The Strange Current-Mass Operator in VERSF

Resolved Balanced Support Readout, Complement Breaking, and the Twentyfold Down-to-Strange Ratio

Keith Taylor VERSF Theoretical Physics Programme — Quark Mass Hierarchy Series Working manuscript, v1.0

Successor to *Quark Masses from One Anchor in VERSF*, *The Strange Baseline in VERSF*, and *Quark Masses from One Anchor II: The Current-Mass Operator and the Trace Audit of the Diagonal Quark Spectrum*.

Summary for the General Reader

The previous quark-mass papers reached a useful but uncomfortable point. The six quark masses could be organised from one down-quark anchor and a short list of dimensionless ratios, but those ratios did not have equal status. The light ratio $m_u/m_d = 6/13$ and the strange ratio $m_s/m_d \approx 20$ were the empirically successful pieces; the two the programme tried to *construct* most explicitly — the charm increment 77/3 and the bottom amplifier $\frac{1}{4}\cdot e^{(16/3)}$ — were the pieces in tension with precision lattice data. That grading set the agenda. The rational next step is not to justify every old formula. It is to derive the one open trace whose value already looks right:

$\mathrm{Tr}\, O_s = 20.$

The earlier strange-baseline note proposed $20 = C(6,3)$, the number of unordered three-channel supports inside a six-channel interface. But it also exposed its own obstruction. If the six primitive channels are interchangeable under the symmetric group Sym(6), then the six single channels form one orbit and the twenty triads also form one orbit, so a single symmetric counting rule gives either 1 (both quotient-counted) or $20/6 = 10/3$ (both raw-counted) — never 20. The real problem is not the arithmetic of C(6,3); it is the **mixed readout rule**:

d : one unresolved invariant baseline, s : twenty resolved support sectors.

This paper gives the first explicit candidate for that rule, and states it in the language of group representations so that its successes and failures become precise rather than verbal. The proposal is that the down quark reads the single Sym(6)-invariant mode of the channel interface — the part that survives when no support boundary is recorded — while the strange quark reads the *full* module of balanced occupied/exterior supports: three occupied channels inside the six, with the remaining three forming the excluded exterior.

The occupied triad and its complement have equal size but are **not** the same physical readout: the occupied side is what is realised; the exterior is what that realisation excludes. So the complement map $T \leftrightarrow I\setminus T$ is not quotiented, while internal ordering inside each side is. The strange readout space is therefore the induced module

$\mathscr{H}_s \cong \mathbb{C}[\, \mathrm{Sym}(6) / (\mathrm{Sym}(3) \times \mathrm{Sym}(3)) \,] \cong \mathbb{C}[\mathcal{S}_3(I)]$, $\dim \mathscr{H}_s = 6! / (3!\cdot 3!) = 20$.

In one sentence: **the strange current-mass operator counts the twenty occupied balanced support atoms of a six-channel interface, while the down anchor reads the single unresolved invariant baseline.**

The claim is bounded. This paper does not derive the absolute mass scale, does not repair the charm or bottom tensions, and does not close the Standard Model. It attempts one precise step: explain why the strange current-mass operator should have trace 20 relative to the down trace 1. The result stays conditional on deriving the support-readout premises from the underlying confinement operator — but the target is now sharp enough to fail cleanly. If the confinement operator yields a complement quotient, an ordered path space, a channel-incidence observable, or a uniform orbit rule, the mechanism is wrong, and §14 says exactly which wrong number each failure produces.

One caveat is stated up front rather than buried. A companion VERSF decomposition writes this *same* ratio multiplicatively — as a large amplifier times a large suppression ($\approx 108 \times 0.18 \approx 20$) — and an additive count of twenty support atoms is not automatically the same object as that multiplicative residue. Reconciling the two is a genuine open problem (§11), not a formality, and the present construction is explicitly a candidate *inside the support-count picture*, not a proof that the support-count picture is the primitive one.

Abstract

Prior VERSF work reduced the diagonal quark mass spectrum to a one-anchor current-mass grid in which the strange baseline enters as $r_s = m_s/m_d \approx 20$. The trace audit ranked $r_s = 20$ as the highest-priority unresolved trace: empirically successful at current lattice precision, structurally unproven. The obstruction is not the identity $C(6,3) = 20$; it is the multiplicity rule. The inherited six-channel interface carries a Sym(6) action transitive on singletons and on triads, so a uniform orbit rule gives 1 and a uniform raw count gives 10/3, never 20.

This paper proposes a candidate strange current-mass operator that resolves the obstruction by making the down and strange readouts **different representations of the same six-channel interface**. Let I be the six-element primitive interface-channel set and $G = \mathrm{Sym}(I) \cong \mathrm{Sym}(6)$. The down baseline is the invariant (trivial-isotypic) mode of the point module,

$\mathscr{H}_d = \mathbb{C}[\mathcal{S}_1(I)]^G$, $\dim \mathscr{H}_d = 1$.

The strange readout is the full balanced occupied/exterior support module,

$\mathscr{H}$_s = ℂ[G / (Sym(3) × Sym(3))] ≅ ℂ[$\mathcal{S}_3$(I)], dim $\mathscr{H}$_s = C(6,3) = 20,

a basis state being an occupied/exterior split I = T ⊔ T^c with |T| = 3 and T marked as the realised support. The complement map T ↦ T^c is **not** quotiented (it exchanges occupied and exterior roles); ordering inside T and inside T^c **is** quotiented (the down-sector baseline reads supports, not fold sequences). The candidate operator is the identity support projector O_s = Σ_{T∈$\mathcal{S}_3$(I)} |T⟩⟨T|, so

Tr_{$\mathscr{H}$_d} O_d = 1, Tr_{$\mathscr{H}$_s} O_s = C(6,3) = 20,

and, under the support-trace unit premise, m_s/m_d = 20.

