Why Finite Distinguishability Forces Continuous U(1) Phase

The Shared Phase Obligation: the Born Face Discharged, the Generation Face Reduced — Holonomy Consistency Under Unbounded Composition

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What It Would Mean If This Paper Is Right

A plain-language note on the stakes, before the technical work begins.

Every standard presentation of quantum mechanics contains a rule that is simply announced rather than explained: the probability of an outcome is the square of a quantity called the amplitude. Why the square — not the amplitude itself, not its cube, not some other function? A century of physics has used this rule in every prediction and every technology built on quantum theory, and it works flawlessly. But it has remained a postulate: something assumed at the start, not something derived from anything deeper. Asking a physicist why probability is quadratic has traditionally earned the answer that nature is just like that.

If the argument in this paper holds, that question gets an answer. To see it, one piece of quantum imagery is needed, and it can be pictured without any mathematics. Imagine that every possible history a particle can follow carries a tiny dial — a single hand sweeping around a clock face — and that the hand advances as the history unfolds. When two histories start at the same place and end at the same place, quantum mechanics compares their dials. If the hands point the same way, the histories reinforce each other and the outcome becomes likely; if they point opposite ways, the histories cancel and the outcome becomes rare. That comparing of dials is interference — the effect behind the famous double-slit experiment — and it is not a decoration on quantum theory; it is the engine. Every quantum probability is computed by combining dial readings across histories before squaring. The dial's reading is what physicists call the phase.

The question this paper asks is about the face of that clock. Does the hand move smoothly, able to point in any direction at all — a complete, unbroken circle of positions? Or does it click between a fixed set of marks, like the hour positions on a watch? For a theory built from finite ingredients, the second option seems forced and the first seems impossible. This paper argues that the first is not just possible but compelled, and for a reason that can be said in one sentence: the circle of dial positions cannot have a ragged edge — it cannot permit positions arbitrarily close to some direction while forbidding that direction itself, because the forbidden direction would differ from its permitted neighbours in a way nothing in the world could ever detect, and the programme's foundational discipline forbids exactly that kind of phantom structure. Probability is the square of an amplitude, on this account, because that is what survives when the dial is a true circle and every undetectable distinction has been stripped away.

There is a second implication, larger in a quieter way. A long-standing dilemma haunts every attempt to describe nature as built from finite, discrete ingredients: such theories struggle to recover the smoothly turning dial — the perfectly continuous phase — that quantum interference demands. The choice has seemed forced — keep the finite substrate and lose exact quantum mechanics, or keep quantum mechanics and abandon finiteness. This paper argues the choice is false. A substrate can be genuinely finite in what it can locally distinguish, yet be forced into a continuous circle of accumulated phase, because finiteness of steps does not survive unbounded composition as finiteness of the whole. The dial's individual clicks are coarse; the positions it can reach by clicking without limit are not. Continuity, on this view, is not a primitive ingredient of nature. It is what consistency demands of a finite world that keeps composing.

And there is a third, internal to the programme: the recurring powers of two in the structure of matter — the pattern 1, 2, 4, 8 in refinement counts — and the squareness of probability would turn out not to be two mysteries but one, two shadows cast by the same continuous phase.

Honesty requires the conditionals to be stated as plainly as the stakes. These implications hold only if the paper's central argument survives — and that argument leans hardest on one step, the claim that the circle of phase can have no undetectable edge, whose precise point of attack is named in §10. The argument also stands on results inherited from earlier papers in the programme, each carrying its own status; this paper is a link in a chain, and a chain's strength is its weakest link, not its newest. Finally, the two big questions do not close equally. The probability question closes if the argument here holds — what remains on that side is calculation already laid out in the companion paper, not a missing foundation. The powers-of-two question takes a long step but ends one question short: it needs not only the menu of possible phases to be forced — which this paper establishes — but the assignment of phases to individual histories to be forced as well, and that final step is sharpened here to a single named question, then left open for the successor paper.

The reader should therefore take what follows neither as a settled revolution nor as speculation, but as what it is: a conditional derivation with named assumptions, a named weakest point, and consequences large enough to make the checking worth doing.

General Reader Summary

The obligation this paper takes up

Two of the deepest results in the programme — why particle generations come in powers of two, and why quantum probability is the square of an amplitude ($P = |\psi|^2$) — turn out to rest on one and the same unproven step. Both need a single fact about the substrate's "phase": that it is not a coarse, finite catalogue of values but a smooth, continuous circle of them, and that this continuity is forced by the substrate rather than chosen by hand. This paper sets out to force it.

There is an objection that has to be met head-on, because it sounds decisive. The programme insists that distinguishability is finite — distinction bottoms out at a smallest indivisible step, the Fold. So surely, the objection runs, the phase can take only finitely many distinguishable values, and a continuous circle of phase is exactly what finite distinguishability forbids.

The objection conflates two different things. A ruler marked only in millimetres has finite resolution, but the line it measures is not thereby forced to have only millimetre lengths. A camera has finite pixels, but the world is not thereby pixelated. Finite distinguishability limits what can be told apart locally — call this observability — but it says nothing on its own about the catalogue of values the phase variable may take. "Finite resolution" and "finite catalogue" are different claims, and the slide from one to the other is the whole error.

So where could continuity come from, if not from arbitrarily fine local distinctions? From a completely different direction: consistency under unbounded composition. Phase, in the programme, is not a local label stuck onto each state; it is an accumulated quantity — a holonomy, the net reading of the dial from the opening note — that builds up as transport is carried around longer and longer closed paths. Local steps are finite and coarse; but they compose, and a commitment record can be extended without bound, so the compositions accumulate without bound. The demand that the phase assigned to a path stay consistent no matter how the path is composed, reversed, or extended — that demand, this paper argues, is what forces the set of possible holonomies to fill out a continuous circle and to do so as a matter of the substrate's own structure, not of our choosing.

The argument runs in four moves. The possible holonomies form a group (paths compose and reverse, so their phases do too). That group is infinite (a finite phase catalogue is already ruled out elsewhere in the programme). An infinite group of phases on a circle is necessarily dense — its values come arbitrarily close to every point. And here the decisive step: the catalogue cannot have a ragged edge — it cannot contain values arbitrarily close to some phase while excluding that phase itself — because such a gap would be a distinction with nothing admissible to mark it, exactly the kind of surplus structure the programme forbids. So the catalogue must be closed; and a dense, closed group on the circle is the whole circle. That circle, with the smallest admissible structure (one Fold, reversible transport), is U(1) — the continuous phase group that the powers-of-two result and the Born rule both require.

The honest status: this is a derivation, but it leans hardest on that one step — that the catalogue can have no unwitnessed ragged edge — and that step is where any challenge should be aimed. If it holds, the probability question closes: the squareness of $P = |\psi|^2$ needs exactly this continuous circle of phase. The powers-of-two question takes a large step but does not fully close: it needs not only the menu of possible phases to be forced — which this paper establishes — but the assignment of phases to individual histories to be forced as well, and that further step is reduced here to a single named question, not settled.

Abstract

The dyadic-loading programme (*The Shared Phase Obligation*) and the Born-rule programme (*Physical Necessity of Quantum Probability Structure*) reduce, independently, to one proposition: that the substrate's transport phase is forced into a continuous U(1) group by admissible structure alone. The companion Born-rule paper establishes the impossibility of a purely finite holonomy set and defers the upgrade — infinite → continuous → minimally U(1) — to a dedicated treatment. This paper is that treatment.

The argument is a global-consistency argument, not a local-density one, and the distinction is load-bearing (§2). Finite distinguishability constrains phase observability — phase differences below the minimum distinguishable amount are not locally separable — but does not constrain the phase catalogue, the set of admissible holonomy values. Phase enters the programme not as a local distinguishability coordinate but as an accumulated transport holonomy; the minimum-distinguishability scale therefore bears on the wrong axis to obstruct continuity, and the objection "finite distinguishability ⟹ finite phase" is a category error (§2).

