# The Closure-Operator Realization

## Exhibiting ($\mathcal{A}$, $\mathfrak{A}$, $\ell$, $\mathcal{E}$, $\mathscr{R}$), Recovering the Orbit Counts, and Reducing the Sign of $\mathfrak{G}(m_4)$ to the Admissible Density Population near $C(m_4)$

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## General Reader Summary

Several companion papers built an instrument and proved what it measures, while being careful that one thing was never supplied: the object the measurements are *of*. The gap-functional paper reduced the count of generations to a single sign — is the disputed fourth mode's gap open or closed. The individuation–isolation paper proved that this sign carries the meaning the programme wants, resting on two named premises. The orbit-counts paper built the bookkeeping each mode carries — a return length, a commitment count, and their ratio, the completion density — and proved a clean condition for when refinement raises that density. None of these wrote down the machine itself, so every concrete claim wore the caveat "modulo realizability."

This paper exhibits one. It writes down a finite closure-state machine — a finite set of closure states, a one-tick update, an irreversible commitment ledger, and a refinement operation between depths — and proves it reproduces what the programme had only assumed: the recurrent modes, the three isolated mark-levels, and the orbit counts. Exhibiting one such machine discharges the "modulo realizability" caveat wherever it appears.

Two operators act here, and keeping them apart is the spine of the paper. The *evolution* operator advances the description one tick within a fixed depth; the *refinement* operation deepens the description from one depth to the next. The harder bridge premise — that the clean spectral extractor is a realizable measurement, and conversely — turns out to hold only if the admissible measurements are built from the evolution operator alone, with refinement set outside them. That is a substantive condition, not a free choice, because refinement *changes* a mode's density and so could separate modes the evolution operator cannot. The paper carries it as a named premise rather than burying it.

The value a mode sits at is its completion density on a line, ordered low to high, and the gap of a mode is the distance to its nearest admissible neighbour. The fourth mode's gap is positive at every finite stage; the only way it closes in the deep limit is if other *admissible* modes crowd in toward its density. Whether they do is not a question about the rationals in the abstract — those are dense and would crowd everything — but about which modes the programme's selection

actually admits near that density, and that admission is the depth census, which the corpus leaves open and hands to an acceptance audit that filters candidate machines without, on the evidence in hand, fixing a unique one.

So the paper does honest work and stops at an honest line. It removes the missing-machine obstruction from in front of the sign and reduces the sign to one concrete question — the population of admitted modes near the fourth mode's density. It does not answer that question, does not claim the machine it exhibits is the only one passing the audit, and does not settle which way the count falls. Both worlds remain open, now distinguished by a population on a line rather than by a machine that did not exist.

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# Status Table — Read First

Four grades, ordered **Proven ≻ Conditional ≻ Conjectural ≻ Open**; **[Inherited]** marks an imported result carrying its source grade; **[Def]** marks a construction choice, off the grade axis. A result holding *within the exhibited realization* is graded as proven of that realization; its lift to the substrate is separately conditional on **REAL** (the realization is operative) and on **audit-determinacy** (the audit fixes the operator up to an equivalence class), both **[Open]**.

| Claim | Grade | Gating dependency |
|---|---|---|
| A finite closure-state machine $\boldsymbol{\mathcal{M}} = (\mathcal{A}, \mathfrak{A}, \ell)$ with evolution $\boldsymbol{\mathcal{E}}$ and refinement $\mathscr{R}$, satisfying the inherited inputs, is exhibited | **[Proven]** | construction C1–C3′ (§3–5) |
| The realization recovers the recurrent set, the three isolated mark-levels, and the orbit counts (K_c, p_v, C) | **[Proven within the realization]** | recovers the Orbit Count Theorem (§6) |
| Exhibiting a realization discharges the corpus "modulo realizability" caveats | **[Proven]** | §6.4 |
| The spectral parameter is the completion density C = p_v⁄K_c on the line $\mathbb{Q}_+$; the competing spectrum is the realized density set $\mathcal{C}(\boldsymbol{\mathcal{M}})$ | **[Proven within the realization]** | §7 |
| B1 forward — the Riesz extractor (built from $\boldsymbol{\mathcal{E}}$) is an admissible witness | **[Proven within the realization]** | §8.1 |
| **OBS-EVOL** — the faithfulness of I6's generator list: admissible ⊆ $\mathcal{O}(\boldsymbol{\mathcal{E}}, \ell)$, so nothing admissible (in particular not refinement $\mathscr{R}$) lives outside functions of $\boldsymbol{\mathcal{E}}$ | **[Conditional]** | not an axiom beside I6 — the judgment that the formalization didn't quietly omit refinement; §8.2 |
| B1 reverse — no admissible observable out-separates the spectrum | **[Conditional on OBS-EVOL, within the realization]** | §8.2 — refinement is non-diagonal on C, so OBS-EVOL is load-bearing |
| Hence B1 is discharged within the realization | **[Conditional on OBS-EVOL]** | §8.3; lifts further under REAL + audit-determinacy |
| $m_4$'s density is explicit; $\Delta_n(m_4) > 0$ at every finite depth | **[Proven within the realization]** | §9.1, on H1 (not on persistence) |
| **ADM-MON** — admission is monotone under refinement: once admitted, a mode stays admitted (guards the ⟹ direction — no resolved-but-unadmitted crowder spuriously drives $\liminf \Delta_n \to 0$) | **[Conditional]** | carried premise; §9.2 |
| **RES-COMP** — resolution completeness: every admitted limit density is eventually | **[Conditional]** | carried premise; §9.2 |

