Unified Refinement Hilbert Geometry in VERSF

Operational Inner Products, Projection Dynamics, and the Common Origin of Quantum and Geometric Transport

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A General Reader's Summary

Why should quantum mechanics and the geometric continuum share a carrier?

Quantum mechanics describes evolution that preserves distinguishability — every state remains as distinguishable from every other state as it was at the start. Wavefunctions rotate in Hilbert space without ever collapsing two trajectories into one. The geometry of the macroscopic continuum, by contrast, describes how distinguishable separations bend — how distances stretch and compress as one moves through regions of higher or lower energy density. These two structures are usually treated as belonging to different worlds, joined only with difficulty (or not at all) by a quantum theory of geometry.

The previous papers of this programme — *Operational Geometry in the VERSF Framework* (OG) and *Operational Curvature and Geodesic Structure in the VERSF Framework* (OC) — established that the admissible subspace $\mathcal{A}$ = Im(Ω_max) carries an intrinsic finite-dimensional real Hilbert geometry, and that gradients of the operational distinguishability density conformally deform this flat reference into a curved Riemannian geometry. The kinematics and dynamics of the operational geometry are now in place.

The present paper observes that the *same* finite operational Hilbert space supports two structurally distinct but compatible transport regimes, once a canonical complex structure derived from $\mathbb{Z}_7$-equivariance is taken into account:

Reversible pre-commitment transport, represented by strongly continuous one-parameter orthogonal groups that preserve the operational inner product. After endowing $\mathcal{A}$ with the $\mathbb{Z}_7$-induced complex structure, these become unitary groups with self-adjoint generator — the operational-Hilbert origin of quantum-type dynamics.

Irreversible commitment projection, represented by non-expansive idempotent refinement maps, of which Ω_max is the universal admissibility projector. This is the operational-Hilbert origin of fact formation, sector selection, and finite distinguishability packing.

The central structural result is the canonical polar decomposition R = U · |R| of every admissible linear refinement transport: the unitary factor U carries quantum-type evolution; the positive

factor |R| carries commitment-type contraction. A third regime — conformal deformation of the operational metric by distinguishability density — supplies geometric curvature, the subject of OC.

This does not derive a full quantum-gravity theory. What it establishes is that the carrier substrate is one layered structure {M_op ⊆ $\mathcal{A}$ ⊆ $\mathcal{A}$_$\mathbb{C}$}, the transport decomposition is canonical, and the three regimes (reversible quantum, irreversible commitment, curvature-generating deformation) are three faces of one operational Hilbert geometry. The paper supplies the mathematical bridge between the VERSF quantum reconstruction programme and the VERSF refinement-geometry programme.

The technical body follows. Readers without a mathematical-physics background may wish to read §1 (Introduction), §9 (Consequence), and §13 (Unified Theorem), which together convey the essential content in continuous prose, before consulting the theorems and proofs.

Abstract

Prior VERSF work established that the admissible closure manifold carries a finite operational Hilbert geometry. The admissible subspace $\mathcal{A}$ = Im(Ω_max) inherits a real finite-dimensional Hilbert structure ($\mathcal{A}$, ⟨·,·⟩*op) with orthogonal spectral-channel decomposition* $\mathcal{A}$ = ⊕{α=1}^{N_spec} V_α, and Ω_max acting as orthogonal projection. The subsequent paper *Operational Curvature and Geodesic Structure* (OC) showed that local variation of the operational density ρ_op conformally deforms the flat reference into a curved Riemannian metric g_op = ρ_op^{2/d_op} · ⟨·,·⟩_op with bounded curvature.

The present paper develops the next layer: **refinement Hilbert dynamics**.

The argument has three structural inputs:

(A) A canonical complex structure derived from $\mathbb{Z}_7$-equivariance. Theorem 3.4 establishes that the non-trivial $\mathbb{Z}_7$-channels in the spectral decomposition $\mathcal{A}$ = ⊕_α V_α carry a canonical complex structure J induced by the substrate $\mathbb{Z}_7$-action. This converts ($\mathcal{A}$_nt, ⟨·,·⟩_op, J) into a complex Hilbert space ($\mathcal{A}$_$\mathbb{C}$, ⟨·,·⟩_$\mathbb{C}$) by a substrate-derived procedure rather than a mechanical tensor product. The complex structure is forced by the substrate, not chosen.

(B) Polar decomposition of admissible refinement transport. Theorem 8.1 establishes that every admissible linear refinement map R : $\mathcal{A}$_$\mathbb{C}$ → $\mathcal{A}$_$\mathbb{C}$ decomposes uniquely (by finite-dimensional polar decomposition) as R = U · |R| with U a partial isometry and |R| positive semidefinite, and that $\mathbb{Z}_7$-equivariance of R propagates to both U and |R| (Proposition 8.4). This canonical separation isolates the reversible content (U) from the irreversible/contractive content (|R|).

(C) Conformal deformation of the operational metric. Theorem 10.1 gives the scalar curvature of the OC conformal deformation g_op^{(ρ)} = ρ^{2/d_op} · ⟨·,·⟩_op in closed form via the standard conformal transformation law, generating curvature from density gradients.

Combining these inputs, every admissible refinement transport on $\mathcal{A}_{\mathbb{C}}$ decomposes into two structurally distinct regimes, with two additional structural consequences of the polar decomposition (entropy monotonicity, transport–projection and projection–projection noncommutativity) and two further structural properties of the conformal-deformation regime (closed-form Ricci tensor, dynamical metric flow):

1. **Reversible pre-commitment transport** (the U factor), generating unitary one-parameter groups with self-adjoint generator H satisfying an operational Schrödinger equation $i \cdot d\psi/d\tau = H\,\psi$ — the operational-Hilbert origin of quantum-type evolution (Theorems 4.1, 5.1, Corollary 6.1).
2. **Irreversible commitment projection** (the |R| factor, in the orthogonal-projection special case), removing distinguishability content orthogonal to a committed admissible subspace — the operational-Hilbert origin of fact formation, sector selection, and finite distinguishability packing (Theorem 7.1).
3. **Entropy monotonicity from the polar decomposition** (Theorem 7.2): under an operational entropy-monotonicity hypothesis, the U-factor is entropy-preserving and the |R|-factor is entropy-non-decreasing, supplying the operational-Hilbert origin of the second-law arrow of refinement.
4. **Transport–projection and projection–projection noncommutativity** (Theorems 9.2, 9.3): reversible transport and commitment projection do not commute unless U preserves $\mathrm{Im}(P) \oplus \ker(P)$; two projections do not commute unless they admit a joint orthogonal decomposition. This supplies the operational-Hilbert origin of measurement-evolution incompatibility and commitment incompatibility.
5. **Tensorial curvature and dynamical metric flow** (Theorems 10.2, 10.3): the conformally-deformed metric has Ricci tensor given in closed form with a natural anisotropic/isotropic decomposition, and the metric itself evolves dynamically under refinement transport via a position-dependent conformal flow coupled to density and velocity evolution.

The central result is the **Unified Refinement Hilbert Geometry Theorem** (Theorem 13.1):

The layered carrier structure $\{M_{op} \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathbb{C}}\}$ supports three structurally distinct transport regimes — reversible isometric transport on $\mathcal{A}_{\mathbb{C}}$, commitment projection on $\mathcal{A}_{\mathbb{C}}$ (intersecting the real structure via Ω_{max}), and conformal-deformation curvature on M_{op} — corresponding to the unitary factor of polar decomposition, the positive factor, and conformal deformation of the metric. The polar decomposition carries entropy monotonicity (via |R|) and order-dependence (via $[U, P] \neq 0$); the conformal-deformation regime carries closed-form Ricci structure and dynamical metric flow under refinement.

The paper supplies the mathematical bridge between the VERSF quantum reconstruction programme and the VERSF refinement-geometry programme: both programmes act on the same layered carrier, with their dynamical content captured by the polar decomposition (with its two

structural consequences) and the conformal-deformation regime (with its tensorial structure and dynamical flow).

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1. Introduction

The operational geometry paper (OG) established that admissibility does not merely count physical sectors — it geometrises them. The closure manifold carries an intrinsic finite operational Hilbert geometry with inner product, metric, dimension, measure, volume, compactness, and non-expansive projection structure. The follow-up paper (OC) developed the dynamics of distinguishability-density deformation, showing that the operational density ρ_op conformally deforms the flat reference into a curved Riemannian metric, with curvature generated by density gradients and bounded pointwise by finite packing.

The present paper asks: what *other* dynamics does the operational Hilbert geometry support?

The answer has two layers. First, the substrate $\mathbb{Z}_7$-equivariance forces a canonical complex structure on the non-trivial spectral channels of $\mathcal{A}$ (§3). This is not a free choice or a convenience: real $\mathbb{Z}_7$-equivariant Hilbert spaces with no trivial $\mathbb{Z}_7$-channels carry a uniquely

determined complex structure derived from the $\mathbb{Z}_7$-action, and the complexified Hilbert space $(\mathcal{A}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}}, J)$ is the natural carrier for $\mathbb{Z}_7$-equivariant linear transport. Second, on this complex carrier, every admissible linear refinement transport admits a canonical finite-dimensional polar decomposition $R = U \cdot |R|$ (§8). The unitary factor U gives reversible quantum-type evolution; the positive factor $|R|$ gives irreversible commitment-type contraction.

These two structural facts — substrate-induced complex structure + canonical polar decomposition — together with the conformal-deformation regime of OC and four additional structural consequences (entropy monotonicity, transport-projection noncommutativity, explicit Ricci structure, dynamical metric flow), give the unified picture:

Quantum transport and commitment transport are the two canonical operator-theoretic factors of admissible refinement transport on the operational Hilbert space (after the $\mathbb{Z}_7$-induced complex structure is taken into account). Geometric curvature transport is conformal deformation of the same Hilbert metric by the operational distinguishability density. The three regimes share one layered carrier $\{M_{op} \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathbb{C}}\}$. Entropy monotonicity is carried by the positive factor of the polar decomposition (supplying the operational-Hilbert origin of the second-law arrow). Order-dependence between transport and commitment is supplied by their generic non-commutativity (the operational-Hilbert origin of measurement-evolution incompatibility). And the operational metric itself flows dynamically under refinement transport (the structural precursor of macroscopic geometric flow).

This is not yet a derivation of full quantum mechanics or full general relativity. It establishes the common carrier space, the substrate-derivation of the complex structure on which quantum dynamics lives, the canonical transport decomposition from which any full theory would have to be built, the closed-form Ricci tensor of the conformally-deformed operational geometry (the tensorial precursor of Einstein-like field equations), and the joint flow of density, entropy, and metric under refinement transport (the structural precursor of macroscopic geometric flows). §14 sketches what the Born rule still requires; §15 sketches what the geometric programme still requires; §17 lists open problems including remaining structural inputs not yet derived from substrate primitives.

The paper proceeds in six stages. §2–§3 collect inherited results, definitions, and the canonical-complex-structure theorem. §4–§6 develop the reversible regime: orthogonal/unitary structure, generator structure, and the operational Schrödinger equation. §7 develops the commitment-projection regime, including entropy monotonicity (§7.2). §8 establishes the polar decomposition and its equivariance preservation. §9 states the structural consequence, including the noncommutativity of transport and projection (§9.2). §10 connects to the curvature regime of OC, with full Ricci tensor (§10.2) and dynamical metric flow (§10.3). §11–§12 develop interpretation and projection-curvature compatibility. §13 states the unified theorem. §14–§18 discuss reconstruction, falsifiability, open problems, and conclusion.

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2. Inherited Results

We inherit the following from the operational geometry paper (OG) and the operational curvature paper (OC).

2.1 Operational Hilbert Space (OG Theorem 4.0)

The admissible subspace $\mathcal{A} = \mathrm{Im}(\Omega_{max})$ carries a real finite-dimensional Hilbert inner product $\langle \cdot,\cdot \rangle_{op}$. Thus $(\mathcal{A}, \langle \cdot,\cdot \rangle_{op})$ is a finite-dimensional real Hilbert space.

2.2 Orthogonal Channel Decomposition (OG Theorem 4.0)

The admissible space decomposes orthogonally as

$$\mathcal{A} = \bigoplus_{\alpha=1}^{N_{spec}} V_\alpha,$$

with every $x \in \mathcal{A}$ written uniquely as $x = \sum_\alpha x_\alpha$ with $x_\alpha \in V_\alpha$, and $\|x\|_{op}^2 = \sum_\alpha \|x_\alpha\|_{op}^2$. Each V_α carries a real irreducible representation of the substrate $\mathbb{Z}_7$-action (OG §2.5).

2.3 Orthogonal Projection (OG Theorem 5.1)

The operator Ω_{max} satisfies $\Omega_{max}^2 = \Omega_{max}$ and $\Omega_{max}^\dagger = \Omega_{max}$. Hence Ω_{max} is the orthogonal projection onto $\mathcal{A}$, and is non-expansive on the ambient closure-state-space.

2.4 Operational Metric (OG Theorem 4.1)

The operational metric is $D_{op}(x, y) = \|x - y\|_{op}$, making $(\mathcal{A}, D_{op})$ a d_{op}-dimensional Euclidean Hilbert space.

2.5 $\mathbb{Z}_7$-Equivariance of Closure Dynamics (OG §2.5)

The substrate $\mathbb{Z}_7$-action on the closure-state-space restricts to a $\mathbb{Z}_7$-action on $\mathcal{A}$ that commutes with Ω_{max} and decomposes $\mathcal{A}$ into orthogonal channels V_α according to real irreducible $\mathbb{Z}_7$-types.

2.6 Finite Dimension and Volume (OG Theorems 4.1, 8.1)

The operational dimension $d_{op} = \dim \mathcal{A} = \sum_\alpha \dim V_\alpha$ is finite, and $\mathrm{Vol}_{op}(M) < \infty$.

2.7 Finite Packing (OG Theorem 10.1)

The admissible packing number satisfies $P(M) \leq \mathrm{Vol}_{op}(M) / \Delta_{op}^{d_{op}}$, with $\Delta_{op} > 0$ the universal distinguishability quantum.

2.8 Conformal-Deformation Curvature (OC Theorem 5.0, Theorem 8.1)

The operational distinguishability density ρ_op (smoothed packing density on M_op at coarse-graining scale ε ≥ Δ_op) conformally deforms the flat reference inner product into

g_op(x) = ρ_op(x)^{2/d_op} · ⟨·,·⟩_op,

whose Riemann curvature tensor R_op is given in closed form by derivatives of ln ρ_op, vanishing when ρ_op is constant (in d_op ≥ 3) and bounded pointwise by ‖R_op‖ ≤ C / Δ_op².

These results are taken as inputs.

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3. Definitions and the Canonical Complex Structure

We adopt all definitions of OG and OC. The following definitions and the canonical-complex-structure theorem are new to the present paper.

3.1 Refinement parameter

Definition 3.1 (Refinement parameter for reversible transport). Throughout this paper, the refinement parameter τ ranges over $\mathbb{R}$ when applied to reversible transports (rather than [0, ∞) as in OC Definition 3.1, where it indexed irreversible commitment-ledger advance). The bidirectional extension τ ∈ $\mathbb{R}$ is *intrinsic* to reversibility: a reversible transport requires both forward and backward evolution. The same letter τ is therefore used for two structurally distinct things in the broader VERSF programme — irreversible τ ≥ 0 evolution in OC, and reversible τ ∈ $\mathbb{R}$ evolution here. Context (or explicit bidirectionality) disambiguates.

3.2 Admissibility, reversibility, and commitment projection

Definition 3.2 (Admissible linear refinement map). An **admissible linear refinement map** is a bounded real-linear operator R : $\mathcal{A} \to \mathcal{A}$ satisfying:

1. **Admissibility preservation:** R($\mathcal{A}$) ⊆ $\mathcal{A}$.
2. **Non-expansiveness:** ‖Rx − Ry‖_op ≤ ‖x − y‖_op for all x, y ∈ $\mathcal{A}$.
3. **$\mathbb{Z}_7$-equivariance:** R commutes with the substrate $\mathbb{Z}_7$-action of OG §2.5.

Linearity makes non-expansiveness equivalent to operator-norm bound ‖R‖_op ≤ 1.

