The Full $\mathfrak{su}(8)$ Weak-Doublet Hessian Return in VERSF

▲ Programme Milestone — Standard Model Flavour Series

Extending the Projected C_3 Hessian Return from the Minimal Committed-Quark Cell to the Full $\mathfrak{su}(8)$ Weak-Doublet Embedding

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General Reader Summary

Quarks — the particles inside protons and neutrons — come in three "generations": three progressively heavier copies of the same basic pair. The weak force can convert one generation into another, and it does so with definite probabilities that physicists collect in a table called the CKM matrix. Most of that table is dominated by one well-known entry, the Cabibbo mixing between the first two generations. But there is also a much smaller entry connecting the first and third generations, and it carries the seed of one of nature's deepest asymmetries: the difference between matter and antimatter.

The previous paper in this series calculated that small entry inside a deliberately simplified model — just the three generations, isolated by hand from everything else. The result was striking. The tiny first-to-third correction was not put in; it fell out, as a by-product. Two adjustments that the theory separately allows turn out to disagree when performed in different orders, and the mismatch is exactly the missing correction — with a definite size and a definite orientation.

Striking, but exposed to an obvious objection:

Maybe the answer only came out that way because the model was small. The real structure has sixty-four independent directions, and the toy calculation ignored most of them. Any one of the ignored directions might reach in and change the result.

This paper confronts that objection head-on, in three steps.

First, a complete inventory. Every one of the sixty-four directions is listed, and each is examined for whether it could interfere with the mixing calculation. No direction is waved away by definition; each gets a specific, stated reason it cannot couple — and the two most dangerous cases, which an earlier draft of this audit missed entirely, are called out and dealt with individually.

Second, two symmetries that force the shape of the answer. Before the theory "reads out" its result, nothing is allowed to tell apart certain unresolved internal phases — like clock faces with no markings, they can be turned freely without changing anything physical. And before mass enters the picture, nothing distinguishes the two partners of a weak pair (the up-type from the down-type role). These two facts, taken as explicit premises, do enormous work: they force the entire mixing structure down to a single number and forbid every one of the extra directions from touching it. Most directions can't couple for the same reason you can't hear a colour — they carry the wrong kind of charge to interact.

Third, one matching condition that pins the size. Symmetries fix the shape of the answer but not its overall scale. The scale agrees with the small model only if the full theory's resting state, restricted to the quark corner, really is what the small model assumed. That cannot be proven from symmetry — so the paper does not pretend to prove it. It is stated openly as a third premise, with its own test.

The conclusion: the small model's answer survives the full structure exactly, provided the three named premises hold. And each premise comes with a tripwire — a specific measurable quantity that goes nonzero if that premise is false, with no overlap between them. Even the ways the result could fail are worked out in advance: a failure of the phase premise flips the answer's sign exactly, with nothing in between; a failure of the matching condition changes only the overall size, while every symmetry check still passes. The heavy numerical computation that once seemed necessary to settle the question is thereby demoted: it no longer decides the answer, it checks the premises.

In plain language:

The elegant result of the small model was not luck. The full machinery is forbidden by its own symmetries from changing the shape of the answer, and one further matching condition — stated openly, with its own test — pins the size. If any of this is wrong, the paper says in advance exactly which dial will move, and by how much.

That is the milestone.

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Abstract

The Weak-Doublet Projection Theorem identified the correct projected flavour object,

$$H\_W = P\_W^\dagger H\_cl P\_W,$$

where $H\_cl = G|\_\{\mathfrak{su}(8)\}$ is the 𝔰𝔲(8)-restricted admissibility Hessian of the substrate free-energy functional and $P\_W = P\_Y P\_R P\_C$ projects onto generation completion, weak role, and mass/readout support. That theorem reduced the CKM/PMNS programme to the computation of the relevant projected matrix elements, explicitly recording that the actual $P\_W H\_cl P\_W$ entries remained owed.

The Projected C_3 Hessian Return then computed the minimal committed-quark branch cell. On the six real branch coordinates

$$x = (s_{12}, a_{12}, s_{23}, a_{23}, s_{31}, a_{31}),$$

the closure, transport, and record-composition second variations returned scalar blocks,

$$G\_C^B = \kappa\_C \cdot I_6,\ G\_T^B = \kappa\_T \cdot I_6,\ G\_R^B = \kappa\_R \cdot I_6,$$

so that

$G\_C_3{}^B = \kappa\_C_3 \cdot I_6$.

This forced C_3 covariance, democratic branch weights, $|\zeta| = b/\sqrt{3}$, the 2↔3 readout survivor, and the returned phase

$\arg \zeta = 5\pi/6$,

giving

$\zeta\_C_3 = (b/\sqrt{3}) \cdot e^{5\pi i/6}$, $\Delta_{13} = ½ \cdot a \cdot \zeta\_C_3$.

The present paper extends that result from the minimal committed-quark C_3 cell to the full $\mathfrak{su}(8)$ weak-doublet embedding. It decomposes the full weak-doublet Hessian into the C_3 branch orbit $\mathcal{B}$, spectator/anchor modes $\mathcal{S}$, gapped role-changing modes $\mathcal{M}$, and removed radial/trace modes $\mathcal{R}$, and — crucially — resolves the branch orbit itself into its C_3 isotypes,

$\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}\_\omega \oplus \mathcal{B}\_\bar{\omega}$,

with $\mathcal{B}_0$ the democratic (trivial-isotype) channel and $\mathcal{B}\_\omega$, $\mathcal{B}\_\bar{\omega}$ the conjugate rotating channels, $\omega = e^{2\pi i/3}$. The full projected Hessian is written in block form,

$G\_W^{(8)} = \begin{pmatrix} G\_\mathcal{BB} \; G\_\mathcal{B}\perp \end{pmatrix} \begin{pmatrix} G\_\perp\mathcal{B} \; G\_\perp\perp \end{pmatrix}$,

with the effective branch return given by the Schur complement

$G\_\mathcal{B}^{eff} = G\_\mathcal{BB} - G\_\mathcal{B}\perp \; G\_\perp\perp^{-1} \; G\_\perp\mathcal{B}$

when spectator modes are integrated out, or by $G\_\mathcal{BB}$ under direct compression. A well-posedness lemma (Lemma 0) records that the ordered compression is automatically Hermitian, that positivity of H_cl makes the Schur complement well-defined on the gapped complement, and that the two readings are ordered in the Loewner sense, $G\_\mathcal{B}^{eff} \preccurlyeq G\_\mathcal{BB}$, with the quantitative leakage bound

$\|G\_\mathcal{BB} - G\_\mathcal{B}^{eff}\| \leq \|G\_\mathcal{B}\perp\|^2 / \lambda\_min(G\_\perp\perp)$.

The main result is a full-embedding survival theorem, stated isotype-by-isotype. If the non-branch $\mathfrak{su}(8)$ modes are generation-singlet, trace/radial, or Axiom-B-gapped role-changing modes, then Schur orthogonality of C_3 isotypes forbids leading leakage into the rotating channels $\mathcal{B}\_\omega \oplus \mathcal{B}\_\bar{\omega}$; singlet spectators may couple only to the democratic channel $\mathcal{B}_0$, and such coupling renormalises the democratic quadrature block. Under one further explicitly stated premise — quadrature isotropy (premise Q): no S↔A mixing within an isotype, a condition the minimal cell returned but the full embedding must be tested for, since the C_3 commutant on the branch space is twelve-dimensional, of which the symmetric family a real Hessian inhabits is seven-dimensional (a real symmetric 2×2 trivial-isotype block plus a Hermitian 2×2 ω block, the $\bar{\omega}$ block fixed by reality) — not two scalars — the full branch return collapses to the two-scalar equivariant form

$$G\_\mathcal{B}^{(8)} = \kappa_0 \cdot \Pi_0 + \kappa\_\omega \cdot (\Pi\_\omega + \Pi\_\bar{\omega}),$$

with the strong pass $\kappa_0 = \kappa\_\omega$ recovering the minimal scalar return. Conditional deviations are tracked isotype-by-isotype: a κ_ω split is recorded structurally (η_iso) and does not reach the quark residue, while the CKM-facing amplitude is carried by the Rayleigh quotient κ_Z along the returned curvature vector, through the two-tier norm diagnostics

$$\rho_8\text{\^{}dir} = \sqrt{(\kappa\_Z\text{\^{}dir} / \kappa\_C_3)},\ \rho_8\text{\^{}eff} = \sqrt{(\kappa\_Z\text{\^{}eff} / \kappa\_C_3)},$$

whose ordering ρ_8^eff $\leq \rho_8$^dir is theorem-backed (Loewner), while $\rho_8 \leq 1$ outright is not — it holds only under the separately checkable hypothesis that direct compression reproduces the minimal cell (premise P-bg of §13).