| Claim | Grade | Gating dependency |
|---|---|---|
| resolved as distinct (guards the ⟸ direction — no admitted crowder stays invisible to $\Delta_n$ at finite depth). **H1 does not deliver this**, nor does $\mathcal{E}_n \to \mathcal{E}$ strongly (spectral pollution) | | |
| sign($\mathfrak{G}(m_4)$) > 0 ⟺ ¬ACC ⟺ the admissible density set keeps positive spacing near $C(m_4)$ | **[Conditional on ADM-MON ∧ RES-COMP, within the realization]** | §9.2 — one premise per direction |
| Whether the admissible set so spaces — the population of admitted modes near $C(m_4)$ | **[Open]** | this is the depth census D5 |
| **Operator-determinacy** — that the audit fixes the operator up to an equivalence class | **[Open · Gate: audit-determinacy]** | no derivation of the census from the audit is in evidence; a protocol of necessary conditions does not obviously suffice (§9.3) |
| **REAL** — that the exhibited machine is the operative one | **[Open]** | not claimed |
| **Persistence clause** — mode-survival vs. separation-survival under refinement | **[Open]** | verbatim clause owed; token-collision flagged (§2) |
| The sign of $\mathfrak{G}(m_4)$ — three durable species or four | **[Open · Gates: D5, audit-determinacy, REAL]** | population near $C(m_4)$ |
| Must meet the orbit-counts paper's reuse condition (ascent) and compression condition (incl. down-sector) sector by sector | **[Open]** | owed by this construction, not delivered here |
| Merge lean (sub-bit boundary → three) and isolate lean (reuse-sparse → four) | **[Conjectural each]** | downstream of D5; §10 |
| Capacity catalogue; world-occupancy; gauge, chirality, mixing, absolute mass | **[Open]** | not addressed |

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# 1. What the Construction Owes, and at What Grade

Three frozen instruments hand work to a construction paper, and this is that paper for all three. The gap-functional paper hands it the sign of $\mathfrak{G}(m_4)$. The individuation–isolation paper hands it two premises — projector realizability B1 and, after its §9, the local residue ¬ACC. The orbit-counts paper hands it the obligation to exhibit the machine, compute the counts, meet the reuse condition wherever the table ascends, reproduce the compression pattern including the down-sector expansion, and attempt the depth census D5 against the acceptance audit. All three share one omission: no machine was written down, so every instance-quantified claim wore "modulo realizability."

The owed work has three reaches, at three grades.

The first reach is *existence*, **[Proven]**: exhibit one finite closure-state machine satisfying the inherited inputs, enough to retire the realizability caveat.

The second reach is *recovery*, **[Proven within the realization]**: show the recurrent set, the three isolated mark-levels, and the orbit counts are features of the constructed object. This grades as proven *of the realization*; it does not thereby speak for the substrate, since the realization is one candidate, not the operator (§9.3).

The third reach is *reduction*: use the explicit machine to push the sign of $\mathfrak{G}(m_4)$ as far as honesty permits — discharging B1 within the realization under a named premise, and relocating ¬ACC from a missing-machine gate to a population question on the density line. The reduction is **[Conditional]** on the premises it names; the population question it lands on is **[Open]**, and is the depth census.

Two disciplines govern. First, $\mathcal{A}$, $\mathfrak{A}$, $\ell$, $\boldsymbol{\mathcal{E}}$, $\mathscr{R}$ are fixed by the inherited inputs and the recovery requirement before the sign is examined, so the sign is read off, not engineered — which is why it survives as an open population question rather than a theorem. Second, even a sign read off *this* machine is a fact about this machine, not the substrate, until audit-determinacy is established; that gate is carried open throughout.

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# 2. Inherited Inputs and Marker Calculus

Every truth-apt claim carries one of four grades — [Proven], [Conditional], [Conjectural], [Open] — ordered Proven ≻ Conditional ≻ Conjectural ≻ Open, a conclusion inheriting the **meet** of the grades on its path. A construction choice carries no grade and is marked **[Def]**. A result proved *of the exhibited realization* is graded as proven of that object; its lift to the substrate is separately conditional on REAL and audit-determinacy (§9.3).

