B2), (B3) of PAMV are all implied by UMP. Conversely, each of (B_i) implies UMP restricted to the corresponding commensurability relation (C-i), in the structural regime where that commensurability relation is non-vacuous. On a* ***fully*** *commensurable carrier, UMP is therefore logically equivalent to the conjunction*

(B1) on (C-comp) pairs ∧ (B2) on (C-emb) pairs ∧ (B3) on (C-ref) pairs.

On a ***partially*** *commensurable carrier, the conjunction → UMP direction fails to extend to incommensurable sectors: the conjunction forces uniformity within each connected component of the commensurability graph, while UMP does the same, so the equivalence is preserved component-wise but the conjunction does not constrain probability scales across components that the partial-commensurability structure does not connect. Full commensurability is therefore the precise structural assumption under which UMP and the conjunction are globally logically equivalent.*

Proof. The forward implications UMP ⟹ (B_i) are Theorems 6.1.1, 6.1.2, 6.1.3.

The converse implications (B_i) ⟹ UMP-restricted-to-(C-i) follow from the PAMV proofs of Theorem 6.2: (B1) plus the minimal core forces uniformity across all pairs commensurable via (C-comp); (B2) does so for (C-emb) pairs; (B3) does so for (C-ref) pairs. Each (B_i) therefore implies UMP restricted to its own commensurability relation.

On a fully commensurable carrier, every commensurable pair is connected by at least one of the three relations, so the conjunction of the three restricted UMPs equals full UMP, and the global logical equivalence holds. On a partially commensurable carrier, the same argument applies within each connected component of the commensurability graph: the conjunction and UMP each yield $C_\alpha = C_\beta$ whenever (α, β) is a commensurable pair, but the closure of those pairwise equalities under transitivity reaches only across the connected component containing the pair. The difference from the fully-commensurable case is therefore not in what the conjunction and UMP assert (they assert the same content) but in whether their assertions propagate to global cross-sector equality; pairwise equalities no longer propagate to global equality across disconnected components, and probability scales between such components remain unconstrained by both the conjunction and UMP. The equivalence is thus component-wise rather than global.

Remark 6.2.1 (Conceptual vs logical depth). The forward direction UMP ⟹ (B_i) is structurally clean: UMP is a single principle on commensurability and each (B_i) is its particular operational expression on one commensurability relation. The converse direction (B_i) ⟹ UMP is partial: each (B_i) recovers UMP only on the specific commensurability relation it witnesses.

On a fully commensurable carrier, however, the conjunction (B1) ∧ (B2) ∧ (B3) restricted to their respective domains is *logically equivalent* to UMP. The unification UMP delivers is therefore not a logical strengthening of the conjunction — UMP and the conjunction have the same logical content on the VERSF carrier. The unification operates at two other levels:

- **Conceptual.** UMP names one principle in place of three. The reader and the framework benefit from organizing the structural content under a single concept (probability invariance under commensurability witnesses) rather than under three named conditions. The previous "three independent derivations" reading is structurally redundant; the "one principle, three witnesses" reading is structurally minimal.
- **Structural.** UMP exhibits the common operational origin of the three bridging conditions via the triple structural failure of Theorem 3.2 (Remark 6.2.4 below). The convergence of the three bridging routes is not a coincidence to be defended but a structural reflection of their common origin in the violation-vs-respect of operational commensurability.

UMP is therefore *conceptually and structurally* deeper than any individual (B_i), and conceptually-organizationally deeper than the conjunction. It is not logically stronger than the conjunction. This is the precise content of the "UMP is deeper than (B_i)" claim throughout the paper, and the answer to PAMV's Open Problem 2.

Remark 6.2.2 (Robustness — two senses). PAMV emphasized that the existence of three structurally distinct bridging conditions gave the framework robustness against challenges to any single condition. The present paper supports this claim, but it requires careful distinction between two senses of robustness:

- **Evidential robustness:** the three commensurability witnesses supply three operationally distinct ways to probe UMP empirically. If an experimental test of (say) compositional multiplicativity returned an anomalous result, the other two windows would remain available as independent probes. The anomalous result could indicate either a genuine violation of UMP, or operational subtleties specific to the compositional test (e.g. tensor-extension issues with the experimental apparatus), and the embedding and refinement tests could help distinguish these. This evidential redundancy is a real practical advantage of having three windows onto one principle.
- **Logical robustness:** UMP logically *entails* all three (B_i) on their respective domains. If a genuine violation of any (B_i) is established — not merely an anomalous test result but a definitively confirmed failure of (B_i) on a commensurable pair — then UMP is violated. UMP cannot be "supported" by (B_j) and (B_k) when (B_i) is established to fail, because UMP entails (B_i). The framework does not have logical redundancy at this level: a single confirmed (B_i) violation on a commensurable pair falsifies the whole framework.

The distinction matters: the framework is *evidentially* robust (three independent operational probes), not *logically* robust (it is not the case that two of three bridging conditions surviving suffices to preserve UMP). UMP-test 1 of §9.2 below makes this sharper.

Remark 6.2.3 (Connection to Theorem 3.2). The triple structural failure of Theorem 3.2 ((F1) compositional, (F2) embedding, (F3) refinement) and the three UMP-windows of Theorem 6.2 ((B1) compositional, (B2) embedding, (B3) refinement) match. This is not coincidental: the three failures and the three windows both arise from the three commensurability relations (C-comp), (C-emb), (C-ref) of Definition 4.1. Theorem 3.2 establishes that violating UMP produces these three failures simultaneously; Theorem 6.2 establishes that UMP repairs all three simultaneously through these three windows. The structural symmetry between negative and positive results is what makes UMP the natural single principle: it is the unique requirement that the three commensurability relations collectively impose on operational measure.

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7. The Universal Born Rule

The combination of PAMV's quadraticity result and the present paper's universality result yields the Born rule as a universal probability rule on the operational carrier.

Theorem 7.1 (Universal Born rule under UMP). *Under the inherited operational structure of PAMV §2, the minimal-core axioms (A1)–(A3), (A5), (A7) of PAMV Definition 3.1, and the Universal Measure Principle (Definition 5.1), the operational probability measure on* $\mathcal{A}_\mathbb{C}$ *is uniquely*

$$\mu(\psi) = \|\psi\|^2_\mathbb{C},$$

and the transition probability between admissible states is

$$P(\psi \to \phi) = |\langle\phi, \psi\rangle_\mathbb{C}|^2$$

for every pair of normalized admissible states $\psi, \phi \in \mathcal{A}_\mathbb{C}$.

Proof. By Lemma 3.1, the minimal core forces $\mu(\psi) = \sum_\alpha F_\alpha(\|\psi_\alpha\|_\mathbb{C})$. By Theorem 5.3, UMP on a fully commensurable carrier forces $F_\alpha(r) = r^2$ uniformly across all α (with the constant fixed to 1 by (A7)). Therefore $\mu(\psi) = \sum_\alpha \|\psi_\alpha\|^2_\mathbb{C} = \|\psi\|^2_\mathbb{C}$ (PAMV §2.3 norm identity). The transition-probability statement follows by PAMV Theorem 7.1: $P(\psi \to \phi) = \mu(P_\phi \psi) = \|P_\phi \psi\|^2_\mathbb{C} = |\langle\phi, \psi\rangle_\mathbb{C}|^2$ for normalized ϕ.