Two strengthenings sharpen the construction beyond the bare triad count. **(i)** The mixed rule is exactly the statement *down keeps only the trivial isotypic component of its module; strange keeps the whole module*: the point module decomposes as $\mathbb{C}^6$ = S^(6) ⊕ S^(5,1) (dims 1+5) and the support module as M^(3,3) = S^(6) ⊕ S^(5,1) ⊕ S^(4,2) ⊕ S^(3,3) (dims 1+5+9+5 = 20), and the four "keep trivial vs keep all" pairings reproduce the earlier selection lattice r_s ∈ {1, 1/6, 10/3, 20}. **(ii)** Equivariance does more than flatten the support weights: the Sym(6)-equivariant operators on $\mathscr{H}$_s form the 4-dimensional Bose–Mesner algebra of the Johnson scheme J(6,3), every element of which has *constant diagonal*, so Tr O_s = 20·a_0 for **any** equivariant operator — independent of how supports couple to one another. The trace therefore depends on a single number, the self-weight a_0, and the support-trace unit premise is precisely a_0 = 1. This collapses the *weighting* risk to one scalar plus Sym(6)-equivariance — but it does nothing for the *space-selection* risk that put the readout on M^(3,3) in the first place, which remains distributed across the conditional premises QM-S1–S4 and S7 (trivial-down, support boundary, central shell, complement breaking, support-not-incidence). Two named premises govern the weights; the choice of space is still six-fold conditional.

The contribution is conditional but exact in form. It replaces "count the twenty triads" with a representation-level support-readout ansatz, and it makes every failure mode a specific wrong number — 1, 10/3, 10, 15, 60, 120 — or else shows that the empirical 20 has no primitive support-trace home (the five-gate eviction, §11). The remaining proof obligations are explicit and listed as QM-S1…S7.

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0. Predictive-Content Ledger

This paper addresses one target only:

QM-S: construct the strange current-mass readout and derive Tr O_s = 20.

It does not derive the absolute quark mass scale, the light split 6/13, or the heavy-sector pieces 77/3 and ¼·e^(16/3). It asks whether the strange trace can be obtained from the six-channel interface structure rather than imported as an empirical baseline.

Object	Value	Status in this paper
Down anchor m_d⋆	external	declared current-mass scheme
Primitive channel set I	|I| = 6	inherited from interface/capacity sector
Down readout space	ℂ[$\mathcal{S}_1$(I)]^Sym(6)	proposed invariant baseline (trivial isotypic)
Down trace	Tr O_d = 1	anchor normalisation + unresolved-baseline premise
Strange shell rank	k = 3	balanced support premise
Strange support space	ℂ[$\mathcal{S}_3$(I)] = M^(3,3)	resolved occupied-support module (full)
Complement quotient	absent	occupied/exterior orientation premise
Internal ordering	absent	support readout, not path readout

Object	Value	Status in this paper
Counting unit (self-weight a_0)	1 per support atom	support-trace unit premise
Strange trace	Tr O_s = C(6,3) = 20	exact from the above premises
Mass ratio	m_s/m_d = 20	conditional trace result
Empirical status	≈ 19.8–20.2; 20 within ~1% (upper edge)	inherited audit
Five-gate reconciliation	not closed	serious remaining bridge

The central discipline remains: **matching a number is not deriving a number.** The paper separates three layers:

1. **Mathematical trace result.** If the down and strange readout spaces are the proposed spaces, the trace ratio is exactly 20. This is proved (§9).
2. **Operator-construction claim.** The strange current-mass operator acts on resolved balanced support atoms — not on a quotient orbit, an ordered path space, a complement-quotiented bipartition, or a channel-incidence space. This is proposed.
3. **Physical derivation still owed.** The confinement operator must *force* the support module, the complement breaking, Sym(6)-equivariance with self-weight a_0 = 1, and the rank-3 selection. These are the real proof targets (QM-S1…S7).

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1. The Inherited Problem

The one-anchor programme writes the diagonal quark masses relative to a declared down-quark anchor m_d⋆. The ratio row is

(m_u, m_d, m_s, m_c, m_b, m_t) = m_d⋆ · (6/13, 1, 20, 3080/13, 5·e^(16/3), 5·e^(16/3)·(6/13)·(77/3)^(3/2)).

The trace audit reorganised this as m_(d_g) = m_d⋆·Tr O_(d_g) and m_(u_g) = m_(d_g)·e^(χ_g), with down-sector operators O_d, O_s, O_b and a same-generation χ-operator. In that language the strange step is Tr O_s = 20.

The strange trace is special. Unlike the charm and bottom pieces, it is **not** a constructed piece in tension with data; it is the open trace whose target value appears correct (m_s/m_d ≈ 19.8–20.2 — 20 sits within about 1% of current same-scheme estimates, ≈1.1% above the FNAL/MILC self-consistent value 92.47/4.675 = 19.78 and ≈0.5% above the m_s/m_ℓ fold, at the upper edge). The audit therefore ranked it the safest and most valuable next derivation: securing it canonises a *correct* number rather than a disfavoured one.

The strange-baseline paper proposed 20 = C(6,3) but exposed its own obstruction. Sym(6) is transitive on the six singletons and on the twenty triads, so a single symmetric count gives either

$| \mathcal{S}_1(I)/\mathrm{Sym}(6) | = | \mathcal{S}_3(I)/\mathrm{Sym}(6) | = 1$, or $| \mathcal{S}_3(I) | / | \mathcal{S}_1(I) | = 20/6 = 10/3$,

never 20. So 20 requires a mixed rule — d quotient/invariant, s resolved — and the present paper turns that mixed rule into an operator statement.

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2. What the Strange Paper Must Prove

A successful strange paper must do more than observe C(6,3) = 20. It must establish the four choices that make 20, rather than a neighbouring value, the readout.

2.1 The down denominator must be one. If the down quark were counted as six singleton sectors, the triad count would give 20/6 = 10/3. The down readout must be a single invariant baseline: $\dim \mathcal{H}_d = 1$. This is stronger than the anchor convention — one can always pick units with $m_d = 1$, but the physical question is whether the readout has *one* baseline mode or six resolved singleton sectors before the anchor is imposed.

2.2 The strange numerator must be twenty. The strange readout must be a direct sum over twenty distinct support atoms — not one triad orbit (1), ten complement pairs (10), fifteen 2-shells (15), sixty channel incidences, or one hundred and twenty ordered triples.

2.3 The trace unit must be common. The ratio $m_s/m_d = \mathrm{Tr}_{\mathcal{H}_s} O_s / \mathrm{Tr}_{\mathcal{H}_d} O_d$ is meaningful only if the trace functional carries a common current-mass unit across $\mathcal{H}_d$ and $\mathcal{H}_s$. The trace audit called this the trace-normalisation premise; this paper sharpens it:

Support-Trace Unit Premise. Once a support sector is an admissible current-mass readout atom, its identity projector contributes one down-normalised trace unit. The down anchor fixes the unit; the strange operator counts how many support atoms carry it.