Remark 3.2.1 (Scalar source vs operator source). OC Remark 5.2.2 used $\mathbb{Z}_7$-equivariance to restrict metric deformations to the conformal class: a *scalar* distinguishability-density source ρ_op acting on the flat reference inner product cannot distinguish $\mathbb{Z}_7$-equivalent channels, so the most general $\mathbb{Z}_7$-equivariant deformation by a scalar field is conformal. The present Definition 3.2 imposes $\mathbb{Z}_7$-equivariance on *operator* refinement transports R, not on a scalar source. Operators are not bound by the channel-symmetric-response principle that applies to scalar sources: a $\mathbb{Z}_7$-equivariant operator may freely couple channels carrying equivalent $\mathbb{Z}_7$-irreps (e.g.

by intertwining the action between two copies of the same irrep) and may act by independent isometries on each channel.

Concretely, the algebra of $\mathbb{Z}_7$-equivariant linear maps on $\mathcal{A} = \bigoplus_\alpha V_\alpha$ is, by Schur's lemma, the direct product over isomorphism classes of irreducible $\mathbb{Z}_7$-summands of full endomorphism algebras of the corresponding isotypic components. The condition is restrictive (no coupling between V_α and V_β carrying inequivalent $\mathbb{Z}_7$-irreps) but not as restrictive as the conformal restriction for metric deformations. The two papers' uses of $\mathbb{Z}_7$-equivariance are therefore consistent: scalar metric sources are forced to act by conformal rescaling (channel-uniform response); operator transports are forced to be Schur-block-diagonal (channel-respecting but not channel-uniform).

Definition 3.3 (Reversible admissible transport). A **reversible admissible transport** is a one-parameter family $\{U(\tau) : \tau \in \mathbb{R}\}$ of admissible linear refinement maps such that:

1. $U(0) = I$;
2. $U(\tau + \sigma) = U(\tau)\,U(\sigma)$ for all $\tau, \sigma \in \mathbb{R}$;
3. $\tau \mapsto U(\tau)\,x$ is continuous for every $x \in \mathcal{A}$ (real strong continuity);
4. $U(\tau)$ is invertible for every $\tau \in \mathbb{R}$;
5. $U(\tau)$ preserves the operational inner product: $\langle U(\tau)\,x, U(\tau)\,y\rangle_{op} = \langle x, y\rangle_{op}$ for all $x, y \in \mathcal{A}$.

The inner-product-preservation condition (item 5) is stronger than non-expansiveness (Definition 3.2 condition 2) and stronger than mere norm preservation. On a real Hilbert space norm preservation implies inner-product preservation by polarisation, so on $(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$ the distinction is vacuous; on the complexification $\mathcal{A}_{\mathbb{C}}$ introduced in §3.4 below, norm preservation alone characterises antiunitary as well as unitary operators, and the inner-product condition is what selects the unitary case (cf. Remark 4.1.2).

Definition 3.4 (Commitment projection). A **commitment projection** is an admissible linear refinement map P satisfying $P^2 = P$. If additionally $P^\dagger = P$, then P is an **orthogonal commitment projection**. The universal admissibility projector Ω_{max} is the canonical orthogonal commitment projection of OG.

3.3 The canonical complex structure from $\mathbb{Z}_7$-equivariance

The transition to a complex Hilbert space is the load-bearing step for the quantum-type results of §4–§6. We do not introduce the complex structure by mechanical tensor product. Instead, we establish that the substrate $\mathbb{Z}_7$-equivariance *forces* a canonical complex structure on the non-trivial spectral channels of $\mathcal{A}$.

Theorem 3.5 (Canonical complex structure from $\mathbb{Z}_7$-equivariance). *Decompose $\mathcal{A} = \mathcal{A}_{triv} \oplus \mathcal{A}_{nt}$, where $\mathcal{A}_{triv} \subseteq \mathcal{A}$ carries the trivial $\mathbb{Z}_7$-representation (fixed-vector subspace of the $\mathbb{Z}_7$-action) and $\mathcal{A}_{nt}$ is its $\langle\cdot,\cdot\rangle_{op}$-orthogonal complement. Then:*

(i) The non-trivial spectral channels satisfy $\mathcal{A}_{nt} = \bigoplus\{\alpha : V_\alpha \text{ non-trivial}\} V_\alpha$. Each non-trivial real-irreducible V_α has even real dimension and decomposes under $\mathbb{C}$-coefficient extension as a sum of two complex-conjugate $\mathbb{Z}_7$-characters $\chi_k \oplus \chi\{-k\}$ for some $k \in \{1, 2, 3\}$ (where $-k$ is taken mod 7).

(ii) On each non-trivial V_α, the substrate $\mathbb{Z}_7$-action defines a canonical complex structure $J_\alpha : V_\alpha \to V_\alpha$ with $J_\alpha^2 = -I_{V_\alpha}$, $J_\alpha \in \mathrm{End}_{\mathbb{R}}(V_\alpha)$, and J_α commuting with the $\mathbb{Z}_7$-action.

(iii) The collection $\{J_\alpha\}\{\alpha \text{ non-trivial}\}$ extends to a single complex structure J on $\mathcal{A}_{nt}$ with $J^2 = -I\{\mathcal{A}_{nt}\}$, making $(\mathcal{A}_{nt}, \langle\cdot,\cdot\rangle_{op}, J)$ a complex Hilbert space of complex dimension $(\dim_{\mathbb{R}} \mathcal{A}_{nt}) / 2$, with complex inner product

$$\langle x, y\rangle_{\mathbb{C}} := \langle x, y\rangle_{op} + i \cdot \langle J x, y\rangle_{op}.$$

(iv) Every $\mathbb{Z}_7$-equivariant real-linear map $R : \mathcal{A}_{nt} \to \mathcal{A}_{nt}$ is complex-linear with respect to J (i.e. commutes with J), and therefore induces a $\mathbb{C}$-linear map $R_{\mathbb{C}} : (\mathcal{A}_{nt}, J) \to (\mathcal{A}_{nt}, J)$.

Proof. (i) The non-trivial real irreducible representations of the cyclic group $\mathbb{Z}_7$ are well-known (see e.g. Serre, *Linear Representations of Finite Groups*, §12.2). For $\mathbb{Z}_7$ prime, the irreducible complex characters are $\chi_k(g) = \exp(2\pi i k / 7)$ for $k = 0, 1, ..., 6$, with χ_0 the trivial character. Real-irreducible representations of $\mathbb{Z}_7$ are: the trivial 1-dim real rep (corresponding to χ_0), and 2-dim real reps obtained as the real forms of the conjugate pairs $\{\chi_k, \chi_{-k}\}$ for $k = 1, 2, 3$ (with $-k \equiv 7 - k \bmod 7$). Each non-trivial V_α is therefore even-dimensional over $\mathbb{R}$ and is the real form of exactly one such conjugate pair.

(ii) Let $g \in \mathbb{Z}_7$ be the generator. On a 2-dim real V_α corresponding to conjugate pair $\{\chi_k, \chi_{-k}\}$, the action of g is by rotation through angle $2\pi k/7$ in some orthonormal basis (rotation, not reflection, by the conjugate-pair structure and the requirement that V_α has no real $\mathbb{Z}_7$-invariant subspaces). Define J_α as rotation through angle $\pi/2$ in the same plane. Then J_α^2 is rotation through π, which is $-I_{V_\alpha}$. Since SO(2) is abelian (any two rotations in the same plane commute), J_α commutes with the $\mathbb{Z}_7$-action: $J_\alpha g = g J_\alpha$. Furthermore J_α is orthogonal (rotation is an isometry), so $J_\alpha^\dagger = J_\alpha^{-1} = -J_\alpha$, i.e. J_α is anti-self-adjoint.

There is a binary orientation choice in defining J_α — rotation by $+\pi/2$ versus rotation by $-\pi/2$ — both satisfying $J_\alpha^2 = -I_{V_\alpha}$ and both commuting with g. Equivalently, the choice corresponds to designating which of the two conjugate characters $\{\chi_k, \chi_{-k}\}$ is taken as χ and which as $\bar{\chi}$ under the resulting complex structure. The two choices are related by complex conjugation and give isomorphic (but not identical) complex Hilbert space structures on V_α. We fix one orientation per non-trivial channel by convention; either choice gives an equivalent complex structure up to overall complex conjugation, and the structural results of the paper are invariant under this choice.

For higher-dimensional V_α (occurring when the isotypic component for $\{\chi_k, \chi_{-k}\}$ has multiplicity ≥ 1), the same construction applies to each 2-dim summand, with $J_\alpha := \bigoplus_i J\{\alpha,i\}$ the direct sum of plane rotations. Commutativity with the $\mathbb{Z}_7$-action is preserved.

(iii) Combine the per-channel structures: $J := \bigoplus_\alpha J_\alpha$ on $\mathcal{A}_{nt} = \bigoplus_\alpha V_\alpha$. Then $J^2 = -I$, $J^\dagger = -J$ (anti-self-adjoint on the real Hilbert space), and J commutes with the $\mathbb{Z}_7$-action by construction. The standard prescription $\langle x, y\rangle_{\mathbb{C}} := \langle x, y\rangle_{op} + i \cdot \langle J x, y\rangle_{op}$ for converting a real Hilbert space with anti-self-adjoint J satisfying $J^2 = -I$ into a complex Hilbert space verifies all complex-Hilbert-space axioms (see e.g. Conway, *A Course in Functional Analysis*, Chapter I, or any standard reference on complex structures on real vector spaces).

(iv) A real-linear map $R : \mathcal{A}_{nt} \to \mathcal{A}_{nt}$ commutes with the $\mathbb{Z}_7$-action iff it commutes with the generator g. Since J is constructed from the same plane structure as g (J = rotation by $\pi/2$ in the same planes where g rotates by $2\pi k/7$), any R commuting with g commutes with J. Hence R is $\mathbb{C}$-linear with respect to J. ■

Remark 3.5.1 (The trivial-channel correction). If $\mathcal{A}_{triv} \neq \{0\}$, the canonical complex structure J is defined only on $\mathcal{A}_{nt}$, not on $\mathcal{A}_{triv}$. The trivial subspace remains a real Hilbert space without complex structure. Two structural options handle this:

(a) **Trivial-channel suppression.** If the dynamical content of the operational system is carried entirely by non-trivial $\mathbb{Z}_7$-channels (i.e. $\mathcal{A}_{triv}$ contributes only static configurations with no operational distinguishability content), then the quantum-type dynamics of §4–§6 is restricted to $\mathcal{A}_{nt}$, and $\mathcal{A}_{triv}$ plays no role.

(b) **Trivial-channel real-structure preservation.** If $\mathcal{A}_{triv}$ carries dynamical content, the natural extension is to retain $(\mathcal{A}_{triv}, \langle\cdot,\cdot\rangle_{op})$ as a real Hilbert space with a separate transport regime, and combine with the complex $(\mathcal{A}_{nt}, J)$ structure into a hybrid layered carrier {real $\mathcal{A}_{triv} \oplus$ complex $\mathcal{A}_{nt}$}.

The choice between (a) and (b) depends on the substrate physics of the trivial $\mathbb{Z}_7$-channel. For the K = 7 closure architecture of VERSF, the natural reading is (a): the closure dynamics is supported on non-trivial $\mathbb{Z}_7$-channels, with the trivial channel either empty or carrying only background structure. See §17 item 1 for the open problem of fully classifying $\mathcal{A}_{triv}$ from substrate primitives.

Remark 3.5.2 (Why not quaternionic). A real Hilbert space carrying a $\mathbb{Z}_7$-equivariant structure could in principle carry a *quaternionic* structure rather than a complex one, via three anti-self-adjoint operators I, J, K satisfying $I^2 = J^2 = K^2 = IJK = -I$. The $\mathbb{Z}_7$-equivariance argument of Theorem 3.5 picks out a *single* anti-self-adjoint structure J per pair of conjugate characters $\{\chi_k, \chi_{\{-k\}}\}$, not a triple. This is because $\mathbb{Z}_7$ is cyclic, with each conjugate pair generating a single 2-dim real irrep — there is no automatic quaternionic structure from the $\mathbb{Z}_7$-action alone. A quaternionic structure would require an additional structural input (e.g. a $\mathbb{Z}_4$-action commuting with the $\mathbb{Z}_7$-action, or a non-commutative symmetry algebra). The K = 7 closure architecture supplies a single cyclic group, so the complex structure is forced and the quaternionic alternative is excluded.

This is a partial answer to the question that Hardy / Masanes–Müller / Chiribella-style quantum reconstruction programmes treat. A full answer would also exclude (or include) more exotic alternatives — e.g. complex-with-superselection, octonionic, modular — and verify that no

additional substrate input forces a different choice. The $\mathbb{Z}_7$-argument given here is the natural substrate-derivation; complete classification is Open Problem 1 of §17.

3.4 The complexified operational Hilbert space

Definition 3.6 (Complexified operational Hilbert space). Under the trivial-channel-suppression option of Remark 3.5.1(a), define

$\mathcal{A}_{\mathbb{C}} := (\mathcal{A}_{nt}, J)$, $\langle \cdot,\cdot \rangle_{\mathbb{C}} := \langle \cdot,\cdot \rangle_{op} + i \cdot \langle J \cdot, \cdot \rangle_{op}$.

$(\mathcal{A}_{\mathbb{C}}, \langle \cdot,\cdot \rangle_{\mathbb{C}})$ is a finite-dimensional complex Hilbert space of complex dimension $d_{op}^{\mathbb{C}} := (\dim_{\mathbb{R}} \mathcal{A}_{nt}) / 2$. **The complex norm coincides with the real norm:** for any $x \in \mathcal{A}_{nt}$,

$\|x\|_{\mathbb{C}}^2 = \langle x, x \rangle_{\mathbb{C}} = \langle x, x \rangle_{op} + i \cdot \langle J x, x \rangle_{op} = \langle x, x \rangle_{op} = \|x\|_{op}^2$,

where the imaginary part $\langle J x, x \rangle_{op}$ vanishes because J is anti-self-adjoint (Theorem 3.5): $\langle J x, x \rangle_{op} = \langle x, J^\dagger x \rangle_{op} = -\langle x, J x \rangle_{op} = -\langle J x, x \rangle_{op}$ (using the symmetry of the real inner product), forcing $\langle J x, x \rangle_{op} = 0$. This norm coincidence is what makes Lemma 8.2 work — non-expansiveness on $(\mathcal{A}_{nt}, \|\cdot\|_{op})$ transfers cleanly to non-expansiveness on $(\mathcal{A}_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$.

$\mathbb{Z}_7$-equivariant real-linear operators on $\mathcal{A}_{nt}$ are automatically $\mathbb{C}$-linear on $(\mathcal{A}_{\mathbb{C}}, J)$ by Theorem 3.5(iv). For real-linear non-expansive operators on $\mathcal{A}_{nt}$, the induced $\mathbb{C}$-linear operators on $\mathcal{A}_{\mathbb{C}}$ are non-expansive under $\|\cdot\|_{\mathbb{C}}$ as well, so non-expansiveness transfers cleanly from $\mathcal{A}_{nt}$ to $\mathcal{A}_{\mathbb{C}}$.

In what follows, "the operational Hilbert space" or "the complexification" refers to $(\mathcal{A}_{\mathbb{C}}, \langle \cdot,\cdot \rangle_{\mathbb{C}})$ without further qualification, with the trivial-channel correction (if needed) handled separately.

Definition 3.7 (Distinguishability-density deformation). A **distinguishability-density deformation** is a smooth positive function $\rho : M_{op} \to \mathbb{R}_{>0}$ (typically $\rho = \rho_{op}$ of OC Definition 3.4) defining the deformed Riemannian metric

$g_{op}^{(\rho)}(x) := \rho(x)^{2/d_{op}} \cdot \langle \cdot,\cdot \rangle_{op}$

on M_{op}. By OC Theorem 5.0, this is the unique conformal rescaling under which $g_{op}^{(\rho)}$-length counts local distinguishability quanta. Note that the conformal deformation lives on the *real* operational image $M_{op} \subseteq \mathcal{A}$, distinct from the complexified Hilbert space $\mathcal{A}_{\mathbb{C}}$ on which the transport regimes of §§4–9 act. The layered structure $\{M_{op} \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathbb{C}}\}$ carries different transport regimes at different layers, with the unified theorem of §13 making the layering explicit.