If committed role-isotype norm matching holds in the full embedding, $\rho_8 = 1$, so

$$\zeta_8 = \zeta\_C_3.$$

A validation computation reported with the paper establishes a caution the minimal language obscured: rotating-channel leakage can deform κ_ω while leaving all three per-branch weights exactly equal, so branch-weight democracy is a necessary but not sufficient pass condition. The sharp diagnostics are the isotype norms themselves.

The paper then propagates the general case,

$$\zeta_8 = \rho_8 \cdot (b/\sqrt{3}) \cdot e^{i\theta_8} + \delta\_leak,$$

through the CKM residue,

$$\Delta_{13}^{(8)} = \tfrac{1}{2} \cdot a \cdot \zeta_8,$$

so that any full-embedding deviation is mapped directly into the CKM ledger; in particular the antipodal phase branch $\theta_8 = 11\pi/6$ acts as the exact sign flip $\Delta_{13} \to -\Delta_{13}$.

A closing symbolic audit then discharges the block conditions at leading order. Two named premises — **P-phase** (invariance of the pre-readout closure functional under the generation root-phase torus T^2) and **P-role** ($\mathbb{Z}_2$ role-swap parity on the committed doublet) — force the pass values: the T^2-invariant symmetric commutant on the branch space collapses to per-plane scalars and, with C_3, to $\kappa \cdot I_6$ (verified by explicit commutant computation), deriving quadrature isotropy and $\kappa_0 = \kappa\_\omega$ rather than assuming them; root-phase charge conservation kills every branch–complement coupling except the gapped role-changing sector and the role-odd off-diagonal sector, and the latter — same C_3 character, same root charge as the branch orbit — is decoupled by P-role alone. A full 64-mode inventory assigns every complement direction an exclusion mechanism, discharging the same-representation spectator debt by enumeration, and complement non-renormalisation becomes a theorem: ρ_8^eff = ρ_8^dir. The value $\rho_8 = 1$ follows under a third named premise, **P-bg** (background matching: the full admissible background restricts on the branch support to the minimal-cell background), whose direct test is the κ_Z^dir = κ_C_3 row of the block table. SM-2 closes at grade **Conditional/symbolic (leading order)** on five named premises, with the numerical block table demoted to confirmation — of P-phase and P-role through the symmetry diagnostics, of P-bg through the direct-reproduction row — and the bound on O(Ω_mix) residuals.

The result is therefore not a new phenomenological fit. It is a gate paper. It either confirms that the minimal C_3 cell was representative, or it identifies the precise 𝔰𝔲(8) leakage, isotype split, norm shift, quadrature mixing, branch-survivor change, or phase flip that forces revision — and the audit of §13 performs the confirmation, conditionally on P-phase, P-role, and P-bg.

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0. Notation, Conventions, and Firewalls

0.1 Closure Hessian and ordered compression

Let

H_cl = G|_{𝔰𝔲(8)}

denote the 𝔰𝔲(8)-restricted Hessian of the substrate admissibility/free-energy functional at the admissible background.

The weak-doublet projected block is written in ordered Hermitian compression form,

H_W = P_W† H_cl P_W, P_W = P_Y P_R P_C,

where

P_C = generation-completion projector, P_R = weak-role projector, P_Y = mass/readout projector.

If the projectors commute on the admissible support, this reduces to the previously used shorthand H_W = P_W H_cl P_W. The daggered compression is retained here because the full 𝔰𝔲(8) embedding must not assume commutativity prematurely — and because the daggered form is automatically Hermitian whatever the ordering (Lemma 0), which is what makes the block formalism of §3 well-posed without a commutativity assumption.

0.2 Frame generator

The weak-frame generator is anti-Hermitian:

Ω_W = i·ε_W·H_W.

The role decomposition is

$\Omega\_W = \Omega_0 \otimes I + \Omega_1 \otimes \tau_1 + \Omega_2 \otimes \tau_2 + \Omega_3 \otimes \tau_3$.

Define

$\Omega\_even = \Omega_0$, $\Omega\_odd = \Omega_3$, $\Omega\_mix = \Omega_1 \otimes \tau_1 + \Omega_2 \otimes \tau_2$.

The clean flavour-frame regime is the regime in which Ω_mix is gapped or subleading.

0.3 Branch orbit

Let

$\boldsymbol{\mathcal{B}} = B_{12} \oplus B_{23} \oplus B_{31}$

be the real six-dimensional Hermitian branch orbit, where each branch plane is

$B\_ij = span\_\mathbb{R}\{S\_ij, A\_ij\}$,

with

$S\_ij = (E\_ij + E\_ji)/\sqrt{2}$, $A\_ij = i\cdot(E\_ij - E\_ji)/\sqrt{2}$.

The branch-coordinate vector is

$x = (s_{12}, a_{12}, s_{23}, a_{23}, s_{31}, a_{31})^T$.

The generation cycle R_3 acts as

$12 \mapsto 23$, $23 \mapsto 31$, $31 \mapsto 12$.

0.4 C_3 isotype decomposition of the branch orbit

The action of R_3 on the three branch planes is the regular representation of C_3 tensored with the plane $\mathbb{R}^2$. Over $\mathbb{C}$ it therefore decomposes into three isotypes, one for each character of C_3:

$\boldsymbol{\mathcal{B}} \otimes \mathbb{C} = \boldsymbol{\mathcal{B}}_0 \oplus \boldsymbol{\mathcal{B}}\_\omega \oplus \boldsymbol{\mathcal{B}}\_\bar{\omega}$, $\omega = e^{2\pi i/3}$,

with isotype projectors

$\Pi_0 = (I + R_3 + R_3^2)/3$, $\Pi\_\omega = (I + \bar{\omega}\cdot R_3 + \omega\cdot R_3^2)/3$, $\Pi\_\bar{\omega} = (I + \omega\cdot R_3 + \bar{\omega}\cdot R_3^2)/3$,

satisfying $\Pi_0 + \Pi\_\omega + \Pi\_\bar{\omega} = I$, $\Pi\_a^2 = \Pi\_a$, $\Pi\_a \Pi\_b = 0$ for $a \neq b$. (Verified by explicit computation on the six-dimensional branch space.)

$\mathcal{B}_0$ is the **democratic channel**: the common-mode combination in which all three branch planes move in phase. The democratic curvature vector Z_{C_3} of the minimal paper lives in $\mathcal{B}_0$. $\mathcal{B}_\omega$ and $\mathcal{B}_{\bar\omega}$ are the **rotating channels**, exchanged by complex conjugation; on the real branch space they combine into a real four-dimensional isotype, and reality forces their curvature norms to be equal, $\kappa_\omega = \kappa_{\bar\omega}$.

This decomposition is not cosmetic. It is the resolution the full-embedding test needs: the minimal paper's scalar return $\kappa_{C_3} \cdot I_6$ is the special case $\kappa_0 = \kappa_\omega$, and every full-embedding deformation of the branch sector is a statement about which isotype moved.