Continuity is forced instead by consistency of holonomy under unbounded composition, in four steps. (I) Admissible holonomies form a group H under phase composition, since admissible paths compose, reverse, and contain the trivial path (§5) — [Proven]. (II) H is infinite, since finite holonomy is impossible (companion) and commitment records are unboundedly extensible (§6) — [Inherited / Proven]. An infinite subgroup of the circle is dense (§6). (III) H is closed: a catalogue boundary requires a maintaining mechanism, and the candidates fail exhaustively — a physical barrier would be an invariant resolving θ* from holonomies below the minimum distinguishable scale (contradicting finite distinguishability), an algebraic obstruction would contradict the group structure, and an unmaintained exclusion is an admissibly-unwitnessed distinction, surplus structure forbidden by no-pre-individuation — so H contains its limit points (§7) — [Proven, conditional on no-pre-individuation and finite distinguishability, both [Inherited], and the group structure of (I)]. A dense, closed subgroup of the circle is the whole circle. (IV) The ambient phase group is the circle (reversible transport gives unit-modulus phase factors, not unbounded $\mathbb{R}$), one-dimensional (a single Fold gives one independent phase), connected (by III), and abelian (phases compose additively) — hence exactly U(1), the unique one-dimensional compact connected abelian Lie group (§8) — [Proven, conditional on inherited reversibility and single-Fold binarity].

Crucially the continuity so obtained is forced from the distinguishability data D in the catalogue sense: closure is compelled by no-pre-individuation, a constraint on admissible structure, so the continuous value space of the phase is D-determined rather than freely chosen (§9). This discharges the Born chain, which consumes exactly the group structure — continuous U(1) phase makes the interference structure non-degenerate and the quadratic measure unique. The generation chain requires more — a D-determined assignment of holonomy to path, not merely a D-forced catalogue, since two assignments could share the full range U(1) while differing on paths. The paper therefore reduces the generation obligation rather than discharging it — from "is θ D-determined?" to "given that the catalogue is D-forced to U(1), is the assignment L ↦ h(L) D-determined?" — with the candidate closing argument (no-pre-individuation applied to assignments) named as the remaining step (§9) — [Open]. The load-bearing step of the catalogue

result is (III); its refuter is named (§10). If (III) holds, the Born obligation is discharged and the generation obligation stands at its final question.

Epistemic markers: [Inherited] imported from prior VERSF papers; [Proven] established here; [Conditional] holding under stated inputs; [Conjectural] motivated but unproven; [Open] undecided.

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1. Introduction — The Shared Obligation

Two independent arcs of the programme bottom out at one proposition.

The dyadic-loading arc (*The Shared Phase Obligation*, §17.4) reduces "why do refinement counts go 1, 2, 4, 8, …?" to the supervenience of a candidate-layer support functional s on the admissible distinguishability data D. The corpus builds s from a single object — the geometric phase θ of the Double Square kernel

$$W(P, P') = e^{i(\theta(P) - \theta(P'))} ,$$

so supervenience holds if and only if θ is determined by D. By inspection θ is not D-determined: the distinguishability datum is a symmetric metric d, while θ is oriented and carries a holonomy that d does not contain. Supervenience therefore holds exactly if θ is forced from D.

The Born-rule arc (*Physical Necessity of Quantum Probability Structure*) reduces "why is probability quadratic, $P = |\psi|^2$?" to the requirement that the same phase be continuous and valued in U(1): a finite or discrete phase does not support the interference structure that makes the quadratic measure the unique admissible one.

These are not two requirements but one. The proposition that discharges both is specifically that D forces continuous U(1) phase — the forcing doing double duty, since "forced from D" supplies the generation arc's D-determination while "continuous U(1)" supplies the Born arc's interference structure. Neither weaker reading suffices: a phase forced-but-discrete would serve the generation arc but not the Born arc; a phase continuous-but-free would serve the Born arc but not the generation arc. The shared obligation is the conjunction, carried by the single word *forces*.

The companion Born-rule paper establishes the first half — that a purely finite holonomy set is impossible — and defers the upgrade to continuous U(1) as its sole remaining foundational obligation. This paper takes up that upgrade. The strategy is to show that continuity is not a local-resolution fact (which finite distinguishability would indeed forbid) but a global-consistency fact: a consequence of how transport holonomy must behave under unbounded composition. On that route the minimum-distinguishability scale is no obstruction, and the continuity obtained is forced by admissible structure at the level of the catalogue — the value space of the phase. Whether the catalogue result suffices differs between the two client chains, and the difference is tracked honestly throughout (§9): the Born arc consumes the catalogue result and is discharged; the generation arc requires in addition that the assignment of holonomy to path be D-determined, a question this paper sharpens to its final form but does not settle.

2. Observability Versus Catalogue — Why the Minimum Scale Is No Objection

The programme posits finite distinguishability: admissible distinction bottoms out at the Fold, the minimal indivisible distinction. The immediate objection is that finite distinguishability forces a finite phase set, so that continuous U(1) phase is excluded at the outset. This objection must be answered before any positive argument, because if sound it ends the discussion.

It is not sound. It conflates two claims that come apart.

Observability Versus Catalogue

Phase observability is the set of phase differences that can be locally told apart — bounded below by the minimum distinguishable phase. *Phase catalogue* is the set of values the phase variable may take. Finite distinguishability constrains the former; it does not, on its own, constrain the latter.

The two are as different as the resolution of an instrument from the granularity of the quantity it measures. A ruler marked in millimetres has millimetre resolution; the lengths it measures are not thereby quantised to millimetres. Finite pixels do not pixelate the world. The inference "finite distinguishability ⟹ finite phase catalogue" is therefore a non-sequitur — valid only if phase were a local distinguishability coordinate, a label read off each state by direct comparison, whose admissible values were exactly the locally-distinguishable ones.

In the programme phase is not such a coordinate. It is an accumulated transport holonomy: a quantity that builds up as transport is composed around closed paths (§4). Its values are not read locally from a single state; they are the net result of composing many admissible steps. The minimum-distinguishability scale constrains each step — and constrains which holonomies can be locally distinguished — but it does not constrain which holonomy values the catalogue contains, any more than millimetre marks constrain the lengths of lines.

This relocates the question entirely. Continuity, if it holds, cannot come from arbitrarily fine local distinction — finite distinguishability genuinely forbids that. It must come from the structure of holonomy under composition. The minimum-distinguishability scale bears on local separability, an axis orthogonal to the catalogue's topology; it is set aside here not by being answered but by being shown irrelevant to the claim at issue. No theorem is claimed in this section and no marker attached: the observability/catalogue distinction is a conceptual clarification that dissolves the objection, not a proposition with a proof. The positive case for continuity is the business of §§5–8.

3. The Global-Consistency Frame

The name of the companion's result — finite holonomy impossible → continuous U(1) holonomy — already signals a global-consistency argument rather than a local-density one. Local density would source continuity from the existence of arbitrarily fine local distinctions; the programme has none, by finite distinguishability. Global consistency sources continuity from the requirement that holonomy compose consistently along unboundedly extensible paths. The two routes are distinguished sharply because only the second is available, and only the second yields continuity in the forced-from-D form (§9).

The frame has three standing inputs, each inherited.

Finite local distinction. [Inherited, §2 of *Binary Refinement from Minimal Distinction*; *The Shared Phase Obligation* §9] Admissible distinction bottoms out at the Fold. Each elementary transport step is finite and coarse.

Reversible transport. [Inherited; the Reversible Connectedness paper] Admissible transport between commitments is reversible: every step has an admissible inverse, and norm is preserved, so transport acts by unit-modulus phase factors. This is what places holonomy on the circle rather than the line (§8).