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4. Reversible Transport is Orthogonal/Unitary

Theorem 4.1 (Reversible admissible transport is orthogonal/unitary). *Let $\{U(\tau) : \tau \in \mathbb{R}\}$ be a reversible admissible transport on $(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$ in the sense of Definition 3.3. Then $U(\tau) \in O(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$ for every $\tau \in \mathbb{R}$. Restricted to $\mathcal{A}_{nt}$ and viewed under the canonical complex structure J of Theorem 3.5, $U(\tau)$ becomes a $\mathbb{C}$-linear unitary operator $U_{\mathbb{C}}(\tau) \in U(\mathcal{A}_{\mathbb{C}})$.*

Proof. By Definition 3.3 condition 5, $U(\tau)$ preserves the operational inner product. Hence $U(\tau)$ preserves the operational norm and is an isometry; by Definition 3.3 condition 4 it is invertible, so it is an orthogonal transformation of $(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$.

By Theorem 3.5(iv), the $\mathbb{Z}_7$-equivariance of $U(\tau)$ (Definition 3.2 condition 3) implies that $U(\tau)$ restricted to $\mathcal{A}_{nt}$ commutes with J, making it $\mathbb{C}$-linear with respect to J. The $\mathbb{R}$-isometry property transfers to $\mathbb{C}$-isometry: for $x, y \in \mathcal{A}_{nt}$,

$\langle U(\tau)\, x, U(\tau)\, y\rangle_{\mathbb{C}} = \langle U(\tau)\, x, U(\tau)\, y\rangle_{op} + i \cdot \langle J\, U(\tau)\, x, U(\tau)\, y\rangle_{op} = \langle U(\tau)\, x, U(\tau)\, y\rangle_{op} + i \cdot \langle U(\tau)\, J\, x, U(\tau)\, y\rangle_{op}$ (since $J\, U(\tau) = U(\tau)\, J$) $= \langle x, y\rangle_{op} + i \cdot \langle J\, x, y\rangle_{op}$ (by inner-product preservation) $= \langle x, y\rangle_{\mathbb{C}}$.

Hence $U(\tau)$ is $\mathbb{C}$-linear and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$-preserving, i.e. unitary on $(\mathcal{A}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$. Invertibility transfers from the real isomorphism. Hence $U_{\mathbb{C}}(\tau) \in U(\mathcal{A}_{\mathbb{C}})$. ■

Remark 4.1.1 (No mechanical complexification needed). Theorem 4.1 does *not* complexify by tensor product. The complex structure on $\mathcal{A}_{nt}$ was *already supplied* by Theorem 3.5 from the substrate $\mathbb{Z}_7$-action. What Theorem 4.1 shows is that the orthogonal real-isometry property of $U(\tau)$, together with its $\mathbb{Z}_7$-equivariance, automatically gives unitarity in the substrate-supplied complex structure. The substrate selects the complex structure; the real isometry property selects the unitary subgroup of $GL(\mathcal{A}_{\mathbb{C}})$ within that complex structure.

Remark 4.1.2 (Inner-product preservation vs norm preservation). The Definition 3.3 reversibility condition is *inner-product* preservation, not merely norm preservation. On a real Hilbert space these are equivalent by polarisation. On the complex Hilbert space $(\mathcal{A}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$, norm preservation alone characterises both unitary and antiunitary operators; requiring full sesquilinear inner-product preservation is what selects the unitary regime. Structurally, the antiunitary alternative corresponds precisely to operators that flip the orientation of J (the binary orientation choice acknowledged in Theorem 3.5(ii)): an antiunitary operator on $(\mathcal{A}_{\mathbb{C}}, J)$ is a unitary operator on $(\mathcal{A}_{\mathbb{C}}, -J)$. Because we derive complex inner-product preservation from real inner-product preservation plus $\mathbb{Z}_7$-equivariance plus the *fixed* orientation choice for J, antiunitary operators are automatically excluded — they violate either the linearity over the chosen J (they are *anti*-linear over J, i.e. linear over $-J$) or the real $\mathbb{Z}_7$-equivariance constraint under the fixed orientation.

Remark 4.1.3 (Reversibility versus non-expansiveness). A non-expansive linear map need not be an isometry — it is one only if it preserves all norms, which by polarisation requires inner-product preservation. Definition 3.3 imposes inner-product preservation as a separate condition precisely because non-expansiveness alone is insufficient. §8 will show that every admissible refinement transport decomposes canonically into a reversible (isometric) component and a non-isometric contractive component, with reversibility characterising precisely the isometric factor.

5. Generator Structure on the Complexification

Theorem 5.1 (Self-adjoint generator). *Let $\{U_{\mathbb{C}}(\tau) : \tau \in \mathbb{R}\}$ be a one-parameter group of unitary operators on $\mathcal{A}_{\mathbb{C}}$ obtained as the $\mathbb{Z}_7$-equivariant complexification of a reversible admissible transport on $\mathcal{A}$ (in the sense of Theorem 4.1). Suppose $\tau \mapsto U_{\mathbb{C}}(\tau)$ is strongly continuous. Then there exists a unique self-adjoint operator H on $\mathcal{A}_{\mathbb{C}}$ such that*

$$U_{\mathbb{C}}(\tau) = \exp(-i\,\tau\,H).$$

Proof.

Step 1: Strong continuity transfers from real to complex. Definition 3.3 condition 3 gives real strong continuity: $\tau \mapsto U(\tau)\,x$ is continuous for every $x \in \mathcal{A}$. For $x \in \mathcal{A}_{nt}$ with complex-norm $\|x\|_{\mathbb{C}}^2 = \|x\|_{op}^2$, continuity of $\tau \mapsto U(\tau)\,x$ in $(\mathcal{A}_{nt}, \|\cdot\|_{op})$ is identical to continuity in $(\mathcal{A}_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$. Hence $\tau \mapsto U_{\mathbb{C}}(\tau)\,x$ is strongly continuous in $\mathcal{A}_{\mathbb{C}}$ for every x.

Step 2: Strong continuity $\Leftrightarrow$ norm continuity in finite dimension. In finite-dimensional $\mathcal{A}_{\mathbb{C}}$, strong continuity of a one-parameter group is equivalent to norm continuity (the operator norm on $\mathrm{End}(\mathcal{A}_{\mathbb{C}})$ is equivalent to any of the strong-operator topologies in finite dimension). Hence $\tau \mapsto U_{\mathbb{C}}(\tau)$ is norm-continuous.

Step 3: Matrix-Lie-group exponential. A norm-continuous one-parameter subgroup of the finite-dimensional matrix Lie group $U(d_{op}^{\mathbb{C}})$ is automatically smooth, and by the standard matrix-Lie-group theorem (Hall, *Lie Groups, Lie Algebras, and Representations*, Theorem 2.14) is the exponential of a unique element K of the Lie algebra:

$$U_{\mathbb{C}}(\tau) = \exp(\tau\,K),$$

with $K \in \mathfrak{u}(d_{op}^{\mathbb{C}}) = \{K \in \mathrm{End}(\mathcal{A}_{\mathbb{C}}) : K^{\dagger} = -K\}$.

Step 4: Conversion to self-adjoint form. Define $H := i\,K$. Then

$$H^{\dagger} = (i\,K)^{\dagger} = -i\,K^{\dagger} = -i\,(-K) = i\,K = H,$$

so H is self-adjoint. Substituting $K = -i\,H$ into $U_{\mathbb{C}}(\tau) = \exp(\tau\,K)$ gives

$$U_{\mathbb{C}}(\tau) = \exp(\tau \cdot (-i\,H)) = \exp(-i\,\tau\,H).$$

Uniqueness of H follows from uniqueness of K and the bijection $H \leftrightarrow K = -i\,H$. ■

Remark 5.1.1 (Stone-type but finite-dimensional). Stone's theorem proper is for unbounded generators on infinite-dimensional Hilbert spaces; the finite-dimensional case used here is the elementary matrix-Lie-group exponential statement. "Stone-type" in this paper refers to the

structural pattern (strongly continuous one-parameter unitary group ↔ self-adjoint generator) rather than to Stone's theorem in its full infinite-dimensional form.

Remark 5.1.2 (Choice of H vs K convention). We adopt $U_{\mathbb{C}}(\tau) = \exp(-i\,\tau\,H)$ with H self-adjoint, the form under which the operational Schrödinger equation of §6 takes its conventional form $i\,\partial_\tau\,\psi = H\,\psi$. The anti-self-adjoint $K = -i\,H$ is the corresponding Lie-algebra generator, related by a factor of $-i$; both forms are equivalent.

Remark 5.1.3 (Refinement parameter as Hamiltonian conjugate). The parameter τ here is the substrate-level refinement parameter (extended bidirectionally to $\mathbb{R}$ for the reversible regime), not the macroscopic time coordinate (which is emergent in VERSF). Whether τ reduces to t under coarse-graining, with H reducing to the macroscopic Hamiltonian, is part of the broader quantum-reconstruction question (§14, §17 items 3 and 5).

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6. Operational Schrödinger Equation

Corollary 6.1 (Operational Schrödinger equation). *For any differentiable trajectory $\psi(\tau) = U_{\mathbb{C}}(\tau)\,\psi(0)$ in $\mathcal{A}_{\mathbb{C}}$ generated by a reversible admissible transport,*

$$i \cdot d\psi/d\tau = H\,\psi.$$

Restoring physical units of action gives

$$i\,\hbar_{op} \cdot d\psi/d\tau = H_{phys}\,\psi,$$

where $\hbar_{op}$ is a substrate-level action quantum and $H_{phys} = \hbar_{op} \cdot H$ is the physical generator.

Proof. Differentiate $\psi(\tau) = \exp(-i\,\tau\,H)\,\psi(0)$ with respect to τ:

$$d\psi/d\tau = -i\,H\exp(-i\,\tau\,H)\,\psi(0) = -i\,H\,\psi(\tau).$$

Multiplying by i gives $i \cdot d\psi/d\tau = H\,\psi$. Rescaling $\tau \mapsto \tau / \hbar_{op}$ and $H \mapsto H / \hbar_{op}$ gives the physical form. ■

Remark 6.1.1 (Identification of $\hbar_{op}$). The substrate-level action quantum $\hbar_{op}$ is the natural conjugate of the refinement parameter τ — it converts τ-evolution rates into Hamiltonian eigenvalue units. Whether $\hbar_{op}$ coincides quantitatively with the macroscopic $\hbar$ under coarse-graining (analogously to the OC open question of whether Δ_{op} coincides with the Planck length ℓ_P) is an open identification question, listed as §17 item 5.

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7. Commitment Projection: Non-expansiveness, Distinguishability Loss, and Entropy Monotonicity

7.1 Non-expansiveness and strict distinguishability loss

Theorem 7.1 (Orthogonal projections are non-expansive and strictly remove orthogonal distinguishability). *Let $P : \mathcal{A} \to \mathcal{A}$ be an orthogonal projection ($P^2 = P$, $P^\dagger = P$). Then:*

(i) $\mathcal{A}$ decomposes as an orthogonal direct sum $\mathcal{A} = \mathrm{Im}(P) \oplus \ker(P)$, with the two summands $\langle \cdot,\cdot \rangle_{op}$-orthogonal.

(ii) For every $x \in \mathcal{A}$, $x = P\,x + (I - P)\,x$ with $\|x\|_{op}^2 = \|P\,x\|_{op}^2 + \|(I - P)\,x\|_{op}^2$.

(iii) P is non-expansive: $\|P\,x - P\,y\|_{op} \le \|x - y\|_{op}$ for all $x, y \in \mathcal{A}$.

(iv) The inequality in (iii) is strict — $\|P\,x - P\,y\|_{op} < \|x - y\|_{op}$ — whenever $\ker(P) \neq \{0\}$ (equivalently, whenever P is a non-trivial projection, $P \neq I$) and the difference $x - y$ has non-zero component in $\ker(P)$.

Proof. (i) Standard for orthogonal projections on a Hilbert space.

(ii) For any x, $x = P\,x + (I - P)\,x$ with $P\,x \in \mathrm{Im}(P)$ and $(I - P)\,x \in \ker(P)$ orthogonal:

$$\|x\|_{op}^2 = \|P\,x\|_{op}^2 + \|(I - P)\,x\|_{op}^2.$$

(iii) Applying (ii) to $x - y$:

$$\|x - y\|_{op}^2 = \|P(x - y)\|_{op}^2 + \|(I - P)(x - y)\|_{op}^2 \ge \|P\,x - P\,y\|_{op}^2.$$

(iv) Equality in (iii) requires $\|(I - P)(x - y)\|_{op} = 0$, i.e. $x - y \in \mathrm{Im}(P) = \ker(I - P)$. If P is non-trivial ($\ker(P) \neq \{0\}$) and $x - y$ has non-zero component in $\ker(P)$, then $(I - P)(x - y) \neq 0$, so the inequality is strict. ■

Remark 7.1.1 (Operational interpretation of distinguishability loss). In the operational reading, $\|x - y\|_{op}$ is the count of distinguishability quanta separating x and y (OC Theorem 5.0). Theorem 7.1(iv) states that orthogonal projection by a non-trivial P strictly decreases this count whenever $x - y$ has any component orthogonal to $\mathrm{Im}(P)$. Projection is therefore not a passive labelling operation: it removes distinguishability content lying outside the committed admissible subspace. The component $(I - P)\,x$ is precisely the part of x that is *not committed* — it carries operational distinguishability that the projection discards.

Remark 7.1.2 (Ω_{max} as the universal projector). The universal admissibility projector Ω_{max} of OG §2.3 is the orthogonal projection onto $\mathcal{A}$ itself, viewed as a subspace of the ambient closure-state-space. Theorem 7.1 applied to Ω_{max} gives: every ambient state x decomposes as $x = \Omega_{max}\,x + (I - \Omega_{max})\,x$, with the inadmissible component $(I - \Omega_{max})\,x$

strictly removed. This is the operational-Hilbert statement of admissibility-as-projection: inadmissible distinguishability is exactly what Ω_max discards.

7.2 Entropy monotonicity of positive refinement contraction

We now connect non-expansiveness of |R| to operational entropy. The connection is structural: the positive factor |R| in the polar decomposition cannot increase the operational distinguishability content of any state, and under the natural hypothesis that reducing distinguishability resolution does not decrease operational entropy, |R| acts as an entropy-non-decreasing operator.

Hypothesis 7.2 (Operational entropy monotonicity under distinguishability reduction). *The operational entropy functional S_op of OG §2.3 and OC Theorem 4.2, extended to a functional S_op : $\mathcal{A}_\mathbb{C} \setminus \{0\} \to \mathbb{R}$ on non-zero states, satisfies the* ***operational coarse-graining monotonicity*** *condition: for every non-expansive admissible operator T on $\mathcal{A}_\mathbb{C}$ and every non-zero $x \in \mathcal{A}_\mathbb{C}$ with $Tx \neq 0$,*

$$S_{op}(Tx) \geq S_{op}(x),$$

with equality if and only if T acts isometrically on x — i.e. $\|Tx\|_\mathbb{C} = \|x\|_\mathbb{C}$ and $\langle Ty, Tx\rangle_\mathbb{C} = \langle y, x\rangle_\mathbb{C}$ for all $y \in \mathcal{A}_\mathbb{C}$ in the cyclic subspace generated by x.

Remark 7.2.0 (Strength and scope of Hypothesis 7.2). Hypothesis 7.2 is *substantially stronger* than the standard thermodynamic statement of entropy non-decrease under coarse-graining. Three features deserve flagging:

(a) **It applies to a single state, not an ensemble.** Standard entropy-monotonicity statements (data-processing inequality, Jensen-type bounds) apply to probability distributions or ensembles. Hypothesis 7.2 asks for monotonicity of S_op evaluated on a single state vector $x \in \mathcal{A}_\mathbb{C}$ under operator action. This requires S_op to be a functional of the *spectral profile* of x relative to the admissible decomposition (e.g. a Shannon-type entropy of the squared amplitudes $|\langle e_\alpha, x\rangle|^2$ across an $\mathbb{Z}_7$-channel-adapted basis), not merely of the norm $\|x\|_\mathbb{C}$.

(b) **The equality clause is strong.** Equality under T iff T acts isometrically on x means S_op is determined by the orbit structure of x under isometric operations — a strong constraint connecting S_op to the symmetric structure of the admissible Hilbert space.

(c) **Specification of S_op on $\mathcal{A}_\mathbb{C}$ is gestural in the present paper.** OC's S_op is a density-functional on M_op (an integral against ρ_op). The extension to $\mathcal{A}_\mathbb{C}$ as a state-functional is described above in structural terms (Shannon-type spectral entropy) but is not fully pinned down. Full specification, together with a substrate-level derivation of Hypothesis 7.2, is §17 item 10 (see also the analogous open problem in OC concerning the S_op-concavity hypothesis used in OC Theorem 4.2). The structural results of §7.2 follow from Hypothesis 7.2 regardless of its precise form, so the development of the paper does not depend on the missing specification — but a complete operational-thermodynamic theory does.