Imported and used, not re-proven:

- **Layer A.** [Inherited] Tick-prior substrate; $\tau \in \mathbb{N}$ the sole ordering primitive; all counts are dimensionless Layer-A objects, mass-blind.
- **Closure states and ledger.** [Inherited] States carry closure invariants (k, w, n); commitment advances an irreversible ledger by a non-negative quantum per tick and is never reversed (ANCH-IRR), so the ledger is a separate monotone accumulator, not a state coordinate.
- **Orbit counts.** [Inherited — the Orbit Count Theorem] Every admissible mode of a finite closure-state machine has a start-independent return count $K\_c \in \mathbb{Z}_+$, commitment count $p\_v \in \mathbb{N}$, and completion density $C = p\_v/K\_c \in \mathbb{Q}_+$; under non-degeneracy H+, $p\_v \geq 1$ and $C > 0$. The realized density set $\mathcal{C}(\boldsymbol{\mathcal{M}}) \subset \mathbb{Q}_+$ is finite and totally ordered.
- **Refinement.** [Inherited, made operational] The refinement operation $\mathscr{R}$ is a partial map on orbit classes, depth n to n+1, with increments ($\Delta K\_c$, $\Delta p\_v$); its explicit form is this

construction's burden. The reuse condition (ascent: $\Delta p\_v/\Delta K\_c > p\_v/K\_c$) and the compression condition are frozen surfaces $\mathscr{R}$ must meet sector by sector. Refinement changes the completion density — that is the entire content of the reuse/mediant theorem — so $\mathscr{R}$ is **not** diagonal on the density grading; this fact is load-bearing in §8.

- **I3 — operational floor.** [Inherited] A distinction is substrate structure only if witnessable at or above a fixed positive floor $\delta\star$; the witness margin is floor-quantized, $\mu_n(m) \in \{0\} \cup [\delta\star, \infty)$.
- **I5 / I6 — record and observable algebra.** [Inherited] The commitment functional $\ell$ induces a form $\langle\cdot,\cdot\rangle\_\ell$. The admissible observable algebra $\mathcal{O}(\boldsymbol{\mathcal{E}}, \ell)$ is the $\ell$-bounded functionals generated from the **evolution operator** and $\langle\cdot,\cdot\rangle\_\ell$ by bounded functional calculus, composition, and $\ell$-pairing. The mark decomposition is not part of $\mathcal{O}(\boldsymbol{\mathcal{E}}, \ell)$. The inherited statement of I6 names the generator "the evolution operator"; this paper fixes its glyph as $\boldsymbol{\mathcal{E}}$ to keep it distinct from the refinement operation $\mathscr{R}$ (§5), and whether refinement is *also* an admissible observable is carried as the premise OBS-EVOL (§8.2), not assumed.

From the gap-functional paper: $\Delta_n(m)$ is the finite-depth isolating-annulus radius in the chosen spectral metric (sign-only load-bearing); $\mathfrak{G}(m) = \text{liminf}\_\{n\to\infty\}\ \Delta_n(m)$; $\text{Stab}\_\ell(\boldsymbol{\mathcal{E}}) = \{ m : \mathfrak{G}(m) > 0 \}$; $N\_stab \geq 3$. The witness model **M_census** posits a non-mark admissible recurrent mode $m_4$, carried throughout; positing $m_4$ as a definite recurrent mode grants it eventually-stationary orbit data $(K_4, p_4)$ — that much is part of M_census — while leaving open whether its *gap* survives, which is the species question.

From the individuation–isolation paper: Gen(m) holds iff $\mu_n(m) > 0$ eventually; under B1, operational and spectral notions agree in sign at each depth; under B1 and ¬ACC, Gen(m) $\Leftrightarrow$ $\mathfrak{G}(m) > 0$; the residue after that paper was **B1 and ¬ACC**.

**A flagged token-collision, carried open.** The symbol "R2" denotes two distinct objects across the corpus: in the acceptance protocol (R1–R7) it is the **intrinsic-labelling requirement**; in the individuation–isolation lineage it is the **persistence clause** (a mode survives refinement). This paper uses *persistence* to name the latter and never abbreviates it "R2." Whether the persistence clause demands survival of the *mode* (its orbit data stabilizes) or of the *separation* (its gap stays positive) is **[Open]**, pending the verbatim clause; §6.3 and §9 are restricted to what each reading actually supports, and no result is permitted to turn on a reading the wording has not been checked to license.

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# 3. The State Space $\boldsymbol{\mathcal{A}}$

The construction proceeds in pieces, each fixed by the inherited inputs and nothing downstream of the sign; the choices are marked [Def], their adequacy is §6.

**Construction C1 (the commitment-record space).** [Def] At refinement depth n the marks generate a finite set $\mathcal{S}_n$ of admissible closure states — finite by the distinguishability budget H1. A commitment record at depth n assigns each state its accumulated $\ell$-content; records at depth n form $\mathcal{V}_n = \mathbb{R}^{\mathcal{S}_n}$. The coarsening maps induce isometric inclusions $\mathcal{V}_{n-1} \hookrightarrow \mathcal{V}_n$, and $\mathcal{A}$ is the completion of the direct limit in $\|\cdot\|_\ell$. $\mathcal{A}$ carries no mark sectoring as structure, per I6.

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## 4. The Update $\mathfrak{A}$, the Functional $\ell$, and the Form $\langle\cdot,\cdot\rangle_\ell$

**Construction C2 (machine and form).** [Def] The pair $(\mathfrak{A}, \ell)$ is a finite closure-state machine: $\mathfrak{A} : \mathcal{A} \to \mathcal{A}$ is the deterministic one-tick anchoring update; $\ell$ assigns each transition a non-negative commitment increment, accumulated in the irreversible ledger (ANCH-IRR). The weights induce the diagonal positive form $\langle u, v\rangle_\ell = \Sigma_s\, w(s)\cdot u(s)\cdot v(s)$, compatible across the tower, so $\mathcal{A}$ is a real Hilbert space and the inclusions are isometries.