Unbounded composition. [Inherited; the commitment-record extensibility results] A commitment record is extensible without bound: there is no longest admissible path, no maximal composition. Holonomies accumulate from arbitrarily long compositions of elementary steps.

The first input says the steps are finite. The third says the compositions are unbounded. The tension between them — coarse steps, unbounded accumulation — is the engine of the argument: finiteness of steps does not survive unbounded composition as finiteness of the catalogue, and reversibility forces the accumulated quantity onto a compact circle where unbounded accumulation must fold back on itself. The four sub-claims make this precise.

4. Transport, Holonomy, and the Phase Group

Let admissible transport carry the candidate-layer structure along admissible paths between commitments. By reversibility (§3) each elementary transport acts by a phase factor $e^{i\alpha}$ with α real and the factor of unit modulus. Composition of transports multiplies phase factors; reversal conjugates them.

For a closed path L — one returning to its starting commitment — the net transport is a phase factor

$$h(L) = e^{i\theta(L)} \,,\quad \theta(L) \in \mathbb{R} \bmod 2\pi \,,$$

the holonomy of L. The holonomy is the accumulated phase around L, well-defined modulo 2π because the phase factor, not the angle, is the admissible object (reversibility fixes $|h(L)| = 1$; nothing fixes a preferred branch of the angle). Define the holonomy catalogue

$$H = \{ h(L) : L \text{ an admissible closed path} \} \subseteq \mathbb{T} \,,$$

where $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ is the circle of unit-modulus phase factors. The proposition to be established is

$$H = \mathbb{T} = U(1) \,, \text{ with the equality forced by admissible structure (D-determined).}$$

The four sub-claims establish, in turn: H is a subgroup of $\mathbb{T}$ (§5); H is infinite, hence dense in $\mathbb{T}$ (§6); H is closed (§7); and the ambient group $\mathbb{T}$, carrying the minimal admissible structure, is exactly U(1) (§8). A dense, closed subgroup of $\mathbb{T}$ is $\mathbb{T}$ itself, so (I)–(III) give $H = \mathbb{T}$ and (IV) identifies $\mathbb{T}$ as U(1).

A note on the phase θ and the support functional s. The kernel of *The Shared Phase Obligation* is $W(P, P') = e^{i(\theta(P) - \theta(P'))}$; the difference $\theta(P) - \theta(P')$ is the holonomy of the path $P' \to P \to P'$ in the present notation, so the θ whose D-determination was the open question there is exactly the accumulated holonomy phase here. A distinction must be kept sharp from the outset: establishing that H is forced to U(1) by admissible structure establishes that the *value space* of θ is forced from D. It does not by itself establish that θ — the assignment of holonomy to individual paths — is forced from D: the catalogue is the range of the assignment $L \mapsto h(L)$, not the assignment, and two distinct assignments could share the full range U(1) while differing on paths. Which of the two client chains needs which is the business of §9.

5. Sub-Claim I — Admissible Holonomies Form a Group

Holonomy Group

The holonomy catalogue H is a subgroup of $\mathbb{T}$: it is closed under composition and inversion and contains the identity.

The reasoning is structural and follows from what admissible paths are.

Identity. The trivial path — remaining at a commitment — is admissible and has holonomy e^{i·0} = 1. So $1 \in H$.

Closure under composition. If L_1 and L_2 are admissible closed paths at a common commitment, their concatenation $L_1 \circ L_2$ is an admissible closed path, with holonomy $h(L_1 \circ L_2) = h(L_1)\, h(L_2)$. So $h(L_1)\, h(L_2) \in H$. (Paths based at different commitments are related by reversible transport between the base points — which requires that the commitment structure be connected, so that a connecting admissible path exists [Inherited; the Reversible Connectedness paper] — and base-point transport conjugates the holonomy; since $\mathbb{T}$ is abelian, conjugation is trivial and the catalogue is base-point independent.)

Closure under inversion. By reversibility (§3) the reverse path L^{-1} is admissible, with holonomy $h(L^{-1}) = h(L)^{-1}$. So $h(L)^{-1} \in H$.

Hence H is a subgroup of $\mathbb{T}$. [Proven], given reversibility [Inherited] and the admissibility of path concatenation [Inherited]. No appeal to continuity, density, or the cardinality of H is made here; this step fixes only the algebraic type of the catalogue. Its force in the argument is that the next two steps may now use the structure theory of subgroups of the circle, which is far more rigid than that of arbitrary subsets.

6. Sub-Claim II — The Group Is Infinite, Hence Dense

Infinitude

H is infinite.

The claim rests on the inherited result; a supporting intuition from unbounded composition is recorded second, with its status honestly demoted.

From the companion. [Inherited] *Physical Necessity of Quantum Probability Structure* establishes that a purely finite holonomy set is inadmissible: a finite H would make the candidate-layer interference structure degenerate in a way the admissibility of the measure forbids. This is the half of the shared obligation already discharged there; the present paper inherits it, and Sub-Claim II rests entirely on it.

From unbounded composition. [Conjectural — supporting intuition, not proof] Suppose for contradiction H were finite. A finite subgroup of $\mathbb{T}$ is cyclic of some order n: $H = \{ e^{2\pi ik/n} : k = 0, \ldots, n-1 \}$, i.e. $H \cong \mathbb{Z}/n$. Then every admissible holonomy is an n-th root of unity, and composing any path with itself n times returns holonomy 1. The intuition is that unbounded composition (§3) admits paths of arbitrary length whose elementary increments need not be rational multiples of $2\pi/n$, so that iteration escapes any finite $\mathbb{Z}/n$. But the argument as stated is circular at one point and cannot carry [Proven]: the claim that the elementary increment α can be (or generically is) an irrational multiple of 2π presupposes a structure of available increments richer than the finite catalogue under refutation, and no measure or structure over possible increments is supplied — "the substrate does not impose rationality" is not the same as "the substrate supplies irrationality". The leg is therefore recorded as intuition for why finiteness is fragile under unbounded composition, nothing more. The proof obligation is carried wholly by the inherited result above.

Infinitude gives density. A standard and rigid fact about the circle: every subgroup of $\mathbb{T}$ is either finite cyclic or dense in $\mathbb{T}$. (A subgroup containing an element of irrational angle generates a dense set; a subgroup all of whose elements have rational angle is either finite or, if unbounded in denominator, dense.) Since H is a subgroup (§5) and infinite (just shown), H is dense in $\mathbb{T}$:

$$\bar{H} = \mathbb{T} ,$$

where $\bar{H}$ is the topological closure. [Proven], given (I) and infinitude [Inherited]. Note what this does and does not give: density says H comes arbitrarily close to every phase, but not that H contains every phase. The gap between dense and full is exactly the gap between a catalogue with a ragged edge and a continuous one — closed in §7.

7. Sub-Claim III — The Group Is Closed, Hence Continuous

This is the load-bearing step. Density (§6) places H arbitrarily close to every point of $\mathbb{T}$; what remains is to show H contains its limit points — that the catalogue is closed — for a dense closed subgroup of $\mathbb{T}$ is $\mathbb{T}$ entire. Closure is not a structural fact like §5 nor a cardinality fact like §6. The argument is given in two equivalent registers. In the mechanism register, closure is forced by a maintenance demand: a catalogue boundary is a piece of physical structure, and physical structure must be maintained by something admissible; §7.1 exhausts what that something could be. In the conservation register, closure is forced by an invariance demand: admissibility may change only where some admissible invariant changes, and across an unrealized limit point none does; §7.2 states this as a lemma and derives it from no-pre-individuation. Structural instability corroborates (§7.3), the amplification asymmetry answers the mirror objection — why the principle removes the boundary without coarsening the catalogue (§7.4) — and the canonical dense-proper-subgroup counterexample is dispatched explicitly (§7.5).