Theorem 7.2 (Born rule as the unique commensurable minimal-core measure). *Under the inherited operational structure of PAMV §2, the minimal-core axioms (A1)–(A3), (A5), (A7), and on a fully commensurable carrier, the Born rule is the unique minimal-core admissible probability measure compatible with operational commensurability across coherence sectors. Equivalently: among all measures satisfying the minimal core, the Born rule is the only one that* *respects the operational substrate-only principle that probability depends on operational distinguishability geometry alone.*

Proof. Direct corollary of Theorem 7.1. UMP is the formalization of the substrate-only principle in the present framework (Remark 5.3.3); the universal Born rule is its unique solution within the minimal-core admissible measures.

Remark 7.2.1 (Quadraticity is not the fingerprint). The structural reading enabled by Theorem 7.2: the Born rule is *not* the unique minimal-core measure on $\mathcal{A}_\mathbb{C}$ (the minimal core admits the parametrized family $\mu_{(C_\alpha), (F_\alpha^{(1)})}$ by Lemma 3.1), and quadraticity alone does not determine it. The Born rule is *the unique commensurable* member of the minimal-core family, picked out by UMP as the only member that does not introduce hidden sector-specific parameters. Universality is therefore the structural fingerprint of the Born rule, with within-sector quadraticity for $d_\alpha \geq 2$ sectors supplied by the minimal-core Cauchy-equation argument and within-sector quadraticity for $d_\alpha = 1$ sectors supplied by UMP via commensurability inheritance. This inverts the conventional emphasis: the deep structural fact about Born probability is universality across the operational carrier, not the squaring formula within a single sector.

8. Interpretation

Convention on the term "universality." Throughout this paper, **universality** of an operational probability measure means *invariance of probability assignment across operationally commensurable coherence sectors* — equivalently, that the measure does not depend on sector-identity data beyond what operational structure encodes. This is the precise sense in which the Born rule is "universal" in the present framework: it is the unique minimal-core admissible measure invariant under all commensurability witnesses on the carrier. The term is *not* used to mean "universal across the carrier as a whole regardless of sector relations" (which would be a stronger and content-less claim) nor "unique globally" (which is what universality, combined with the minimal core, *yields* — not what it asserts).

8.1 The structural architecture

The deep architecture this paper isolates can be stated compactly:

Object	Role
Minimal-core geometry	determines local within-sector quadraticity (on $d_\alpha \geq 2$ sectors)
ODG-preserving structure	supplies the global commensurability witnesses across sectors
UMP	the invariance principle: probability is unchanged under commensurability witnesses

The Born rule is then the unique fixed point of this architecture:

Local within-sector geometry + *Global cross-sector commensurability = Born universality.*

Within each coherence sector of dimension ≥ 2, the minimal-core probability axioms supply within-sector quadraticity via orthogonal additivity and continuity (PAMV Theorem 6.1; the within-sector Cauchy-equation argument). Across sectors and on $d_\alpha = 1$ sectors, operational commensurability supplies the cross-sector invariance via UMP. Neither component alone determines the Born rule — the minimal core admits a multi-parameter family of measures (Lemma 3.1), and commensurability without any within-sector structure has nothing to relate. The Born rule is the unique measure compatible with both layers simultaneously.

8.2 The three structural layers

The structural reframing of probability that this paper delivers can be stated in three layers, refining the architecture above.

Layer 1: Within-sector quadraticity from minimal-core orthogonality. Inside any single coherence sector V_α with $d_\alpha \geq 2$, the operational measure must take the form $C_\alpha \|\psi_\alpha\|^2_{\mathbb{C}}$. This is supplied by PAMV's minimal-core argument: orthogonal additivity on complex-orthogonal vector pairs within V_α, combined with continuity, forces the Cauchy-equation solution $G_\alpha(t) = C_\alpha t$ and hence $F_\alpha(r) = C_\alpha r^2$. No deeper substrate principle is needed to establish quadraticity within a sector of dimension ≥ 2; it follows from operational additivity and regularity alone.

Layer 2: Cross-sector and $d_\alpha = 1$ quadraticity from operational commensurability. Across sectors, and on $d_\alpha = 1$ sectors, the minimal core leaves the per-sector functions and constants free. UMP supplies the principle that probability is invariant under operational commensurability witnesses; this forces both (a) quadraticity on $d_\alpha = 1$ sectors via their commensurability with $d_\beta \geq 2$ sectors (Lemma 5.2), and (b) cross-sector uniformity of the quadratic constants (Theorem 5.3). PAMV's three bridging conditions (B1), (B2), (B3) are three structurally distinct operational witnesses of UMP — three ways the commensurability of sectors manifests in operational tests.

Layer 3: Born universality. The combination of within-sector quadraticity (from minimal-core or commensurability inheritance, depending on dimension) and cross-sector commensurability yields the Born rule as a universal probability law on the operational carrier. Universality — not quadraticity — is the deepest structural content; within-sector quadraticity for $d_\alpha \geq 2$ is a derived consequence of operational additivity, while quadraticity for $d_\alpha = 1$ sectors and uniformity across sectors are the genuinely substantive structural inputs supplied by UMP.

8.3 Geometric intuition

Operational commensurability can be understood geometrically as the requirement that different coherence sectors inhabit *one shared distinguishability geometry* rather than disconnected local geometries. Once sectors participate in shared operational structure — composability into common composites, embeddability into common ambient spaces, joint refinement-relabelability — probability assignment can no longer consistently remain sector-relative. The mathematical impossibility of sector-relative probability under commensurability (Theorem 3.2) is the rigorous form of a geometric intuition: *one geometry admits one probability measure*. UMP is the statement of this geometric fact at the level of operational structure.

The conceptual picture this delivers is the following. Probability is operational distinguishability volume, as PAMV §9 already framed it. Within each coherence sector of dimension ≥ 2, this volume is measured quadratically (the squared norm) by within-sector orthogonal-additivity geometry. Across sectors and on $d_\alpha = 1$ sectors, the volumes are *commensurable* — they are measured in a common unit, not in sector-specific units, with the $d_\alpha = 1$ quadratic form inherited from higher-dimensional commensurable sectors via UMP. The unit-commensurability is what makes probability universal; the quadratic measurement within sectors is what makes probability geometric.

A theory in which sector-specific units were permitted — one in which different coherence sectors carried different intrinsic probability scales, or in which $d_\alpha = 1$ sectors carried non-quadratic per-sector functions — would be a theory in which probability was not really *a* measure on the operational carrier but a family of sector-relative measures glued together. UMP excludes this. The Born rule is the assertion that probability is genuinely one measure on the whole admissible operational geometry, not a sector-relative collection.

A useful way to state the reframing: PAMV asked "why is quantum probability quadratic?" and answered with within-sector orthogonal additivity plus a bridging principle for cross-sector uniformity. The present paper asks "why is quantum probability *universal and quadratic everywhere*?" and answers with operational commensurability across coherence sectors. The first question is largely settled by within-sector geometry on $d_\alpha \geq 2$ sectors; the second is the genuine structural question that the Born-rule reconstruction is fundamentally about.

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9. Predictions and Falsifiability

Following the §12 structure of PAMV, we distinguish generic from substrate-specific predictions.