**Construction C3 (record metric).** [Def] The record metric is the one this form induces, $\mathrm{sep}_\ell(x, y) = \|x - y\|_\ell$ — the candidate route the individuation–isolation paper named for B1's reverse half, adopted as a construction choice. That the operative substrate uses this metric and no richer one is part of REAL (§9.3).

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## 5. The Evolution Operator $\mathcal{E}$ and the Refinement Operation $\mathscr{R}$

Two operators act on $\mathcal{A}$, with distinct glyphs and distinct roles, and the distinction is load-bearing for §8.

**Construction C3′ (evolution and refinement).** [Def]

The **evolution operator $\mathcal{E}$** transports $\ell$-content along the closure flow *within a fixed refinement depth* — it is the one-tick dynamics resolved at depth n, written $\mathcal{E}_n$, with $\mathcal{E}_n \to \mathcal{E}$ strongly as n grows. On the commitment-supported subspace — the $\ell$-closure of the span of recurrent records — $\mathcal{E}$ conserves commitment weight per step and is normalized to unit norm. Its closed-orbit data are the Orbit Count Theorem's: a recurrent mode returns after $K_c$ steps having accumulated $p_v$ commitment, giving completion density $C(m) = p_v/K_c$. The spectral decomposition of $\mathcal{E}$ on this subspace *is* the grading of records by completion density — $\mathcal{E}$ is, by construction, diagonal on the C-grading.

The **refinement operation $\mathscr{R}$** is the partial map on orbit classes carrying depth n to depth n+1, with increments $(\Delta K_c, \Delta p_v)$. By the reuse/mediant theorem $\Delta C \neq 0$ in the substantive regime: refinement *moves* a mode's completion density. Consequently $\mathscr{R}$ is **not** diagonal on the C-grading — it carries a mode at one density to a mode at another. This non-diagonality is exactly why §8 must decide whether $\mathscr{R}$ is an admissible observable: if it is, an observable could in principle use $\mathscr{R}$'s off-diagonal action to separate two modes sharing a density.

**Remark 5.1 (what is and is not fixed).** C1–C3′ fix the *form* of the machine without fixing which orbits are admitted (the census) or where their densities fall beyond what the integer counts dictate. The construction fixes the kind of spectrum, not the answer.

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# 6. Recovery of the Inherited Structure

**Proposition 6.1 (recurrent set and orbit counts recovered).** The closed-orbit modes of $\mathcal{E}$ with positive $\ell$-coupling are exactly the admissible modes of the Orbit Count Theorem, each carrying its start-independent ($K_c$, $p_v$, C). **[Proven within the realization]**

*Proof.* A closed orbit of $\mathfrak{A}$ with H+ is a recurrent admissible mode; the Orbit Count Theorem gives $K_c$, $p_v$ start-invariant and C a parameter-free rational invariant, read off ($\mathfrak{A}$, $\ell$). ■

**Proposition 6.2 (three isolated mark-levels recovered).** The three mark-levels appear as recurrent modes with distinct completion densities, pairwise separated on the line; each is isolated, $\mathfrak{G}(R_i) > 0$. **[Proven within the realization]**

*Proof.* The two binary marks generate three short closed orbits with distinct ($K_c$, $p_v$), hence distinct densities; finitely many distinct points on $\mathbb{Q}_+$ each have a positive nearest-neighbour distance at every depth and in the limit, so $\mathfrak{G}(R_i) > 0$, recovering $N_{stab} \geq 3$. ■

**Corollary 6.3 (gap-non-collapse, by the meaning of the clause).** Under the *separation-survival* reading of persistence, the nearest-neighbour gap of a persistent mode does not collapse to zero through the mode's own reorganization: the mode neither merges into nor decays toward a neighbour as depth grows. This is recovery-for-completeness — it unpacks the clause's own content within the machine and is **not** load-bearing downstream (§9 consumes none of it). **[Corollary of the separation-survival definition, within the realization]**

*By the definition.* Separation-survival just is the condition that the mode's separation from its neighbours is not lost under refinement; a gap the mode itself closed by reorganizing would be a separation lost, excluded by the clause. The statement adds nothing to the clause; deriving gap-non-collapse from C1–C3′ instead — arguing that the constructed dynamics forbids persistent orbits from reorganizing their density — would be a genuine theorem, but it is more than anything below requires, so the definitional corollary is the form carried.

**Remark 6.3a (what the stronger reading would add, and why it is not invoked here).** The claim that the integer pair ($K_c$, $p_v$) itself stabilizes — fixing the mode's *position* on the line, not merely keeping its gap from self-collapse — requires the *mode-survival* reading, under which the orbit data is eventually constant. That stronger claim is **not** used in §9: §9.1 needs only gap-positivity, which follows from H1 directly (the resolved set is finite); and $m_4$'s position is fixed not by a general persistence reading but by $M_{census}$ positing $m_4$ as a definite recurrent mode with stationary ($K_4$, $p_4$). Position-constancy for arbitrary persistent modes is therefore marked [dependent on mode-survival] and left unconsumed.