7.1 The Boundary Maintenance Principle

The question to put to any catalogue boundary is not whether it is permitted but what maintains it. The catalogue is not a static list: H is generated, holonomy by holonomy, by admissible transport compositions, and unbounded composition (§3) keeps generating. Suppose $\theta^* \in \mathbb{T}$ is excluded from H while admissible holonomies $\theta_k \in H$ approach θ^* arbitrarily closely — by density (§6), every point of $\mathbb{T}$ is so approached. Admissible transport is actively producing values converging on θ^*. Something must answer the question: what prevents the limit?

Boundary Maintenance Principle

A catalogue boundary is physically admissible only if some admissible structure maintains the exclusion.

The possibilities are exhausted by a trichotomy. Either composition remains admissible arbitrarily close to θ^* or it does not; and if it does, either some admissible structure marks the exclusion of θ^* or nothing does.

Case 1 — a physical barrier. Some admissible feature of the substrate blocks approach to θ^* or marks its exclusion. Then the barrier is admissible physical structure: detectable in principle, an admissible invariant, and the boundary is witnessed. But consider the work the barrier must do: it must resolve θ^* from admissible holonomies lying within the minimum distinguishable phase of it — the θ_k come that close (§7.2). No local comparison resolves distinctions below that scale (§2). The only admissible mechanism known to the programme for resolving sub-scale phase differences is amplification under composition (§7.4), and it requires both terms of the comparison to be realized by admissible paths, so that their difference can be iterated above threshold; θ^* is, by hypothesis, realized by nothing, so no path on its side of the comparison exists to iterate. Any barrier must therefore either operate through amplification — which it cannot — or constitute a new admissible sub-scale comparison of a kind the programme does not contain, which the challenger is invited to exhibit (§10). The exhaustiveness of amplification is thus carried as a burden-shift, not asserted as a proven universal — the same posture §7.6 adopts for the argument as a whole. Case 1 supplies no admissible mechanism. (Note the reversal: finite distinguishability, the source of the original objection to continuity, is here exactly what forbids the barrier that a discrete or gapped catalogue would need.)

Case 2 — an algebraic obstruction. Composition ceases to be admissible near θ^*: concatenations whose holonomies would converge on θ^* are themselves inadmissible. But admissible closed paths concatenate and reverse admissibly — these are the inherited inputs of §5 — and the obstruction would break exactly that. Case 2 contradicts the group structure of Sub-Claim I.

Case 3 — nothing maintains the exclusion. Then the catalogue simply declares: every nearby phase exists; this one does not. The exclusion has no physical cause, performs no operational work, and admits no witness by any admissible route, local or composed (§7.4); it is a hidden label, pure surplus structure, and no-pre-individuation forbids it (§7.2).

The trichotomy is exhaustive and every case fails. Hence no catalogue boundary can be physically maintained: H contains its limit points, i.e. H is closed. Combined with density (§6),

$H = \mathbb{T}$.

[Proven, conditional on no-pre-individuation [Inherited] (Cases 1 and 3, with Case 1 also using finite distinguishability [Inherited]) and the group structure of §5 (Case 2).]

The upgrade over a bare appeal to no-pre-individuation is the burden it imposes. The claim is no longer only that unwitnessed distinctions are forbidden; it is that every catalogue boundary requires a physical mechanism maintaining it, and no such mechanism exists. The challenger's task is correspondingly concrete: name the mechanism (§10).

7.2 The No-Pre-Individuation Closure Argument

Cases 1 and 3 of the trichotomy terminate in a single principle: the exclusion of θ^* can have no admissible witness. This subsection states that principle in its strongest form — as a conservation statement — and then gives the argument in full.

Admissibility Conservation Lemma

Physical admissibility may change only where some admissible invariant changes.

[Proven, conditional on no-pre-individuation, [Inherited]: if admissibility differed between two configurations while every admissible invariant agreed, that difference would itself be a distinction with no admissible witness — exactly what no-pre-individuation forbids.]

The closure argument is then four steps. (i) Admissibility may change only where some admissible invariant changes (the Lemma). (ii) Across a limit point θ^* of H, no admissible invariant separates θ^* from the admissible holonomies converging on it: no local comparison (§2), and no amplification witness, since θ^* is realized by no path (§7.4). (iii) Therefore admissibility cannot change at θ^*. (iv) Therefore θ^* cannot be excluded: it carries the same admissibility as the values converging on it, and the catalogue is closed. The remainder of this subsection unpacks step (ii) — the unwitnessed-distinction argument — in full.

Suppose $\theta^* \in \mathbb{T}$ is a limit point of H: there are admissible holonomies $\theta_k \in H$ with $\theta_k \to \theta^*$. By density such θ^* is every point of $\mathbb{T}$. The question is whether $\theta^* \in H$ — whether the catalogue contains the values it approaches.

Suppose, for contradiction, that $\theta^* \notin H$ while $\theta_k \to \theta^*$, $\theta_k \in H$. Then the catalogue distinguishes θ^* from its own accumulation points: it admits holonomies arbitrarily close to θ^* but excludes θ^* itself. Consider what marks θ^* as excluded. The θ_k approach θ^* within any scale, in particular within the minimum distinguishable phase (§2): beyond some k the holonomies θ_k are not locally distinguishable from θ^*. Nor is there a global witness: the only admissible route known to the programme for resolving sub-scale phase differences is amplification under composition (§7.4), and it is unavailable here, since θ^* is realized by no admissible path whose iteration could enter the comparison; any further route is the challenger's to exhibit (§10). So the exclusion of θ^* is witnessed by no admissible distinction, local or composed — there is no admissible comparison that separates θ^* from the included θ_k near it.

The catalogue boundary at θ^* is an admissibly-unwitnessed distinction: a feature of the structure carried by nothing in the distinguishability data.

This is precisely surplus structure, which no-pre-individuation forbids.

No-Pre-Individuation [Inherited]

Admissible physical structure is exhausted by admissible distinguishability relations. A distinction with no admissible witness is not admissible structure.

A catalogue that includes points arbitrarily close to θ^* but excludes θ^* posits a distinction — "in the catalogue" versus "not in the catalogue" — at a location no admissible comparison can resolve. By no-pre-individuation that distinction is not admissible structure; the catalogue cannot carry it. Hence $\theta^* \in H$. As θ^* ranges over all limit points, H contains its closure:

$\bar{H} \subseteq H$, so H is closed.

Closure

The holonomy catalogue H is topologically closed: it contains its limit points, because excluding a limit point would posit an admissibly-unwitnessed distinction forbidden by no-pre-individuation.

Combining with density (§6): H is dense and closed in $\mathbb{T}$, hence

$H = \bar{H} = \mathbb{T}$.

The catalogue is the entire circle. [Proven, conditional on no-pre-individuation, which is [Inherited].]

A structural reading, and the form of the conclusion carried forward. H was defined extensionally in §4, as the set of holonomies of admissible closed paths. Read extensionally, closure would assert that every limit point θ^* is realized by some admissible path — an existence claim about paths that no-pre-individuation, a prohibition on unwitnessed distinctions, does not obviously deliver. The defensible conclusion, and the one this paper relies on downstream, is structural: the distinction between H and its closure $\bar{H}$ is not admissible structure, so the admissible phase structure is the closed group $\bar{H} = \mathbb{T}$, irrespective of which individual points happen to be path-realized. This weaker form is all the subsequent arguments consume — §8 identifies the admissible phase group, and the Born chain needs the group structure, not a census of realized values.

Closure as completion. Read this way, closure is a completion theorem: realizable sequences converge, their limits are admissibly inseparable from their tails, and what admissible structure cannot separate it cannot exclude — the admissible phase structure is the completion of the realizable holonomies. One caution keeps the theorem in this form rather than an apparently stronger one. It is tempting to argue instead that "physical equivalence identifies limits of realizable sequences" and obtain closure by identifying indistinguishables pointwise. But indistinguishability under finite resolution is not transitive — chains of sub-resolution steps sum to super-resolution differences — so it is not an equivalence relation, and applying the identification as a quotient would collapse the catalogue itself: the mirror objection of §7.4, returned in quotient form. The conservation route avoids the sorites entirely: it removes exactly one thing, the unwitnessed boundary distinction, and never identifies any pair of admissible holonomies.