9.1 Generic prediction inherited from PAMV

The Sorkin third-order interference vanishing (PAMV §12.1) is inherited and unaffected. It follows from orthogonal additivity (A2) alone — the within-sector axiom — and tests the additivity backbone of any framework imposing (A2). UMP adds no further content to the Sorkin test directly, since the Sorkin quantity I_3 is a within-sector observable and UMP is a cross-sector principle. However, UMP does generate stronger cross-sector tests below. Experimental verification of Sorkin null at high precision (Sinha et al. 2010, *Science* 329, 418) supports (A2); the UMP-extended tests supply sharper cross-sector probes.

9.2 UMP-specific predictions

These are predictions that test the universality content of UMP beyond what the bridging conditions of PAMV individually deliver.

UMP-test 1 (the principal novel falsifier): joint validity of compositional, embedding, and refinement consistency. This is the test that most clearly distinguishes UMP from PAMV's three-route reading and supplies the principal novel falsifiability content UMP provides over PAMV.

Under PAMV's three-route reading, any single bridging condition holding suffices to close the uniqueness argument; failure of one (B_i) is not by itself a falsifier of the framework, because the remaining bridging conditions could still establish C_α = C uniformly. The three bridging conditions were operationally independent assumptions any one of which sufficed.

Under UMP, by contrast, *all three commensurability windows must hold jointly on a fully commensurable carrier*. The reason: UMP is a single principle on commensurability, and on a fully commensurable carrier it is logically equivalent to the conjunction of all three (B_i) on their respective domains (Theorem 6.2). Failure of any single (B_i) on a commensurable pair therefore falsifies UMP — and by Theorem 6.2's conjunction-equivalence, falsifies the entire UMP framework even though two of the three windows still report success.

This is genuine added empirical content. PAMV's framework was *evidentially* robust against single-(B_i) failures (the other two routes still gave C_α = C); UMP's framework is *not* logically robust against single-(B_i) failures on fully commensurable carriers (because UMP entails the conjunction, and the conjunction's truth requires all three conjuncts). Detection of a stable operational setting in which (say) compositional multiplicativity holds but isosymmetric embedding consistency fails on a (C-emb)-commensurable pair would falsify UMP while leaving each of PAMV's individual three-route arguments structurally available. This is the principal novel falsifier UMP supplies.

Schematic experimental protocol. The analogous experimental shape to the Sinha et al. Sorkin-null protocol — which defined an operational procedure for testing within-sector additivity (A2) by comparing two-slit and three-slit interference patterns in an optical interferometer — would be a *cross-sector* protocol testing the joint validity of (B1), (B2), (B3) on the same physical system. Schematically: prepare a physical system supporting two operationally identifiable coherence sectors V_α, V_β commensurable in at least two of the three senses (the VERSF substrate makes this generic, as Remark 4.4.2 establishes), then run three coordinated probes on the same system —

- **(B1)-probe:** prepare composite ($\psi \otimes \phi$) preparations and test whether the composite measurement statistics factor multiplicatively into the individual-sector statistics (with character-matching) as Theorem 6.1.1 predicts;
- **(B2)-probe:** prepare states in V_α and in their isosymmetric embedded image ι(V_α) ⊆ V_β and test whether the measurement statistics match;

- **(B3)-probe:** measure the same physical system using two distinct admissible orthogonal decompositions of the V_α ⊕ V_β subspace (a "decomposition-switching" protocol) and test whether the total probability is decomposition-invariant.

The (B3)-probe is the most delicate to realize operationally. The (C-ref) witness W : V_α ⊕ V_β → V_α' ⊕ V_β' is a decomposition-admissible (U-dec) unitary that may mix inequivalent ℤ_7-character sectors — by definition (§2), such unitaries are *not* transport-admissible when α ≠ β. The (B3)-probe therefore does not correspond to a physical transport between the two sector-decompositions; it corresponds to measuring the same physical state using two different admissible *measurement frames* on the V_α ⊕ V_β subspace, where the frame choice itself reflects a different operational refinement of the same admissible geometry. In standard quantum-mechanical terms the analogue would be measuring a fixed state in two different orthonormal bases that span the same 2-dimensional subspace; the (B3)-probe asks whether the total probability is invariant under this basis-switch. What kind of physical measurement realizes a (U-dec) decomposition-switching when the two decompositions cross inequivalent ℤ_7-character sectors is a non-trivial operational design question that the VERSF substrate would need to specify; for the present paper's purposes, the (B3)-probe should be understood as the *abstract operational test* that the (C-ref) commensurability relation calls for, with the concrete experimental realization deferred to a substrate-specific operational protocol. This is the (C-ref) leg's experimental subtlety, and we flag it rather than glossing it.

Under UMP, all three probes must succeed jointly on the same system; under PAMV's three-route reading, any one succeeding suffices. The novel UMP content is the *joint-success requirement*: a system on which one probe definitively failed while the others succeeded would falsify UMP while leaving PAMV's three-route framework structurally intact (since one of the remaining bridging conditions would still force C_α = C). The Sinha et al. protocol provides a methodological template — defining operational equivalents of the abstract additivity condition — for what such a cross-sector joint-probe experiment would look like; the detailed experimental design would need to be developed against the specific operational signatures the VERSF substrate predicts.

UMP-test 2: Universality across newly-coupled sectors. UMP predicts that whenever two previously incommensurable sectors become operationally coupled — e.g. through the introduction of an admissible composition rule connecting them, or through an embedding becoming operationally realizable — their probability scales must equalize. This is in principle observable: if a substrate were to permit dynamically-emerging sector commensurability, UMP predicts dynamically-emerging probability-scale uniformity across the newly-coupled sectors. Detection of stable scale-mismatch across operationally commensurable sectors, persistent under all three commensurability windows, would falsify UMP.

UMP-test 3: Sector-relative probability under operational isolation. UMP only forces invariance under commensurability witnesses, hence equal scales on commensurable sectors. If a substrate carrier contained operationally isolated sectors (sectors with no compositional, embedding, or refinement relation to others), UMP would *permit* those sectors to carry independent per-sector functions. This is a positive structural prediction: probabilistic universality is not unconditional but is mediated by operational commensurability. Detection of an operationally isolated sector in the VERSF substrate carrying a probability scale identical to the commensurable sectors would be a structural success for the conjecture that the VERSF substrate is fully commensurable (Remark 4.4.2); detection of an isolated sector carrying a different scale would be consistent with UMP but would indicate that the carrier is not fully commensurable.

UMP-test 4: Detection of triple structural failure under sector-relativity. Theorem 3.2 predicts that any operational measure with non-uniform sector constants must violate compositional neutrality (F1), embedding consistency (F2), and refinement invariance (F3) *simultaneously*. Experimental detection of (F1) without (F2) and (F3), or any combination violating the joint-failure prediction, would falsify the triple-structural-failure theorem and indicate that the three commensurability relations do not all share the common structural origin asserted in Remark 3.2.1.

9.3 General falsifiers

Each of the following would falsify the framework:

- demonstration of a fully commensurable carrier in which cross-sector probability scales differ;
- demonstration that any one of (B1), (B2), (B3) fails on a commensurable pair (UMP-test 1; this is the principal novel falsifier);
- demonstration of stable probability assignments that depend on sector identity beyond what operational distinguishability geometry alone can encode;
- demonstration that the triple structural failure (F1)–(F3) of Theorem 3.2 separates — i.e. that one of the three failures can occur in isolation while the other two do not — which would invalidate the shared-structural-origin claim of Remark 3.2.1;
- demonstration of a substrate-derived principle from which UMP itself can be derived (which would not falsify UMP but would resolve Open Problem 1 of §10 and supersede the present paper's structural status of UMP).