These three features together mean Hypothesis 7.2 is *more demanding* than typical entropy-monotonicity hypotheses, and its acceptance is structurally on a par with admissibility itself. The hypothesis is taken as an operational primitive in the present paper; subsequent work in the broader programme is responsible for deriving it from substrate primitives or for replacing it with a weaker but sufficient condition.

Theorem 7.2 (Entropy monotonicity under positive refinement contraction). *Let $R : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be an admissible linear refinement map with polar decomposition $R = U \cdot |R|$. Under Hypothesis 7.2:*

(i) The unitary factor U is entropy-preserving on its initial space: $S_{op}(U x) = S_{op}(x)$ for all non-zero $x \in \ker(R)^{\perp}$.

(ii) The positive factor $|R|$ is entropy-non-decreasing: $S_{op}(|R| x) \geq S_{op}(x)$ for all non-zero $x \in \mathcal{A}_{\mathbb{C}}$ with $|R| x \neq 0$, with equality if and only if $|R|$ acts as the identity on x — equivalently, x lies in the eigenspace of $|R|$ with eigenvalue 1.

(iii) The full refinement $R = U |R|$ satisfies $S_{op}(R x) \geq S_{op}(x)$ on non-zero x with $R x \neq 0$, with equality if and only if $|R|$ acts as the identity on x.

Proof. (i) U is a partial isometry (Theorem 8.1), isometric on its initial space $\ker(R)^{\perp}$. Since U is isometric on the whole of $\ker(R)^{\perp}$, it is isometric on every cyclic subspace within $\ker(R)^{\perp}$, hence the cyclic-subspace qualification of Hypothesis 7.2's equality clause is automatically satisfied. By that equality clause, $S_{op}(U x) = S_{op}(x)$ for non-zero $x \in \ker(R)^{\perp}$.

(ii) By Proposition 8.3, $|R|$ is non-expansive. By Hypothesis 7.2, $S_{op}(|R| x) \geq S_{op}(x)$ for non-zero x with $|R| x \neq 0$. Equality requires $|R|$ to be isometric on x. Since $|R|$ is positive semidefinite, isometric action on x means $|R| x$ has the same norm as x, equivalently x lies in the eigenspace of $|R|$ with eigenvalue 1 (the only spectral component on which $|R|$ preserves norm).

(iii) $S_{op}(R x) = S_{op}(U |R| x) = S_{op}(|R| x) \geq S_{op}(x)$, using (i) on the first equality (U acts isometrically on $\mathrm{Im}(|R|) \subseteq \ker(R)^{\perp}$) and (ii) on the inequality. Equality follows the same condition as (ii). ■

Remark 7.2.1 (Operational interpretation of |R|). Theorem 7.2 gives the operational interpretation of the positive factor $|R|$: it is not merely a contraction operator but an **entropy-non-decreasing refinement operator**. The polar decomposition therefore separates admissible refinement into:

- **reversible entropy-preserving transport (U)** — the unitary factor, generating quantum-type evolution with no change in operational entropy;
- **irreversible entropy-non-decreasing contraction (|R|)** — the positive factor, generating commitment-type evolution that either preserves entropy (when $|R|$ acts as identity on x) or strictly increases it (when $|R|$ has eigenvalues < 1 in directions x has support on).

This supplies the **operational-Hilbert origin of the second-law arrow of refinement**: irreversibility is not added to the framework as an independent postulate but emerges from the spectral structure of the positive factor in the polar decomposition. Where |R| has eigenvalue 1, refinement is reversible; where |R| has eigenvalue < 1, refinement is strictly irreversible and entropy strictly increases.

Remark 7.2.2 (Orthogonal projection as extremal entropy increase). For an orthogonal projection P, |P| = P has spectrum {0, 1}, with eigenvalue 1 on Im(P) and eigenvalue 0 on ker(P). Theorem 7.2(ii) then gives: S_op(P x) ≥ S_op(x) with strict inequality whenever x has any component in ker(P). Orthogonal projection is therefore the *extremal* form of entropy-non-decreasing refinement: components orthogonal to Im(P) are fully discarded, and the entropy increase reflects the maximum-distinguishability-loss that can occur under non-expansive refinement.

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8. Polar Decomposition of Admissible Refinement Transport

We now show that every admissible refinement transport on the complexification splits canonically into a reversible isometric component and a positive contractive component, *and* that $\mathbb{Z}_7$-equivariance propagates separately to each factor.

Theorem 8.1 (Polar decomposition). *Let R : $\mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be a bounded $\mathbb{C}$-linear operator on the complex Hilbert space $\mathcal{A}_{\mathbb{C}}$. Then R admits a unique polar decomposition*

$$R = U \cdot |R|,$$

where:

(i) |R| := (R^† R)^{1/2} is positive semidefinite;

(ii) U is a partial isometry with initial space ker(|R|)^⊥ = ker(R)^⊥ and final space Im(R), uniquely determined by R when U is set to vanish on ker(R);

(iii) If R is invertible, then U is unitary on $\mathcal{A}_{\mathbb{C}}$.

Proof. Standard finite-dimensional polar decomposition theorem (Horn–Johnson, *Matrix Analysis*, Theorem 7.3.1). R^† R is self-adjoint and positive semidefinite, so |R| = (R^† R)^{1/2} exists by the spectral theorem. Define U on Im(|R|) by U(|R| x) := R x; this is well-defined because $\||R|\,x\|^2 = \langle x, R^\dagger R\, x\rangle = \|R\,x\|^2$. Extend U by zero on Im(|R|)^⊥ = ker(R) to get a partial isometry on $\mathcal{A}_{\mathbb{C}}$. Uniqueness follows from the well-definedness and the prescribed vanishing on ker(R). Invertibility of R gives ker(R) = {0}, so |R| is positive definite and U is unitary on all of $\mathcal{A}_{\mathbb{C}}$. ■

Lemma 8.2 (Real-to-complex non-expansiveness transfer). *Let R : $\mathcal{A} \to \mathcal{A}$ be a real-linear non-expansive map (operator-norm bound on $(\mathcal{A}, \langle\cdot,\cdot\rangle_{op})$) that is $\mathbb{Z}_7$-equivariant, hence (by*

Theorem 3.5(iv)) restricts to a $\mathbb{C}$-linear map $R_{\mathbb{C}} : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ on the complex structure of $\mathcal{A}_{nt}$. Then $R_{\mathbb{C}}$ is non-expansive on $(\mathcal{A}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$.

Proof. For $x \in \mathcal{A}_{nt}$ with $\|x\|_{\mathbb{C}}^2 = \|x\|_{op}^2$,

$\|R_{\mathbb{C}}\, x\|_{\mathbb{C}}^2 = \|R\, x\|_{op}^2 \le \|R\|_{op}^2 \cdot \|x\|_{op}^2 \le \|x\|_{op}^2 = \|x\|_{\mathbb{C}}^2$.

Hence $\|R_{\mathbb{C}}\|_{\mathbb{C}} \le 1$, i.e. $R_{\mathbb{C}}$ is non-expansive. ■

Lemma 8.2 ensures that the polar decomposition of Theorem 8.1, when applied to the $\mathbb{C}$-linear extension of an admissible refinement map (Definition 3.2), preserves the non-expansiveness condition. We can therefore apply Theorem 8.1 unambiguously to admissible refinement maps regarded as $\mathbb{C}$-linear operators on $\mathcal{A}_{\mathbb{C}}$.

Proposition 8.3 (Spectral norm bound on $|R|$). *If R is non-expansive on $\mathcal{A}_{\mathbb{C}}$, then $\||R|\|_{op} \le 1$, with equality holding on the eigenspace of $|R|$-eigenvalue 1.*

Proof. $\||R|\, x\|^2 = \langle x, R^\dagger R\, x\rangle = \|R\, x\|^2 \le \|x\|^2$ since R is non-expansive. Hence $|R|$ is non-expansive, with the spectral characterisation immediate. ■

Proposition 8.4 ($\mathbb{Z}_7$-equivariance preservation under polar decomposition). *Let $R : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be a $\mathbb{Z}_7$-equivariant $\mathbb{C}$-linear operator. Then both U and $|R|$ in the polar decomposition $R = U \cdot |R|$ are $\mathbb{Z}_7$-equivariant.*

Proof. Let $g \in \mathbb{Z}_7$ denote the substrate $\mathbb{Z}_7$-generator, viewed as a unitary operator on $\mathcal{A}_{\mathbb{C}}$ (which it is, by Theorem 3.5(ii) and the orthogonality of the $\mathbb{Z}_7$-action). Define $\rho_g : \mathrm{End}(\mathcal{A}_{\mathbb{C}}) \to \mathrm{End}(\mathcal{A}_{\mathbb{C}})$ by $\rho_g(A) := g\, A\, g^{-1}$.

Equivariance of $R^\dagger R$. Since R commutes with g, so does $R^\dagger$:

$R^\dagger g = (g^{-1} R)^\dagger = (R\, g^{-1})^\dagger = g\, R^\dagger$ (using $g^{-1} = g^\dagger$ since g is unitary).

Hence $R^\dagger R$ commutes with g:

$R^\dagger R \cdot g = R^\dagger g R = g\, R^\dagger R$.

Equivariance of $|R|$. $|R| = (R^\dagger R)^{1/2}$ is a continuous function of $R^\dagger R$ via the spectral theorem (specifically, the operator square root commutes with any operation that commutes with $R^\dagger R$). Since g commutes with $R^\dagger R$ and g is unitary,

$g\, |R|\, g^{-1} = (g\, R^\dagger R\, g^{-1})^{1/2} = (R^\dagger R)^{1/2} = |R|$.

Hence $|R|$ commutes with g, i.e. $|R|$ is $\mathbb{Z}_7$-equivariant.

Equivariance of U. Since R = U |R|, both R and |R| are $\mathbb{Z}_7$-equivariant. On the subspace ker(R)^⊥ where |R| is invertible (i.e. on the initial space of U), we have U = R |R|^{−1}. Therefore

U g = R |R|^{−1} g = R g |R|^{−1} = g R |R|^{−1} = g U on ker(R)^⊥,

using that |R|^{−1} commutes with g (since |R| does and g is unitary). On ker(R), U = 0 by convention (Theorem 8.1(ii)), so U g x = 0 = g U x for x ∈ ker(R). Hence U is $\mathbb{Z}_7$-equivariant on all of $\mathcal{A}_{\mathbb{C}}$. ■

Remark 8.4.1 (Why this matters). Proposition 8.4 is essential to the structural argument of the paper: it shows that the polar decomposition R = U · |R| preserves the admissibility condition ($\mathbb{Z}_7$-equivariance) on each factor separately. Without this, the canonical decomposition would in principle break admissibility, even if R itself satisfies Definition 3.2. With Proposition 8.4, both U and |R| are admissible linear refinement maps in the sense of Definition 3.2: U is non-expansive (it is a partial isometry, hence ‖U‖ ≤ 1), admissibility-preserving (since R is and the decomposition respects the $\mathbb{Z}_7$-decomposition), and $\mathbb{Z}_7$-equivariant.

Corollary 8.5 (Reversibility ⇔ |R| = I). *Let $R : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be a $\mathbb{Z}_7$-equivariant bounded $\mathbb{C}$-linear operator obtained as the complexification of an admissible linear refinement map on $\mathcal{A}$. Then the following are equivalent:*

(i) R preserves the operational inner product on $\mathcal{A}$ (equivalently, R is the value at a single τ of some reversible admissible transport family);

(ii) R^† R = I on $\mathcal{A}_{\mathbb{C}}$;

(iii) |R| = I;

(iv) R = U is unitary on $\mathcal{A}_{\mathbb{C}}$.

Proof. (i) ⟹ (ii): ⟨R x, R y⟩ = ⟨x, y⟩ for all x, y means ⟨x, R^† R y⟩ = ⟨x, y⟩, hence R^† R = I. (ii) ⟹ (iii): |R| = (R^† R)^{1/2} = I^{1/2} = I. (iii) ⟹ (iv): R = U · I = U, and U is a partial isometry with full domain (since ker(|R|) = ker(I) = {0}), hence unitary. (iv) ⟹ (i): unitary operators preserve the inner product.

The clause in (i) regarding embedding in a one-parameter family reflects that Definition 3.3 defines a *family*, not a single map. An inner-product-preserving single R is unitary on $\mathcal{A}_{\mathbb{C}}$, and unitary operators on a finite-dimensional complex Hilbert space are surjectively reached by the exponential map: every U ∈ U(d_op^ℂ) can be written as U = exp(K) for some K ∈ 𝔲(d_op^ℂ) (Hall, *Lie Groups, Lie Algebras, and Representations*, Theorem 11.10, using compactness and connectedness of U(n)). Such K is *not* unique — different K satisfying exp(K) = R differ by integer multiples of 2πi in any eigenvalue — and the choice of K determines a specific reversible admissible transport family R(τ) := exp(τK) with R(1) = R. Hence a single inner-product-preserving R can be embedded in *at least one* (in fact infinitely many) reversible admissible

transport families; the present statement (i) is "R is the value at some τ of some reversible family," which is correctly delivered by this embedding. ■

Remark 8.5.1 (Canonical separation). Theorem 8.1 + Proposition 8.4 + Corollary 8.5 together supply the canonical operator-theoretic separation between the two transport regimes. Every admissible refinement transport R splits, uniquely, into:

- a **partial isometry U** — the reversible-content factor, preserving distinguishability on its initial space and mapping it isometrically to its final space, $\mathbb{Z}_7$-equivariantly;
- a **positive contraction |R|** — the irreversible-content factor, encoding the spectral magnitude of refinement, $\mathbb{Z}_7$-equivariantly.

If |R| = I, then R = U is purely reversible (unitary). If |R| has eigenvalue 0 along some directions, those directions are annihilated (the strongest form of commitment projection). If |R| has eigenvalues in (0, 1), those directions are *attenuated* — partial commitment, intermediate between full reversibility and orthogonal projection.

Remark 8.5.2 (Orthogonal projections in the decomposition). For an orthogonal projection P with $P^2 = P$, $P^\dagger = P$:

$P^\dagger P = P^2 = P$,

so |P| = P^{1/2} = P (the unique positive semidefinite square root of P, which has eigenvalues 0 and 1 only). The partial-isometry factor is then determined by the requirement U(|P| x) = P x for $x \in \mathrm{Im}(|P|)$: on Im(|P|) = Im(P), U acts as the identity; on $\mathrm{Im}(|P|)^\perp = \ker(P)$, U is zero by the polar-decomposition convention. So U is precisely the partial isometry onto Im(P), which numerically *equals* P (as an operator on $\mathcal{A}_{\mathbb{C}}$): U = P. The polar decomposition of P is therefore $P = P \cdot P$, where the first P is the partial-isometry factor (onto Im(P)) and the second P is the positive factor.

In this decomposition, U = P is non-trivial as a partial isometry, but it is *not* a reversible admissible transport in the sense of Definition 3.3 unless P = I (because the corollary requires |P| = I, which forces P = I). Thus orthogonal projections sit on the |R|-side of the canonical separation in the sense that their reversibility content is *non-unitary* (a non-trivial partial isometry, not a unitary), and they carry intrinsic contractive content. The U-factor and |R|-factor of a projection are numerically equal but structurally distinct: U is the "shape" of the projection, |R| is the "strength" — and for an orthogonal projection these coincide.

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9. Consequence: Quantum and Commitment Dynamics as the Two Factors, and Their Noncommutativity

9.1 The structural bridge

The polar decomposition theorem plus equivariance preservation gives the structural bridge.

A general admissible refinement transport R contains both regimes:

$R = U \cdot |R|$.

The unitary factor U preserves operational distinguishability on its initial space:

$\|U\,x - U\,y\|_{op} = \|x - y\|_{op}$ for $x, y \in \ker(R)^{\perp}$.

The positive factor |R| changes operational distinguishability, non-expansively:

$\||R|\,x - |R|\,y\|_{op} \leq \|x - y\|_{op}$,

with equality if and only if $x - y$ lies in the eigenspace of |R| with eigenvalue 1.

By Proposition 8.4, both U and |R| are individually $\mathbb{Z}_7$-equivariant, hence individually admissible refinement maps. The canonical decomposition is therefore *within* the class of admissible transports, not outside it.

Quantum-like reversible evolution = isometric (U) factor of admissible refinement transport. Commitment-projection / refinement contraction = positive (|R|) factor of admissible refinement transport.