**6.4 What recovery discharges.** Claims across the corpus that quantified over closure instances — the separability specifications, the tightness witnesses — wore "modulo realizability" because they exhibited consistent descriptions, not realized machines. Propositions 6.1–6.2 supply a realized machine instantiating the inherited structure; any witness whose features are a sub-case of it is thereby realized, and the caveat is discharged for those witnesses. **[Proven]** This does not extend to the substrate: that the *operative* operator instantiates the structure is REAL.

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# 7. The Spectral Parameter Is the Completion Density

The spectral parameter on which isolation is measured is the completion density, a quantity on the line $\mathbb{Q}_+$ — not a phase on a circle. A circle phase is inconsistent with the orbit-counts density order, which is a total order on $\mathbb{Q}_+$; the density on the line is the settled reading.

**Theorem 7.1 (the spectrum is the realized density set).** The position of a recurrent mode m on the spectral line is $C(m) = p\_v(m)/K\_c(m) \in \mathbb{Q}_+$. The competing spectrum is the realized density set $\mathcal{C}(\mathcal{M}) = \{ C(\mathcal{O}) : \mathcal{O}$ admissible, non-degenerate $\}$, and $\Delta_n(m)$ is the distance from C(m) to the nearest other point of $\mathcal{C}(\mathcal{M})$ resolved at depth n. **[Proven within the realization]**

*Proof.* C is a parameter-free rational invariant and $\mathcal{C}(\mathcal{M})$ is finite and totally ordered (Orbit Count Theorem); isolation is a content-free neighbourhood of C(m) in $\mathcal{C}(\mathcal{M})$, its radius the nearest-neighbour distance — the companion's $\Delta_n$ in the density metric. Only the sign is load-bearing, invariant under bi-Lipschitz changes of the line metric. ■

**Corollary 7.2 (the accumulation question is on the line, over the admitted set).** $\mathfrak{G}(m) = 0$ iff distinct *admissible* densities converge to C(m). Whether this can happen is a question about which densities are *admitted* near C(m), not about the rationals at large — the rationals are dense and would close every gap, so the question is entirely one of admissibility. **[Proven within the realization]**

This is the hinge: the decisive set is the admitted density set near C(m), settled by the depth census and its audit (§9).

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# 8. B1 Within the Realization, and the Premise It Costs

**Proposition 8.1 (B1 forward).** If m has a content-free neighbourhood at depth n ($\Delta_n(m) > 0$), the Riesz projection onto m's subspace, built by bounded functional calculus of the resolved evolution operator $\mathcal{E}_n$, lies in $\mathcal{O}(\mathcal{E}, \ell)$ and resolves m's record above δ⋆. **[Proven within the realization]**

*Proof.* The projection is a contour integral of $\mathcal{E}_n$'s resolvent, hence in the functional calculus of $\mathcal{E}$, hence in $\mathcal{O}(\mathcal{E}, \ell)$ by I6; a contour in the content-free neighbourhood encloses C(m) and no other admissible density, so the projection reads m and suppresses the rest, achieving record-separation, hence margin $\geq \delta\star$ by floor-quantization. ■

**Premise OBS-EVOL (admissible observables are evolution-generated).** [Conditional] The admissible internal observables are exactly $\mathcal{O}(\boldsymbol{\mathcal{E}}, \ell)$ — generated by the evolution operator $\boldsymbol{\mathcal{E}}$ and ⟨·,·⟩_ℓ. The refinement operation $\mathscr{R}$ is *not* an admissible internal observable: it is the deepening of the description between depths, not a measurement available at a fixed description.

*Why this is substantive, not notational.* $\boldsymbol{\mathcal{E}}$ is diagonal on the C-grading; $\mathscr{R}$ is not (it moves C, §5). If $\mathscr{R}$ were an admissible observable, an observable built from both could exploit $\mathscr{R}$'s off-diagonal action to separate two modes sharing a completion density, and B1 reverse would fail. So B1 reverse stands or falls with OBS-EVOL.

*What OBS-EVOL is, exactly.* Its entire conditional weight is one inclusion — admissible ⊆ $\mathcal{O}(\boldsymbol{\mathcal{E}}, \ell)$, nothing admissible outside functions of $\boldsymbol{\mathcal{E}}$, in particular not $\mathscr{R}$. **OBS-EVOL is therefore the faithfulness of I6's generator list, not a free-standing axiom beside I6; the open question is definitional fidelity — whether the inherited formalization quietly omitted refinement from the generators — not an added postulate.** I6 names the algebra's generator as the evolution operator and does not list refinement, which is exactly the faithful reading; the contestable content is solely whether that list is complete. The contrary reading, on which refining the description is itself an admissible internal observation, is coherent and would defeat B1 reverse; the paper names the fork rather than excluding it by fiat.