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10. Open Problems

1. **Substrate-level derivation of UMP.** Whether UMP can itself be derived from a more primitive substrate-level principle — e.g. from finite packing structure (PAMV Open Problem 1), or from the substrate $\mathbb{Z}_7$-equivariance directly. The present paper treats UMP as a structural input on the operational measure; whether it descends from packing or character-theoretic structure at a deeper level is the natural next question.

Candidate sketch. UMP might follow from the observation that the packing-volume measure of OG Theorem 10.1 is intrinsically carrier-geometric. The packing inequality $|\Sigma(M)| \leq \mathrm{Vol\_op}(M) / \Delta\_\mathrm{op}^{d\_\mathrm{op}}$ depends on Vol_op(M) and Δ_op — both of which are substrate-derived quantities not sector-indexed. A probability measure compatible with packing-volume proportionality (in the sense of PAMV §9, Proposition 9.1) would be forced to inherit the carrier-geometric structure of the packing measure, i.e. could not introduce sector-specific scaling beyond what the packing-volume already encodes. Sector-specific scaling factors C_α ≠ C_β would constitute additional structural data the packing inequality does not supply, and would violate the consistency between probability and packing-volume that PAMV Proposition 9.1 asserts. If this argument can be made rigorous — turning the packing-volume identification of PAMV §9 from a definitional convention into a structural constraint on probability assignment — UMP would follow as a derived consequence of finite-packing geometry rather than as a structural axiom.

Three-link structural chain. The candidate derivation decomposes into three structurally distinct links, each of which would need to be made rigorous:

- *Link (i).* PAMV Open Problem 10's strengthening of Vol_op(χ) = ‖χ‖²_ℂ from definitional convention to derived theorem — establishing that the packing-volume measure is a structurally determined function of the carrier rather than a convention adopted for convenience.
- *Link (ii).* Derivation of probability-volume proportionality from the strengthened Vol_op identification plus finite packing geometry — establishing that any admissible operational probability measure must agree with packing-volume up to overall normalization, not merely that PAMV Proposition 9.1 stipulates this proportionality definitionally.
- *Link (iii).* Inheritance of sector-independence from Vol_op to probability — establishing that since Vol_op is a substrate-derived (and therefore sector-independent) quantity under link (i), the probability measure forced into proportionality with it under link (ii) cannot carry sector-specific scaling, yielding UMP.

Link (i) is PAMV's Open Problem 10. Link (iii) is essentially immediate once links (i) and (ii) hold (it is the substrate-only-principle of Remark 5.3.3 applied to the proportionality of link (ii)). Link (ii) is the genuinely new structural work the present paper's Open Problem 1 calls for: deriving probability-volume proportionality from the strengthened packing structure rather than stipulating it. Each link admits independent attack; the natural sequence is to address link (i) first (resolving PAMV Open Problem 10), then link (ii) (the genuinely new step), with link (iii) following as a corollary.

Coupling to PAMV Open Problem 10. PAMV's Open Problem 10 (strengthening Vol_op(χ) = ‖χ‖²_ℂ from convention to theorem) and the present paper's Open Problem 1 are intertwined precisely through this three-link chain. Resolving PAMV's Open Problem 10 supplies link (i) directly. Combined with link (ii) — the new structural step — it would supply UMP as a corollary via link (iii). Resolution of PAMV's Open Problem 10 is therefore necessary but not sufficient for closing the present paper's Open Problem 1; the genuinely new structural work the present paper calls for is link (ii), the proportionality-from-rigour step that connects substrate-derived packing geometry to operational probability assignment.

2. **The deeper origin of commensurability itself.** Why does the inherited VERSF carrier supply full commensurability (Remark 4.4.2)? The three commensurability relations (C-comp), (C-emb), (C-ref) are properties of the ℤ_7-character lattice and the operational

refinement structure. The structurally cleanest framing recognizes three distinct possibilities, not a binary forced-vs-contingent dichotomy:

- *(2a) Forced by substrate symmetry alone.* If ℤ_7-character-lattice structure plus admissible-transport structure suffices to guarantee full commensurability for any substrate of this symmetry type, then UMP becomes essentially automatic on any ℤ_7-equivariant admissible operational carrier. The (C-comp) relation comes for free from the character lattice, (C-emb) from the natural dimensional ordering, and (C-ref) from any non-trivial admissible refinement. This would make full commensurability a structural consequence of the symmetry choice.
- *(2b) Forced by the K = 7 closure architecture specifically.* If the K = 7 minimal-fact architecture (rather than ℤ_7 alone) is what supplies full commensurability — e.g. if the seventh-power closure used in Lemma 5.0a is a structural reflection of the K = 7 closure rather than just the character-lattice arithmetic — then UMP is contingent on the K = 7 selection (PAMV's Open Problem 11) rather than on ℤ_7 symmetry alone. The character-lattice closure and the K = 7 closure architecture would then be reflections of a single deeper principle.
- *(2c) Contingent feature of the URHG carrier.* If neither (2a) nor (2b) holds — i.e. neither symmetry nor K = 7 closure forces full commensurability — then UMP is a non-trivial structural axiom on the inherited URHG carrier that might not hold on other admissible operational carriers. The full commensurability of the VERSF substrate would then be a contingent feature of the specific URHG-derived structure, with no general structural guarantee.

Distinguishing among (2a), (2b), (2c) is the substantive open question. Each possibility has different implications for the present paper's framework: (2a) would make UMP nearly automatic and downstream from substrate symmetry; (2b) would tie UMP to the K = 7 architecture and parallel PAMV's Open Problem 11; (2c) would preserve UMP's status as a substantive structural axiom independent of substrate symmetry and K = 7 architecture. The natural attack is to identify which features of the URHG carrier are actually used in establishing full commensurability per Remark 4.4.2, and whether those features generalize to other ℤ_7-equivariant or K = 7-closure substrates.

3. **UMP under partial commensurability.** Remark 5.4 notes that UMP under partial commensurability permits incommensurable sector-classes to carry distinct probability scales. The structure of such "multi-component" admissible measures — which would be a structural generalization of standard superselection sectors — and whether they could be physically meaningful is open. The VERSF substrate as inherited does not produce this case, but it is structurally available, and a deeper substrate theory permitting operational isolation between sector-classes would have admissible measures that decompose into UMP-uniform components glued at incommensurable boundaries.
4. **Operational entropy and commensurability.** Whether operational entropy on the admissible carrier — defined in the corpus's thermodynamic-unfolding programme — interacts with UMP in a structurally non-trivial way. Candidate: under UMP, operational entropy would inherit cross-sector universality directly, with no separate bridging required, since entropy would be a function of the universal probability measure. The

entropic-unfolding route to the Born rule (PAMV §8 (v)) might then admit a clean UMP-reformulation: the iso-entropic limit operates on commensurable sector-classes, and the bare quadratic measure recovered in that limit is the UMP-universal one.