Without this premise, the result is dimension counting, not a mass-ratio derivation.

2.4 The five-gate reading must be reconciled. The one-anchor basis treats $m_s/m_d = 20$ as a primitive trace; the five-gate G/M_χ decomposition treats it as a residue of a large amplifier and a large suppression (§11). A serious strange paper must not hide this conflict.

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3. Six-Channel Support Algebra and the Isotypic Picture

Let $I = \{1,\dots,6\}$ be the inherited primitive interface-channel set and $G = \mathrm{Sym}(I) \cong \mathrm{Sym}(6)$. For each k, the rank-k support shell is

$\mathcal{S}$_k(I) = { T ⊂ I : |T| = k },

with support module ℂ[$\mathcal{S}$_k(I)] carrying the permutation action σ|T⟩ = |σT⟩. Because G is transitive on $\mathcal{S}$_k(I) for every k = 1,…,5, the orbit quotient $\mathcal{S}$_k(I)/G is a single point, so **symmetry alone never distinguishes k = 1 from k = 3 by orbit count.** The distinction must come from the readout operator.

The decisive reframing of this paper is that "quotient vs resolve" is not vague — it is the choice of *which isotypic components of the permutation module the readout keeps.* Two facts (verified by hook-length and the Johnson scheme):

- The **point module** is ℂ[$\mathcal{S}_1$(I)] = $\mathbb{C}^6$ = M^(5,1), and it decomposes as

  $\mathbb{C}^6$ ≅ S^(6) ⊕ S^(5,1), dims 1 + 5 = 6.

  Its Sym(6)-invariant subspace is the trivial component S^(6), one-dimensional, spanned by the all-channels average.

- The **balanced support module** is ℂ[$\mathcal{S}_3$(I)] = M^(3,3), and it decomposes as

  ℂ[$\mathcal{S}_3$(I)] ≅ S^(6) ⊕ S^(5,1) ⊕ S^(4,2) ⊕ S^(3,3), dims 1 + 5 + 9 + 5 = 20.

A readout that keeps **only the trivial component** of a module returns dimension 1 (= number of orbits). A readout that keeps the **whole module** returns its full dimension. The mixed rule is therefore exactly:

down → trivial component of the point module (dim 1); strange → the full balanced support module (dim 20).

The four "trivial-only vs full" pairings reproduce — now with a representation-theoretic meaning — the selection lattice the earlier papers wrote by hand:

down readout	strange readout	r_s = dim $\mathcal{H}$_s / dim $\mathcal{H}$_d
trivial of M^(5,1) (1)	full M^(3,3) (20)	**20** (the proposal)
trivial of M^(5,1) (1)	trivial of M^(3,3) (1)	1
full M^(5,1) (6)	full M^(3,3) (20)	10/3
full M^(5,1) (6)	trivial of M^(3,3) (1)	1/6

The "uniform" rules (both trivial → 1; both full → 10/3) are exactly the two obstruction values. The proposal is the one mixed choice **(trivial-down, full-strange)**. So the open problem is now stated with maximal precision: *derive why the confinement operator projects the down readout onto the trivial isotypic component while keeping the full strange module.* The physical narrative — gauge labels before a boundary, resolved sectors after — is below, but its content is this isotypic asymmetry.

4. The Down Readout: Invariant Depth-One Baseline

The down quark is the unexcited down-sector baseline — not yet a resolved support shell. Its readout space is the invariant subspace of the point module:

$\mathscr{H}_d = \mathbb{C}[\mathcal{S}_1(I)]^G = S^{(6)}$, $\dim \mathscr{H}_d = 1$,

spanned by $\Omega_d = (1/\sqrt{6}) \Sigma_{i\in I} |\{i\}\rangle$. The operator is the identity on this line, $O_d = I_{\mathscr{H}_d}$, so **$\mathrm{Tr}_{\mathscr{H}_d} O_d = 1$.**

4.1 Why not six. The raw point module $\mathbb{C}^6$ has dimension six. If the down quark resolved the six labels as distinct current-mass sectors, the denominator of m_s/m_d would carry multiplicity six and the strange ratio would be 10/3, not 20.

Unresolved Baseline Principle. At depth one a primitive channel label is not yet a confined support record; it is a coordinate inside the symmetric interface. The down readout sees the invariant baseline mode $S^{(6)}$, not the standard component $S^{(5,1)}$ carrying the six distinguishable occupations.

In isotypic terms: the down readout discards $S^{(5,1)}$ and keeps $S^{(6)}$. This must eventually be derived from the confinement operator (QM-S1); here it is the first structural premise.

4.2 Anchor normalisation versus physical dimension. The anchor convention sets the numerical unit of mass; it does not decide the physical dimension of the readout space. The claim is not "$m_d\star = 1$ in chosen units" but **$\dim \mathscr{H}_d = 1$**, because the physical down readout is the invariant singleton mode. This is precisely the denominator required for $m_s/m_d = 20$, and precisely where the derivation fails if depth one is physically resolved.

5. The Support-Resolving Transition

The mixed rule needs a physical trigger: why is depth one trivial-only while the strange state is full?

The proposed trigger is **support-boundary formation**. A current-mass readout counts stable support atoms of a confined partial closure, not abstract channel labels. Before a boundary is committed, channel labels are redundant coordinates and only the invariant mode survives; once a boundary is committed, the occupied support is a recorded sector.

Support-Resolving Rule. A channel subset $T \subset I$ is counted as a current-mass sector only when the confinement operator records the boundary between T and its exterior $I\setminus T$. Before such a boundary is recorded, the readout retains only the invariant channel mode (the trivial component).

The down baseline lies before this transition; the strange state is proposed to be the first state after it. The same Sym(6) symmetry then produces two multiplicity behaviours without contradiction: before boundary formation it removes labels (project to S^(6)); after boundary formation it relates physical sectors of equal weight (keep all of M^(3,3) and act as a degeneracy symmetry). In isotypic shorthand:

unrecorded label : $\mathbb{C}[\mathcal{S}_1(I)] \to S^{(6)}$; recorded support atom : $\mathbb{C}[\mathcal{S}_3(I)] \to \mathbb{C}[\mathcal{S}_3(I)]$ (identity).