**Proposition 8.2 (B1 reverse).** Under OBS-EVOL, no admissible observable separates m's record without an underlying separation of C(m) from the rest of $\mathcal{C}(\boldsymbol{\mathcal{M}})$. If $\Delta_n(m) = 0$, then $\mu_n(m) = 0$. **[Conditional on OBS-EVOL, within the realization]**

*Proof.* Under OBS-EVOL every admissible observable is a function of $\boldsymbol{\mathcal{E}}$ built with ℓ-pairing; such a function acts diagonally on the C-grading ($\boldsymbol{\mathcal{E}}$'s spectral decomposition), multiplying the component at density c by f(c). Two modes at the same value — m with C(m) an embedded or accumulation point of $\mathcal{C}(\boldsymbol{\mathcal{M}})$, the case $\Delta_n(m) = 0$ — are scaled identically and cannot be separated; by C3 the record metric is the ℓ-metric, in which identical scaling preserves indistinguishability. Hence $\mu_n(m) = 0$. The step that fails without OBS-EVOL is precisely the diagonality: $\mathscr{R}$, were it admissible, is non-diagonal on the C-grading and could separate co-valued modes. ■

**Theorem 8.3 (B1 discharged within the realization, under OBS-EVOL).** Within the machine and under OBS-EVOL, B1 holds in both directions. **[Conditional on OBS-EVOL; lifts further under REAL + audit-determinacy]** The companion carries B1 as a two-directional Conditional; under OBS-EVOL it reduces to this one premise.

*Proof.* Propositions 8.1 (unconditional within the realization) and 8.2 (under OBS-EVOL). ■

Within this machine and under OBS-EVOL the species reading rests on ¬ACC alone, B1's earlier conditionality now localized to one premise about what counts as an admissible observation. The further lift to the substrate requires REAL and audit-determinacy (§9.3).

## 9. The Fourth Mode, and the Sign Reduced to a Population Question

**9.1 $m_4$ made explicit.** By M_census, $m_4$ is a non-mark admissible recurrent mode with definite ($K_4$, $p_4$), distinct from the three mark-levels' data, and density $C(m_4) = p_4/K_4 \in \mathbb{Q}_+$; its position is stationary because M_census posits it as a definite mode (Remark 6.3a). By H1 the resolved density set at any fixed depth is finite, so $C(m_4)$ has a positive nearest-neighbour distance there:

**$\Delta_n(m_4) > 0$ at every finite depth n. [Proven within the realization, on H1 — independent of the persistence reading]**

**Premise ADM-MON (monotonicity of admission — guards ⟹).** [Conditional] Admission is monotone under refinement: a mode admitted by the census at some depth remains admitted at all greater depths. Equivalently, no mode resolved-and-admitted near $C(m_4)$ at depth n is later pruned by the census.

*What it guards.* $\Delta_n$ is computed on the set *resolved* at depth n, while $\mathfrak{G} = \liminf \Delta_n$ and the census D5 acts on the *limit* and may prune. Without ADM-MON, a mode resolved-and-near at depth n but ultimately *not* admitted could pull $\Delta_n$ down along a subsequence, sending $\liminf \Delta_n \to 0$ even when the admitted limit set is sparse. ADM-MON stops this spurious crowding, securing the ⟹ direction of the equivalence (a positive admitted gap ⟹ positive $\mathfrak{G}$). It is plausible — admission is a selection criterion that should not oscillate with depth — but it is a premise.

**Premise RES-COMP (resolution completeness — guards ⟸).** [Conditional] Every density admitted in the limit near $C(m_4)$ is *eventually resolved as distinct*: for each admitted limit competitor there is a depth beyond which $\Delta_n$ registers it. Equivalently, no admitted crowder remains invisible to the finite-depth separation.

*What it guards, and why it is genuinely separate.* ADM-MON closes only one direction. The converse leak is dual: a true crowder that separates from $C(m_4)$ only under deep refinement stays invisible to $\Delta_n$ at finite n, so $\liminf \Delta_n$ could remain positive while the admitted set actually crowds — breaking ⟸ (sparse-looking $\Delta_n$ while the admitted set is dense). RES-COMP rules this out. **It is not delivered by H1**, which gives only finiteness of the resolved set at each depth, not that each admitted limit density enters the resolved set by some finite depth — a deep-resolving crowder is compatible with finiteness at every depth. **Nor can it be leaned on $\mathcal{E}_n \to \mathcal{E}$ strongly**: strong operator convergence does not control spectral resolution — spectral pollution and missed eigenvalues are the standard failure — so the §5 convergence statement cannot carry RES-COMP. It is a genuine spectral-approximation premise of the same character as ADM-MON, carried separately because the two guard opposite implications and are independently contestable.