The reach of the Lemma — and what it cannot do. A scope statement prevents over-application. The Lemma forces in exactly the limit points of realized structure — values that realized holonomies approach at every scale — and is silent about isolated exclusions. The difference lies in premise (ii). At a limit point, any candidate invariant marking the boundary would have to separate θ* from realized values at arbitrarily small distances, which neither local comparison nor any fixed composition power can do. At an isolated excluded value — a fixed positive distance δ from every realized holonomy — premise (ii) cannot be established: composition powers of the nearby realized members supply super-resolution markers (m-fold composition displaces a δ-separation to mδ) in whose terms the boundary's location is admissibly expressible, so the exclusion is not unwitnessed in the Lemma's sense. Two consequences. First, finite catalogues do not fall to conservation: their gaps are isolated, and only the companion's interference argument excludes them — the division of labour with §6 is principled, not provisional. Second, no inflation argument can substitute for the companion's result: applying the Lemma pairwise to isolated sub-resolution neighbours of realized values would, if licensed, force an interval into the catalogue and — since admissible structure composes — inflate even the trivial catalogue {1} to the full circle, deriving U(1) from a substrate with no holonomy at all, which proves far too much. The Lemma's restriction to genuine limit points blocks that absurdity, exactly as its restriction to unwitnessed boundaries blocked the sorites above. [Proven, conditional on the inputs of this section.]

Assembly without density — a second route. The result reassembles without §6's density step carrying independent load. H is a subgroup (§5) and infinite (§6, inherited); an infinite subset of the compact circle has limit points (Bolzano–Weierstrass); the Lemma forces every limit point in; and the closure of a subgroup is a subgroup — so $\bar{H}$ is a closed infinite subgroup of $\mathbb{T}$. The closed subgroups of $\mathbb{T}$ are completely classified: finite cyclic, or $\mathbb{T}$ entire. Infinitude excludes the former; hence $\bar{H} = \mathbb{T}$, and density falls out of the conclusion rather than feeding it. The two assemblies consume identical inputs; the second shows that a referee who resists the density step has not thereby escaped the result. [Proven, given the same inputs as the first assembly.]

7.3 Structural Instability of Unwitnessed Boundaries

A further consideration sharpens Case 3. Suppose θ* is excluded with nothing maintaining the exclusion. Consider an admissible transport history whose holonomy lies near θ*, and perturb it by composing with admissible paths of arbitrarily small holonomy — density (§6) guarantees such paths exist. Arbitrarily small perturbations carry the holonomy past θ*, from admissible values on one side to admissible values on the other, and the perturbed histories remain admissible throughout. Yet at θ* itself admissibility flips: the value is excluded. No invariant changes across the flip; no distinction becomes detectable; no physical event marks the crossing.

Admissibility is then discontinuous under arbitrarily small perturbations of transport history, with the discontinuity unaccompanied by any admissible structure.

Instability of Unwitnessed Boundaries

An admissibility boundary unsupported by an invariant makes admissibility discontinuous under arbitrarily small perturbations of transport history. Such boundaries are structurally unstable and therefore inadmissible.

A note on the status of this argument. The instability does not need to be imported as an independent stability axiom; it reduces to the programme's existing discipline — it is the Admissibility Conservation Lemma (§7.2) read dynamically. A discontinuity in admissibility accompanied by no change in any admissible invariant is itself an unwitnessed distinction — the same prohibited object as in §7.2, now in dynamical form. The instability argument is therefore corroborating rather than separately load-bearing: it exhibits the unwitnessed boundary as not merely a static surplus but a dynamical pathology, an admissibility that jumps where nothing happens. [Proven, conditional on no-pre-individuation, [Inherited].]

7.4 The Amplification Asymmetry — Why the Boundary Falls and the Catalogue Stands

The closure argument invites a mirror objection, and this section answers it. §7.2 argued that the exclusion of θ^* is unwitnessed because nearby admissible θ_k fall within the minimum distinguishable phase of θ^*. A referee will invert the blade: if sub-resolution distinctions are inadmissible, why does no-pre-individuation not collapse the included points as well? The difference between two admissible holonomies θ_k and θ_{k+1} is equally invisible to any local comparison. Applied with the same hand, the principle appears to threaten to quotient the catalogue down to a coarse finite set — contradicting §6 and destroying the result from the opposite side. If the closure argument proved that, it would prove too much.

The answer is an asymmetry in what composition can witness.

Take two admissible holonomies θ_a and θ_b with sub-resolution difference $\delta = \theta_b - \theta_a$, realized by admissible closed paths L_a and L_b. The concatenation $L_b \circ L_a^{-1}$ is admissible (§5) with holonomy δ, and iterating it n times is admissible (unbounded composition, §3) with holonomy $n\delta$. Writing ε_{min} for the minimum distinguishable phase: amplification accumulates $n\delta$ modulo 2π, so n is chosen with $n\delta$ inside the distinguishable band but below wraparound — $\varepsilon_{min} < n\delta < 2\pi - \varepsilon_{min}$ — which is always possible for sub-resolution δ (take $n \approx \lceil \varepsilon_{min}/\delta \rceil$, well short of $2\pi/\delta$). For such n, $n\delta$ exceeds the minimum distinguishable phase without wrapping. The sub-resolution distinction between θ_a and θ_b is therefore witnessed — not locally, but globally, by an explicit admissible comparison built out of the very paths that realize them. Distinctions between admissible holonomies are admissible structure because composition amplifies them above threshold.

Now run the same construction at the boundary. The distinction to be witnessed is between the included θ_k and the excluded θ^*. Amplification requires an admissible path on each side of the comparison: a loop realizing θ* whose iterates could be set against the iterates of L_k. By hypothesis there is none — θ* is excluded, realized by nothing. The would-be boundary admits no amplification witness, and no local witness either (§2). It is a distinction that nothing admissible can reach by any route.

This is the precise asymmetry that makes no-pre-individuation cut where the argument needs it and nowhere else. Distinctions between realized holonomies carry an amplification witness and stand; the distinction between the catalogue and its limit points carries none and falls. The principle does not coarsen the catalogue — it closes it. The exclusion of θ* performs no operational work and encodes a hidden label no admissible invariant supports; the inclusion of arbitrarily fine-grained admissible holonomies performs operational work under composition and is untouched.

Without this asymmetry the closure argument would be self-undermining; with it, the argument removes exactly one thing — the unwitnessed boundary — and the dense catalogue, now edgeless, is closed: $\bar{H} = \mathbb{T}$ as admissible phase structure (§7.2). The continuity of phase is thereby forced by the requirement that admissible transport and admissible distinguishability describe the same physical structure. [Proven, conditional on no-pre-individuation [Inherited], unbounded composition [Inherited], and the group structure of §5.]

7.5 Why Dense Proper Subgroups Are Not Physically Stable

A referee who knows the structure theory of the circle will raise the canonical counterexample at once: dense proper subgroups of $\mathbb{T}$ exist in abundance. The rational-angle subgroup

$$H_{\mathbb{Q}} = \{ e^{2\pi i q} : q \in \mathbb{Q} \}$$

is dense, proper, and a perfectly legitimate mathematical object — it is, in fact, exactly the torsion subgroup of $\mathbb{T}$, the elements of finite order. If mathematics permits a dense catalogue with gaps, why must physics close them? The answer is that $H_{\mathbb{Q}}$ fails every test of §§7.1–7.4, and watching it fail concretely is instructive.