This is the clean mathematical statement that avoids hand-waving. Quantum dynamics and commitment dynamics are not separate postulates. They are the two canonical operator-theoretic factors of admissible transport on the operational Hilbert space (after the $\mathbb{Z}_7$-induced complex structure is taken into account), given by finite-dimensional polar decomposition.

Remark 9.1.1 (Two factors of transport, plus metric deformation). The polar decomposition has exactly two factors: U and |R|. The third regime of the unified theorem (§13) — curvature from conformal deformation — is *not* a factor of R but a deformation of the underlying metric $\langle\cdot,\cdot\rangle_{op}$ into $g_{op}^{(\rho)}$. The two-factor structure of refinement transport and the metric-deformation structure of curvature are distinct mathematical operations on the operational Hilbert geometry, and §13 makes the relationship explicit through the layered carrier structure.

9.2 Noncommutativity of reversible transport and commitment projection

The polar decomposition exhibits reversible and commitment regimes as two factors of the *same* transport. A separate question is: when these regimes are applied independently and in different orders, do they commute? The answer is generically no, with a clean characterisation of the commuting case.

Theorem 9.2 (Transport–projection noncommutativity). *Let $U : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be a unitary admissible transport (a value of a reversible admissible transport family, or any unitary $\mathbb{Z}_7$-*

equivariant $\mathbb{C}$-linear operator on $\mathcal{A}_{\mathbb{C}}$), and let $P : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be an orthogonal commitment projection. Then:

(i) $[U, P] = UP - PU$ vanishes if and only if U preserves the orthogonal decomposition $\mathcal{A}_{\mathbb{C}} = \mathrm{Im}(P) \oplus \ker(P)$, i.e. $U(\mathrm{Im}(P)) \subseteq \mathrm{Im}(P)$ and $U(\ker(P)) \subseteq \ker(P)$.

(ii) For generic U and P satisfying neither containment, $[U, P] \neq 0$.

Proof. (i) Write $\mathcal{A}_{\mathbb{C}} = \mathrm{Im}(P) \oplus \ker(P)$ as an orthogonal direct sum. For $x \in \mathrm{Im}(P)$, $Px = x$, so $UPx = Ux$ and $PUx = P(Ux)$. These agree iff $Ux \in \mathrm{Im}(P)$, i.e. iff $U(\mathrm{Im}(P)) \subseteq \mathrm{Im}(P)$. For $x \in \ker(P)$, $Px = 0$, so $UPx = 0$ and $PUx = P(Ux)$. These agree iff $Ux \in \ker(P)$, i.e. iff $U(\ker(P)) \subseteq \ker(P)$. The commutator $[U, P]$ vanishes on all of $\mathcal{A}_{\mathbb{C}}$ iff both containments hold.

(ii) Suppose U does not preserve the decomposition. Then either $U(\mathrm{Im}(P)) \not\subset \mathrm{Im}(P)$ or $U(\ker(P)) \not\subset \ker(P)$. In the first case, there exists $x \in \mathrm{Im}(P)$ with $Ux \notin \mathrm{Im}(P)$, so $PUx \neq Ux = UPx$, giving $[U, P]x \neq 0$. The second case is symmetric. ■

Remark 9.2.1 (Operational origin of order-dependence). Theorem 9.2 establishes that *the order in which reversible transport and commitment projection are applied generically matters*. The composition UP (transport-then-project) differs from PU (project-then-transport) unless U respects the decomposition imposed by P.

This is the operational-Hilbert origin of an order-dependence phenomenon familiar in quantum mechanics under a different name: the noncommutativity of measurement and unitary evolution. In the standard QM reading, $[H, P_{\text{measurement}}] \neq 0$ says that performing a measurement at different times in a Hamiltonian evolution generically gives different outcomes. In the operational-Hilbert reading derived here, $[U, P] \neq 0$ says that performing commitment-projection at different stages of refinement-transport generically gives different post-refinement states. The two readings are structurally identical.

Theorem 9.3 (Projection–projection noncommutativity). *Let $P_1, P_2 : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ be orthogonal commitment projections. The following are equivalent:*

(i) $[P_1, P_2] = P_1 P_2 - P_2 P_1 = 0$;

(ii) $P_1 P_2 = P_2 P_1$ is itself an orthogonal projection (necessarily onto $\mathrm{Im}(P_1) \cap \mathrm{Im}(P_2)$);

*(iii) P_1 and P_2 admit a **joint orthogonal decomposition** of $\mathcal{A}_{\mathbb{C}}$ — i.e. $\mathcal{A}_{\mathbb{C}}$ splits as an orthogonal direct sum of four common eigenspaces of $\{P_1, I - P_1\} \times \{P_2, I - P_2\}$.*

Proof. (i) $\Longrightarrow$ (ii). Suppose $[P_1, P_2] = 0$. Then $P_1 P_2 = P_2 P_1$, and

$$(P_1 P_2)^2 = P_1 P_2 P_1 P_2 = P_1 P_1 P_2 P_2 = P_1 P_2$$

(using commutativity and idempotence). Also

$(P_1 P_2)^\dagger = P_2^\dagger P_1^\dagger = P_2 P_1 = P_1 P_2$.

Hence $P_1 P_2$ is self-adjoint and idempotent, i.e. an orthogonal projection. Its image is $\mathrm{Im}(P_1) \cap \mathrm{Im}(P_2)$: for $x \in \mathrm{Im}(P_1 P_2)$, $x = P_1 P_2 x$ lies in $\mathrm{Im}(P_1)$, and $x = P_2 P_1 x$ lies in $\mathrm{Im}(P_2)$; conversely, for $x \in \mathrm{Im}(P_1) \cap \mathrm{Im}(P_2)$, $x = P_1 x = P_1 (P_2 x) = P_1 P_2 x$.

(ii) $\Longrightarrow$ (iii). Define the four subspaces

$W_{11} := \mathrm{Im}(P_1) \cap \mathrm{Im}(P_2)$, $W_{10} := \mathrm{Im}(P_1) \cap \ker(P_2)$, $W_{01} := \ker(P_1) \cap \mathrm{Im}(P_2)$, $W_{00} := \ker(P_1) \cap \ker(P_2)$.

The orthogonal decomposition $\mathcal{A}_{\mathbb{C}} = W_{11} \oplus W_{10} \oplus W_{01} \oplus W_{00}$ follows from commutativity of P_1 and P_2 (each pair of subspaces in the decomposition is orthogonal under one of the projections). On W_{11}, P_1 and P_2 both act as identity; on W_{10}, P_1 acts as identity and P_2 as zero; on W_{01}, P_1 acts as zero and P_2 as identity; on W_{00}, both act as zero. This is the joint orthogonal decomposition.

(iii) $\Longrightarrow$ (i). Under the joint decomposition, $P_1 P_2$ acts as identity on W_{11} and as zero on $W_{10} \oplus W_{01} \oplus W_{00}$, and $P_2 P_1$ does the same. Hence $P_1 P_2 = P_2 P_1$, i.e. $[P_1, P_2] = 0$. ■

Remark 9.3.1 (Incompatibility of distinct commitments). Theorem 9.3 supplies the operational-Hilbert origin of **incompatibility of distinct commitments**: committing to admissible subspace $V_1 = \mathrm{Im}(P_1)$ and then to $V_2 = \mathrm{Im}(P_2)$ generically produces a different final state than the reverse order, unless V_1 and V_2 admit a common four-fold orthogonal decomposition of $\mathcal{A}_{\mathbb{C}}$. Generic V_1 and V_2 (with $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$ and $\mathrm{Im}(P_1) \cap \mathrm{Im}(P_2)^\perp \neq \{0\}$) do not admit such a decomposition, and the two commitment orders give different final states.

Combined with Theorem 9.2 (transport–projection non-commutativity), the picture is: non-commuting transports, non-commuting projections, and non-commuting transport-projection compositions. This supplies the substrate-level structural origin of the incompatibility phenomena that quantum mechanics encodes via non-commuting observables. The operational Hilbert geometry inherits a non-commutative algebraic structure not from quantum-mechanical postulate but from the very fact that admissible operations act on a non-trivial decomposition.

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10. Curvature from Hilbert-Metric Deformation: Scalar, Ricci, and Metric Flow

10.1 Scalar curvature of the conformal deformation

We now show how geometric curvature enters without changing the underlying Hilbert carrier space. This section operates on the real operational image M_op ⊆ $\mathcal{A}$, with the conformal-deformation regime distinct from the transport regimes of §§4–9 which operate on $\mathcal{A}$_ℂ.

The flat operational Hilbert metric is the inherited inner product

g_op^{(0)} = ⟨·,·⟩_op.

A distinguishability-density deformation (Definition 3.7) by ρ : M_op → ℝ_{>0} gives

g_op^{(ρ)}(x) = ρ(x)^{2/d_op} · g_op^{(0)}.

Setting u(x) := (1/d_op) · ln ρ(x), so ρ^{2/d_op} = e^{2u}, the deformed metric takes the conformal form

g_op^{(ρ)} = e^{2u} · g_op^{(0)}.

Since g_op^{(0)} is flat, g_op^{(ρ)} is a conformal deformation of a flat reference metric. The Riemann curvature tensor of g_op^{(ρ)} is given in closed form by OC Theorem 8.1; the scalar curvature follows from the standard conformal-rescaling formula.

Theorem 10.1 (Scalar curvature of distinguishability-density deformation). *Let d := d_op ≥ 3, let g_0 = ⟨·,·⟩_op be the flat operational Hilbert metric, and let g_ρ = e^{2u} g_0 with u = (1/d) ln ρ for a smooth positive density ρ. Then the scalar curvature of g_ρ is*

R_ρ = −e^{−2u} · [2(d−1) Δ̄_0 u + (d−1)(d−2) · ‖∇̄_0 u‖²],

where ∇̄_0 and Δ̄_0 denote the gradient and Laplacian with respect to the flat reference metric g_0. Substituting u = (1/d) ln ρ:

R_ρ = −ρ^{−2/d} · [(2(d−1)/d) · Δ̄_0 (ln ρ) + ((d−1)(d−2)/d²) · ‖∇̄_0 (ln ρ)‖²].

Proof. The standard scalar-curvature transformation law under conformal rescaling g_ρ = e^{2u} g_0 in dimension d (Besse, *Einstein Manifolds*, §1.J, equation 1.161) gives

R_ρ = e^{−2u} · [R_0 − 2(d−1) Δ̄_0 u − (d−1)(d−2) · ‖∇̄_0 u‖²].

For the flat reference, R_0 = 0, so

R_ρ = −e^{−2u} · [2(d−1) Δ̄_0 u + (d−1)(d−2) · ‖∇̄_0 u‖²].

Substituting u = (1/d) ln ρ, with e^{−2u} = ρ^{−2/d}, Δ̄_0 u = (1/d) Δ̄_0 (ln ρ), and ‖∇̄_0 u‖² = (1/d²) ‖∇̄_0 (ln ρ)‖², gives the stated form. ■

Remark 10.1.1 (Sign-convention consistency with OC Theorem 8.1). OC Theorem 8.1 gives the (1,3)-form Riemann tensor R_op^a_{bcd} with the last term carrying −‖∇ψ‖² (ψ = u in this

paper's notation). Contracting to obtain the Ricci tensor ($R_{bd} = R^a_{bad}$) and tracing to obtain the scalar curvature, the resulting expression matches the conformal-rescaling formula above with consistent signs. Specifically: the $(d-1)(d-2)$ coefficient in R_ρ arises from the d-dimensional trace of the Kronecker-tensor terms in OC Theorem 8.1's curvature formula, and the negative overall prefactor matches OC's sign convention. Verification of the trace calculation is a standard exercise in the conformal-rescaling formulas of Besse op. cit. and need not be repeated here; what matters is that both papers use the same sign convention and the scalar curvature is consistent across them.

Remark 10.1.2 (Two-dimensional case). In $d_{op} = 2$ the prefactor $(d-1)(d-2)$ vanishes and the formula reduces to $K = R/2 = -e^{-2u}\,\Delta_0 u$, recovering the 2D Gaussian-curvature expression noted in OC Remark 8.2.1. The 2D exception of OC Theorem 8.2 ($R_\rho = 0$ admits non-constant harmonic ρ) is recovered here as the existence of non-constant solutions to $\Delta_0 u = 0$.

Remark 10.1.3 (Connection to the OC pointwise bound). The pointwise bound $\|R_{op}\| \le C / \Delta_{op}^2$ of OC Theorem 9.1 implies a corresponding bound on the scalar curvature: $|R_\rho| \le d_{op} \cdot C / \Delta_{op}^2$. Lemmas 9.0 and 9.0' of OC bound the gradient and Hessian of $\ln \rho$ pointwise by $1/\Delta_{op}$ (with density-contrast prefactor), and these bounds directly bound the two terms in Theorem 10.1.

10.2 Explicit Ricci tensor of the distinguishability-deformed metric

The scalar curvature of Theorem 10.1 is the trace of the operational Ricci tensor. The full Ricci tensor, in closed form, supplies the natural tensorial bridge to Einstein-like geometric dynamics in the broader programme.

Theorem 10.2 (Ricci tensor of the operational conformal deformation). *Let $d := d_{op} \ge 3$, let $g_0 = \langle\cdot,\cdot\rangle_{op}$ be the flat operational Hilbert metric, and let $g_\rho = e^{2u}\, g_0$ with $u = (1/d) \ln \rho$ for a smooth positive density ρ. Then the Ricci tensor of g_ρ, expressed as a symmetric (0,2)-tensor on M_{op}, is*

$$\mathrm{Ric}_\rho = -(d-2)\cdot(\nabla_0^2 u - \bar{d}u \otimes \bar{d}u) - [\Delta_0 u + (d-2)\cdot\|\nabla_0 u\|^2]\cdot g_0,$$

where ∇_0^2, $\bar{d}$, Δ_0 and $\|\cdot\|$ are computed with respect to the flat reference metric g_0 (Definition 3.5). Substituting $u = (1/d) \ln \rho$:

$$\mathrm{Ric}_\rho = -((d-2)/d)\cdot\nabla_0^2(\ln\rho) + ((d-2)/d^2)\cdot\bar{d}(\ln\rho)\otimes\bar{d}(\ln\rho) - T(\rho)\cdot g_0,$$

where $T(\rho)$ is the trace-source scalar

$$T(\rho) := (1/d)\cdot\Delta_0(\ln\rho) + ((d-2)/d^2)\cdot\|\nabla_0(\ln\rho)\|^2.$$

Proof. The standard conformal transformation law for the Ricci tensor under $g_\rho = e^{2u}\, g_0$ in dimension $d \ge 3$ is (Besse, *Einstein Manifolds*, §1.J, equation 1.159 / 1.160):

$$\mathrm{Ric}_\rho = \mathrm{Ric}_0 - (d-2)\cdot(\nabla_0^2 u - du\otimes du) - [\Delta_0 u + (d-2)\cdot\|\nabla_0 u\|^2]\cdot g_0.$$

For the flat reference, Ric_0 = 0, so the first stated form follows immediately. The second form follows by substitution u = (1/d) ln ρ, with $\nabla_0^2 u = (1/d) \cdot \nabla_0^2 (\ln \rho)$, $\bar{d}u \otimes \bar{d}u = (1/d^2) \cdot \bar{d}(\ln \rho) \otimes \bar{d}(\ln \rho)$, $\bar{\Delta}_0 u = (1/d) \cdot \bar{\Delta}_0 (\ln \rho)$, and $\|\nabla_0 u\|^2 = (1/d^2) \cdot \|\nabla_0 (\ln \rho)\|^2$. ■

Consistency check (Remark 10.2.1). Tracing Ric_ρ with $g_\rho^{ab} = e^{-2u} (g_0)^{ab} = \rho^{-2/d} (g_0)^{ab}$ should recover the scalar curvature R_ρ of Theorem 10.1. Computing:

$R_\rho = \rho^{-2/d} \cdot (g_0)^{ab} \cdot (Ric_\rho)_{ab} = \rho^{-2/d} \cdot [-((d-2)/d) \cdot \bar{\Delta}_0 (\ln \rho) + ((d-2)/d^2) \cdot \|\nabla_0 (\ln \rho)\|^2 - d \cdot T(\rho)] = \rho^{-2/d} \cdot [-((d-2)/d) \cdot \bar{\Delta}_0 (\ln \rho) + ((d-2)/d^2) \cdot \|\nabla_0 (\ln \rho)\|^2 - \bar{\Delta}_0 (\ln \rho) - ((d-2)/d) \cdot \|\nabla_0 (\ln \rho)\|^2] = \rho^{-2/d} \cdot [-(2(d-1)/d) \cdot \bar{\Delta}_0 (\ln \rho) - ((d-1)(d-2)/d^2) \cdot \|\nabla_0 (\ln \rho)\|^2]$,

matching Theorem 10.1 exactly. The sign convention and dimensional factors are consistent between Theorems 10.1 and 10.2.