5. **Infinite-dimensional UMP.** Extension of UMP to infinite-dimensional carriers. The full-commensurability assumption (Remark 4.4.2) may need refinement in infinite dimensions where compositional and embedding relations among infinitely many sectors carry technical subtleties (in particular, convergence of the channel-decomposition sum $\mu(\psi) = \sum_\alpha C_\alpha \|\psi_\alpha\|^2_{\mathbb{C}}$ as a series rather than a finite sum).
6. **UMP and decoherence.** Whether decoherence — operational coupling between admissible coherence sectors and inadmissible degrees of freedom — admits a UMP-compatible description, with environmental sectors entering the commensurability graph dynamically. The structural shape of the question: when an environmental sector becomes operationally accessible (through decoherence-induced coupling), does UMP automatically extend probability commensurability to include it, and does this match the standard treatment of measurement-induced state-reduction?
7. **Lorentzian extension.** Whether UMP survives the Riemannian-to-Lorentzian operational signature transition conjectured in OC and URHG. Lorentzian signature changes the inner-product structure (one negative direction); the commensurability relations may require modification under the changed admissibility geometry.
8. **Strengthening Theorem 6.2's converse.** The converse direction $(B_i) \Longrightarrow$ UMP-restricted-to-(C-i) is partial: each (B_i) implies UMP only on its corresponding commensurability relation. A stronger result would establish that any single (B_i) plus auxiliary structural conditions implies UMP across all three commensurability relations. This would weaken the conjunction-equivalence of Theorem 6.2 to a single-condition equivalence, with practical implications for empirical testing: combined with UMP-test 1 of §9.2 (which already establishes the falsifying direction — a single (B_i)-test failing on a commensurable pair suffices to falsify UMP), such a strengthening would make a single (B_i)-test simultaneously confirmatory (if it passes, UMP holds across all three relations) and falsifying (if it fails, UMP fails). The asymmetry of the current framework is that falsification is single-test but confirmation requires all three; achieving symmetric single-test status for both falsification and confirmation is the open structural challenge.

Structural shape of the candidate strengthening. The natural auxiliary condition exploits the *composability* of commensurability witnesses across types. On a fully commensurable carrier (Remark 4.4.2), the three commensurability relations are not structurally independent: a sector V_α connected to V_β via a (C-emb) embedding is generically also connected to *some* V_γ via a (C-comp) relation (when $\alpha + \beta \not\equiv 0 \bmod 7$), and to *some* V_δ via a (C-ref) refinement. The structural shape of the converse strengthening would then be: given (B_i) on its corresponding (C-i) commensurability witnesses, plus the structural fact that *every* (C-j) witness ($j \neq i$) factors operationally through compositions of (C-i) witnesses with other operational structure already on the carrier, the constraint (B_i) propagates to UMP-restricted-to-(C-j).

Concretely, in the most tractable direction: (B2) (isosymmetric embedding consistency) plus the structural fact that every (C-comp) commensurability witness $\sigma_{\{\alpha,\beta\}}$ factors operationally as the composition of a (C-emb) embedding into the tensor-extension carrier with a substrate-character identification — would yield UMP-comp from UMP-emb. *A subtlety here: the (C-*

comp) witness $\sigma_{\alpha,\beta}$ lives between the original carrier and the tensor-extension carrier (a different carrier from $\mathcal{A}_\mathbb{C}$), while (C-emb) witnesses as defined in §4.1 live within a single carrier. The proposed factorization therefore requires the cross-carrier embedding structure — what counts as a (C-emb) witness between distinct admissible carriers, what its admissibility conditions are, and how character-matching identifications compose with such cross-carrier embeddings — to be defined precisely. This definitional work is part of the open structural content of Open Problem 8; the present paper does not supply it. The analogous shape for (B3) → (B1) and (B1) → (B2/3) routes would need to be worked out. The general structural pattern: *commensurability witnesses compose; if the auxiliary structure of the carrier (the operational data of §2) supports witness-composition, then a single (B_i) plus that auxiliary structure suffices*.

This sketch is *not* a proof; the witness-composition statements need to be made precise and the propagation rigorously argued. But it indicates where structural progress is available: the open question is whether the VERSF carrier's operational data is rich enough to support witness-composition in the required sense, and whether that composition propagates invariance with no leakage. If yes, Open Problem 8 closes with a single-test confirmation theorem; if no, the asymmetry between confirmation (three-test) and falsification (single-test) is intrinsic to the framework. Settling the question either way would be a meaningful structural advance.

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11. Conclusion

The previous paper (PAMV) established that admissible operational geometry plus the minimal-core probability axioms forces probability to take a sector-decomposed quadratic form (on sectors of dimension ≥ 2), with three structurally distinct bridging conditions — compositional multiplicativity, isosymmetric embedding, refinement stability — each independently closing the uniqueness argument. The structural coincidence of three convergent bridging routes was flagged as PAMV's Open Problem 2: do the three conditions reduce to a single deeper principle?

The present paper answers in the affirmative, with appropriate precision about what "deeper" means. We have introduced operational commensurability between coherence sectors (Definition 4.1, three relations supplying three structural witnesses) and the Universal Measure Principle stating that probability must be invariant under every commensurability witness (Definition 5.1, three operational invariance clauses). The central technical content:

- **The triple structural failure.** Theorem 3.2 establishes that sector-relative probability assignments (non-uniform $(C_\alpha)_\alpha$) simultaneously break compositional neutrality, embedding consistency, and refinement invariance, with all three failures arising from a single shared structural origin: probability depending on sector-identity rather than on operational geometry. The negative result motivates UMP as the unique positive principle repairing that shared origin.
- **UMP forces uniform quadraticity.** Theorem 5.3: under the minimal core plus UMP on a fully commensurable carrier, every per-sector function is quadratic (with $d_\alpha = 1$ quadraticity supplied by UMP via Lemma 5.2 / 5.2′ / 5.2″) and the quadratic constants

agree across all sectors (uniformity supplied by UMP across the commensurability graph).

- **The three bridging conditions are UMP-instances.** Theorem 6.2: each of (B1), (B2), (B3) follows from UMP applied to the corresponding commensurability witness; on a fully commensurable carrier the conjunction (B1) ∧ (B2) ∧ (B3) is logically equivalent to UMP, with UMP delivering conceptual and structural unification rather than logical strengthening (Remark 6.2.1). The principal novel content beyond PAMV is the joint-validity falsifier (UMP-test 1 of §9.2): under UMP, all three bridging conditions must hold jointly on a fully commensurable carrier; failure of any single (B_i) on a commensurable pair falsifies UMP.
- **The Born rule is the unique commensurable minimal-core measure.** Theorem 7.1: under the inherited operational structure plus UMP, $\mu(\psi) = \|\psi\|^2_{\mathbb{C}}$ and $P(\psi \to \phi) = |\langle \phi, \psi \rangle_{\mathbb{C}}|^2$ uniquely. Theorem 7.2: the Born rule is the unique minimal-core admissible measure compatible with operational commensurability.

The structural reframing this delivers is significant. The deepest content of the Born-rule reconstruction is *not* quadraticity within sectors of dimension ≥ 2, which is already largely forced by within-sector orthogonal additivity and continuity. It is *universality across sectors* (and the corresponding quadraticity on $d_\alpha = 1$ sectors via commensurability inheritance), which UMP supplies as the structural requirement that probability depend only on operational distinguishability geometry and not on coherence-sector identity.