The second arrow is intentionally the identity on the full support module: the group remains present as a degeneracy symmetry of the shell, not as a gauge redundancy of labels.

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6. The Strange Readout: Balanced Occupied/Exterior Support Shell

The strange quark is proposed to be the first down-sector confined closure whose support is internally resolved. A support state is not a bare subset but an occupied/exterior split $I = T \sqcup E$, $E = I \setminus T$. The first self-contained boundary is the balanced split $|T| = |E|$, and since $|I| = 6$ this gives $|T| = 3$. So the strange shell is $\mathcal{S}_3(I)$, and a support atom is an oriented split $I = T \sqcup T^c$ with $|T| = |T^c| = 3$ and T marked occupied.

The stabiliser of a chosen occupied triad T is $G_T = \mathrm{Sym}(T) \times \mathrm{Sym}(I \setminus T) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$, so the resolved strange space is the induced module

$\mathcal{H}_s = \mathbb{C}[\, G / G_T \,] \cong \mathrm{Ind}_{\mathrm{Sym}(3)\times\mathrm{Sym}(3)}^{\mathrm{Sym}(6)} \mathbb{1}$, $\dim \mathcal{H}_s = |\mathrm{Sym}(6)| / (|\mathrm{Sym}(3)|\cdot|\mathrm{Sym}(3)|) = 6!/(3!\cdot 3!) = 20$.

This representation-level statement is the main formal improvement over the bare count: the strange states are twenty not because a binomial coefficient was noticed, but because the support module is the induced module of occupied balanced 3+3 splits.

6.1 Why rank three. The cleanest selection argument is through the complement involution $c : T \mapsto T^c$ itself — the same map whose *breaking* gives 20 rather than 10 (§6.2). It is worth being explicit that c plays two different roles at two different levels, because that is exactly where the construction is most easily mistaken for sleight of hand.

At the level of *which shell*, c acts on the shell lattice by $k \mapsto 6 - k$, pairing {0,6}, {1,5}, {2,4} and fixing {3}. So k = 3 is the **unique complement-fixed shell**. At the level of *which subset within the shell*, c acts on the twenty triads, and since no 3-set meets its own complement ($T \cap T^c = \emptyset$, so $T = T^c$ is impossible), it acts **freely**, pairing the twenty into ten complementary pairs — a pairing then broken by orientation (occupied vs exterior). The coherent position is therefore *shell* *index is structural, occupation is physical*: the complement map selects the shell by fixing it, and is broken within the shell by the occupied/exterior marking.

Three independent characterisations of $k = 3$ then coincide: it is **balanced** ($k = 6 - k$); it is the **largest** shell ($C(6,3) = 20 > C(6,2) = 15 > C(6,1) = 6$, from shell sizes 1, 6, 15, 20, 15, 6, 1); and it is the **complement-fixed** shell. The balanced-support premise records this:

Balanced Support Premise. The first resolved down-sector confinement support is the unique complement-fixed shell of the six-channel interface — equivalently the balanced and largest shell — namely $k = 3$.

This rejects $k = 1$ (the unresolved baseline), $k = 2, 4$ (unbalanced), and the full $k = 0, 6$ closures (no partial boundary). It is not yet derived from a closure Hessian, transfer operator, or confinement functional — that is QM-S3.

6.2 Why the central shell does not force complement quotienting. The central shell is self-complementary in size, which creates the complement hazard: if only the unordered partition $\{T, I\setminus T\}$ were physical, the count would be $(1/2)\cdot C(6,3) = 10$. The proposal avoids this by making the split *oriented* — T occupied with $I\setminus T$ exterior is a different readout from $I\setminus T$ occupied with T exterior. The complement map reverses what is realised; it does not relabel the same state. Hence the relevant subgroup is $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$, not its complement-exchange extension $(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) \rtimes \mathbb{Z}_2 = \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$: inducing the trivial off the wreath product $\mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ gives the dimension-10 unordered-bipartition module, while inducing off $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$ gives the dimension-20 oriented module. Complement breaking is exactly the choice of the smaller subgroup — the coset space already distinguishes occupied from exterior, so no extra premise is needed beyond *not* extending by the swap.

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7. The Strange Current-Mass Operator and the Johnson-Scheme Commutant

On $\mathscr{H}_s = \mathbb{C}[\mathcal{S}_3(I)]$ define support projectors $P_T = |T\rangle\langle T|$ and the candidate operator

$$O_s = \Sigma_{T\in\mathcal{S}_3(I)} P_T = I_{\mathscr{H}_s}, \quad \mathrm{Tr}_{\mathscr{H}_s} O_s = |\mathcal{S}_3(I)| = 20.$$

The construction is short; the physical content is the claim that the strange current-mass observable is the identity support projector — **the strange mass readout counts each admissible balanced support atom once.** Two questions decide whether that claim is forced or merely chosen.

7.1 Why flat weights — and why this is more robust than it looks. A general support-diagonal operator is $O_s = \Sigma_T w_T P_T$. $\mathrm{Sym}(6)$ permutes the supports transitively, so equivariance forces $w_T = w$ for all T. But equivariance says something stronger and more useful than "the diagonal is flat." The $\mathrm{Sym}(6)$-equivariant operators on $\mathscr{H}_s$ form the **Bose–**

Mesner algebra of the Johnson scheme J(6,3) — a 4-dimensional commutative algebra spanned by the adjacency operators $A_0 = I$, A_1, A_2, A_3, where A_d connects supports sharing exactly $3-d$ channels (intersection 3, 2, 1, 0). Every off-diagonal adjacency operator has zero diagonal (a support shares all three channels only with itself), so for *any* equivariant operator $O_s = \Sigma_d a_d A_d$,

$\mathrm{Tr}\, O_s = a_0 \cdot 20$, independent of a_1, a_2, a_3.

This is a genuine robustness result the bare construction missed: the trace cannot be spoiled by how supports couple to one another (shared-channel interactions), because those couplings are purely off-diagonal. **The trace depends on exactly one number, the self-weight a_0.** Equivariance fixes flatness automatically; the only remaining content is the scale.