Remark 10.2.2 (Tensorial vs scalar curvature response). Theorem 10.2 supplies more than Theorem 10.1: it gives the *full tensorial* curvature response of the operational metric to distinguishability-density gradients, not merely its trace. The two structural components are:

- A **traceless tensorial piece** $-(d-2) \cdot (\nabla_0^2 (\ln \rho)/d - \bar{d}(\ln \rho) \otimes \bar{d}(\ln \rho)/d^2)$, which contributes anisotropic curvature (direction-dependent under the operational metric — different components in different tangent directions at a fixed point) and vanishes when ln ρ has zero Hessian and zero gradient — i.e. when ρ is constant.
- An **isotropic trace piece** $-T(\rho) \cdot g_0$, which contributes uniform scalar curvature in all tangent directions at each point (proportional to g_0).

Terminological clarification. Here "anisotropic" and "isotropic" refer to the *direction-dependence in the tangent space at a fixed point* (whether the Ricci tensor acts differently on different tangent directions), not to *spatial homogeneity vs inhomogeneity* in the cosmology sense (whether matter distribution is uniform across space). In the present substrate-level framework both kinds of dependence can vary: the traceless piece is anisotropic at every point where $\nabla_0^2 (\ln \rho)$ is non-zero, and is spatially inhomogeneous when ρ varies across M_op. The two notions of (in)homogeneity are independent.

This is an **anisotropic/isotropic curvature split**, not an Einstein/cosmological-constant split per se. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - (1/2) R g_{\mu\nu}$ is the trace-reversed Ricci tensor; both its traceless and trace parts contribute to the Einstein equation. The cosmological constant Λ appears as $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$, i.e. as an *additional* term, not as the trace component of Ric.

What the present split does deliver, structurally: under coarse-graining and (open) Lorentzian-signature emergence, the *traceless* part of Ric naturally maps to anisotropic curvature contributions of the macroscopic Einstein tensor (i.e. to the parts of $G_{\mu\nu}$ driven by anisotropic stress-energy), while the *isotropic* trace part naturally maps to a sum of contributions including both isotropic stress-energy (perfect-fluid-like) and a cosmological-constant-like term, depending on the precise coarse-graining of T(ρ). The Ricci decomposition is therefore the

substrate-level precursor of the Einstein/cosmological-constant decomposition in the eventual macroscopic field equation, with the precise correspondence determined by (open) coarse-graining of the operational stress-energy tensor. This is one formulation of OC Open Problem 5 (here §17 item 12).

10.3 Operational metric flow under refinement transport

The refinement continuity equation of OC Theorem 10.3 — $\partial_\tau \rho + \nabla_0 \cdot (\rho\, v) = 0$ — induces a dynamical evolution of the operational metric $g_{op}^{(\rho)}$ itself, since $g_{op}^{(\rho)}$ depends on ρ. This makes the operational geometry not static but dynamical under refinement transport.

Theorem 10.3 (Operational metric flow). *Let $\rho(x, \tau)$ evolve under the operational continuity equation of OC Theorem 10.3,*

$$\partial_\tau \rho + \nabla_0 \cdot (\rho\, v) = 0,$$

with smooth velocity field $v(x, \tau)$ on M_{op}. Then the conformally-deformed operational metric $g_{op}^{(\rho)} = \rho^{2/d_{op}} \cdot g_0$ evolves according to

$$\partial\tau\, (g_{op}^{(\rho)})_{ab} = -(2/d_{op}) \cdot [\nabla_0 \cdot v + v \cdot \nabla_0 (\ln \rho)] \cdot (g_{op}^{(\rho)})_{ab}.$$

This is a position-dependent conformal flow: the metric undergoes pure pointwise rescaling, with rate function

$$\lambda(x, \tau) := -(2/d_{op}) \cdot [\nabla_0 \cdot v + v \cdot \nabla_0 (\ln \rho)]\,(x, \tau).$$

*The metric flow is **not autonomous**: the rate λ depends on the velocity field v and the density ρ at the current instant, not directly on the current metric. Closing the system requires an additional dynamical law for v — typically a specification such as $v = \nabla_0 S_{op}$ for entropy-gradient refinement (OC Theorem 4.2). Under such closure, the metric flow becomes part of a joint flow on the triple $(\rho, v, g_{op}^{(\rho)})$, with ρ evolving via continuity, v evolving (or remaining fixed) via the chosen dynamical law, and $g_{op}^{(\rho)}$ evolving via Theorem 10.3. This is the substrate-level analogue of the coupled Einstein–matter system in continuum geometric theory, where the metric does not evolve independently of the matter fields it carries.*

Proof. Differentiate $g_{op}^{(\rho)} = \rho^{2/d_{op}}\, g_0$ with respect to τ, holding g_0 fixed:

$$\partial\tau\, (g_{op}^{(\rho)})_{ab} = (2/d_{op}) \cdot \rho^{(2/d_{op}) - 1} \cdot (\partial\tau\, \rho) \cdot (g_0)_{ab}.$$

By the continuity equation,

$$\partial_\tau \rho = -\nabla_0 \cdot (\rho\, v) = -\rho \cdot \nabla_0 \cdot v - v \cdot \nabla_0 \rho.$$

Substituting:

$\partial\tau\ (g_op^{(\rho)})\{ab\} = -(2/d_op) \cdot \rho^{2/d_op\} \cdot [\nabla_0 \cdot v + (v \cdot \nabla_0\ \rho) / \rho] \cdot (g_0)\{ab\} = -(2/d_op) \cdot \rho^{2/d_op\} \cdot [\nabla_0 \cdot v + v \cdot \nabla_0\ (\ln \rho)] \cdot (g_op^{(\rho)\})_\{ab\}$,

using $\rho^{2/d_op\}\ (g_0)\{ab\} = (g_op^{(\rho)\})\{ab\}$ in the last step. ■

Remark 10.3.1 (Refinement as joint density-entropy-metric flow). Theorem 10.3 shows that operational geometry itself evolves dynamically under refinement transport. Refinement flow is therefore *simultaneously*:

- **density evolution** (OC Theorem 10.3: $\partial_\tau\ \rho = -\nabla_0 \cdot (\rho\ v)$);
- **entropy evolution** (Theorem 7.2 plus continuity: S_op evolves non-decreasingly along the irreversible component);
- **metric evolution** (Theorem 10.3 here: $\partial_\tau\ g = \lambda \cdot g$ with λ given above);

with all three driven by the same velocity field v. The operational geometry is not a static stage on which refinement acts — it is itself part of the dynamical content of refinement.

Remark 10.3.2 (Asymptotic flatness recovery). In the asymptotic $\tau \to \infty$ limit under the entropy-gradient refinement flow (OC Theorem 4.2, Corollary 10.4), ρ approaches the constant operational equilibrium density $1/Vol_op(M)$. At that point $\nabla_0\ (\ln \rho) = 0$, and if $v = \nabla_0\ S_op$ approaches zero (which it does at the entropy-maximum, where S_op is stationary), then $\lambda \to 0$ and the metric stops flowing. The asymptotic operational metric is the flat reference g_0 (up to constant rescaling), recovering OC Remark 10.4.1.

Remark 10.3.3 (Structural relation to continuum geometric flows). The operational metric flow $\partial_\tau\ g = \lambda\ g$ is structurally a conformal flow — pointwise rescaling of the metric without changing the conformal class. Conformal flows of this type appear in continuum geometry as the *Yamabe flow* and related families:

- Yamabe flow: $\partial_t\ g = -(R_g - \bar{r}_g)\ g$, where R_g is the scalar curvature and $\bar{r}_g$ its volume-averaged mean. The driving function is curvature-derived.
- Conformal Ricci flow: $\partial_t\ g = -(R_g + n)\ g$, a conformal projection of Ricci flow.

Theorem 10.3's operational metric flow is driven by the transport divergence $\nabla_0 \cdot v + v \cdot \nabla_0\ (\ln \rho)$ rather than directly by curvature. For the entropy-gradient flow $v = \nabla_0\ S_op$ (suppressing the κ_op coefficient of OC Definition 10.2 for simplicity), the divergence is $\bar{\Delta}_0\ S_op + \nabla_0\ S_op \cdot \nabla_0\ (\ln \rho)$, which couples the entropy Laplacian to the density-gradient. Whether this operational conformal flow reduces, under coarse-graining, to a Yamabe-type or Ricci-type flow on the emergent 4-manifold is a structural conjecture of the broader programme.

Arrow-of-time compatibility. The operational metric flow is forward in τ only, since the underlying continuity equation $\partial_\tau\ \rho + \nabla_0 \cdot (\rho\ v) = 0$ comes from forward refinement transport (OC Definition 3.1, $\tau \in [0, \infty)$). Continuum geometric flows like Ricci and Yamabe are typically run in the smoothing direction (forward in t), with their arrow of time fixed by the parabolic structure of the flow equation rather than by an intrinsic substrate-level irreversibility. The

operational metric flow inherits its directionality from the entropy-monotonicity of refinement transport (§7.2, Theorem 7.2 plus Hypothesis 7.2): refinement transport is entropy-non-decreasing, hence intrinsically τ-asymmetric, hence the induced metric flow inherits the τ-asymmetry. The structural similarity to Yamabe/Ricci flow plus the substrate-derived directionality is a notable feature: the operational framework supplies an *origin* for the arrow of time of geometric flow, whereas in continuum geometry the directionality is a feature of the equation chosen rather than derived. The explicit reduction is §17 item 11.

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11. Interpretation: Curvature from Distinguishability-Density Gradients

The scalar-curvature formula of Theorem 10.1 makes a precise structural statement:

Operational curvature is generated by gradients and Laplacians of the distinguishability density.

If ρ is constant on M_op, then ∇_0 (ln ρ) = 0 and Δ¯_0 (ln ρ) = 0, so R_ρ = 0. Hence:

Uniform distinguishability density ⟹ flat operational geometry.

Conversely, non-uniform distinguishability density generically generates curvature, with uniqueness of the converse in dimension d_op ≥ 3 supplied by OC Theorem 8.2.

This is the bridge to the geometric programme. Curvature is *not* added to the Hilbert geometry as an independent dynamical field. It is generated by the operational distinguishability density acting on the flat reference inner product through conformal deformation. The Hilbert carrier space is unchanged; what changes is the metric the carrier space carries.

Remark 11.1 (Three structurally distinct sources of curvature). In the broader VERSF programme, curvature can arise from three structurally distinct mechanisms on the operational Hilbert geometry:

1. **Conformal deformation by ρ_op** (this paper, Theorem 10.1; OC Theorem 5.2) — local variation of admissible sector density produces curvature confined to the conformal class.
2. **$\mathbb{Z}_7$-equivariance breaking** (open, OC Open Problem 3) — channel-mixing deformations would lift the conformal restriction, producing curvature outside the conformal class. Such deformations are excluded under unbroken $\mathbb{Z}_7$-equivariance (OC Remark 5.2.2, present paper Remark 3.2.1).
3. **Coarse-graining** (OC §11) — at scales L > Δ_op, the smoothing produces a C^{1,α} Riemannian limit whose curvature is inherited from but not identical to the substrate-level curvature.

The present paper addresses (1) in the Hilbert-geometric formulation, with (2) reserved for the post-$\mathbb{Z}_7$-equivariance regime and (3) inherited from OC.

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12. Projection–Curvature Compatibility

We now examine how commitment projection interacts with distinguishability-density-induced curvature.

Theorem 12.1 (Projection localises distinguishability support). *Let $P : \mathcal{A} \to \mathcal{A}$ be an orthogonal projection with image $V := Im(P) \subseteq \mathcal{A}$, and let $\rho : M_{op} \to \mathbb{R}_{>0}$ be a smooth distinguishability density. For each $y \in V$, let $F_y := P^{-1}(y) \cap M_{op}$ be the fibre over y (an affine subset of $\mathcal{A}$ parallel to $\ker P$). Define the* ***fibre-interior subset*** *$V^\circ := \{y \in V : F_y \subseteq int(M_{op})\}$, the points whose fibre lies entirely in the interior of M_{op}. Define the projected density $\rho_P : V \to \mathbb{R}_{\geq 0}$ by*

$$\rho_P(y) := \int_{F_y} \rho(x)\, d\mathcal{H}^{d_{op} - \dim V}(\ker P)(x),$$

where $d\mathcal{H}^{d_{op} - \dim V}_{\ker P}$ is the $(d_{op} - \dim V)$-dimensional Hausdorff measure on the affine fibre F_y (well-defined since M_{op} is bounded). Then:

(i) $\mathrm{supp}(\rho_P) \subseteq P(\mathrm{supp}\, \rho)$;

(ii) ρ_P is non-negative everywhere, and smooth on V°. At points $y \in V \setminus V^\circ$ where the fibre intersects ∂M_{op}, ρ_P may have lower regularity, with the set of such points of codimension ≥ 1 in V.

(iii) The scalar curvature R_{ρ_P} of the conformally-deformed metric $g_{\rho_P} = \rho_P^{2/\dim V} \cdot \langle \cdot, \cdot \rangle_{op,V}$ on V° (where $\langle \cdot, \cdot \rangle_{op,V}$ is the restriction of $\langle \cdot, \cdot \rangle_{op}$ to V) is non-zero only where ρ_P varies smoothly on V° with non-vanishing gradient or non-vanishing Laplacian.

Proof. (i) If $x \in F_y$ implies $\rho(x) = 0$ for all x in the fibre, then $\rho_P(y) = 0$ by the integral definition.

(ii) Non-negativity is immediate. Smoothness on V° follows from smoothness of ρ on $\mathrm{int}(M_{op})$ (OC Theorem 5.1, applied to ρ_{op}) and from smooth parametrisation of fibres over y when the fibre stays entirely in the interior of M_{op}. Specifically, for $y \in V^\circ$, a small neighbourhood of y in V has all fibres staying in $\mathrm{int}(M_{op})$, and the fibre-parametrised integral $\int_{F_y} \rho\, d\mathcal{H}$ is smooth in y by standard results on smoothness of parametrised integrals over fixed-shape domains (e.g. Folland, *Real Analysis*, §2.4). At $y \in V \setminus V^\circ$ the fibre touches ∂M_{op} and the integration domain changes shape as y varies, generically giving Lipschitz or Hölder rather than C^∞ regularity. The set $V \setminus V^\circ$ corresponds to fibres tangent to or crossing ∂M_{op}, which is a codimension-≥ 1 set in V under mild geometric hypotheses on ∂M_{op}.

(iii) By Theorem 10.1 applied on V° with the flat induced metric $\langle\cdot,\cdot\rangle_{op,V}$, $R\{\rho_P\}$ is given by gradient and Laplacian of ln ρ_P on V°, which vanish wherever ρ_P is locally constant or has vanishing first/second derivatives. ■

Corollary 12.2 (Constant density projects to fibre-measure-weighted density). *If $\rho \equiv \rho_0$ is constant on M_op, then on V° the projected density ρ_P satisfies*

$$\rho_P(y) = \rho_0 \cdot \mathscr{H}^{d_op - \dim V}(F_y).$$

Therefore ρ_P inherits any non-uniformity from the y-dependence of the fibre measure of M_op under P. If M_op is fibre-uniform under P — meaning $\mathscr{H}^{d_op - \dim V}(F_y)$ is constant on V° — then ρ_P is constant on V° and $R_{\rho_P} = 0$ there. In the general case, ρ_P is not constant even when ρ is constant, and any residual curvature $R_{\rho_P} \neq 0$ is geometric — inherited from the shape of M_op under projection, not from ρ itself.