One further piece of structure must be kept visible. Each isotype carries **quadrature multiplicity two**: $\mathcal{B} \otimes \mathbb{C}$ is the regular representation of C_3 tensored with the quadrature plane, so the commutant of the C_3 action is not two scalars but three 2×2 Hermitian blocks — one per character, acting on the (S, A) multiplicity plane, with reality identifying the ω and $\bar\omega$ blocks. (Verified by explicit computation: the nullspace of ad_{R_3} on real 6×6 matrices is twelve-dimensional, and its intersection with the symmetric matrices — the family a real Hessian actually inhabits — is seven-dimensional.) **Quadrature isotropy** — premise Q below — is the condition that each block is scalar: no S↔A mixing within an isotype. The minimal cell returned quadrature isotropy as part of its scalar result; the full embedding must be *tested* for it, not credited with it. It enters as an explicit premise of Theorems 2 and 6, with its own diagnostic (η_{quad}), falsifier (F12), and conditional-pass route (§12.1).

0.5 Full $\mathfrak{su}(8)$ complement

The full projected weak-doublet space decomposes as

$$\mathcal{H}_W^{(8)} = \mathcal{B} \oplus \mathcal{S} \oplus \mathcal{M} \oplus \mathcal{R}.$$

Here:

- $\mathcal{B}$ is the C_3 branch orbit;
- $\mathcal{S}$ denotes spectator/anchor/generation-singlet modes;
- $\mathcal{M}$ denotes role-changing (τ_1, τ_2) modes;
- $\mathcal{R}$ denotes radial, trace, or u(1)-removed modes.

The branch complement is

$$\mathcal{B}^\perp = \mathcal{S} \oplus \mathcal{M} \oplus \mathcal{R}.$$

0.6 CKM firewall

This paper does not use CKM data to select any block, sign, phase, norm, isotype weight, or survivor.

The only CKM-side input inherited here is the previously derived role-odd generator structure,

$$\Omega_q = \begin{pmatrix} 0 & a & c\cdot e^{i\varphi} \\ -a & 0 & b \\ -c\cdot e^{-i\varphi} & -b & 0 \end{pmatrix},$$

with a, b, c, φ treated as inherited quark-generator data from the prior flavour stack.

0.7 PMNS firewall

This paper does not compute the PMNS leakage trace, octant sign, or neutrino kernel. Those belong to SM-3, SM-6, and SM-24. The present paper is the quark-side full 𝔰𝔲(8) Hessian return.

0.8 Claim-level firewall

This paper does not claim that the entire 𝔰𝔲(8) substrate Hessian has been derived from first principles. It computes the consequences of the full weak-doublet embedding under explicitly stated block-admissibility premises and identifies the exact matrix elements a final microscopic computation must return, with quantitative pass bounds.

0′. Predictive-Content Ledger

Object	Prior status	Status here
$H_W = P_W^\dagger H_{cl} P_W$	inherited object	used as full embedding object; Hermiticity and positivity of the compression proven (Lemma 0)
Minimal C_3 Hessian return	derived in minimal cell	target to test
$\mathcal{B} = B_{12} \oplus B_{23} \oplus B_{31}$	minimal branch orbit	embedded into full 𝔰𝔲(8) block
Isotype decomposition $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_\omega \oplus \mathcal{B}_{\bar\omega}$	implicit before	made explicit; carries the sharp pass conditions
Quadrature isotropy (premise Q)	implicit in minimal scalar return	promoted to explicit, testable premise with diagnostics η_{quad} and η_{SA} and falsifier F12; C_3 commutant honestly stated as twelve-dimensional, seven-dimensional on the symmetric family a Hessian inhabits
Full 𝔰𝔲(8) complement $\mathcal{B}^\perp$	open	decomposed into $\mathcal{S}$, $\mathcal{M}$, $\mathcal{R}$
Spectator leakage $G_{\mathcal{B}\perp}$	owed	named test with quantitative bound
Loewner ordering $G_{\mathcal{B}}^{eff} \preccurlyeq G_{\mathcal{B}\mathcal{B}}$	absent before	derived here; consistency check between readings
Effective branch return $G_{\mathcal{B}}^{eff}$	owed	block formula derived; well-posedness proven

Object	Prior status	Status here
C_3 covariance in full embedding	owed	survival theorem under isotype orthogonality
Democratic branch weights	minimal-cell derived	shown necessary but **not sufficient**; isotype norms are the sharp test
Norm diagnostics ρ_8^dir, ρ_8^eff	absent before	two-tier diagnostics on the Rayleigh amplitude κ_Z; ordering ρ_8^eff ≤ ρ_8^dir theorem-backed; outright $\rho_8 \le 1$ demoted to checkable hypothesis
Survivor channel 2↔3	minimal-cell derived	full readout condition
Phase branch σ_φ^q	minimal-cell positive	full return condition; antipodal branch shown to be the exact flip $\zeta \to -\zeta$
ζ_8 = ζ_C_3	open	central survival theorem
Δ_{13}^{(8)} = ½·a·ζ_8	exact algebra	propagated to CKM ledger
Full numerical block table	owed	appendix format specified; skeleton code validated on synthetic blocks
Root-phase covariance (P-phase)	absent before	named, falsifiable premise; derives Q and κ_0 = κ_ω (Lemma 7); falsifier F13
Committed role-swap parity (P-role)	implicit in Theorem 3 role isotropy	named $\mathbb{Z}_2$ premise; decouples the role-odd same-charge sector (Lemma 8′); falsifier F14
Full 64-mode inventory with exclusion mechanisms	owed (Debt 2)	printed (§13.1); discharges the same-representation audit conditionally on the inherited support (F15)
Complement non-renormalisation ρ_8^eff = ρ_8^dir	absent before	theorem (Lemma 8)
$\rho_8 = 1$	hypothesis (role-isotype matching)	conditional on P-bg (background matching); tested by the Debt-1 direct-reproduction row (F16)
Background matching (P-bg)	implicit	named premise; the one pass value the symbolic audit cannot derive; falsifier F16
SM-2 gate status	open	closed at Conditional/symbolic grade (§13, Theorem 9); numerics confirmatory

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0″. Inherited Infrastructure

The present paper inherits three key results.

First, the Weak-Doublet Projection Theorem identifies

$H\_W = P\_W H\_cl P\_W$

as the correct object for CKM and PMNS analysis and separates exact algebra, projection theorems under axioms, and the substrate computation still owed.

Second, the Projected C_3 Hessian Return computes the minimal committed-quark cell. It returns the scalar branch Hessian, democratic branch weights, $|\zeta| = b/\sqrt{3}$, the $2\leftrightarrow3$ survivor, and the CKM residue

$\Delta_{13} = \frac{1}{2}\cdot a\cdot\zeta\_C_3$.

That minimal result is the benchmark this paper tests against the full $\mathfrak{su}(8)$ embedding.

Third, the Master Ledger defines SM-2 as a Tier A risk gate: the full $\mathfrak{su}(8)$ weak-doublet Hessian return must confirm or correct the minimal C_3 cell. If the full calculation contradicts the minimal cell, the CKM branch may need revision.

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1. Purpose and Claim Level

The purpose of this paper is to answer:

Was the minimal C_3 Hessian return representative of the full weak-doublet $\mathfrak{su}(8)$ substrate, or was it a lucky local cell?

The target is to compute, or reduce to finite block tests,

$P\_W^\dagger H\_cl P\_W$

over the full weak-doublet embedding.

The claim level is:

Conditional full-embedding return theorem.

The paper proves that the minimal C_3 result survives the full embedding if and only if specific block-return conditions hold:

1. the branch orbit $\boldsymbol{\mathcal{B}}$ is invariant under the full projected return;
2. spectator leakage into the rotating isotypes $\boldsymbol{\mathcal{B}}\_\omega \oplus \boldsymbol{\mathcal{B}}\_\bar{\omega}$ vanishes at leading order;
3. the full effective branch Hessian is C_3-equivariant and quadrature-isotropic (premise Q), with isotype norms κ_0 and κ_ω, and the democratic-channel norm satisfies role-isotype matching;
4. the committed role-isotype norm remains b^2;

5. the mass/readout complement still leaves the 2↔3 survivor;
6. the Hermitian phase-root sign remains $\sigma_\varphi^q = +1$.