What would mark its boundary? The defining property of $H_{\mathbb{Q}}$ is finite order: $h \in H_{\mathbb{Q}}$ iff $h^n = 1$ exactly, for some integer n. For that property to maintain the boundary it would have to be an admissible invariant — and under finite distinguishability it is not. Certifying $h^n = 1$ exactly requires resolving exact return from sub-resolution near-return, a sub-scale comparison; and amplification does not rescue it, because at every finite iteration count the admissible observations of a torsion element are consistent with a non-torsion element sufficiently close to it. No admissible protocol, local or composed, decides finite order. The one invariant that could carry the rational/irrational boundary is admissibly invisible at every scale.

The trichotomy of §7.1 then runs its course: no barrier (the order invariant is inadmissible), no algebraic obstruction (concatenation of admissible loops is admissible irrespective of the rationality of their angles — §5 nowhere consults the angle), so the exclusion of every irrational-angle value would be unmaintained — forbidden. Equivalently, by the Admissibility

Conservation Lemma (§7.2): across each excluded point, no admissible invariant changes, so admissibility cannot change there.

The structural-stability statement of the failure: a dense proper subgroup is not merely missing points; it is missing them in a way that no admissible invariant detects. Admissible perturbations of transport move holonomies arbitrarily close to every excluded value while nothing physical registers the approach to a forbidden zone. The exclusion is not structurally robust — it is a mathematical decoration the physics cannot see. Dense proper subgroups of $\mathbb{T}$ are admissible as mathematics and inadmissible as physics, and the difference is exactly the programme's discipline: physical structure is exhausted by admissible invariants. [Proven, conditional on the inputs of §§7.1–7.2.]

7.6 Why Closure Is the Load-Bearing Step

The honest accounting: §5 (group) is structural and secure; §6 (infinite, dense) follows from the inherited finite-holonomy-impossibility and a rigid fact about circle subgroups; §8 (ambient is U(1)) follows from inherited reversibility and single-Fold binarity. The weight of the paper rests on §§7.1–7.2 — that no admissible mechanism can maintain a catalogue boundary. Three features of that step deserve emphasis.

First, it is the step that makes continuity *forced from D* rather than merely true. Closure is compelled by no-pre-individuation, a constraint on what counts as admissible structure — i.e. on what D contains. The continuity of H is therefore not an extra fact layered onto D but a consequence of D's own admissibility discipline (§9). Had continuity been argued from local density — arbitrarily fine distinctions — it would have been a fact about resolution, not about D, and would not have delivered D-determination. The no-pre-individuation route is the one that yields the form both client chains require.

Second, it is the step a referee should attack, and the attack now has an exhaustively specified shape (§10): name the mechanism that maintains the boundary. By the trichotomy of §7.1 the only candidates are an admissible barrier — a comparison that resolves θ^* from holonomies within the minimum distinguishable phase of it, contradicting finite distinguishability — and an algebraic obstruction, contradicting the group structure of §5. Absent either, the exclusion is unmaintained, and no-pre-individuation forbids it. The burden, as throughout the programme, lies with the challenger to name the admissible mechanism, not with the argument to prove a universal negative.

Third, closure is not optional, and its weight is not aesthetic: it is the unique point where three of the programme's chains intersect. If closure fails — if the admissible phase structure is a gapped dense subgroup rather than the connected circle — then the Born derivation fails (the companion requires continuous U(1), and a dense proper subgroup is disconnected and is not it); support supervenience cannot close even if the assignment question (§9) were settled favourably, since a gapped value space re-opens a degree of freedom the supervenience argument must exclude; and with it the dyadic-loading derivation fails. Closure is a necessary condition of all three results — necessary, not sufficient, for the generation pair, which additionally awaits the [Open] assignment step of §9 — and that convergence is why this one step carries the scrutiny it does.

8. Sub-Claim IV — The Group Is Minimally U(1)

Sub-claims (I)–(III) give H = $\mathbb{T}$, the full circle of unit-modulus phase factors. It remains to identify $\mathbb{T}$, equipped with the minimal admissible structure, as exactly U(1) — and to confirm there is no smaller or different continuous group the substrate could supply.

U(1) is characterised as the unique one-dimensional compact connected abelian Lie group. The four properties are each forced.

Compact (the circle, not the line $\mathbb{R}$). Reversible transport (§3) acts by unit-modulus phase factors: the admissible object is h = e^{iθ}, not θ itself. The angle θ is defined only modulo 2π, because no admissible structure selects a branch — selecting one would be an unwitnessed distinction, forbidden as in §7. So the phase group is the compact circle $\mathbb{T}$, not the non-compact line $\mathbb{R}$. The unboundedly-many compositions of §6 do not run off to infinity along a line; they fold back onto the circle, which is precisely why unbounded composition produces density rather than divergence.

Connected. By §7 the group is the whole circle, which is connected. (Equivalently: closure plus density removes every gap, and a circle with no gaps is connected.) Connectedness is what distinguishes the continuous U(1) from its dense-but-disconnected proper subgroups, and it is exactly what §7's closure step secures.

One-dimensional (U(1), not a torus $\mathbb{T}^n$). The phase accumulated by transport is generated by the elementary admissible distinction, the Fold, which is single and binary [Inherited, *Binary Refinement* §2]. One Fold contributes one independent phase degree of freedom; there is no second independent admissible phase coordinate, since a second would require a second independent minimal distinction, which the single-Fold architecture does not supply. The inference from a single binary distinction to a single phase dimension deserves to be flagged rather than absorbed: the step is that each independent phase coordinate must be generated by transport across an independent minimal distinction, so that the count of independent phases is bounded by the count of independent Folds. That generating relation is carried at its source [Inherited, *Binary Refinement* §2] and is consumed here, not re-derived. Made explicit, the chain is: one independent Fold ⟹ one independent transport generator (every elementary admissible transport acts by a phase increment along the single Fold's direction of distinction, and independent generators would require independent minimal distinctions) ⟹ a one-parameter phase group (the group generated by the increments of a single generator) ⟹ the unique one-dimensional compact connected abelian Lie group, which is U(1). No smoothness is presupposed in saying so, but one qualifier cannot be dropped: compact connected abelian one-dimensional groups include exotic alternatives — solenoids, inverse limits of circles under winding maps — that are not U(1) and are excluded in the standard classification only by the Lie (equivalently, locally connected) condition. The substrate supplies that exclusion for free rather than by assumption: the phase group is realized from the outset as unit-modulus complex numbers (§4), because reversible transport acts by complex phase factors, and a compact subgroup of $\mathbb{C}^*$ lies on the unit circle — it is a closed subgroup of U(1), hence finite cyclic or U(1) entire, and no solenoid admits such a realization. Continuity (§7) rules out the finite cyclic case; what remains is exactly U(1). The exotic case is thus avoided traceably — by reversibility's complex action —

not by an unexamined regularity assumption. So the holonomy group is generated by one circle — U(1) — not a product $\mathbb{T}^n$ of several. A higher torus would be surplus structure beyond the single Fold.

Abelian. Phase factors on the circle commute: e^{iα} e^{iβ} = e^{iβ} e^{iα}. Equivalently the kernel W(P, P′) = e^{i(θ(P) − θ(P′))} composes additively in θ. The holonomy group is abelian automatically, consistent with U(1).

Minimality

The phase group is U(1): compact (reversibility), connected (closure, §7), one-dimensional (single Fold), abelian (additive phase composition). It is the minimal continuous phase group consistent with the inherited structure — no smaller continuous group carries a reversible one-Fold holonomy, and any larger one ($\mathbb{R}$, $\mathbb{T}^n$) requires structure the substrate does not supply.

[Proven, conditional on inherited reversibility and single-Fold binarity.]

9. What Is Forced From D — One Chain Discharged, One Reduced

The result — the admissible phase structure is U(1), in the structural form of §7.2 — is established. What it discharges must now be stated with care, because the two client chains consume the result at different depths, and the difference is the catalogue/assignment distinction of §4.