7.2 The single real premise: $a_0 = 1$. The support-trace unit premise is therefore not "the twenty weights are equal" (automatic) but **$a_0 = 1$** in down-normalised units: one realised support atom carries one down-normalised current-mass quantum. With $a_0 = 1$ the trace is 20 — and this holds for *any* equivariant observable, not only the identity projector; the identity is the special representative with $a_1 = a_2 = a_3 = 0$, but unit self-weight alone fixes the trace regardless of the off-diagonal couplings. With $a_0 = w \neq 1$, $\mathrm{Tr}\, O_s = 20w$ and the binomial count alone does not give the ratio. The premise is essential, not cosmetic, and it is now isolated as a single scalar — the cleanest possible form for the open problem QM-S5.

7.3 The one assumption hiding upstream: equivariance itself. The robustness of §7.1 holds *given* that the mass observable is Sym(6)-equivariant — i.e. that the six channels are genuinely interchangeable. That is inherited from the interface sector's uniform 1/6 allocation, but it is a premise: if the physical confinement operator breaks Sym(6) (channels not equivalent), the diagonal need not be constant and $\mathrm{Tr}\, O_s \neq 20$. So the trace rests on two things only — Sym(6)-equivariance (inherited) and $a_0 = 1$ (new) — and nothing else about the spectrum of O_s matters.

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8. Complement Breaking, Unorderedness, and Support-Atom Normalisation

The strange trace is fixed by three quotient decisions, now each phrased as a falsifiable choice.

8.1 Internal permutations are quotiented. Inside a realised triad, labels have no order: {1,2,3} is the same occupied support as {2,1,3}. The construction quotients by Sym(3) inside the occupied support and by Sym(3) inside the exterior, producing the denominator $3!\cdot3!$ in $6!/(3!\cdot3!)$.

8.2 Occupied/exterior exchange is not quotiented. The map $T \mapsto I\setminus T$ exchanges realised support with exterior complement; since current-mass readout reads what is realised, it changes the physical state. Hence no extra factor of 2: $6!/(3!\cdot3!) = 20$ rather than $6!/(2\cdot3!\cdot3!) = 10$. This is complement breaking. Operationally — to make "not quotiented" a readout property rather than an assertion — complement breaking means there exists a current-mass readout functional B with $B(T) = -B(T^c)$, i.e. a probe living in the complement-antisymmetric sector (the -1 eigenspace of the involution $T \leftrightarrow T^c$, which has dimension 10). The existence of such a sign-flipping B certifies that the complement map is a physical inequivalence, not a gauge redundancy of the readout. Equivalently: the complement-symmetric subspace of $M^{(3,3)}$ has dimension 10, so if the readout factored through it the trace would collapse to 10; keeping the full 20 means the mass observable is sensitive to the complement-antisymmetric sector as well, and QM-S4 is precisely the demand that the confinement operator supply such a B.

8.3 Supports, not incidences, carry the unit. The unit is attached to a support atom T, not to each channel occurrence (T, i). So the count is $|\mathcal{S}_3(I)| = 20$, not $\Sigma_T |T| = 60$. A channel-incidence operator $N_s = \Sigma_T \Sigma_{i \in T} |T,i\rangle\langle T,i|$ would trace to 60; the proposal rejects it because a confined strange support is stabilised as a triad support, not as three additive channel occupations. This is a sharp falsifier (§14.6): if the confinement operator decomposes strange current mass additively over channel incidences, the 20-trace mechanism fails.

8.4 Support readout, not ordered-path readout. An ordered triad would give $P(6,3) = 6 \cdot 5 \cdot 4 = 120$. The operator is invariant under permutations inside the occupied support and inside the exterior. Ordering belongs to fold orientation and χ-access (where ABC vs CBA may matter); the strange down-sector baseline is a support-readout problem, not a path-ordering one. This separation must be derived (QM-S7); if the strange operator carries ordered access histories, the count changes.

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9. Candidate Strange Current-Mass Trace Theorem

Theorem (Candidate Strange Current-Mass Trace). Let I be a six-element primitive interface-channel set with $G = \mathrm{Sym}(I)$. Assume:

1. The down quark is the unresolved depth-one baseline, $\mathcal{H}_d = \mathbb{C}[\mathcal{S}_1(I)]^G$ with $O_d = I_{\mathcal{H}_d}$.
2. The strange quark is the first resolved balanced support, with oriented split $I = T \sqcup T^c$, $|T| = |T^c| = 3$.
3. The readout distinguishes occupied support from exterior complement, so T is not identified with T^c.
4. The readout is unordered inside the occupied support and inside the exterior.
5. The mass observable is Sym(6)-equivariant with self-weight $a_0 = 1$, i.e. each resolved support atom contributes one down-normalised trace unit.

Then $\mathcal{H}_s \cong \mathbb{C}[\ \mathrm{Sym}(6)/(\mathrm{Sym}(3)\times\mathrm{Sym}(3))\ ]$, the support module has dimension 20, and (taking the identity support projector as the canonical candidate, $O_s = I_{\mathcal{H}_s}$)

$\mathrm{Tr}_{\mathcal{H}_s} O_s = 20$, hence $m_s/m_d = 20$.

Proof. Since G acts transitively on $\mathcal{S}_1(I)$, the invariant singleton subspace is one-dimensional, so $\mathrm{Tr}_{\mathcal{H}_d} O_d = 1$. For the strange state a basis vector is an oriented split $I = T \sqcup T^c$ with $|T| = 3$ modulo internal permutations of each block; the stabiliser is Sym(3) × Sym(3). Assumption 3 excludes the complement-exchange $\mathbb{Z}_2$; assumption 4 excludes ordering inside either block. Hence the number of support atoms is $|\mathrm{Sym}(6)|/(|\mathrm{Sym}(3)|\cdot|\mathrm{Sym}(3)|) = 6!/(3!\cdot 3!) = 20$. By assumption 5 the observable is equivariant, so (§7.1) it lies in the Johnson-scheme algebra $O_s = \Sigma_d a_d A_d$ with constant diagonal a_0; since A_1, A_2, A_3 have zero diagonal, $\mathrm{Tr}_{\mathcal{H}_s} O_s = 20\cdot a_0$ independent of the off-diagonal couplings, and with $a_0 = 1$ this gives $\mathrm{Tr}_{\mathcal{H}_s} O_s = 20$. (The identity support projector is the canonical representative, the special case $a_1 = a_2 = a_3 = 0$; unit self-weight alone does not force it.) The trace-normalised ratio is $\mathrm{Tr}\, O_s / \mathrm{Tr}\, O_d = 20/1 = 20$. ■

9.1 What the theorem proves. Given the proposed readout spaces and the unit self-weight, the trace ratio is exactly 20. It sharpens the mixed rule to $\mathcal{H}_d$ = trivial component of $M^{(5,1)}$, $\mathcal{H}_s$ = full $M^{(3,3)}$ — an explicit representation-level asymmetry, not "quotient down, count strange."