If all pass, then

$$\zeta_8 = \zeta_{C_3}.$$

If any fail, the paper gives the exact deformation,

$$\zeta_8 = \rho_8 \cdot (b/\sqrt{3}) \cdot e^{i\theta_8} + \delta_{leak},$$

and propagates it through

$$\Delta_{13}^{(8)} = \tfrac{1}{2} \cdot a \cdot \zeta_8.$$

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2. The Full $\mathfrak{su}(8)$ Embedding Problem

The minimal cell isolated only the committed-quark C_3 branch orbit. That was legitimate as a local calculation, but it leaves five possible full-embedding failures.

2.1 Spectator leakage

Additional $\mathfrak{su}(8)$ directions may couple into the C_3 branch orbit:

$$G_{\mathcal{B}\perp} \neq 0.$$

If so, the minimal scalar Hessian can be deformed by spectator mixing. The isotype resolution of §0.4 splits this danger in two: leakage into $\mathcal{B}_0$ renormalises one scalar and is survivable; leakage into $\mathcal{B}_\omega \oplus \mathcal{B}_{\bar{\omega}}$ deforms the rotating structure and is not.

2.2 Branch anisotropy and its blind spot

The full embedding may break branch democracy:

$$w_{12} \neq w_{23} \neq w_{31}.$$

If this happens, $|\zeta| = b/\sqrt{3}$ is no longer returned. But the converse fails, and this is the trap: a rotating-channel deformation can shift κ_ω while leaving $w_{12} = w_{23} = w_{31}$ exactly, because the per-plane weights are traces that average over isotypes. (Demonstrated explicitly on synthetic blocks in the validation run of Appendix C.) Equal weights therefore do **not** certify the minimal return. The pass conditions of §12 are stated on the isotype norms, where no such blind spot exists.

2.3 Readout survivor shift

The mass/readout projection may return a different leading branch:

2↔3 ⟶ 1↔2 or 3↔1.

If 1↔2 survives, Cabibbo protection fails. If 3↔1 survives, the triangle correction is inserted directly.

2.4 Isotype splitting

Even with zero leakage, the full embedding's own branch block may return

$\kappa_0 \neq \kappa\_\omega$.

This is a genuinely new failure mode invisible to the minimal cell, which had no complement to source it. It does not split the branches and does not move the survivor; it renormalises the democratic channel and is tracked exactly by ρ_8 (§6). It is recorded here as a failure of the *strong* pass and a trigger for the *conditional* pass.

2.5 Quadrature mixing

Even with C_3 equivariance fully intact, the full embedding may mix the S_ij and A_ij quadratures within an isotype. Such a deformation commutes with the generation cycle, leaves all three per-branch weights exactly equal, leaves the trace-averaged isotype norms unchanged, and at first order leaves even the Rayleigh amplitude κ_Z unchanged — it is invisible to η_anis, η_iso, ℓ_spec, and ρ_8 simultaneously. (Demonstrated on synthetic blocks in the validation run of Appendix C: a uniform within-plane quadrature splitting — κ·I + ε·σ_x on each plane — defeats every trace-based diagnostic, registering on η_quad as $\varepsilon/\bar{\kappa}$, while the soft returned direction does not drift off the democratic S-line but snaps to the $(S - A)/\sqrt{2}$ quadrature diagonal for any $\varepsilon \neq 0$.) It is detected only in the quadrature basis, by the 2×2 isotype blocks themselves — and it has two channels: off-diagonal S↔A mixing, detected by η_quad, and diagonal S/A anisotropy (κ_S ≠ κ_A with no mixing), which is equally equivariant, equally invisible to η_iso and η_anis, invisible even to η_quad, and registers only as η_SA ≠ 0 and $\kappa\_Z \neq \kappa_0$. Once either is present, the residue must be rebuilt from κ_Z along the actually returned vector rather than from trace-averaged norms. Quadrature isotropy is therefore carried below as an explicit premise (Q), not an inheritance from the minimal cell — which is, after all, the object under test.

This paper exists because any of these failures would show that the minimal C_3 result was not representative.

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3. Full $\mathfrak{su}(8)$ Decomposition

3.1 Branch/complement split

Let

$\mathcal{H}_W^{(8)} = \mathcal{B} \oplus \mathcal{B}^\perp$.

Write the full Hessian as

$G_W^{(8)} = \begin{pmatrix} G_{\mathcal{B}\mathcal{B}} & G_{\mathcal{B}\perp} \\ G_{\perp\mathcal{B}} & G_{\perp\perp} \end{pmatrix}$.

The minimal calculation computed

$G_{\mathcal{B}\mathcal{B}}^{min} = \kappa_{C_3} \cdot I_6$.

The full question is whether $G_{\mathcal{B}}^{full}$ remains C_3-equivariant — and if so, with which isotype norms — after the complement is included.

3.2 Direct compression versus integrated complement

There are two legitimate readings.

Reading A — Direct projected branch return

The branch return is simply

$G_{\mathcal{B}}^{dir} = P_{\mathcal{B}} G_W^{(8)} P_{\mathcal{B}} = G_{\mathcal{B}\mathcal{B}}$.

This is the correct reading if spectator/complement modes are not dynamically eliminated but merely projected away.

Reading B — Effective branch return

If the complement modes are integrated out as constrained non-propagating modes, the effective branch return is the Schur complement,

$G_{\mathcal{B}}^{eff} = G_{\mathcal{B}\mathcal{B}} - G_{\mathcal{B}\perp} G_{\perp\perp}^{-1} G_{\perp\mathcal{B}}$.

This is the more conservative reading: even spectator modes can deform the visible branch Hessian if they mix with it before elimination.

3.3 Lemma 0 — Well-posedness, Loewner monotonicity, and the leakage bound

The block formalism above is used freely in what follows; this lemma earns that use.

(i) Hermiticity of the ordered compression. For any operator P_W (projector or not),

$$H_W^\dagger = (P_W^\dagger H_{cl} P_W)^\dagger = P_W^\dagger H_{cl}^\dagger P_W = H_W,$$

since H_{cl} is Hermitian. The ordered compression is therefore Hermitian without any commutativity assumption on P_Y, P_R, P_C — which is precisely why the daggered form of §0.1 is retained.

(ii) Positivity and invertibility on the complement. If $H_{cl} \geq 0$ at the admissible background (second variation at an admissibility minimum), then $H_W \geq 0$, since $\langle x, H_W x\rangle = \langle P_W x, H_{cl} P_W x\rangle \geq 0$. The Schur complement of Reading B is well-defined provided $G_{\perp\perp}$ is invertible on $\mathcal{B}^\perp$; this is exactly the content of the Axiom-B gap condition — gapped complement modes have $\lambda_{min}(G_{\perp\perp}) > 0$ — so Reading B is well-posed in precisely the regime where integrating out is legitimate. Where a complement mode is gapless, it may not be integrated out, and Reading A is the only admissible reading for that mode.

(iii) Loewner monotonicity. With $G_{\perp\perp} \succ 0$, the correction term $G_{\mathcal{B}\perp} G_{\perp\perp}^{-1} G_{\perp\mathcal{B}}$ is positive semidefinite, hence

$$G_{\mathcal{B}}^{eff} \preccurlyeq G_{\mathcal{B}\mathcal{B}}$$

in the Loewner order. Integrating out a gapped complement can only soften the branch curvature, never stiffen it. This gives a free consistency check on any computed block table: $\kappa_0^{eff} \leq \kappa_0^{dir}$ and $\kappa_\omega^{eff} \leq \kappa_\omega^{dir}$ componentwise on isotypes; a table violating either inequality contains an error independent of all physics.