**Theorem 9.2 (the sign is a population predicate).** Within the realization and under ADM-MON ∧ RES-COMP,

$\mathfrak{G}(m_4) > 0 \Leftrightarrow \neg$ACC at $C(m_4) \Leftrightarrow$ **the admissible density set $\mathcal{C}$ keeps a positive minimum distance from $C(m_4)$ in the deep limit.**

**[Conditional on ADM-MON ∧ RES-COMP, within the realization]**

*Proof.* The moat is positive at every finite depth (§9.1) and the target $C(m_4)$ is stationary (§9.1). **(⟸)** Suppose the admitted set keeps a positive minimum distance δ from $C(m_4)$. By RES-COMP every admitted competitor is eventually resolved, so no admitted competitor closer than δ hides from $\Delta_n$; the eventual resolved-nearest competitor is an admitted one at distance ≥ δ, giving $\liminf \Delta_n \geq \delta > 0$. **(⟹)** Suppose the admitted set crowds $C(m_4)$ (¬ACC fails): distinct admitted densities converge to it. By ADM-MON these competitors, once admitted, persist in the resolved set, so the resolved-nearest competitor is eventually one of them and $\Delta_n \to 0$ along a subsequence, giving $\liminf \Delta_n = 0$. The two directions give the equivalence. Each premise guards exactly one: without ADM-MON, ⟹ leaks (resolved-but-unadmitted crowding); without RES-COMP, ⟸ leaks (admitted-but-unresolved crowding). ■

**9.3 What Theorem 9.2 does and does not do, and why the gate stays double.** It does not return the sign. It converts the sign into one question: is the admitted density set sparse or dense near $C(m_4)$? Two gates stand between this and a verdict.

The first is the **depth census D5** — which recurrent modes are admitted. The orbit-counts paper states D5 is [Open], "governed by the Selection Audit frozen in the operator specification."

The second is **audit-determinacy**. The audit is an acceptance protocol — construction requirements, intrinsic labelling, output requirements, an input fence, a freeze, staged unblinding, publication requirements, together with S1–S5 — i.e. *necessary conditions a candidate machine must pass*. A protocol of necessary conditions filters the candidate space. A uniqueness theorem would read: *any operator satisfying the audit is unique up to equivalence X*; no such converse is in evidence in the inherited material. The honest claim is therefore the sufficient one and no more: **no derivation of the census from the audit is in evidence, and a protocol of necessary conditions does not obviously suffice to determine the operator.** (The corpus's "governed by," not "determined by," is consistent with this and is logged as suggestive, not probative; the openness of D5 is *not* offered as evidence that the audit fails to determine it, since a quantity can be determined-in-principle yet underived-in-practice — determinacy and derivedness are distinct, and conflating them is the very error this paper otherwise polices.)

**Operator-determinacy is [Open · Gate: audit-determinacy].** Every result graded "within the realization" speaks for *this* machine and lifts to the substrate only under both REAL and audit-determinacy. A different admitted machine passing the same audit could carry a different population near $C(m_4)$, hence a different sign.

**Remark 9.4 (why the construction cannot be tuned to the answer).** The machine's form was fixed before $m_4$ was placed in it, and the admitted population near $C(m_4)$ is set by the census and audit — frozen, prior instruments the constructor does not author. So the construction cannot deliver three or four by fiat; its survival as an open question is the certificate that it was not engineered.

## 10. The Two Leans, Each Conjectural

Two structural intuitions bear on the population near $C(m_4)$, pulling opposite ways; each is named, graded [Conjectural], and fenced.

**The merge lean (toward three).** If admitted modes crowd $C(m_4)$ so that the nearest gap is sub-$\delta^\star$, the boundary is sub-bit and hence absent by I3, the witness margin is zero, and $m_4$ merges into the band — three durable species, $m_4$ a gap-closed resonance. This reads I3 at its load-bearing point: the floor is on *separations*, so a sub-floor boundary is a non-distinction. Its debt is that a merge needs something to merge into — it requires the admitted population near $C(m_4)$ to be dense, a claim about D5. **[Conjectural · Gate: D5]**

**The isolate lean (toward four).** If the Selection Audit admits only a sparse, reuse-selected chain near $C(m_4)$, the density keeps a positive gap and $m_4$ is a fourth durable species. The orbit-counts finding that the charged columns *ascend and expand* — gaps widening with depth — weakly favours sparseness at the generation levels, but only conditionally: that finding rides on the log-linear bridge connecting measured mass-gaps to density-gaps, and it concerns the generation levels themselves, not whatever modes might be admitted near $C(m_4)$. **[Conjectural · Gate: D5; rides on the log-linear bridge]**

Two readings are set aside as unsupported. A reading on which long orbits fail to survive refinement through a per-tick compounding hazard — and so fail to crowd — presumes the compounding form of the persistence clause, which the wording has not been checked to license (§2); it is not available. A reading that counts all rationals near $C(m_4)$ as competitors is superseded by Corollary 7.2: the competitors are the *admitted* densities, not the rationals at large. Neither lean rises above [Conjectural], and both are downstream of D5 and audit-determinacy.

## 11. What This Settles, and What It Leaves Open

The realizability debt is paid: a finite closure-state machine realizing the inherited inputs is exhibited (§3–5), recovers the recurrent set, the three isolated mark-levels, and the orbit counts (§6), and discharges the "modulo realizability" caveat on the corpus's witnesses (§6.4).

B1 is discharged within the realization under one premise (§8): with the evolution operator $\mathcal{E}$ distinguished from the refinement operation $\mathscr{R}$, the Riesz extractor lies in $\mathcal{O}(\mathcal{E}, \ell)$ (B1 forward, unconditional within the realization), and — under OBS-EVOL, that admissible observables are evolution-generated and exclude refinement — functions of $\mathcal{E}$ act diagonally on the density grading and cannot separate co-valued modes (B1 reverse). Because refinement moves the density, OBS-EVOL is load-bearing, not cosmetic; the species reading rests, within this machine, on OBS-EVOL and ¬ACC.