9.2 What the theorem does not prove. It does not prove from first principles that the strange quark *must* use $\mathcal{H}_s$. It assumes the unresolved-baseline, balanced-support, complement-breaking, unordered-support, and equivariance + unit ($a_0 = 1$) premises. The honest status is **candidate current-mass trace derivation, not closed first-principles derivation.** Each possible failure now has a definite mathematical form (§14).

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10. Relation to the One-Anchor Quark Grid

If the theorem holds, the down and strange entries become $m_d = m_d^\star$ and $m_s = 20\cdot m_d^\star$; the rest of the row is unchanged:

$m_u = (6/13)\cdot m_d^\star$, $m_c = 20\cdot(154/13)\cdot m_d^\star = (3080/13)\cdot m_d^\star$, $m_b = 5\cdot e^{16/3}\cdot m_d^\star$, $m_t = 5\cdot e^{16/3}\cdot(6/13)\cdot(77/3)^{3/2}\cdot m_d^\star$.

What changes is the *status* of the strange factor: from imported empirical baseline → candidate support-trace result, and (if the confinement operator later forces the premises) → derived current-mass trace. Because the strange baseline multiplies the strange, charm, bottom, and top entries, securing it stabilises a large part of the diagonal hierarchy. It does not fix the charm and bottom tensions — those remain separate repair problems — but it removes the weakest-by-provenance input from the grid.

There is an internal cross-check that the strain really does live downstream of the strange factor, not in it. Granting $m_s/m_d = 20$ exactly, the grid's charm entry predicts $m_c/m_s = (3080/13)/20 = 154/13 = 11.846$, against the FLAG $N_f = 2+1+1$ average 11.766(30) — a +0.080 excess, about +2.7σ. So the one ratio in the grid that *contains* the strange baseline reaffirms that the tension sits in the charm numerator (the 154/13 amplifier), not in the strange baseline itself. This is independent corroboration of the audit's ranking: strange is the safe trace to canonise, and charm is where repair is owed.

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11. Relation to the Five-Gate G/M_χ Decomposition

The sharpest conceptual risk is internal. In the five-gate decomposition, m_s/m_d is not primitive but a residue,

$$m_s/m_d = (G_2/G_1) \cdot e^{(M_{\chi,2} - M_{\chi,1})} \approx 108 \times 0.18 \approx 19.4,$$

a large cross-generation amplifier times a maintenance-sector suppression. Since 108 is not a binomial coefficient C(6,k), the triad count has no obvious home in the amplifier; since the maintenance term is a suppression, it has none there either.

But the tension is sharper than a mismatch of *form*, and it should be stated as bluntly as the numbers allow: the two VERSF readings do not agree numerically. The support trace gives exactly 20.00; the five-gate residue gives 108 × 0.18 = 19.44. These differ by 2.9% — larger than either's distance from the data, since against the FNAL/MILC ratio 19.78 the support value is +1.1% and the five-gate value is −1.7%, so the two VERSF predictions *straddle* the measurement. If 108 and 0.18 are exact structural outputs of the five-gate construction, then 19.44 ≠ 20.00 is a one-operator-two-values contradiction: the same ratio cannot have two exact VERSF values, and QM-S6 must then *resolve* the discrepancy, not merely translate between two forms of one number. If instead 108 and 0.18 carry tolerances, the five-gate paper must state them, and the claim downgrades to "the two readings agree at the few-percent level." Either way, as things stand the two pictures cannot both be exact descriptions of the same operator, and this paper does not assume they are.

11.1 Why a "change of basis" is not enough. The naive reconciliation — that the support trace and the G/M_χ product are two coordinate decompositions of the same readout — understates the problem, because the two are structurally different kinds of object. A support count is **additive**: $\mathrm{Tr}\, O_s = \Sigma_T 1 = 20$, the trace of a projector. The G/M_χ form is **multiplicative**: a product of two large factors. Trace is basis-independent but it is *not* multiplicative ($\mathrm{Tr}(AB) \neq \mathrm{Tr}\, A \cdot \mathrm{Tr}\, B$ in general), so no change of basis turns an additive count of twenty unit-eigenvalue atoms into a product 108 × 0.18. The genuine content of the open problem is therefore stronger than "different coordinates":

Trace-Factorisation Problem (sharpened). Exhibit a single strange current-mass operator that simultaneously (i) is the identity on the 20-dimensional support module (additive count 20) and (ii) admits the multiplicative G/M_χ reading as an amplifier-times-suppression on a different factorisation of the *same* space — *and* whose two readings agree numerically, which at present they do not (20.00 vs 19.44). So the resolution must do two things: explain *why* an additive support count would equal a multiplicative residue (not a corollary of trace invariance), and close the 2.9% gap, by correcting the support count, sharpening the five-gate factors, or establishing that one of the two is only approximate.

11.2 What this paper claims about the tension. It does not solve it. If the one-anchor trace basis is primitive for current-mass readout, the strange trace has a clean support-count candidate (this paper). If the five-gate basis is primitive and 20 is only a residue, the binomial interpretation may have no home, even though $m_s/m_d \approx 20$ remains empirically true (the five-gate eviction, §14.8). The next bridge is the trace-factorisation theorem above; until it exists, the support-trace and five-gate pictures are competing descriptions, and this paper is explicitly a candidate solution to QM-S *inside the one-anchor trace basis*.

A necessary first step — prior even to the factorisation theorem — is to quote the companion paper's *exact* forms for the two five-gate factors, because the contradiction-versus-tolerance fork above cannot be adjudicated without them. The two questions are sharp: is $G_2/G_1 = 108$ an exact product of structural integers, and is $e^{(M_{\chi,2} - M_{\chi,1})} = 0.18$ a closed exponential with a specified exponent? If both are exact, the residue is exactly $108 \cdot e^{(M_{\chi,2}-M_{\chi,1})}$, and whether that equals 20 or differs from it is a decidable arithmetic fact — at two significant figures it reads 19.44, a genuine $19.44 \neq 20.00$ contradiction. If either factor carries a tolerance, the residue inherits a band, and the claim downgrades to overlap-at-the-few-percent-level. The paper does not pretend to know which; it flags that the two-significant-figure values 108 and 0.18 are placeholders for forms the five-gate paper must supply, and that supplying them is the first sub-task of QM-S6.