(iv) Quantitative leakage bound. In operator norm,

$$\|G_{\mathcal{B}\mathcal{B}} - G_{\mathcal{B}}^{eff}\| = \|G_{\mathcal{B}\perp} G_{\perp\perp}^{-1} G_{\perp\mathcal{B}}\| \leq \|G_{\mathcal{B}\perp}\|^2 / \lambda_{min}(G_{\perp\perp}).$$

(Verified numerically, including exact saturation in the rank-one case.) The bound converts the qualitative pass condition "leakage small" into an arithmetic one: the measured off-block norm and the measured complement gap jointly bound every possible deformation of the branch return. ■

Reading

Nothing in §§4–9 needs the complement to be computed exactly. It needs two numbers per complement sector: the coupling norm $\|G_{\mathcal{B}\perp}\|$ and the gap $\lambda_{min}(G_{\perp\perp})$. Lemma 0(iv) does the rest.

3.4 Full-embedding diagnostic

Define

$G\_\mathcal{B}^{(8)} = G\_\mathcal{B}^{dir}$ (direct-compression reading), or $G\_\mathcal{B}^{eff}$ (integrated-complement reading).

The full $\mathfrak{su}(8)$ test is:

$G\_\mathcal{B}^{(8)} \overset{?}{=} \kappa_0 \cdot \Pi_0 + \kappa\_\omega \cdot (\Pi\_\omega + \Pi\_\bar{\omega})$,

with the strong pass $\kappa_0 = \kappa\_\omega \equiv \kappa_8$ recovering the minimal scalar form $\kappa_8 \cdot I_6$.

---

4. Isotype-Resolved Spectator Orthogonality

Theorem 1 — Complement leakage is confined to the democratic channel at leading order

Let the full complement decompose as

$\mathcal{B}^\perp = \mathcal{S} \oplus \mathcal{M} \oplus \mathcal{R}$,

where $\mathcal{S}$ carries generation-singlet spectator modes, $\mathcal{M}$ carries role-changing (τ_1, τ_2) modes, and $\mathcal{R}$ carries trace/radial modes removed by $\mathfrak{su}(8)$.

Assume:

1. the pre-readout Hessian is C_3-equivariant;
2. $\mathcal{B}$ decomposes under C_3 as $\mathcal{B}_0 \oplus \mathcal{B}\_\omega \oplus \mathcal{B}\_\bar{\omega}$ per §0.4;
3. $\mathcal{S}$ and $\mathcal{R}$ are C_3-singlet or removed;
4. $\mathcal{M}$ is Axiom-B-gapped at leading flavour-frame order.

Then:

(a) leakage into the rotating channels vanishes at leading order,

$(\Pi\_\omega + \Pi\_\bar{\omega}) \cdot G\_\mathcal{B}\perp = 0 + O(\Omega\_mix)$;

(b) leakage into the democratic channel is not forbidden,

$\Pi_0 \cdot G\_\mathcal{B}\mathcal{S} \neq 0$ in general,

but by C_3-equivariance and Lemma 0(iii) its entire effect on the branch return is a positive-semidefinite, C_3-singlet correction supported on $\mathcal{B}_0$ — a 2×2 PSD block on the democratic quadrature plane. Under quadrature isotropy (premise Q of Theorem 2) that block is a single scalar shift $\delta\kappa_0$; without Q it additionally contributes democratic-channel quadrature mixing, recorded by η_{quad}. In neither case is it a branch splitting or a rotating-channel deformation.

Proof

(a) A leading coupling G: $\mathcal{S} \to \mathcal{B}_\omega$ would be a C_3-equivariant map from the trivial isotype into the ω isotype. By Schur orthogonality of inequivalent C_3 characters, every such map vanishes; likewise for $\mathcal{B}_{\bar{\omega}}$. A coupling from $\mathcal{R}$ vanishes because $\mathcal{R}$ consists of trace/radial modes removed by the $\mathfrak{su}(8)$ restriction. A coupling from $\mathcal{M}$ is not character-forbidden but is role-changing and therefore enters only at $O(\Omega_{mix})$ by the inherited role-gap condition.

(b) The map $\mathcal{S} \to \mathcal{B}_0$ intertwines two copies of the trivial isotype, so Schur permits it; this is exactly why the democratic channel is the one place the full embedding can talk to the minimal cell. The induced correction to the branch return is $\Delta = G_{\mathcal{BS}}\, G_{\mathcal{SS}}^{-1}\, G_{\mathcal{SB}} \geq 0$ (Lemma 0(iii)) with support in range(Π_0) by part (a). A C_3-equivariant PSD operator supported on the democratic channel is a 2×2 PSD form on the quadrature multiplicity plane of $\mathcal{B}_0$ — not automatically a scalar. Under premise Q it is $\delta\kappa_0 \cdot I_2$ with $\delta\kappa_0 \geq 0$, acting on the visible common-mode direction as the single scalar $\delta\kappa_0$; if Q fails, its off-diagonal part is exactly the democratic-channel η_{quad} content of §12.1. Either way it has no matrix elements in $\mathcal{B}_\omega \oplus \mathcal{B}_{\bar{\omega}}$ and therefore cannot split branch weights or deform the rotating structure. ■

Reading

The earlier, unresolved statement "singlet spectators cannot deform the branch orbit" was simultaneously too strong and too weak: too strong because singlet spectators *can* couple to the branch orbit's own singlet channel, and too weak because it did not say what that coupling costs. The isotype-resolved statement is exact on both counts: the rotating channels are Schur-protected; the democratic channel is exposed, and its exposure is a PSD quadrature block — one non-negative scalar under premise Q — absorbed without residue into ρ_8. The full $\mathfrak{su}(8)$ embedding is dangerous only if the complement contains an ungapped mode in the ω isotype **and** the committed weak-role grade — a coincidence of two quantum numbers, which is exactly what the Debt-2 audit must search for.

---

5. Full Branch Return

Theorem 2 — Full $\mathfrak{su}(8)$ branch return has two-scalar isotype form under quadrature isotropy

Assume, in addition to Theorem 1 and the inherited minimal return $G_{\mathcal{BB}}^{min} = \kappa_{C_3} \cdot I_6$:

(Q) Quadrature isotropy. The full pre-readout branch block acts scalarly on the quadrature multiplicity plane within each isotype: no S↔A mixing.

(Premise Q is discharged at leading order by Lemma 7 of §13: under root-phase covariance it is derived, together with $\kappa_0 = \kappa\_\omega$. It is retained here as a standalone premise so that Theorem 2 stands independently of the §13 symmetry premises.)

Then the full branch return has the form

$$G\_\boldsymbol{\mathcal{B}}^{(8)} = \kappa_0 \cdot \Pi_0 + \kappa\_\omega \cdot (\Pi\_\omega + \Pi\_\bar{\omega}) + O(\Omega\_mix),$$

with $\kappa\_\omega = \kappa\_\bar{\omega}$ forced by reality, and

$\kappa_0 = \kappa\_C_3 + \delta\kappa_0$, $\delta\kappa_0$ from democratic-channel renormalisation (Theorem 1(b)), $\kappa\_\omega = \kappa\_C_3 + \delta\kappa\_\omega$, $\delta\kappa\_\omega$ from the full embedding's own rotating-channel curvature.

The **strong pass** is $\kappa_0 = \kappa\_\omega$ ($= \kappa_8$) with Q holding, returning the minimal scalar form $\kappa_8 \cdot I_6$. **Conditional pass A** is $\kappa_0 \neq \kappa\_\omega$ with Q holding: the deviation is confined to the isotype norms and the visible curvature is renormalised through the democratic channel alone (§6). **Conditional pass B** is Q failing with C_3 equivariance intact: the residue must be rebuilt from κ_Z (§6, F12). The **fail** is any residual not C_3-equivariant above the $O(\Omega\_mix)$ budget.

Proof

By Theorem 1, leading leakage into the rotating channels vanishes, so in the Schur-complement reading

$$G\_\boldsymbol{\mathcal{B}}\text{^eff} = G\_\boldsymbol{\mathcal{B}\mathcal{B}} - \Delta, \; \Delta \succeq 0 \text{ supported on } \boldsymbol{\mathcal{B}}_0,$$

and in the direct reading $G\_\boldsymbol{\mathcal{B}}\text{^dir} = G\_\boldsymbol{\mathcal{B}\mathcal{B}}$; both differ from $G\_\boldsymbol{\mathcal{B}\mathcal{B}}$ only within the democratic channel, up to $O(\Omega\_mix^2)$ rotating-channel corrections.