The spectral parameter is the completion density on the line (§7), consistent with the orbit-counts density order; the competing spectrum is the realized density set; the moat is a distance between rationals on $\mathbb{Q}_+$.

The sign's residue is sharpened and relocated, not resolved (§9). Under ADM-MON and RES-COMP — admission monotone under refinement (so no resolved-but-unadmitted crowder spuriously closes $\Delta_n$) and every admitted limit density eventually resolved (so no admitted crowder hides from $\Delta_n$) — the sign of $\mathfrak{G}(m_4)$ is the predicate that the admitted density set keep positive spacing near $C(m_4)$. The two premises guard opposite directions of the equivalence and neither is delivered free: ADM-MON is a claim about the census, RES-COMP a spectral-approximation claim that H1 and strong operator convergence do not supply. The missing-machine obstruction is removed; what remains is a population question on the line, doubly gated by the depth census D5 (which modes are admitted) and by audit-determinacy (whether the audit fixes the operator at all). Both gates are [Open].

Falsifiability is preserved. World A (three species, $C(m_4)$ crowded sub-bit) and World B (four species, $C(m_4)$ sparsely isolated) remain open, now distinguished by the admitted population near one rational on the line.

What this construction owes the orbit-counts instrument is untouched here: that $\mathscr{R}$'s computed increments meet the reuse condition wherever the table ascends, and reproduce the compression pattern including the down-sector expansion. That obligation is named, not met.

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# 12. What This Paper Does Not Do

It does not claim the machine is unique or canonical. The exhibited machine is *a* realization passing the inherited requirements, not *the* operator. Audit-determinacy is **[Open]**, with no converse uniqueness theorem in evidence. Everything graded "within the realization" lifts only under REAL and audit-determinacy.

It does not discharge B1 unconditionally. B1 reverse rests on OBS-EVOL — that refinement is not an admissible internal observable — which is contestable, since refining the description is arguably available to an internal observer; the contrary reading defeats B1 reverse, and the paper names the fork rather than closing it.

It does not establish the headline reduction unconditionally. Theorem 9.2 rests on ADM-MON ∧ RES-COMP — one premise per direction: without monotone admission, resolved-but-unadmitted crowding can break the ⟹ direction; without resolution completeness, admitted-but-unresolved crowding can break the ⟸ direction. RES-COMP in particular is not delivered by H1 or by $\mathcal{E}_n \to \mathcal{E}$ strongly, and is carried as a spectral-approximation premise in its own right.

It does not decide D5, and so does not return the sign. The admitted population near $C(m_4)$ is the depth census, **[Open]**; the sign of $\mathfrak{G}(m_4)$ is **[Open · Gates: D5, audit-determinacy, REAL]**.

It does not settle the persistence clause's reading — mode-survival vs. separation-survival is **[Open]**; §6.3 uses only the weak reading, and the position-constancy a stronger reading would give is marked and unconsumed.

It does not compute counts against the measured table, and so does not meet the reuse or compression conditions for the physical $\mathscr{R}$; those are named obligations. It does not establish a canonical spectral magnitude — only the sign is load-bearing — and does not reach the capacity catalogue, world-occupancy, gauge, chirality, mixing, or absolute-mass structure.

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# 13. Conclusion

The instruments were built and their meanings proved, but none wrote down the machine, so each wore "modulo realizability," and the deepest gate on the census read simply: construct the operator.

This paper constructs one — a finite closure-state machine with a commitment-weighted record space, a deterministic irreversible update, an evolution operator, and a refinement operation — and proves it reproduces the recurrent set, the three isolated mark-levels, and the orbit counts the recurrence theory had only postulated. Exhibiting it discharges the realizability caveat for the witnesses. Distinguishing the two operators is what makes the bridge premise tractable: the evolution operator is diagonal on the density grading, the refinement operation is not, and the harder half of the bridge holds exactly when the admissible measurements are built from evolution alone — a premise the paper carries openly, because the contrary reading, on which refining the description is itself an admissible observation, would defeat it.

What the machine cannot do, it makes precise. Mode positions are completion densities on a line, and the gap of the fourth mode is the distance to its nearest *admitted* neighbour. The gap closes exactly when admitted modes crowd that density — and whether they do is the depth census, left open and handed to an acceptance audit that filters candidate machines without having been shown to fix a unique one. So the sign stands behind two open gates: which modes are admitted near the fourth mode's density, and whether the audit determines the operator at all. The reduction holds under two premises guarding opposite directions — monotone admission and resolution completeness, the latter a spectral-approximation claim that finiteness-per-depth and strong operator convergence do not supply; the bridge holds under an evolution-only-observable premise; all are named, none is hidden.

What the census means was settled by the companions; what the apparatus rests on is narrowed here to three named premises, two open gates, and a persistence clause whose wording is owed; what the census *is* — three durable species or four — remains the one sign left to compute, now standing behind admissibility and determinacy rather than behind a machine that did not exist. The instrument is built. It does not yet report. And the reason it does not is no longer that it is missing, but that what it admits, and whether it is one machine or many, are open and correctly labelled as such.