Proof. The expression $\rho_P(y) = \rho_0 \cdot \mathscr{H}^{d_op - \dim V}(F_y)$ follows directly from the integral definition with ρ constant. The fibre measure $\mathscr{H}^{d_op - \dim V}(F_y)$ varies with y in general (e.g. if M_op is a generic bounded set, the fibre measure is non-constant in y as the fibre slices through different cross-sections of M_op). It is constant if M_op has a product structure $M_op = V_subset \times W$ with $W \subseteq \ker P$, or under analogous fibre-uniformity hypotheses. The residual y-dependence of ρ_P, when present, is generated by the geometry of M_op rather than by the density ρ, and the corresponding curvature R_{ρ_P} reflects this geometric inheritance. ■

Remark 12.2.1 (Projection compatibility, not equivalence). Theorem 12.1 establishes that orthogonal projection acts coherently on the conformal-deformation structure: projecting the density and then taking curvature gives a well-defined result, with curvature support confined to the projected support. This is *compatibility*, not commutativity — there is no claim that projection commutes with the curvature operator. The constant-density case (Corollary 12.2) makes precise the residual geometric contribution: even when ρ is uniform, projection can produce a non-uniform ρ_P reflecting the geometry of M_op under P, and the resulting curvature is geometric inheritance rather than density-driven.

Remark 12.2.2 (Interaction with the §17 open problems). Whether the bounded-curvature estimate of OC Theorem 9.1 transfers cleanly under projection — i.e. whether $\|R_{\rho_P}\| \leq C / \Delta_op^2$ inherits the bound from $\|R_\rho\| \leq C / \Delta_op^2$ — is non-automatic because fibre integration can amplify density-contrast ratios. Explicit estimation is §17 item 4.

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13. The Unified Refinement Hilbert Geometry Theorem

We now state the central structural theorem of the paper, integrating the three transport regimes into a single statement over the **layered carrier structure**

$$M_op \subseteq \mathcal{A} \subseteq \mathcal{A}_\mathbb{C},$$

where ($\mathcal{A}_\mathbb{C}$, J) is the substrate-derived complexification of Theorem 3.5.

Theorem 13.1 (Unified Refinement Hilbert Geometry Theorem). *Let ($\mathcal{A}$, $\langle\cdot,\cdot\rangle_{op}$) be the finite operational Hilbert geometry of OG Theorem 4.0, let J be the canonical complex structure on $\mathcal{A}_nt$ induced by the substrate $\mathbb{Z}_7$-action (Theorem 3.5), and let ($\mathcal{A}_\mathbb{C}$, $\langle\cdot,\cdot\rangle_\mathbb{C}$) = (($\mathcal{A}_nt$, J), $\langle\cdot,\cdot\rangle_{op}$ + i $\langle J\,\cdot,\cdot\rangle_{op}$) be the resulting complex Hilbert space.* ***Assume the trivial-channel-suppression option of Remark 3.5.1(a)****: the operationally dynamical content of $\mathcal{A}$ is supported on $\mathcal{A}_nt$, with $\mathcal{A}_triv = \mathcal{A} \ominus \mathcal{A}_nt$ carrying only static background configurations (or empty). The transport regimes (II)–(V) then act on $\mathcal{A}_nt \subseteq \mathcal{A} \subseteq \mathcal{A}_\mathbb{C}$ via the canonical complex structure, and the conformal-deformation regime (VI) acts on the operationally dynamical subset $M_op \cap \mathcal{A}_nt$. The following structural facts hold on this layered carrier:*

(I) ***Canonical complex structure ($\mathcal{A} \to \mathcal{A}_\mathbb{C}$ layer).*** *The complex structure J on $\mathcal{A}_nt$ is forced by the substrate $\mathbb{Z}_7$-equivariance (Theorem 3.5); it is not chosen. Every $\mathbb{Z}_7$-equivariant real-linear admissible refinement map on $\mathcal{A}$ induces a $\mathbb{C}$-linear non-expansive operator on $\mathcal{A}_\mathbb{C}$ (Lemma 8.2).*

(II) ***Canonical decomposition ($\mathcal{A}_\mathbb{C}$ layer).*** *Every $\mathbb{Z}_7$-equivariant $\mathbb{C}$-linear admissible refinement map $R : \mathcal{A}_\mathbb{C} \to \mathcal{A}_\mathbb{C}$ admits a unique polar decomposition*

$$R = U \cdot |R|,$$

with U a partial isometry and $|R|$ positive semidefinite, both $\mathbb{Z}_7$-equivariant (Theorem 8.1, Proposition 8.4).

(III) ***Reversible regime ($\mathcal{A}_\mathbb{C}$ layer).*** *R is the value at a single τ of a reversible admissible transport family if and only if $|R| = I$, in which case $R = U$ is unitary on $\mathcal{A}_\mathbb{C}$. Strongly continuous one-parameter groups of reversible admissible transports are generated by a self-adjoint operator H via $U_\mathbb{C}(\tau) = \exp(-i\,\tau\,H)$, and trajectories satisfy the operational Schrödinger equation $i \cdot d\psi/d\tau = H\,\psi$ (Corollary 8.5, Theorem 5.1, Corollary 6.1).*

(IV) ***Commitment regime ($\mathcal{A}_\mathbb{C}$ layer, intersecting the real structure).*** *Orthogonal commitment projections P are non-expansive and strictly remove distinguishability content orthogonal to Im(P) whenever P is non-trivial (Theorem 7.1). Ω_max is the universal admissibility projector and the canonical commitment projection of OG; viewed on $\mathcal{A}_\mathbb{C}$, Ω_max has polar decomposition $\Omega_max = \Omega_max \cdot \Omega_max$ with the partial-isometry factor coinciding with the positive factor (Remark 8.5.2).*

(V) ***Distinguishability-distance preservation/reduction.*** *Reversible transport preserves operational distance: $D_op(U\,x, U\,y) = D_op(x, y)$. Commitment projection cannot increase operational distance: $D_op(P\,x, P\,y) \le D_op(x, y)$, with strict reduction when P is non-trivial and $x - y$ has component in $\ker(P)$.*

(VI) ***Curvature regime (M_op layer).*** *Distinguishability-density deformation of the operational Hilbert metric by $\rho : M_op \to \mathbb{R}_{>0}$ produces a conformally-deformed Riemannian metric*

$g_{op}^{(\rho)}(x) = \rho(x)^{2/d_{op}} \cdot \langle \cdot, \cdot \rangle_{op}$,

on M_{op}, with scalar curvature (Theorem 10.1) in dimension $d_{op} \geq 3$:

$$R_\rho = -\rho^{-2/d_{op}} \cdot \left[\frac{2(d_{op}-1)}{d_{op}} \cdot \Delta_0 (\ln \rho) + \frac{(d_{op}-1)(d_{op}-2)}{d_{op}^2} \cdot \|\nabla_0 (\ln \rho)\|^2 \right].$$

Curvature vanishes when ρ is constant; non-zero density gradients generate curvature in $d_{op} \geq 3$.

(VII) ***Projection–curvature compatibility (M_{op} ↔ V layer).*** *Orthogonal projection acts coherently on the conformal-deformation structure: the projected density ρ_P confines curvature support to $P(\mathrm{supp}\ \rho)$, and constant ρ projects to ρ_P proportional to the fibre measure of M_{op} under P, with vanishing curvature when M_{op} is fibre-uniform (Theorem 12.1, Corollary 12.2).*

(VIII) ***Entropy monotonicity under the positive factor.*** *Under Hypothesis 7.2 (operational entropy monotonicity under distinguishability reduction), the polar-decomposition factors split entropy behaviour cleanly: U is entropy-preserving, $|R|$ is entropy-non-decreasing, and the full refinement $R = U|R|$ satisfies $S_{op}(R\,x) \geq S_{op}(x)$ with equality if and only if $|R|$ acts as the identity on x (Theorem 7.2). The positive factor $|R|$ therefore carries the operational-Hilbert origin of the second-law arrow of refinement.*

(IX) ***Transport–projection and projection–projection noncommutativity.*** *Reversible transport and commitment projection generically do not commute: $[U, P] = 0$ if and only if U preserves the orthogonal decomposition $\mathcal{A}_{\mathbb{C}} = \mathrm{Im}(P) \oplus \ker(P)$ (Theorem 9.2). Two orthogonal projections P_1, P_2 commute if and only if they admit a joint orthogonal decomposition of $\mathcal{A}_{\mathbb{C}}$ into the four common eigenspaces of $\{P_1, I - P_1\} \times \{P_2, I - P_2\}$ (Theorem 9.3). These supply the operational-Hilbert origin of order-dependence between transport and commitment and of incompatibility of distinct commitments — the structural analogues of measurement–evolution noncommutativity and observable incompatibility in quantum mechanics.*

(X) ***Full Ricci tensor in closed form.*** *The Ricci tensor of the conformally-deformed metric $g_{op}^{(\rho)}$ in dimension $d_{op} \geq 3$ is given in closed form (Theorem 10.2) as the sum of a traceless tensorial piece in $(\nabla_0^2 \ln \rho, \bar{d} \ln \rho \otimes \bar{d} \ln \rho)$ plus an isotropic trace piece $T(\rho) \cdot g_0$. The traceless/trace split is the structural precursor of the Einstein/cosmological-constant split of macroscopic field equations.*

(XI) ***Operational metric flow under refinement transport.*** *Under the operational continuity equation $\partial_\tau \rho + \nabla_0 \cdot (\rho v) = 0$ (OC Theorem 10.3), the operational metric undergoes a position-dependent conformal flow*

$$\partial\tau\, (g_{op}^{(\rho)})_{ab} = \lambda(x, \tau) \cdot (g_{op}^{(\rho)})_{ab}, \quad \lambda = -(2/d_{op}) \cdot [\nabla_0 \cdot v + v \cdot \nabla_0 (\ln \rho)],$$

(Theorem 10.3). The operational geometry is therefore not a static stage but a dynamical object, evolving jointly with density and entropy under refinement transport.

The layered carrier {M_op ⊆ $\mathcal{A}$ ⊆ $\mathcal{A}$_ℂ} therefore supports three structurally distinct transport regimes — reversible isometric transport on $\mathcal{A}$_ℂ, commitment projection on $\mathcal{A}$_ℂ intersecting M_op via Ω_max, and conformal-deformation curvature on M_op (now equipped with full Ricci structure and dynamical metric flow) — corresponding respectively to the unitary factor of polar decomposition, the positive factor, and the conformal deformation of the metric. The three regimes share one layered carrier, with the entropy arrow (VIII) carried by the |R|-factor and the order-dependence (IX) generated by the non-commutativity of U and P.

Proof. (I) is Theorem 3.5 + Lemma 8.2. (II) is Theorem 8.1 + Proposition 8.4. (III) combines Corollary 8.5, Theorem 5.1, and Corollary 6.1. (IV) is Theorem 7.1 + OG §2.3 + Remark 8.5.2. (V) follows from unitary/orthogonal invariance and Theorem 7.1. (VI) is Theorem 10.1. (VII) is Theorem 12.1 + Corollary 12.2. (VIII) is Theorem 7.2. (IX) is Theorem 9.2 + Theorem 9.3. (X) is Theorem 10.2. (XI) is Theorem 10.3. ■

Remark 13.1.1 (Three regimes, one layered carrier). The unified theorem makes precise the central structural claim of the paper: the operational geometry of admissibility is *one layered* mathematical object — M_op (real subset, where conformal deformation lives), $\mathcal{A}$ (real Hilbert space, the substrate of operational distinguishability), and $\mathcal{A}$_ℂ (complex Hilbert space derived from $\mathbb{Z}_7$-equivariance, where reversible transport and commitment projection live). The three regimes operate on different layers:

- the **conformal-deformation regime** (VI) acts on the metric of M_op;
- the **reversible regime** (III) acts on $\mathcal{A}$_ℂ as unitary one-parameter groups;
- the **commitment regime** (IV) acts on $\mathcal{A}$_ℂ as orthogonal projections, with the universal projector Ω_max intersecting the real structure.

The bridging structure is: every refinement transport on $\mathcal{A}$_ℂ has a canonical polar decomposition (regimes II + III) that respects $\mathbb{Z}_7$-equivariance; every refinement of the metric on M_op is conformal under the same $\mathbb{Z}_7$-equivariance. The shared carrier is the layered structure with shared substrate symmetry.

Remark 13.1.2 (Scope of "no further structure"). The three regimes (II, III, VI) plus their compatibility (V, VII) exhaust the canonical *linear* transport content of admissible refinement on $\mathcal{A}$_ℂ, together with the *conformal* metric-deformation content on M_op under unbroken $\mathbb{Z}_7$-equivariance. There is no further linear dynamical structure on $\mathcal{A}$_ℂ outside this scheme without breaking either $\mathbb{Z}_7$-equivariance or admissibility/non-expansiveness. Non-linear admissible transport (e.g. OC's entropy-gradient flow Φ_τ, which is non-linear in the trajectory but generated by a linear velocity field on $\mathcal{A}$) lies outside the present framework and is treated separately in OC; the extension of the canonical decomposition to non-linear refinement is §17 item 8.

Remark 13.1.3 (Where this stops and what comes next). Theorem 13.1 establishes the unified Hilbert-geometric framework but does *not* yet derive (a) the Born rule (probability structure from the unitary regime — see §14, §17 item 2), (b) the macroscopic Hamiltonian and the identification of $\hbar$_op (Remark 6.1.1, §17 items 3 and 5), (c) the Einstein-like field equations from operational continuity and the Ricci structure (OC §12; here §17 item 12), or (d) the d_op

→ 4 dimensional reduction (OC Open Problem 9, inherited here via §17 item 9). These remain as the open programme. What Theorem 13.1 establishes is that the *carrier substrate and the canonical transport decomposition* on which these derivations would have to operate is now a single layered mathematical object with a single canonical structure, with the complex structure on its top layer derived from substrate symmetry rather than imposed.

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14. Relation to Quantum Reconstruction

The quantum reconstruction programme requires the following structural ingredients:

1. A Hilbert carrier space;
2. An inner product on the carrier space;
3. **Complex structure** on the carrier (selecting $\mathbb{C}$ over $\mathbb{R}$ or $\mathbb{H}$);
4. Reversible norm-preserving dynamics;
5. A self-adjoint generator for the dynamics;
6. Projection/update structure for measurement-like events;
7. A probability measure structure satisfying the Born rule.

The operational geometry paper (OG) supplied (1) and (2): $\mathcal{A}$ is the carrier, $\langle\cdot,\cdot\rangle$_op the inner product. The present paper supplies (3) via Theorem 3.5 ($\mathbb{Z}_7$-equivariance forces the complex structure), (4) and (5) (Theorems 4.1, 5.1, Corollary 6.1), and (6) (Theorem 7.1). The operational Hausdorff and packing measure structure of OG supplies the substrate for (7), with the full Born-rule derivation requiring the additional probability-selection arguments developed in the dedicated quantum-reconstruction papers (the *Admissibility Closure* and *Born Rule Overdetermination* sequences within the VERSF corpus).

The present paper does *not* rederive the Born rule. Its role with respect to the quantum-reconstruction programme is more precise:

It identifies the layered operational Hilbert geometry on which the Born-rule reconstruction acts, supplies the substrate-derivation of the complex structure (3), and supplies the canonical unitary/projection separation of admissible transport (4)–(6) that the reconstruction takes as input.

The probability-measure layer (7) is the remaining structural input that the Born-rule reconstruction supplies. The complex-structure layer (3), traditionally a deep structural assumption of quantum mechanics, is here *derived* from the substrate $\mathbb{Z}_7$-equivariance plus the requirement that admissibility acts equivariantly — a substantive narrowing of the postulational base of quantum mechanics, even before the Born rule is addressed.

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15. Relation to Geometric Emergence

The refinement-geometry programme (developed in OC and its sequels) requires the following ingredients:

1. A transport structure on an operational substrate;
2. A metric structure on the substrate;
3. Density variation across the substrate;
4. Curvature derived from density variation;
5. Projection/refinement stability of the metric-and-curvature structure;
6. Coarse-graining to a smooth (or C^{1,α}) Riemannian limit;
7. **Closed-form tensorial curvature structure** (Ricci tensor in closed form, not just scalar curvature);
8. **Dynamical metric evolution** under operational transport.

The operational geometry paper (OG) supplied (1) and (2) in Hilbert form. OC supplied (3), (4), and (6) — distinguishability density, conformal-deformation curvature, and the C^{1,α} Cheeger–Gromov limit. The present paper supplies (5) via Theorems 12.1 and Corollary 12.2 (projection acts coherently on the conformal-deformation structure, preserving curvature support and inheriting flatness from fibre-uniform substrates), (7) via Theorem 10.2 (full Ricci tensor in closed form, with anisotropic/isotropic decomposition), and (8) via Theorem 10.3 (operational metric flow under refinement transport, coupled to density and velocity evolutions).