The block $G\_\boldsymbol{\mathcal{B}\mathcal{B}}$ itself must commute with the R_3 action by pre-readout C_3 equivariance. The commutant of the C_3 regular action on the branch space is twelve-dimensional; restricted to real symmetric operators — the family a Hessian inhabits — it is seven-dimensional (verified by explicit nullspace computation): a real symmetric 2×2 block on the trivial-isotype quadrature plane plus a Hermitian 2×2 block on the ω plane, the $\bar{\omega}$ block fixed by reality. This family is genuinely larger than two scalars, and no inheritance from the minimal cell may be used to shrink it: the minimal cell's quadrature symmetry is part of what this paper tests. Premise Q, imposed as an explicit testable condition, makes each block scalar, and reality then reduces the family to

$$G\_\boldsymbol{\mathcal{B}\mathcal{B}} = \kappa_0' \cdot \Pi_0 + \kappa\_\omega' \cdot (\Pi\_\omega + \Pi\_\bar{\omega}).$$

Composing with the democratic-channel correction Δ shifts κ_0' only. Hence the displayed form, with the minimal cell recovered as $\kappa_0 = \kappa\_\omega = \kappa\_C_3$ when $\delta\kappa_0 = \delta\kappa\_\omega = 0$. ■

Reading

Under premise Q, the full $\mathfrak{su}(8)$ embedding has exactly two dials it can turn on the branch sector — one per isotype — and Schur orthogonality plus the role gap disconnects every complement mode from the rotating dial. Without Q there is a third, equivariant dial: within-isotype quadrature mixing, which is weight-democratic and trace-invisible, and is priced separately by η_{quad} (§2.5, F12). Even so, the embedding cannot split the three branches against each other (that would require breaking C_3 equivariance itself, a fail of premise 1), and it cannot deform the rotating structure through singlet spectators. What it can do is renormalise the democratic channel, and §6 prices that exactly.

---

6. Norm Diagnostic and the ρ_8 Factor

The minimal C_3 paper fixed

$|\zeta_{C_3}| = b/\sqrt{3}$.

The visible curvature vector Z_{C_3} lives in the democratic channel $\mathcal{B}_0$ (§0.4). Provided the *returned* curvature vector remains in $\mathcal{B}_0$ — which follows from the same Theorem 1 premises that protect the Hessian, but is a checkable condition, not a given — the full-embedding amplitude is controlled by the democratic channel, not by the branch-averaged norm. The ledger-facing quantity is the Rayleigh quotient along the returned vector,

$\kappa_Z = \langle Z, G_{\mathcal{B}}^{(8)} Z\rangle / |Z|^2$,

which equals the trace-averaged κ_0 under quadrature isotropy (premise Q) and remains the correct amplitude when Q fails, where the trace average does not. Define the norm diagnostic in two tiers, one per reading:

$\rho_8^{dir} = \sqrt{\kappa_Z^{dir} / \kappa_{C_3}}$, $\rho_8^{eff} = \sqrt{\kappa_Z^{eff} / \kappa_{C_3}}$,

with $\rho_8 \equiv \rho_8^{eff}$ whenever Reading B applies. Then

$|\zeta_8| = \rho_8 \cdot (b/\sqrt{3})$.

An isotype split $\kappa_\omega \neq \kappa_0$ with intact democratic channel leaves $|\zeta_8|$ unchanged; it is recorded by η_{iso} (§12, Appendix B) as a structural deviation from the strong pass, but it does not move the CKM residue — *so long as $Z \in \mathcal{B}_0$*. The conditionality matters: the readout projector Π_{23} is a branch-plane projector, not an isotype projector, so if the returned curvature vector itself acquires rotating-channel components (vector-level δ_{leak}), Π_{23} transmits κ_ω content into the residue. That route is closed by Theorem 1's premises and its residual is priced inside δ_{leak} in §10; §6's "only the democratic dial reaches the ledger" is exactly as strong as those premises, no stronger.

Theorem 3 — Norm survival under committed role-isotype matching

If the full $\mathfrak{su}(8)$ embedding preserves the committed-quark role-isotype metric between the role-even common-mode branch and the role-odd 2↔3 transport branch, then

$\rho_8 = 1$,

hence

$|\zeta_8| = b/\sqrt{3}$.

Proof

In the committed quark doublet, both weak-role partners are closure-committed. The role metric before mass/readout is isotropic:

$Q_{role} = \kappa_{role} \cdot (|X_u|^2_K + |X_d|^2_K)$.

Passing to even/odd coordinates,

$X_{even} = (X_u + X_d)/\sqrt{2}$, $X_{odd} = (X_u - X_d)/\sqrt{2}$,

is an orthogonal transformation, so

$Q_{role} = \kappa_{role} \cdot (|X_{even}|^2_K + |X_{odd}|^2_K)$.

Thus the role-even common-mode branch and the role-odd committed branch share the same norm isotype. The inherited role-odd 2↔3 coefficient b fixes the total common-mode branch budget:

$|Z_{C_3}|^2_K = b^2$.

The democratic vector $Z_{C_3} \in \mathcal{B}_0$ projects equally onto the three Killing-orthogonal branch planes, so the budget divides three ways:

$3 \cdot |\zeta_8|^2 = b^2$,

so

$|\zeta_8| = b/\sqrt{3}$. ■

Reading

The full embedding does not get to choose a new C_3 amplitude unless it breaks the committed role-isotype matching — which is a statement about the role metric, not about spectators. Spectator effects enter only as $\delta\kappa_0$ in Theorem 2, and role-isotype matching is precisely the condition $\delta\kappa_0 = 0$ on the visible channel. If the matching is broken, $\rho_8 \neq 1$ records it, and Lemma 0(iii) supplies a direction — for the *leakage step*: the Schur correction is PSD, so $\kappa\_Z\hat{}eff \leq \kappa\_Z\hat{}dir$ and hence $\rho_8\hat{}eff \leq \rho_8\hat{}dir$, always. Democratic-channel leakage can only shrink the residue *relative to the direct block*; that statement is arithmetic. What is not arithmetic is $\rho_8 \leq 1$ outright: the full embedding's own direct block may return $\kappa\_Z\hat{}dir \neq \kappa\_C_3$ intrinsically, in which case $\rho_8\hat{}dir > 1$ is legitimate physics. The stronger bound $\rho_8\hat{}eff \leq 1$ holds exactly under the additional hypothesis $\kappa\_Z\hat{}dir = \kappa\_C_3$ — the statement that direct compression reproduces the minimal cell — which is defensible but must be *checked*, and is therefore recorded as a named row of the Debt-1 block table rather than assumed. A computed $\rho_8\hat{}eff > \rho_8\hat{}dir$ is an arithmetic error, not a discovery; a computed $\rho_8\hat{}dir \neq 1$ is a discovery, not an error.

---

7. Full Readout Survivor

The pre-readout branch orbit is democratic. The visible CKM correction is not democratic because P_Y acts after the orbit has formed.

Let Π_{12}, Π_{23}, Π_{31} be the branch-plane projectors on $\boldsymbol{\mathcal{B}}$. The readout complement is

$\Pi\_Y\hat{}B = I\_B - \Pi\_Cab - \Pi\_tar,$

where

$\Pi\_Cab = \Pi_{12}$, $\Pi\_tar = \Pi_{31}$.

Thus

$\Pi\_Y\hat{}B = \Pi_{23}$.

Theorem 4 — Full readout survivor is 2↔3

If the full $\mathfrak{su}(8)$ readout preserves Cabibbo protection and forbids direct target insertion, then the leading role-even common-mode branch surviving mass/readout is

2↔3.

Proof

There are three democratic common branches before readout: 12, 23, 31.

The 12 branch is removed because it is the protected Cabibbo doorway. A common-mode 12 survivor would directly shift the dominant 1↔2 generator and spoil the inherited Cabibbo scale.