The conditional status of the present result mirrors PAMV's. UMP is a structural input, not derived from absolute first principles; whether it can be derived from more primitive substrate-level structure — in particular from the packing-volume identification of PAMV Open Problem 10 — is Open Problem 1 of §10, where a candidate derivation sketch is offered. Resolving PAMV's Open Problem 10 would convert UMP from structural input to derived consequence, the cleanest possible closure of the present paper's structural status — and a structural priority for the next stage of the programme. The deeper substrate-origin questions — why the inherited VERSF carrier supplies full commensurability at all, why $\mathbb{Z}_7$, why the $K = 7$ closure architecture — remain as PAMV left them.

Essential novelty. Where PAMV identified the structural gap that admissible unitary invariance alone could not close, and supplied three independent ways to close it, the present paper identifies *one* principle that closes the gap and shows the three PAMV bridging conditions to be its three operational manifestations. The structural symmetry made visible:

Triple commensurability relations (C-comp), (C-emb), (C-ref) ↔ *Triple structural failures* (F1), (F2), (F3) under sector-relativity ↔ *Triple bridging witnesses* (B1), (B2), (B3) under UMP.

The triplet is the operational signature of one underlying structural requirement, not coincidence. The answer to PAMV's Open Problem 2 is the operational restatement of UMP, the structural unification it delivers (conceptual, structural, and falsifiability-sharpening on a fully commensurable carrier), and the structural reframing of universality — rather than quadraticity — as the deepest content of the Born-rule reconstruction.

The Born rule is not the unique quadratic measure on operational coherence geometry; it is the unique *commensurable* one. Where PAMV showed that within-sector quadraticity follows from operational additivity, the present paper shows that cross-sector universality follows from operational commensurability. The two together — quadraticity from minimal-core within-sector geometry on $\dim \geq 2$ sectors, and universality from UMP across the commensurability graph — uniquely specify the Born rule on the operational carrier. The reframing is structural: universality across operationally commensurable coherence sectors, not the squaring formula itself, is the genuinely substantive content of the Born-rule reconstruction. The three bridging conditions of PAMV converge because they are three operational windows onto a single underlying commensurability requirement; their convergence is not coincidence but structural reflection of common origin. Probability is universal operational distinguishability volume; what makes it universal is that operationally related sectors cannot consistently carry independent probability scales.

Appendix A. Detailed Functional-Equation Proofs

This appendix contains the full Pexider/Cauchy functional-equation manipulations supporting the structural sketches of Lemmas 5.2′ and 5.2″ in the main text. The arguments are standard applications of continuous-solution theory for functional equations (Aczél, *Lectures on Functional Equations and Their Applications*, Chapters 3–4; Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Chapter XIII). They are deferred to this appendix to keep the main text focused on the conceptual content rather than the technical machinery.

A.1 Full proof of Lemma 5.2′ ($d_\alpha = d_\beta = 1$ quadraticity from UMP-ref)

Setup. The (C-ref) relabeling W_θ on the 2-dimensional subspace $V_\alpha \oplus V_\beta$ is a continuous family of decomposition-admissible unitaries, with the explicit parametrization given in the proof of (F3) in Theorem 3.2:

$W_\theta e_\alpha = \cos(\theta) \cdot e_{\alpha'} + \sin(\theta) \cdot e_{\beta'}$, $W_\theta e_\beta = -\sin(\theta) \cdot e_{\alpha'} + \cos(\theta) \cdot e_{\beta'}$.

Under the sector-attachment convention (Theorem 3.2 proof of (F3), elevated to a well-posedness condition there), $F_{\alpha'} = F_\alpha$ and $F_{\beta'} = F_\beta$. UMP-ref applied to $\psi = e_\alpha$ gives $\mu(W_\theta e_\alpha) = \mu(e_\alpha) = F_\alpha(1)$ for every $\theta \in [0, \pi/2]$, so

$F_\alpha(\cos \theta) + F_\beta(\sin \theta) = F_\alpha(1)$ for all $\theta \in [0, \pi/2]$. (★)

(Continuity from axiom (A5) makes both sides continuous in θ; F_α and F_β are continuous and non-negative with $F_\alpha(0) = F_\beta(0) = 0$ by (A1).)

Step 1 (boundary equality). Setting $\theta = \pi/2$ in (★) gives $F_\alpha(0) + F_\beta(1) = F_\alpha(1)$, so $F_\beta(1) = F_\alpha(1)$. Denote this common value by $C := F_\alpha(1) = F_\beta(1) \geq 0$.

Step 2 (extension to rescaled inputs). UMP-ref applies not only to $\psi = e_\alpha$ at unit norm but to every state in $V_\alpha \oplus V_\beta$. In particular, apply UMP-ref to the rescaled state $\psi = \lambda \cdot e_\alpha$ for $\lambda \in (0, 1]$. Computing as in Step 1's argument:

$\|(W_\theta (\lambda \cdot e_\alpha))\{\alpha'\}\|^2_\mathbb{C} = \lambda^2 \cos^2\theta$, $\|(W_\theta (\lambda \cdot e_\alpha))\{\beta'\}\|^2_\mathbb{C} = \lambda^2 \sin^2\theta$,

so

$F_\alpha(\lambda \cos \theta) + F_\beta(\lambda \sin \theta) = F_\alpha(\lambda)$ for all $\theta \in [0, \pi/2]$, $\lambda \in (0, 1]$. ($\bigstar_\lambda$)

The corresponding rescaled-state equation applies to $\psi = \lambda \cdot e_\beta$ by symmetry:

$F_\beta(\lambda \cos \theta) + F_\alpha(\lambda \sin \theta) = F_\beta(\lambda)$ for all $\theta \in [0, \pi/2]$, $\lambda \in (0, 1]$. ($\bigstar'_\lambda$)

Step 3 (Pexider pair on the squared-argument variable). Introduce the change of variable $P(t) := F_\alpha(\sqrt{t})$ and $Q(t) := F_\beta(\sqrt{t})$ for $t \in [0, 1]$. (We rename the functions P, Q rather than φ, ψ to avoid notational clash with the state vector ψ that appears throughout the surrounding theory.) Substituting $t = \lambda^2 \cos^2\theta$, $s = \lambda^2 \sin^2\theta$ (so $t + s = \lambda^2$) into ($\bigstar_\lambda$) gives

$P(t) + Q(s) = P(t + s)$ for all $t, s \geq 0$ with $t + s \leq 1$. ($\diamond$)

Equation ($\diamond'$) arises analogously from UMP-ref applied to $\psi = \lambda \cdot e_\beta$ rather than $\psi = \lambda \cdot e_\alpha$: computing the W_θ-image of e_β using the second column of W_θ gives $\|(W_\theta (\lambda \cdot e_\beta))\{\alpha'\}\|^2_\mathbb{C} = \lambda^2 \sin^2\theta$ and $\|(W_\theta (\lambda \cdot e_\beta))\{\beta'\}\|^2_\mathbb{C} = \lambda^2 \cos^2\theta$, so ($\bigstar'_\lambda$) reads $F_\beta(\lambda \cos \theta) + F_\alpha(\lambda \sin \theta) = F_\beta(\lambda)$. The same substitution $t = \lambda^2 \cos^2\theta$, $s = \lambda^2 \sin^2\theta$ converts this into

$Q(t) + P(s) = Q(t + s)$ for all $t, s \geq 0$ with $t + s \leq 1$. ($\diamond'$)

The two equations ($\diamond$) and ($\diamond'$) form a Pexider-type system on [0, 1]. (They are not related by $\alpha \leftrightarrow \beta$ relabeling of ($\diamond$); they are genuinely the two different input states e_α and e_β acted on by the same W_θ.)