What is forced. Every step that does positive work in §§5–8 is a constraint on admissible structure — on D. Reversibility, finite-holonomy-impossibility, single-Fold binarity, and no-pre-individuation are all statements about what the distinguishability data admit. What they force is the *value space* of the phase: the catalogue of admissible holonomies is the continuous group U(1), and this is compelled by the admissibility discipline D already carries — the closure step (§7), the load-bearing one, runs through no-pre-individuation, the very principle stating that admissible structure is exhausted by D. The catalogue is D-determined, not chosen.

What is not yet forced. The catalogue is the range of the holonomy assignment L ↦ h(L); it is not the assignment. Two distinct assignments θ and θ′ could each generate the full catalogue U(1) while differing on individual paths. The generation chain needs the assignment: *The Shared Phase Obligation* §17.4 requires s = s(θ) to supervene on D, which is a condition on the phase differences θ(P) − θ(P′) entering the kernel — on values at paths, not on the value space. A D-forced catalogue with a free assignment would leave s carrying a degree of freedom beyond D, and dyadic loading would not close. The inference "the steps constrain admissible structure, therefore θ = θ(D)" is not available: the steps constrain the range of θ, not its values on paths.

The Born chain: discharged. *Physical Necessity of Quantum Probability Structure* consumes the group structure of the phase: it requires continuous U(1) phase to make the interference structure non-degenerate and the quadratic measure the unique admissible one. That is exactly the catalogue result, and the structural form of §7.2 suffices. Hence $P = |\psi|^2$ in the form that paper requires. [Proven, conditional on the inherited inputs of §§3, 6, 7.]

The generation chain: reduced, not discharged. The present result advances the generation obligation from "is θ D-determined?" to the sharper residue: *given that the value space of θ is D-forced to U(1), is the assignment of holonomy to path D-determined?* The candidate closing argument has a natural shape — no-pre-individuation applied to assignments: two holonomy assignments that induce identical admissible distinguishability data differ by no admissible witness, hence by no admissible structure, hence are physically identical, making the gauge-invariant content of the assignment (the closed-loop holonomies) D-determined. But that is a different argument from the one this paper makes, with its own refuter — exhibit two assignments, identical on D, differing on the holonomy of some admissible closed path (§10) — and it is recorded here as the named remaining step, not claimed. [Open.]

The near-misses, restated honestly. *Forced-but-discrete* is excluded by §§6–7: the catalogue is neither finite nor a gapped dense subgroup but the full continuous circle. *Continuous-but-free* is excluded at the catalogue level — continuity is compelled by no-pre-individuation, not chosen — but remains live at the assignment level until the [Open] step above closes. The shared obligation of *The Shared Phase Obligation* §17.4 is therefore discharged on its Born face and reduced to a single named question on its generation face.

Empirical standing. The result makes contact with observation, and the contact runs in the falsifying direction, which is the honest one. Continuous U(1) phase is among the most stringently probed structures in physics — multi-path interferometry, Aharonov–Bohm phase shifts, the unbroken gauge symmetry of electromagnetism — and no granularity or gap in the phase circle has ever been resolved. A finite catalogue would generically leave signatures at the scale of its spacing; none is observed. Two cautions keep this paragraph honest. Consistency is not confirmation: standard quantum mechanics also predicts exact U(1), so observation constrains this derivation and the postulate it replaces equally, selecting neither. And the dense-proper-subgroup alternatives (§7.5) are observationally slippery — a dense gapped catalogue approximates the full circle to any finite experimental resolution — so the empirical lever bears almost entirely against finite catalogues and says nearly nothing against dense ones. That is precisely why the argument against dense gaps had to be structural (§7) rather than empirical, and the observation is recorded here as a consistency check and a standing falsification channel, not as support. [Observational consistency; no confirmatory weight claimed.]

10. What Would Refute This

Each sub-claim has a precise refuter; naming them makes the result a defined target rather than a hope.

Against (I), the group structure (§5). Exhibit an admissible closed path whose reverse is inadmissible, or a pair of admissible closed paths whose concatenation is inadmissible — breaking closure under inversion or composition. Standing: secured by inherited reversibility and the admissibility of concatenation; no such path is expected.

Against (II), infinitude/density (§6). Overturn the companion's finite-holonomy-impossibility result — Sub-Claim II rests entirely on it. Standing: the result is established at its source and inherited here at that status. (Exhibiting a substrate reason the elementary increment must be a rational multiple of 2π would bear only on the demoted supporting intuition of §6, which carries no proof weight; the obligation is untouched by it.)

Against (III), closure (§7) — the load-bearing refuter. Exhibit a mechanism that maintains a catalogue boundary. By the Boundary Maintenance trichotomy (§7.1) there are only two candidates: an admissible barrier — a specific invariant that resolves a limit phase θ^* from the holonomies accumulating within the minimum distinguishable phase of it, which contradicts finite distinguishability (§2, §7.1) — or an algebraic obstruction under which compositions approaching θ^* cease to be admissible, which contradicts the group structure (§5). An exclusion maintained by neither is unmaintained, and no-pre-individuation forbids it outright (§7.2). If a mechanism could nonetheless be exhibited, H may be dense-but-not-closed and the phase a dense proper subgroup rather than the continuous circle — continuous-but-incomplete, which serves neither chain cleanly. Standing: no mechanism is known; any candidate barrier is either an admissible datum (in which case it is part of D, and the phase remains D-determined) or inadmissible (forbidden). The burden is on the challenger to name the mechanism, not on the argument to prove a universal negative.

Against (IV), minimality/U(1) (§8). Exhibit a second independent admissible phase coordinate (forcing a torus $\mathbb{T}^n$ rather than U(1)), or an admissible selection of a branch of θ (forcing $\mathbb{R}$ rather than the circle). Standing: the single-Fold architecture supplies one phase; branch selection is an unwitnessed distinction forbidden as in §7.

Beyond (I)–(IV): the reduced generation obligation (§9). The assignment question carries its own refuter, recorded for completeness though it belongs to the successor argument rather than to this paper's claims: exhibit two holonomy assignments that induce identical admissible distinguishability data yet differ on the holonomy of some admissible closed path. Such a pair would show the assignment underdetermined by D even with the catalogue forced, and the generation chain would remain open.

An empirical refuter, recorded. An interference experiment resolving discrete or gapped structure in the phase circle — granularity at any spacing — would refute the conclusion outright (§9, Empirical standing). None is observed; the channel stands open.

The catalogue result stands or falls, in practice, on (III). The other three are secured by inherited structure; (III) is the one new substantive claim, and the programme's no-pre-individuation discipline is what carries it.

Tightness — no input is idle. The derivation is conditional on its inherited inputs, and the conditionality can itself be audited: each input is necessary, in the strong sense that removing any one admits a concrete failure of the conclusion. Drop reversibility, and transport need not act by unit-modulus complex factors: the accumulated quantity lives in $\mathbb{C}$ rather than on the circle, compactness fails, unbounded composition produces divergence rather than density — dissipative transport with $|h| < 1$ spirals to 0 — and, the complex realization gone with it, the solenoids of §8 become available as ambient structure. Drop unbounded composition, and a bounded set of coarse compositions is a perfectly consistent finite catalogue; worse, the amplification witness of §7.4 is lost wherever the required iteration count exceeds the bound, the asymmetry collapses, and the conservation argument would coarsen the catalogue as readily as close it — unbounded composition is what makes closure cut in one direction only. Drop no-pre-individuation, and every dense proper subgroup becomes admissible: $H_{\mathbb{Q}}$ of §7.5 is then a model, carrying its rational/irrational boundary as a bare hidden label, and the Born chain degenerates on a disconnected phase group. Drop the companion's finite-holonomy-impossibility, and finite cyclic catalogues survive untouched: their gaps are isolated, the Conservation Lemma is silent there (§7.2, the reach of the Lemma), and nothing else excludes them. Drop single-Fold binarity, and nothing selects one circle from $\mathbb{T}^n$. Five inputs, five distinct failure modes, no redundancy: the derivation has no slack, and its conditionality is structure to be audited rather than apologized for.