The 31 branch is removed because it is the direct triangle target. A common-mode 31 survivor would insert the desired 1↔3 correction by hand rather than generate it as a noncommutation residue.

The remaining branch is 23. Therefore

$\Pi_Y^B Z_{C_3} = Z_{23}$. ■

Reading

The survivor rule is inherited logic, but in the full embedding it is a computable property of the full P_Y block, not a stipulation: §12.4 requires the three projections $\Pi_Y^B Z_{12} = 0$, $\Pi_Y^B Z_{31} = 0$, $\Pi_Y^B Z_{23} = Z_{23}$ to be verified explicitly from the constructed readout operator. Theorem 4 states what the verification must return; Debt 4 owns the construction.

---

8. Full Phase Return

The minimal C_3 paper fixed the phase on the Hermitian side before the i-rotation into the anti-Hermitian frame generator.

In the full embedding define the quark phase-branch invariant:

$\sigma_\varphi^q = \operatorname{sgn} \operatorname{Im} \zeta_8$.

The half-transport condition is

$h^2/|h|^2 = e^{2\pi i/3}$.

The two roots are

$e^{i\pi/3}, e^{i4\pi/3}$,

the two square roots of the C_3 rotation eigenvalue $\omega = e^{2\pi i/3}$; the choice between them is a $\mathbb{Z}_2$ holonomy datum of the half-transport lift, not a continuous parameter. Hermitian-side minimal phase length selects the principal root,

$h/|h| = e^{i\pi/3}$.

After the i-rotation,

$\arg \zeta_8 = \pi/2 + \pi/3 = 5\pi/6$.

Theorem 5 — Phase survival under Hermitian minimal lift

If the full $\mathfrak{su}(8)$ embedding preserves Hermitian-side minimal half-transport, then

$\sigma_\varphi^q = +1$, $\arg \zeta_8 = 5\pi/6$.

If the embedding selects the antipodal root, then

$\arg \zeta_8 = \pi/2 + 4\pi/3 = 11\pi/6$,

and since $11\pi/6 - 5\pi/6 = \pi$, the deformation is exact:

$\zeta_8 \longrightarrow -\zeta_8$, $\Delta_{13}^{(8)} \longrightarrow -\Delta_{13}^{(8)}$,

with $\sigma_\varphi^q = -1$. (Verified: $e^{11\pi i/6}/e^{5\pi i/6} = -1$ identically.)

Reading

The phase is not a cosmetic sign, and the full embedding cannot deform it gently: the phase branch is a two-valued holonomy invariant, so the only alternatives are exact survival and exact reversal of the CP residue. There is no intermediate CP deformation available from the phase sector — any partial phase shift would violate the half-transport condition itself. This makes F8 a sharp, binary falsifier: the computed σ_φ^q either matches or flips.

---

9. Full $\mathfrak{su}(8)$ Hessian Return Theorem

Theorem 6 — Full $\mathfrak{su}(8)$ Weak-Doublet Hessian Return

Assume:

1. $H_W = P_W^\dagger H_{cl} P_W$ is the ordered Hermitian weak-doublet compression, with $H_{cl} \geq 0$ at the admissible background (Lemma 0);
2. the full branch orbit $\mathcal{B}$ embeds into the $\mathfrak{su}(8)$ weak-doublet block with the C_3 isotype structure $\mathcal{B}_0 \oplus \mathcal{B}_\omega \oplus \mathcal{B}_{\bar{\omega}}$;
3. quadrature isotropy (premise Q of Theorem 2) holds on the full pre-readout branch block;
4. complement modes are generation-singlet, trace/radial removed, or Axiom-B-gapped at leading order;
5. no ungapped complement mode inhabits the ω isotype at the committed weak-role grade (no same-representation spectator);

6. committed role-isotype norm matching survives in the full embedding ($\delta\kappa_0 = 0$ on the visible channel);
7. mass/readout removes the Cabibbo doorway and direct triangle target;
8. Hermitian-side minimal half-transport selects $\sigma_\varphi^q = +1$.

(Under the §13 audit, premise 3 is discharged by Lemma 7 under P-phase; premise 6 is discharged by Lemma 8 for its complement part, with the direct-block value supplied by P-bg; premises 4–5 are discharged by the §13.1 inventory, conditionally on P-phase, P-role, and the inherited support F15.)

Then the full $\mathfrak{su}(8)$ weak-doublet Hessian returns

$$\zeta_8 = (b/\sqrt{3})\cdot e^{5\pi i/6} = \zeta_{C_3},$$

and consequently

$$\Delta_{13}^{(8)} = \tfrac{1}{2}\cdot a\cdot \zeta_8 = \tfrac{1}{2}\cdot a\cdot \zeta_{C_3}.$$

Proof

By Theorem 1, complement leakage into the rotating channels vanishes at leading order under premises 4–5, and democratic-channel leakage is a 2×2 PSD block on the democratic quadrature plane, collapsed to a single non-negative scalar by premise 3. By Theorem 2 with premise 3, the full branch return has the two-scalar isotype form

$$G_{\mathcal{B}}^{(8)} = \kappa_0\cdot\Pi_0 + \kappa_\omega\cdot(\Pi_\omega + \Pi_{\bar\omega}) + O(\Omega_{mix}).$$

By Theorem 3 with premise 6, the democratic channel is unrenormalised: $\rho_8 = 1$, so $|\zeta_8| = b/\sqrt{3}$. (Any κ_ω deviation is invisible to the residue by §6.)

By Theorem 4 with premise 7, mass/readout selects the 2↔3 branch as the unique Cabibbo-protected non-target survivor.

By Theorem 5 with premise 8, the phase branch is the principal root: $\arg \zeta_8 = 5\pi/6$.

Therefore $\zeta_8 = (b/\sqrt{3})\cdot e^{5\pi i/6} = \zeta_{C_3}$. The BCH commutator identity inherited from the CKM curvature algebra gives

$$[\Omega_0(\zeta_8), \Omega_q]_{13} = -a\cdot\zeta_8,$$

and under the convention-invariant split form the leading CKM triangle residue is

$$\Delta_{13}^{(8)} = \tfrac{1}{2}\cdot a\cdot\zeta_8 = \tfrac{1}{2}\cdot a\cdot\zeta_{C_3}. \blacksquare$$

Reading

The theorem is a conditional identity, and every condition is a finite block test with a named diagnostic: leakage (ℓ_{spec}, bounded by Lemma 0(iv)), isotype integrity (η_{iso}), quadrature integrity (η_{quad}, η_{SA}), democratic-channel norm (ρ_8^{dir}, ρ_8^{eff}), survivor (readout table), phase (σ_φ^q). The minimal C_3 cell was representative exactly when all six diagnostic groups return their pass values — and each failure mode deforms the CKM ledger in a distinct, pre-computed direction.

---

10. Deformation Ledger: If the Full Embedding Does Not Return the Minimal Cell

A serious SM-2 paper must not pretend the full embedding can only confirm the minimal result. It must state exactly how failure propagates.

Define the full returned curvature as

$$\zeta_8 = \rho_8 \cdot (b/\sqrt{3}) \cdot e^{i\theta_8} + \delta_{leak},$$

where ρ_8 is the democratic-channel norm factor, θ_8 is the full phase branch, and δ_{leak} is a non-C_3 or rotating-channel leakage contribution. Then

$$\Delta_{13}^{(8)} = \tfrac{1}{2} \cdot a \cdot (\rho_8 \cdot (b/\sqrt{3}) \cdot e^{i\theta_8} + \delta_{leak}).$$

10.1 Norm deformation

If $\rho_8 \neq 1$, the minimal C_3 amplitude was not fully representative. The correction is multiplicative:

$$|\Delta_{13}^{(8)}| = \rho_8 \cdot |\Delta_{13}^{C_3}|.$$

Lemma 0(iii) constrains the leakage step, not the total: $\rho_8^{eff} \leq \rho_8^{dir}$ always, so integrating out the complement can only shrink the residue relative to the direct block. Whether ρ_8 itself exceeds unity is a property of the direct block — κ_Z^{dir} versus κ_{C_3} — and is decided by the Debt-1 table, not by the lemma.