Step 4 (reduction to single Cauchy equation). Define $D(u) := P(u) - Q(u)$ on [0, 1]. Subtracting ($\diamond$) − ($\diamond'$):

$[P(t) + Q(s)] - [Q(t) + P(s)] = P(t + s) - Q(t + s)$, $D(t) - D(s) = D(t + s)$ for all $t, s \geq 0$ with $t + s \leq 1$. ($\bigstar\bigstar\bigstar$)

Setting $t = s$ in ($\bigstar\bigstar\bigstar$) gives $D(2t) = 0$ for all $t \in [0, 1/2]$, hence $D(u) = 0$ for all $u \in [0, 1]$; therefore $P \equiv Q$ on [0, 1].

Substituting $Q = P$ into ($\diamond$):

$P(t) + P(s) = P(t + s)$ for all $t, s \geq 0$ with $t + s \leq 1$. ($\diamond^*$)

This is the additive Cauchy equation on $[0, 1]$.

Step 5 (continuous solution). P is continuous on $[0, 1]$ (from F_α continuous and $\sqrt{\cdot}$ continuous). By Kuczma's theorem on continuous additive functions on $[0, \infty)$ (the same theorem invoked in PAMV Theorem 6.1(i); the restriction to $[0, 1]$ is standard), every continuous additive P on $[0, 1]$ satisfies $P(t) = Kt$ for some constant $K \in \mathbb{R}$. By positivity (A1), $K \geq 0$. The boundary condition $P(1) = C$ (from Step 1, where $F_\alpha(1) = C$, so $P(1) = F_\alpha(\sqrt{1}) = F_\alpha(1) = C$) gives $K = C$.

Step 6 (conclusion on [0, 1]). Therefore $P(t) = Ct$ for $t \in [0, 1]$, which gives $F_\alpha(r) = P(r^2) = Cr^2$ for $r \in [0, 1]$. By $Q \equiv P$, $F_\beta(r) = Cr^2$ on the same range.

Step 7 (extension to r > 1). UMP-ref's invariance applies to states in the *full* subspace $V_\alpha \oplus V_\beta$ with no restriction on norm — Definition 5.1's UMP-ref clause asserts $\mu(W\psi) = \mu(\psi)$ "for every state $\psi \in V_\alpha \oplus V_\beta$", which includes states of arbitrary norm. For $\psi = \lambda \cdot e_\alpha$ with $\lambda \geq 1$, the same argument as Step 2 then applies with the unrestricted-norm content of UMP-ref, giving equation ($\bigstar_\lambda$) at $\lambda \geq 1$; the substitution $t' = \lambda^2 \cos^2\theta$ ranges over $[0, \lambda^2]$, extending P additively to $[0, \lambda^2]$. Continuity of F_α and the Cauchy equation extend to give $P(t) = Ct$ for all $t \geq 0$, hence $F_\alpha(r) = Cr^2$ for all $r \geq 0$. Same for F_β.

Reference. The Pexider-pair-to-Cauchy reduction used in Step 4 is the standard treatment of ($\diamond$) and ($\diamond'$); see Aczél, *Lectures on Functional Equations and Their Applications*, Chapter 3 §3.1 (the Cauchy equation) and Chapter 4 (Pexider equations and their reductions). The subtraction step that eliminates the P/Q distinction and reduces the pair to a single Cauchy equation is standard for Pexider-type systems with shared boundary conditions. Positivity (A1) selects the $K \geq 0$ (equivalently $C \geq 0$) branch among the formal real solutions.

A.2 Full proof of Lemma 5.2″ ($d_\alpha = d_\beta = 1$ quadraticity from UMP-comp)

Setup. We proceed in three steps, avoiding any forward dependence on Lemma 5.0 for the $d_\alpha = 1$ sectors and using UMP-comp's multiplicative content directly.

Step 1 (unit-norm coefficient relation). For each sector V_γ on the character lattice, the value $F_\gamma(1)$ is a single well-defined non-negative number — call it $\kappa_\gamma := F_\gamma(1) \geq 0$. This is well-defined on every sector regardless of whether F_γ is quadratic on $r \neq 1$ (the minimal core supplies continuity and $F_\gamma(0) = 0$, so $F_\gamma(1)$ exists as a value of the continuous F_γ at the input $r = 1$).

UMP-comp applied to unit-norm $\psi \in V_\alpha$, $\phi \in V_\beta$ gives, using the tensor-extension multiplicativity of Definition 5.1:

$$\mu_{\text{tensor}}(\psi \otimes \phi) = \mu(\psi) \cdot \mu(\phi) = \kappa_\alpha \cdot \kappa_\beta.$$

The right-hand side of UMP-comp's identification is $\mu_{\text{original}}(\sigma_{\{\alpha,\beta\}}(\psi \otimes \phi))$ at unit norm, which equals $\kappa_{\{\alpha+\beta \bmod 7\}}$. Combining:

$\kappa_{\alpha+\beta \bmod 7} = \kappa_\alpha \cdot \kappa_\beta$ for all $\alpha, \beta \in \{1, \ldots, 6\}$ with $\alpha + \beta \not\equiv 0 \bmod 7$. (♦)

This is the same multiplicative-homomorphism relation as Lemma 5.0's (♣), but at unit norm and using only the well-defined κ_γ values rather than presupposing quadraticity. Applying Lemma 5.0a (seventh-power closure on the $\mathbb{Z}_7$-character lattice) to the family (κ_γ), which satisfies the (♠) hypothesis as (♦), gives $\kappa_\gamma = 1$ for all γ. Therefore $F_\gamma(1) = 1$ on every sector of the character lattice, including the $d_\gamma = 1$ sectors V_α and V_β.

Step 2 (extension to non-unit norm via multiplicative functional equation). Step 1 fixes $F_\gamma(1) = 1$ but does not yet fix F_γ on $r \neq 1$. We now apply UMP-comp at non-unit-norm composites. For $\psi \in V_\alpha$, $\phi \in V_\beta$ with $\|\psi\|_\mathbb{C} = r$, $\|\phi\|_\mathbb{C} = s$ (both ≥ 0), tensor-extension multiplicativity gives

$$\mu_{\text{tensor}}(\psi \otimes \phi) = F_\alpha(r) \cdot F_\beta(s),$$

and UMP-comp's identification gives

$$\mu_{\text{original}}(\sigma_{\alpha,\beta}(\psi \otimes \phi)) = F_{\alpha+\beta \bmod 7}(rs)$$

(using $\|\sigma_{\alpha,\beta}(\psi \otimes \phi)\|_\mathbb{C} = rs$ from the isotypic identification preserving norms multiplicatively). Therefore

$$F_\alpha(r) \cdot F_\beta(s) = F_{\alpha+\beta \bmod 7}(rs) \text{ for all } r, s \geq 0. \quad (◊)$$

This is a multiplicative functional equation relating the F_γ across the character lattice.