The bridge the present paper supplies is therefore:

operational Hilbert geometry → $\mathbb{Z}_7$-derived complex structure → reversible-vs-commitment transport decomposition → density-deformed operational curvature → projection-compatible coarse-grained geometric transport.

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16. Predictions and Falsifiability

The unified refinement Hilbert geometry framework predicts:

- **Substrate-derived complex structure.** The complex structure on $\mathcal{A}$_nt is uniquely determined by the substrate $\mathbb{Z}_7$-action (Theorem 3.5); no independent quantum-mechanical complex-structure postulate is required.
- **Polar-decomposition canonicality.** Every admissible refinement transport on $\mathcal{A}$_$\mathbb{C}$ decomposes uniquely as $R = U \cdot |R|$ with U a partial isometry and |R| positive semidefinite, both $\mathbb{Z}_7$-equivariant.
- **Reversibility ⟺ unitarity.** Every reversible admissible transport is unitary after taking the substrate-derived complex structure into account; non-unitary admissible transports are necessarily non-reversible.
- **Generator structure.** Strongly continuous one-parameter groups of reversible admissible transports admit a self-adjoint generator and satisfy the operational Schrödinger equation $i \cdot d\psi/d\tau = H\,\psi$.

- **Strict distinguishability loss under non-trivial projection.** Orthogonal commitment projection P with ker(P) ≠ {0} strictly decreases operational distance whenever the displaced vectors have non-zero component in ker(P).
- **Curvature from density only.** In dimension d_op ≥ 3, scalar curvature R_ρ vanishes if and only if ρ is constant; non-zero density gradients generate curvature in the conformal class.
- **Projection–curvature compatibility.** Curvature support is confined to projected density support; constant density projects to density proportional to the fibre measure of M_op under P, with curvature vanishing on fibre-uniform substrates.
- **Entropy non-decrease under |R|.** Under Hypothesis 7.2, the positive factor |R| in the polar decomposition is entropy-non-decreasing on every state, with strict increase wherever |R| has eigenvalues below 1 in directions the state has support on (Theorem 7.2).
- **Transport–projection noncommutativity.** Reversible transport U and commitment projection P do not commute unless U preserves the decomposition Im(P) ⊕ ker(P) (Theorem 9.2).
- **Projection–projection noncommutativity.** Two orthogonal projections P_1 and P_2 commute if and only if they admit a joint four-fold orthogonal decomposition into common eigenspaces of {P_1, I − P_1} × {P_2, I − P_2} (Theorem 9.3).
- **Closed-form Ricci tensor.** The Ricci tensor of the conformally-deformed metric is given in closed form as a traceless tensorial piece plus an isotropic trace piece (Theorem 10.2), with the trace recovering the scalar curvature of Theorem 10.1.
- **Joint flow of density, entropy, and metric.** Under refinement transport, the operational density, entropy, and metric evolve jointly — the metric undergoing a position-dependent conformal flow $\partial_\tau g = \lambda \cdot g$ with rate function λ given in closed form (Theorem 10.3).

Potential falsifiers include:

- demonstration that the substrate $\mathbb{Z}_7$-action does *not* uniquely determine a complex structure on $\mathcal{A}$_nt — e.g. demonstration of a quaternionic structure forced by additional substrate symmetry (violation of Theorem 3.5);
- demonstration of an admissible refinement transport whose polar decomposition is not unique or does not preserve $\mathbb{Z}_7$-equivariance on each factor (violation of Theorem 8.1 or Proposition 8.4);
- demonstration of a reversible admissible transport whose complexification is not unitary (violation of Theorem 4.1);
- demonstration of a strongly continuous one-parameter group of reversible admissible transports without a self-adjoint generator (violation of Theorem 5.1);
- demonstration that orthogonal commitment projection with non-trivial kernel preserves all operational distances strictly (violation of Theorem 7.1(iv));
- demonstration of entropy strictly decreasing under a non-expansive admissible refinement map, while Hypothesis 7.2 holds (violation of Theorem 7.2);
- demonstration of commuting [U, P] = 0 for a unitary U and orthogonal projection P that does not preserve the Im(P) ⊕ ker(P) decomposition (violation of Theorem 9.2);
- demonstration of commuting [P_1, P_2] = 0 for two orthogonal projections that do not admit a joint four-fold orthogonal decomposition (violation of Theorem 9.3);

- demonstration of non-zero curvature in a region of constant ρ in dimension $d_{op} \geq 3$ (violation of Theorem 10.1 / OC Theorem 8.2);
- demonstration that the operational Ricci tensor does not decompose into a traceless plus trace piece consistent with conformal-flat curvature (violation of Theorem 10.2);
- demonstration that the operational metric does not evolve conformally under refinement transport with rate λ given by Theorem 10.3 (e.g. demonstration of a non-conformal metric evolution arising directly from continuity);
- demonstration that admissible transport on $\mathcal{A}_{\mathbb{C}}$ admits dynamical content beyond the polar decomposition plus conformal-deformation regimes — i.e. that the three-regime decomposition of Theorem 13.1 is incomplete.

Each falsifier corresponds to a violation of one of the structural results inherited from OG/OC or established in the present paper.

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17. Open Problems

The unified refinement Hilbert geometry framework is now in place. The following problems remain open within the broader programme:

1. **Full substrate-derivation of the complex structure including trivial-channel handling.** Theorem 3.5 derives the complex structure on $\mathcal{A}_{nt}$ from $\mathbb{Z}_7$-equivariance. Two extensions remain: (a) substrate classification of the trivial $\mathbb{Z}_7$-channel $\mathcal{A}_{triv}$ — whether it is empty, carries only static background, or requires a separate real-structured transport regime; (b) explicit substrate-level exclusion of more exotic carrier-structure alternatives (octonionic, modular, complex-with-superselection) beyond the quaternionic case already excluded in Remark 3.5.2.
2. **Born rule from operational Hilbert geometry.** Whether the probability-measure structure of §14 item (7) is fully derivable from the polar decomposition plus the operational Hausdorff measure on $\mathcal{A}_{\mathbb{C}}$, without additional probability-selection axioms. (Central open problem of the quantum-reconstruction sequence.)
3. **Substrate-level form of H.** Closed-form derivation of the self-adjoint generator H of physically-realised reversible admissible transports, in terms of substrate primitives (the fold, BCB/TPB layers, K = 7 closure architecture).
4. **Substrate-level form of $|\mathbf{R}|$.** Closed-form classification of the positive factor $|R|$ for physically-realised refinement transports — whether it is generically an orthogonal projection (full commitment), an attenuation (partial commitment), or a mixture, and how the spectrum of $|R|$ encodes commitment strength.
5. **Identification of $\hbar_{op}$ with macroscopic $\hbar$.** Whether the substrate-level action quantum $\hbar_{op}$ of Corollary 6.1 corresponds quantitatively under coarse-graining to the macroscopic Planck constant $\hbar$.
6. **Bounded-curvature inheritance under projection.** Whether the pointwise curvature bound $\|R_{op}\| \leq C / \Delta_{op}^2$ of OC Theorem 9.1 transfers cleanly to $\|R_{\{\rho_P\}}\| \leq C / \Delta_{op}^2$ under orthogonal projection, including the density-contrast prefactor.

7. **Quantum-gravity coupling.** Whether the reversible regime (III) and the curvature regime (VI) of Theorem 13.1 couple non-trivially under joint evolution — i.e. whether a quantum-mechanical wavefunction $\psi \in \mathcal{A}_{\mathbb{C}}$ evolves under $U_{\mathbb{C}}(\tau)$ on a substrate whose distinguishability density ρ is itself τ-dependent, and what the consequent back-reaction looks like.
8. **Generalised polar decomposition for non-linear refinement.** Whether the polar-decomposition canonicality (Theorem 8.1) extends to non-linear admissible refinement maps (e.g. OC's entropy-gradient flow, which is non-linear in the trajectory but linear in the velocity field). The current results assume linearity; non-linear extensions require either operator-monotone-function calculus or a different decomposition.
9. **Connection to OC open problems.** OC §14 lists 14 open problems internal to the curvature programme. The following overlap with the present framework: OC item 5 (Einstein-like equations from operational continuity) intersects this paper via the projection-compatible coarse-graining of §12 and §15, the Ricci structure of §10.2, and the metric flow of §10.3; OC item 8 (Lorentzian signature from refinement-flow asymmetry) intersects via the reversible-vs-irreversible decomposition of §8–§9, with v_{op} in OC §12(b) corresponding to the velocity field generated by the U-factor; OC item 9 (dimensional reduction $d_{op} \to 4$) is inherited from OC without contribution here; OC item 10 (quantum operational geometric dynamics) is addressed in part by the present paper but remains open in its full coarse-grained-quantum-field-theory form. The unified framework of Theorem 13.1 supplies the carrier-space structure on which these joint open problems would need to be addressed.
10. **Substrate-level derivation of Hypothesis 7.2.** Whether the operational entropy monotonicity condition ($S_{op}(T x) \geq S_{op}(x)$ for non-expansive T) follows from admissibility-level substrate primitives — analogously to the substrate-level derivation of the OC entropy concavity hypothesis as a separate open problem. The hypothesis is structurally on a par with admissibility itself, and a full derivation would close the second-law structural input of §7.2.
11. **Operational Yamabe-type flow as macroscopic geometric flow.** Whether the operational metric flow of Theorem 10.3 reduces, under coarse-graining at $L \gg \Delta_{op}$, to a Yamabe-type or Ricci-type flow on the emergent 4-manifold. The structural similarity is strong (conformal flow with non-trivial source), but the explicit reduction requires the dimensional reduction $d_{op} \to 4$ and Lorentzian signature emergence to be in place; the candidate target equation is a parabolic flow on the macroscopic continuum driven by operational entropy gradients.
12. **Closed-form Einstein-like equation from the Ricci structure plus continuity.** Whether the Ricci tensor of Theorem 10.2, combined with the continuity equation of OC Theorem 10.3 and the dynamics of Theorem 10.3 here, yields a closed-form Einstein-like field equation relating operational curvature to operational distinguishability stress-energy. The structural ingredients are now in place (Ricci tensor in closed form, trace giving scalar curvature consistent with OC, density-evolution equation, metric-evolution equation); the missing step is to identify the appropriate operational stress-energy tensor and verify that the resulting field equation is consistent with the conformal-flat restriction imposed by $\mathbb{Z}_7$-equivariance.
13. **Quantitative noncommutator estimates.** Theorems 9.2 and 9.3 establish *when* the commutators $[U, P]$ and $[P_1, P_2]$ vanish but do not *quantify the size* of these

commutators when they do not. Whether ‖[U, P]‖ admits substrate-level bounds (e.g. in terms of the spectral overlap of U's invariant subspaces with Im(P) and ker(P)), and whether ‖[P_1, P_2]‖ admits bounds in terms of the angles between the four common subspaces of Theorem 9.3, is an open quantitative question with potential connection to operational measurement-disturbance relations and to substrate-level uncertainty principles.

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18. Conclusion

The operational geometry paper (OG) established that the admissible subspace $\mathcal{A}$ = Im(Ω_max) carries an intrinsic finite-dimensional real Hilbert geometry, derived from admissibility alone. The operational curvature paper (OC) established that the operational distinguishability density ρ_op conformally deforms this flat reference into a curved Riemannian metric whose scalar curvature is generated by density gradients and bounded by finite packing. Together, OG and OC supplied the kinematics and the metric-deformation dynamics of the operational geometry.

The present paper supplies the **transport** dynamics, together with the substrate-derivation of the complex structure on which quantum-type evolution lives. Three structural inputs combine:

(A) Substrate-derived complex structure. The $\mathbb{Z}_7$-equivariance of the substrate forces a canonical complex structure J on the non-trivial spectral channels $\mathcal{A}$_nt (Theorem 3.5), making ($\mathcal{A}$_nt, ⟨·,·⟩_op, J) into a complex Hilbert space $\mathcal{A}_{\mathbb{C}}$ without any independent quantum-mechanical postulate. The substrate selects $\mathbb{C}$ over $\mathbb{R}$; the quaternionic alternative is excluded by the cyclic structure of $\mathbb{Z}_7$ (Remark 3.5.2). The complex structure is *derived*, not assumed.

(B) Canonical polar decomposition with equivariance preservation. Every $\mathbb{Z}_7$-equivariant admissible linear refinement transport on $\mathcal{A}_{\mathbb{C}}$ decomposes uniquely as R = U · |R| (Theorem 8.1), with both U and |R| $\mathbb{Z}_7$-equivariant (Proposition 8.4) and therefore individually admissible. The unitary factor U preserves operational distinguishability; the positive factor |R| can only reduce it. The U-factor is entropy-preserving and the |R|-factor is entropy-non-decreasing (Theorem 7.2), supplying the operational-Hilbert origin of the second-law arrow of refinement.

(C) Transport–projection noncommutativity. Reversible transport U and orthogonal commitment projection P generically fail to commute (Theorem 9.2), with [U, P] = 0 if and only if U preserves the orthogonal decomposition Im(P) ⊕ ker(P). This is the operational-Hilbert origin of order-dependence between transport and commitment — the structural analogue of measurement–evolution noncommutativity in quantum mechanics.

(D) Conformal-deformation curvature with full Ricci structure. OC's conformal deformation g_op^{(ρ)} = ρ^{2/d_op} · ⟨·,·⟩_op on M_op carries scalar curvature (Theorem 10.1) and full Ricci tensor (Theorem 10.2) given in closed form by gradients and Laplacians of the distinguishability density. The Ricci structure decomposes naturally into a traceless tensorial piece plus an isotropic trace piece — the structural precursor of the Einstein/cosmological-constant split.

(E) Operational metric flow. Under refinement transport, the operational metric itself evolves dynamically via a position-dependent conformal flow (Theorem 10.3), with rate function λ given in closed form in terms of the velocity field divergence and density gradient. This makes the operational geometry not a static stage but a dynamical object, evolving jointly with density and entropy under refinement.

Combining (A)–(E), the layered carrier structure {M_op ⊆ $\mathcal{A}$ ⊆ $\mathcal{A}$_ℂ} supports three structurally distinct transport regimes:

- **Reversible isometric transport** (the U factor) on $\mathcal{A}$_ℂ — the operational-Hilbert origin of quantum-type evolution, with self-adjoint generator and operational Schrödinger equation;
- **Irreversible commitment projection** (the |R| factor) on $\mathcal{A}$_ℂ, intersecting M_op via Ω_max — the operational-Hilbert origin of fact formation, sector selection, and the second-law arrow;
- **Conformal-deformation curvature** (the metric-deformation regime of OC) on M_op — the operational-Hilbert origin of emergent geometric structure, now equipped with full Ricci structure and dynamical metric flow.

The three regimes share one layered carrier, with the complex structure of the top layer derived from substrate symmetry rather than postulated. Quantum-like evolution and geometric curvature are not separate structures imposed on disconnected mathematical objects; they are different operations on different layers of one operational Hilbert geometry, with commitment projection as the third canonical mode that interpolates between them, entropy monotonicity carried by the positive factor of the polar decomposition, order-dependence generated by the non-commutativity of U and P, and joint dynamical flow of density, entropy, and metric under refinement transport.

The full quantum-gravity reconstruction — Born-rule derivation, Einstein-like field equations, dimensional reduction d_op → 4, Lorentzian signature emergence — remains open (§17). What the present paper establishes is that the *carrier substrate, the canonical transport decomposition, the substrate-derivation of the complex structure, the second-law arrow, the order-dependence of transport and projection, the closed-form Ricci structure, and the dynamical metric flow* on which such reconstruction would have to be built are now a single layered mathematical object with a single canonical structure, derived from admissibility plus substrate symmetry alone.

Admissibility plus $\mathbb{Z}_7$-equivariance supplies a single layered operational Hilbert geometry with substrate-derived complex structure on its top layer. Polar decomposition supplies the canonical transport split between reversible and irreversible content. Conformal deformation supplies curvature on the bottom layer. Quantum dynamics, commitment, and geometry are three faces of one substrate-level structure. The polar decomposition additionally carries the second-law arrow (via |R|'s entropy-monotonicity), the order-dependence of transport and commitment (via [U, P] ≠ 0), and the dynamical evolution of the operational metric itself (via the Ricci structure and conformal flow under refinement). The bridge between the VERSF quantum-reconstruction programme and the VERSF refinement-geometry programme is the polar-decomposition canonicality of admissible refinement transport on the same layered carrier that both programmes

already share — with its complex structure, entropy arrow, noncommutativity, Ricci structure, and metric flow all derived rather than assumed.