11. What the Paper Establishes

Established (conditionally, with the conditions named):

- The minimum-distinguishability scale is no objection to continuous phase: it constrains observability, not catalogue (§2) — a conceptual clarification dissolving the objection, carried without a marker.
- Admissible holonomies form a group $H \subseteq \mathbb{T}$ (§5). [Proven], given inherited reversibility and connectedness.
- H is infinite, hence dense in $\mathbb{T}$ (§6). [Inherited] for infinitude (the companion's finite-holonomy-impossibility); [Proven] for density given infinitude, by the structure theory of circle subgroups. The standalone composition argument for infinitude is demoted to [Conjectural] supporting intuition, its circularity stated in the text.
- H is closed, in the structural form: no admissible mechanism can maintain a catalogue boundary — a barrier would have to resolve below the minimum distinguishable scale outside the only known amplification mechanism (§7.4), an algebraic obstruction would contradict the group structure, and an unmaintained exclusion is surplus structure forbidden by no-pre-individuation — so the admissible phase structure is the closed group $\bar{H} = \mathbb{T}$ (§7). [Proven, conditional on no-pre-individuation and finite distinguishability, both [Inherited]] — the load-bearing step.
- The amplification asymmetry (§7.4): sub-resolution distinctions between realized holonomies are witnessed globally by composition, while the boundary distinction admits no witness by any route — so no-pre-individuation closes the catalogue without coarsening it. [Proven, conditional on the inputs of §7.4.]
- Dense + closed $\Rightarrow$ the admissible phase structure is $\mathbb{T}$; the ambient group, with minimal admissible structure, is exactly U(1) — compact (reversibility), connected (closure), one-dimensional (single Fold), abelian (§8). [Proven, conditional on inherited reversibility and single-Fold binarity]
- The continuity of the catalogue is forced from D (§9): every positive step constrains admissible structure, the closure step runs through no-pre-individuation, so the value space of θ is D-determined. This is the full forced-from-D form the Born chain requires, and the catalogue half of what the generation chain requires.

- The result discharges the Born obligation (§9): continuous U(1) ⟹ $P = |\psi|^2$ in the form that paper requires. It reduces the generation obligation to its final form — given a D-forced catalogue, is the assignment L ↦ h(L) D-determined? — with the candidate argument and its refuter named.

Not established (open or out of scope):

- the D-determination of the holonomy assignment L ↦ h(L) (§9) — the reduced generation obligation; the catalogue is forced, the assignment is not yet, and the candidate closing argument (no-pre-individuation applied to assignments) is named with its refuter (§10); [Open] — the successor paper's burden;
- the closure step's universality (§7, §10) — secured by the Boundary Maintenance trichotomy and no-pre-individuation but standing or falling on whether any admissible mechanism for a catalogue boundary can be exhibited; the named refuter and the one place to press within this paper;
- the inherited inputs themselves — reversibility, connectedness, finite-holonomy-impossibility, no-pre-individuation, single-Fold binarity — each carried at its source paper's status;
- the detailed interference calculation of the Born chain (the step from continuous U(1) to the explicit quadratic measure) — that is the companion's remaining internal work, downstream of the continuity established here;
- higher-rank or non-abelian gauge structure — out of scope; this paper concerns the single-Fold candidate-layer phase only, which is one-dimensional and abelian.

The honest summary: this paper does not assume continuous phase and does not derive it from arbitrarily fine local distinction. It derives a continuous U(1) phase *catalogue* from consistency of holonomy under unbounded composition — group structure, density, closure, and minimal ambient — with closure (the one substantive new step) forced by no-pre-individuation, so that the catalogue is D-determined. On that route the minimum-distinguishability scale is no obstruction. The result discharges the Born-rule obligation in full and reduces the dyadic-loading obligation to the single named assignment question.

The continuity result is therefore not derived from arbitrarily fine local distinguishability, but from the impossibility of maintaining a physically meaningful boundary inside a dense holonomy catalogue. Continuity emerges as the completion required by admissibility itself.

The architecture of the argument, in one chain, with the open node marked:

```
Admissibility of the measure
        ⟹ (companion) finite holonomy impossible ⟹ H infinite

H infinite + subgroup (§5) ⟹ H dense in 𝕋

Finite distinguishability (no sub-scale witness)
+ unrealized limit point (no amplification witness)
        ⟹ no admissible invariant changes there (Conservation Lemma)
        ⟹ no boundary can stand ⟹ H closed (structural form)
```

```
Dense + closed ⟹ admissible phase structure = 𝕋
Reversible (ℂ-realized), single-Fold ⟹ exactly U(1)

U(1) ⟹ Born obligation discharged

U(1) + [Open: assignment L ↦ h(L) D-determined]
        ⟹ support supervenience ⟹ dyadic loading
```

Each arrow is an explicit step with a named refuter (§10); the one arrow not yet earned is marked.

12. Conclusion

The dyadic-loading programme and the Born-rule programme converged, independently, on one proposition: that the substrate's transport phase is forced into a continuous U(1) group by admissible structure alone. This paper discharges the catalogue half of that proposition in full — the value space of the phase is forced — and with it the Born-rule obligation; the generation obligation it reduces to one sharply named question.

The objection that finite distinguishability forbids continuous phase is a category error: finite distinguishability constrains what can be locally told apart, not the catalogue of holonomy values, and phase in the programme is accumulated transport holonomy, not a local label. Continuity therefore cannot come from arbitrarily fine local distinction — and need not. It comes from consistency under unbounded composition: admissible holonomies form a group; the group is infinite, hence dense on the circle; no mechanism — barrier, obstruction, or bare stipulation — can maintain a catalogue boundary, hence the catalogue is closed; and a dense closed subgroup of the circle is the whole circle, which — reversible, single-Fold, connected, abelian — is exactly U(1).

The decisive step is closure, and it is decisive in two senses. It is where the weight rests — the other three sub-claims are secured by inherited structure — and it is what makes the catalogue's continuity forced from D, since closure is compelled by no-pre-individuation, a discipline on admissible structure itself. That is why this route, and not a local-density one, does the work it does: it yields continuity in the forced-from-D form at the catalogue level, ruling out the forced-but-discrete reading entirely and the continuous-but-free reading at the level of the value space, and so closes the Born chain outright while bringing the generation chain to its final step — whether the assignment of holonomy to path, and not only its range, is D-determined. The phase catalogue is continuous because it can have no unwitnessed edge; it is D-determined because that edgelessness is forced by D's own admissibility; it is U(1) because that is the minimal group a reversible single-Fold holonomy can fill.

A useful way to view the result is that continuity is not being inserted into the substrate. It is being recovered from consistency. The substrate begins with finite distinguishability and finite local commitments. What drives continuity is the existence of arbitrarily long compositions of those commitments. Local distinction remains finite; global transport does not. The continuous phase catalogue is therefore not a primitive ingredient but the completion forced by requiring that admissible transport, admissible distinguishability, and admissible structure remain mutually consistent. In that sense U(1) is not assumed. It is the smallest phase structure left standing once every unsupported distinction has been removed.

The catalogue of phase is continuous, U(1), and forced from the distinguishability data: no holonomy may be excluded while its neighbours are admitted, for that boundary would be a distinction nothing admissible can witness. With that established, the squareness of probability is settled in the form the Born paper requires, and the recurrence of powers of two stands one question from settled — whether the assignment of phase to history, and not only its menu of values, is the substrate's to fix. The menu, at least, it was never free to withhold.