Step 3 (continuous solution: uniform exponent on the character lattice). Take logarithms after restricting to $r, s > 0$: define $g_\gamma(u) := \log F_\gamma(e^u)$ for $\gamma \in \{1, \ldots, 6\}$ and $u \in \mathbb{R}$ (well-defined where $F_\gamma > 0$). Equation (◊) becomes

$$g_\alpha(u) + g_\beta(v) = g_{\alpha+\beta \bmod 7}(u + v) \text{ for all } u, v \in \mathbb{R}, \text{ all } \alpha, \beta \text{ with } \alpha + \beta \not\equiv 0 \bmod 7. \quad (◊′)$$

This is the Cauchy Pexider multiplicative-additive equation across the character lattice. (Note on domain: (◊) is established for $r, s \geq 0$; the logarithmic transformation requires $r, s > 0$, and (◊′) holds on all $(u, v) \in \mathbb{R}^2$ via $u = \log r$, $v = \log s$. The $v = 0$ substitution used below corresponds to $s = 1$, i.e. unit-norm ϕ, which is admissible. The boundary $r = 0$ or $s = 0$ corresponds to $F_\gamma(0) = 0$ by axiom (A1) and is handled separately in Step 5.) Continuous solutions to (◊′) on $\mathbb{R}$ are characterized by linearity: $g_\gamma(u) = k_\gamma u + c_\gamma$ for constants $k_\gamma \in \mathbb{R}$ and $c_\gamma \in \mathbb{R}$. Boundary condition $F_\gamma(1) = 1$ (Step 1) gives $g_\gamma(0) = \log F_\gamma(1) = \log 1 = 0$, hence $c_\gamma = 0$ for every γ.

Substituting $g_\gamma(u) = k_\gamma u$ into (◊′): $k_\alpha u + k_\beta v = k_{\alpha+\beta \bmod 7}(u + v)$ for all $u, v \in \mathbb{R}$. Setting $v = 0$: $k_\alpha u = k_{\alpha+\beta \bmod 7} u$ for all u, hence $k_\alpha = k_{\alpha+\beta \bmod 7}$ for all α, β with $\alpha + \beta \not\equiv 0 \bmod 7$. Fixing α and varying β over $\{1, \ldots, 6\} \setminus \{7-\alpha \bmod 7\}$ shows k_α equals k_γ for the corresponding range of γ; iterating over all $\alpha \in \{1, \ldots, 6\}$ forces $k_1 = k_2 = \ldots = k_6$, call the common value k. Substituting back into (◊′): $k(u + v) = k(u + v)$, which holds identically. So we have established

$F_\gamma(r) = r^k$ uniformly across the character lattice, for some single exponent $k \in \mathbb{R}$.

The unit-norm normalization $F_\gamma(1) = 1$ (Step 1) is consistent with any k since $1^k = 1$, and so does not fix k by itself.

Step 4 (fixing $k = 2$). The exponent k requires one further structural input to be fixed at $k = 2$. There are two cases:

Case (i) — Generic case. If at least one sector V_δ in the character lattice has $d_\delta \geq 2$, then PAMV Theorem 6.1(i) (the within-sector Cauchy-equation argument from minimal-core orthogonal additivity) forces $F_\delta(r) = C_\delta r^2$. Combined with $F_\delta(1) = \kappa_\delta = 1$ from Step 1, this gives $C_\delta = 1$, hence $F_\delta(r) = r^2$. So on the $d_\delta \geq 2$ sector, $g_\delta(u) = \log(e^{2u}) = 2u$, hence $k_\delta = 2$. The uniformity established in Step 3 — that k is a single exponent shared by all V_γ on the character lattice — then transports $k = 2$ from V_δ to every other V_γ on the character lattice, including the $d_\gamma = 1$ sectors V_α and V_β. Therefore $F_\gamma(r) = r^2$ for all $\gamma \in \{1, ..., 6\}$.

Case (ii) — Minimal-multiplicity case. If every sector on the character lattice has $d_\gamma = 1$, no within-sector Cauchy-equation argument is available to anchor the exponent. In this case, Step 3's analysis establishes $F_\gamma(r) = r^k$ uniformly with the exponent k undetermined; UMP-comp alone, plus the minimal core, does not fix $k = 2$ in this case. An additional structural input — for instance, a second-moment normalization, or the (C-emb) embedding into some $d_\gamma \geq 2$ sector elsewhere in the carrier outside the character lattice neighborhood of α and β — would be required.

Scope. For the VERSF substrate as inherited from URHG, the operational coherence-sector decomposition supplies at least some $d_\gamma \geq 2$ sectors (this is part of the inherited structure, Remark 4.4.2's setup, where $\mathcal{A}_\mathbb{C}$ has dimension $d_{op}^{\mathbb{C}} \geq 2$ and the sectors are operationally distinguished irreducible blocks). Case (i) therefore applies in the VERSF setting, and Lemma 5.2″ delivers $F_\gamma(r) = r^2$ without needing additional input. The minimal-multiplicity edge case (ii) is flagged here for completeness and to delimit Lemma 5.2″'s scope honestly; for a carrier in which every sector were one-dimensional, an additional structural input would be required.

Step 5 (extension to $r \geq 0$). In Case (i): $k = 2$ is fixed, so $g_\gamma(u) = 2u$, i.e. $F_\gamma(e^u) = e^{2u} = (e^u)^2$, i.e. $F_\gamma(r) = r^2$ for all $r > 0$ on every sector of the character lattice, including the $d_\gamma = 1$ sectors V_α and V_β. Extending continuously to $r = 0$ via $F_\gamma(0) = 0$ (axiom (A1)): $F_\alpha(r) = F_\beta(r) = r^2$ for all $r \geq 0$. In Case (ii), the lemma's conclusion holds with k determined by whatever additional structural input is supplied; without such input, the lemma establishes the uniform-power form $F_\gamma(r) = r^k$ but not $k = 2$.

Reference. The functional equation (◊) and its logarithm (◊′) is the standard Cauchy Pexider multiplicative-additive equation, treated in Aczél, *Lectures on Functional Equations and Their Applications*, Chapter 3 (Cauchy equation in multiplicative form) and Chapter 4 (Pexider variants). Continuous positive solutions on $(0, \infty)$ are characterized by power functions; in Case (i), normalization by a $d_\gamma \geq 2$ within-sector Cauchy-equation argument fixes the exponent to 2.

Non-circularity note. The proof of Lemma 5.2″ does not invoke Lemma 5.0 and does not presuppose F_γ to be quadratic on d_γ = 1 sectors. Step 1 operates entirely on the unit-norm values κ_γ = F_γ(1), which are well-defined for every continuous F_γ; the seventh-power closure at unit norm fixes κ_γ = 1 without circularity. Step 2 extends this to non-unit norm using UMP-comp's tensor-extension multiplicativity, which is a structural input from Definition 5.1 (Remark 5.1.2), not a derived property of F_γ. Step 3 invokes only continuous-solution theory for the Cauchy-Pexider multiplicative-additive equation, which is standard. Step 4 (Case (i)) uses PAMV Theorem 6.1(i) on a d_δ ≥ 2 sector of the character lattice, which is a forward application of the minimal core to a higher-dimensional sector — not a backward dependence on quadraticity at d = 1. The argument is therefore self-contained on the character lattice, with Case (i) covering the VERSF setting and Case (ii) honestly flagged as needing additional input.