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12. Empirical Status

The ratio m_s/m_d is a clean target because same-scheme mass ratios cancel most of the common QCD renormalisation structure (up to electromagnetic and isospin-breaking conventions). Folded through the light ratios,

$$m_s/m_d = (m_s/m_\ell) \cdot (1 + m_u/m_d)/2, \quad m_\ell = (m_u + m_d)/2,$$

with $m_s/m_\ell \simeq 27.23$ and $m_u/m_d = 6/13$ gives $m_s/m_d \simeq 27.23 \cdot (1 + 6/13)/2 \simeq 19.90$. The target 20 is within about one percent of current same-scheme estimates: roughly 0.5% high under this m_s/m_ℓ + 6/13 fold, and about 1.1% high relative to the FNAL/MILC central ratio 92.47/4.675 = 19.78 — comfortably the best of the open traces, but at the upper edge, not dead-centre.

This paper does not improve the empirical comparison; it improves the structural target. Empirical success does not prove the triad mechanism — it only makes it worth pursuing. The right standard is: **does the closure operator independently force the support module whose trace is 20?** If yes, the match becomes meaningful; if no, it remains a striking but underived coincidence.

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13. What Is Proved, Conditional, and Open

13.1 Proven here. Given $\mathscr{H}_d = \mathbb{C}[\mathcal{S}_1(I)]^{Sym(6)}$ and $\mathscr{H}_s = \mathbb{C}[Sym(6)/(Sym(3)\times Sym(3))]$, the trace ratio is exactly Tr O_s / Tr O_d = 20. The representation-theoretic count, the isotypic decompositions (1+5 = 6; 1+5+9+5 = 20), and the Johnson-scheme trace identity Tr $O_s = 20\cdot a_0$ are exact.

13.2 New structural content. A precise strange-operator candidate $O_s = I_{\mathscr{H}_s}$ on the induced module of occupied balanced splits, specifying: the stabiliser Sym(3)×Sym(3); the absence of the complement-exchange $\mathbb{Z}_2$; the absence of internal ordering; the support atom as trace unit; the invariant singleton mode as the down denominator; and — newly — that equivariance reduces the entire trace to the single self-weight a_0, with $a_0 = 1$ the one operative premise.

13.3 Conditional premises. Unresolved down baseline (depth one keeps the trivial component); support-resolving transition (strange = first post-boundary state); balanced support selection (3+3, not 2+4, 1+5, or full closure); occupied/exterior orientation (complement breaking); Sym(6)-equivariance with self-weight $a_0 = 1$; trace-factorisation bridge to the five-gate product.

13.4 Open problems.

- **QM-S1 — derive the invariant down readout.** Show from the depth-one confinement operator that the down baseline keeps only the trivial component $S^{(6)}$, not the full point module.
- **QM-S2 — derive support-boundary formation.** Show why strange is the first down-sector state with resolved support atoms.
- **QM-S3 — derive central-shell selection.** Show the first strange support is the balanced 3+3 shell (closure Hessian / transfer operator).
- **QM-S4 — derive complement breaking.** Show current-mass readout distinguishes occupied support from exterior complement (full $M^{(3,3)}$, not its complement-symmetric subspace).
- **QM-S5 — derive the self-weight a_0 = 1.** Show the equivariant mass observable has self-coupling exactly one in down-normalised units (the entire trace reduces to this scalar).
- **QM-S6 — prove trace factorisation.** Reconcile the additive support count 20 with the multiplicative five-gate product (§11). The first sub-task is to quote the companion paper's exact forms for G_2/G_1 and $e^{(M_{\chi,2}-M_{\chi,1})}$, since whether $19.44 \neq 20$ is a genuine contradiction or a tolerance overlap cannot be decided until those forms are fixed.
- **QM-S7 — connect to the explicit hadronic confinement operator.** Replace the representation-level ansatz with an operator derived from confinement geometry, including the support-vs-incidence and unordered-vs-ordered separations.

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14. Falsification Conditions

Each failure produces a specific wrong number, which is the point of stating the mechanism representationally. But the conditions below are not all falsifiers in the same sense, and listing them flat would flatter the falsifiability. They fall into three epistemic tiers:

- **Tier I — mechanism falsifiers (§14.1–14.7).** Each is of the form "if the confinement operator does X, the trace is the wrong number Y." These become *decidable* once the explicit confinement operator is known; until then they are open structural risks with definite resolutions.
- **Tier II — intra-theory basis competition (§14.8).** The five-gate eviction is not a falsification of VERSF, nor even strictly of this mechanism; it is a question of which VERSF basis is primitive, decidable only by building QM-S6. If it goes against the support picture, this *account* is evicted while VERSF and the value 20 survive.
- **Tier III — external empirical rejection (§14.9).** Only this tier is a genuine confrontation with measurement that could falsify the value itself rather than one of its derivations.

14.1 Down multiplicity failure. If the depth-one readout keeps the full point module (six sectors), the denominator is 6 and the ratio is $20/6 = 10/3$.

14.2 Uniform orbit failure. If down and strange both keep only the trivial component, both traces are 1 and $r_s = 1$.

14.3 Noncentral shell failure. If the strange operator resolves pairs, 4-shells, or singletons rather than balanced triads, the counts are 15, 15, or 6, not 20.

14.4 Complement quotient failure. If the operator identifies $T \sim I\setminus T$ (keeps only the complement-symmetric subspace of $M^{(3,3)}$), the trace is 10, not 20.

14.5 Ordered path failure. If the readout counts ordered triples, the count is $P(6,3) = 120$, or an ordered quotient.

14.6 Channel-incidence failure. If the readout is per channel incidence rather than per support atom, the natural count is $3 \cdot C(6,3) = 60$.

14.7 Wrong self-weight (refined). Under Sym(6)-equivariance the twenty supports *cannot* be unequally weighted (the diagonal is forced constant; §7.1), so "unequal weighting" is not an available failure. The real failures are scalar or symmetry-level: if the self-weight $a_0 \neq 1$, the trace is $20 \cdot a_0$; and if the confinement operator breaks Sym(6) so the mass observable leaves the Johnson-scheme algebra, the diagonal need not be constant and the trace need not be 20. The robust core is that, *given equivariance*, only a_0 matters.

14.8 Five-gate eviction. If the five-gate decomposition is the primitive current-mass decomposition and m_s/m_d is only a residue with no support-trace interpretation, the binomial assignment fails even though $m_s/m_d \approx 20$ remains empirically true.

14.9 Empirical rejection. If future same-scheme lattice determinations, with electromagnetic and isospin budgets, move m_s/m_d decisively off 20, the value is phenomenologically rejected and the mechanism is wrong however elegant.

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15. Conclusion

The one-anchor programme left the strange baseline as its highest-value unresolved input, and the trace audit made the situation precise: $\mathrm{Tr}\, O_s = 20$ is empirically successful but structurally unproven, while the constructed charm and bottom pieces need repair rather than justification. The rational next step is to derive the strange trace, and this paper gives the first explicit candidate.

The down quark is assigned to the invariant depth-one mode, $\mathscr{H}_d = \mathbb{C}[\mathcal{S}_1(I)]^{\mathrm{Sym}(6)}$ (the trivial component of the point module), so $\mathrm{Tr}\, O_d = 1$. The strange quark is assigned to the first balanced occupied/exterior support shell, $\mathscr{H}_s = \mathbb{C}[\mathrm{Sym}(6)/(\mathrm{Sym}(3)\times\mathrm{Sym}(3))] \cong \mathbb{C}[\mathcal{S}_3(I)]$ (the full balanced support module), so $\mathrm{Tr}\, O_s = 20$. The complement is not quotiented (it exchanges occupied with exterior); internal order is quotiented (the baseline reads supports, not paths); the unit is attached to each support atom. The core claim is

$m_s/m_d = 20$ because $\mathrm{Tr}\, O_s = \dim \mathrm{Ind}_{\mathrm{Sym}(3)\times\mathrm{Sym}(3)}^{\mathrm{Sym}(6)} \mathbb{1} = 20$, with $\mathrm{Tr}\, O_d = 1$.

Two strengthenings make this more than a relabelled triad count. First, the mixed rule is now an exact isotypic statement — trivial component for down, full module for strange — whose two "uniform" alternatives are precisely the obstruction values 1 and 10/3. Second, equivariance reduces the entire strange trace to a single self-weight: $\mathrm{Tr}\, O_s = 20\cdot a_0$ for any Sym(6)-equivariant operator, independent of inter-support couplings, so the operative *weighting* premise collapses to the scalar $a_0 = 1$. This should not be mistaken for collapsing the whole derivation to one scalar: it pins the weights, but the choice of the space $M^{(3,3)}$ itself still rests on the conditional premises QM-S1–S4 and S7. The weighting risk reduces to a_0; the space-selection risk remains distributed across those five open problems.

The claim is bounded. This is not a Standard Model derivation, an absolute-mass derivation, or a solution to CKM mixing or the heavy-quark tensions. But it attaches one measured Standard Model mass ratio to a definite operator with definite ways to succeed and fail. The next technical task is exact:

derive $\mathscr{H}_s = \mathrm{Ind}_{\mathrm{Sym}(3)\times\mathrm{Sym}(3)}^{\mathrm{Sym}(6)} \mathbb{1}$ and the self-weight $a_0 = 1$ from the strange confinement operator.

If that operator delivers the support module and unit self-weight, QM-S closes. If it delivers a complement quotient (→ 10), an ordered path space (→ 120), a channel-incidence trace (→ 60), the symmetric orbit rule (→ 1), or a broken-symmetry diagonal, the mechanism fails cleanly. That is the right form of progress: a Standard Model parameter is no longer merely fitted or admired; it is attached to a specific current-mass readout operator — and, just as importantly, to a single scalar premise and a single representation-theoretic choice that future work must either force or refute.

References

VERSF programme

- Keith Taylor, *Quark Masses from One Anchor in VERSF.* One-anchor calculator; identification of m_s/m_d = 20 as the highest-leverage inherited ratio.
- Keith Taylor, *The Strange Baseline in VERSF: Six-Channel Triad Resolution and the Twentyfold Down-to-Strange Ratio.* Triad-shell proposal; selection-freedom enumeration; mixed multiplicity obstruction.
- Keith Taylor, *Quark Masses from One Anchor II: The Current-Mass Operator and the Trace Audit of the Diagonal Quark Spectrum.* Operator-trace organisation; empirical grading of the open traces; the two-part QM-S condition (right space + flat weighting).
- Keith Taylor, *From Charge-Sector Maintenance to Quark Masses in VERSF.* Five-gate decomposition; G/M_χ basis; primitive-versus-composite tension for m_s/m_d.
- Keith Taylor, *Entropic Confinement and Effective Quark Masses in the VERSF Framework.* Quarks as confined partial closures; current vs constituent mass.
- Keith Taylor, *Finite Distinguishability and Local Capacity Competition* and related interface/capacity papers. Six primitive local interface channels; uniform primitive allocation (Sym(6) symmetry).
- Keith Taylor, *The Substrate Stiffness Hierarchy in VERSF.* Substrate-stiffness and localisation architecture; bottom-sector amplifier context.
- Keith Taylor, *The Single-Scheme χ Audit* and *Uniform Binary Admissibility and the Half-Lazy Closure Operator.* χ-profile audit and conditional halving law.

Lattice and quark-mass references

- Y. Aoki et al. (FLAG), *FLAG Review 2024*, Phys. Rev. D **113**, 014508 (2026), published 30 January 2026, DOI 10.1103/nfzp-p5dn; arXiv:2411.04268 [hep-lat]. N_f = 2+1+1 average m_c/m_s = 11.766(30); m_s/m_ℓ = 27.23(10).
- A. Bazavov et al. (Fermilab Lattice, MILC, TUMQCD), *Up-, down-, strange-, charm-, and bottom-quark masses from four-flavor lattice QCD*, Phys. Rev. D 98, 054517 (2018), arXiv:1802.04248.
- A. Bussone et al. (ETM), *Mass of the b-quark and B-decay constants from N_f = 2+1+1 twisted-mass lattice QCD*, Phys. Rev. D 93, 114505 (2016), arXiv:1603.04306.
- Y. Maezawa, P. Petreczky, *Quark masses and strong coupling constant in 2+1 flavor QCD*, Phys. Rev. D 94, 034507 (2016), arXiv:1606.08798.