10.2 Phase deformation

If $\theta_8 = 11\pi/6$ instead of $5\pi/6$, the CP orientation flips exactly:

$$\Delta_{13}^{(8)} = -\Delta_{13}^{C_3}.$$

There is no intermediate phase deformation (Theorem 5): the phase sector is $\mathbb{Z}_2$.

10.3 Isotype deformation

If $\kappa\_\omega \neq \kappa_0$ with $\rho_8 = 1$, the strong pass fails but the CKM residue is unmoved. This deviation is structural: it signals that the full embedding distinguishes the rotating channels from the democratic one, which the minimal cell could not see, and it must be recorded because downstream PMNS work (SM-3, SM-24) reads the rotating channels that the quark residue does not.

10.4 Leakage deformation

If $\delta\_leak \neq 0$, the full embedding contributes non-minimal branch content. The CKM residue is no longer purely the C_3 common-mode return, and Lemma 0(iv) bounds the damage: $|\delta\_leak|$ is controlled by $\|G\_\mathcal{B}\perp\|^2/\lambda\_min(G\_\perp\perp)$ on the rotating channels.

10.5 Quadrature deformation

If $\eta\_quad \neq 0$, the branch block remains C_3-equivariant but mixes quadratures within an isotype. This deformation is invisible to §§10.1–10.4 as stated — weight-democratic, trace-invariant, and (in the pure-mixing channel) κ_Z-invariant — while relocating the soft returned direction onto the quadrature diagonal. The residue must be rebuilt from κ_Z along the actually returned vector, and the minimal cell's quadrature symmetry is shown to have been local rather than representative.

10.6 Survivor deformation

If the survivor is not 2↔3, the CKM mechanism changes qualitatively:

- 1↔2 survivor: Cabibbo doorway contaminated;
- 3↔1 survivor: triangle correction inserted directly;
- mixed survivor: full readout no longer minimal.

---

11. Numerical Minimal Benchmark

Using the inherited rational values

$a = 9/40$, $b = 81/2000$,

the minimal returned curvature is

$\zeta\_C_3 = (b/\sqrt{3})\cdot e^{5\pi i/6} = -81/4000 + i\cdot(27\sqrt{3}/4000)$.

Numerically,

$\zeta\_C_3 = -0.0202500 + 0.0116913\cdot i$, $|\zeta\_C_3| = b/\sqrt{3} = 0.0233827$.

The CKM residue is

$\Delta_{13} = \frac{1}{2}\cdot a\cdot\zeta\_C_3 = -729/320000 + i\cdot(243\sqrt{3}/320000)$.

Numerically,

$\Delta_{13} = -0.00227813 + 0.00131528\cdot i$, $|\Delta_{13}| = 0.00263055$.

(All four exact fractions and all decimal values re-verified by independent computation for this manuscript.)

The full $\mathfrak{su}(8)$ result must reproduce these numbers if SM-2 is to close without modifying the CKM ledger.

---

12. Full $\mathfrak{su}(8)$ Block Tables Required

The full microscopic calculation must output the following tables before interpretation. Following the symbolic audit of §13, these tables are confirmatory rather than decisive: their pass values are derived at leading order under P-phase and P-role — with the direct-reproduction row supplied by P-bg — so their role is to test all three premises directly and to bound the $O(\Omega\_mix)$ residuals.

12.1 Branch block

$$G\_\mathcal{BB} = \begin{pmatrix} G_{12,12} & G_{12,23} & G_{12,31} \\ G_{23,12} & G_{23,23} & G_{23,31} \\ G_{31,12} & G_{31,23} & G_{31,31} \end{pmatrix},$$

where each entry is a 2×2 real quadrature block on B_ij, together with the isotype norms (real parts taken; the isotype traces are guaranteed real only for equivariant G),

$\kappa_0 = \mathrm{Re}\ \mathrm{tr}(\Pi_0\ G\_\mathcal{BB}\ \Pi_0)/\mathrm{tr}(\Pi_0)$, $\kappa\_\omega = \mathrm{Re}\ \mathrm{tr}(\Pi\_\omega\ G\_\mathcal{BB}\ \Pi\_\omega)/\mathrm{tr}(\Pi\_\omega)$,

the quadrature diagnostics η_quad (maximal off-diagonal norm of the three 2×2 isotype blocks in the quadrature basis, relative to $\bar{\kappa}$) and η_SA (maximal $|B\_k[S,S] - B\_k[A,A]|/\bar{\kappa}$, the diagonal S/A anisotropy), and the Rayleigh amplitude κ_Z along the returned curvature vector.

Strong pass: $G\_\mathcal{BB} = \kappa_8\cdot I_6$ (equivalently $\eta\_iso = 0$, $\eta\_quad = 0$, and $\eta\_SA = 0$). Conditional pass A (isotype split, F11): $G\_\mathcal{BB} = \kappa_0\cdot\Pi_0 + \kappa\_\omega\cdot(\Pi\_\omega + \Pi\_\bar{\omega})$, $\eta\_iso \neq 0$, $\eta\_quad = 0$, $\eta\_SA = 0$.

Conditional pass B (quadrature deformation, F12): C_3-equivariant with $\eta_{quad} \neq 0$ or $\eta_{SA} \neq 0$; the residue is then rebuilt from κ_Z, not from trace-averaged κ_0. Acceptable residual: + $O(\Omega_{mix})$, bounded per §12.3. Fail: any non-equivariant residual above the $O(\Omega_{mix})$ budget.

Warning common to all rows: equal per-branch weights detect none of the above. They are compatible with the strong pass, with both conditional passes, and with equivariance failure alike (§2.2, §2.5): the weights falsify in one direction only (F2), and the isotype and quadrature diagnostics carry the entire confirmation load.

12.2 Spectator leakage block

$G_{\mathcal{B}\perp}$, together with $\lambda_{min}(G_{\perp\perp})$ on each gapped complement sector.

Pass condition:

$(\Pi_\omega + \Pi_{\bar{\omega}})\cdot G_{\mathcal{B}\perp} = 0$, and Π_0-channel Schur correction scalar,

with the total deformation certified by Lemma 0(iv):

$\|G_{\mathcal{B}\mathcal{B}} - G_{\mathcal{B}}^{eff}\| \leq \|G_{\mathcal{B}\perp}\|^2/\lambda_{min}(G_{\perp\perp})$.

Consistency check (free): $G_{\mathcal{B}}^{eff} \preccurlyeq G_{\mathcal{B}\mathcal{B}}$ in the Loewner order; equivalently $\kappa_0^{eff} \leq \kappa_0^{dir}$ and $\kappa_\omega^{eff} \leq \kappa_\omega^{dir}$. Violation indicates computational error, not physics.

Fail condition: the Schur correction contains rotating-channel (Π_ω, $\Pi_{\bar{\omega}}$) components above the $O(\Omega_{mix})$ budget.

12.3 Role-mixing block

$G_{\mathcal{B}\mathcal{M}}$, with the bound

$\|\Omega_{mix}\| / (\|\Omega_{even}\| + \|\Omega_{odd}\|) \leq \varepsilon_{role}$

reported explicitly (Debt 3 fixes ε_{role} from the projection theorem's gap).

12.4 Readout survivor table

The full P_Y block, constructed in the full embedding (Debt 4), must verify explicitly:

$\Pi_Y^B Z_{12} = 0$, $\Pi_Y^B Z_{31} = 0$, $\Pi_Y^B Z_{23} = Z_{23}$.

12.5 Phase branch table

The full Hessian must return $\sigma_\varphi^q = +1$.

Pass: arg $\zeta_8 = 5\pi/6$. Fail branch: arg $\zeta_8 = 11\pi/6$, i.e. $\Delta_{13} \to -\Delta_{13}$ exactly.

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13. Symbolic Full-Embedding Closure Audit

The preceding sections reduce the full $\mathfrak{su}(8)$ weak-doublet question to a finite set of block diagnostics. This section performs the symbolic audit that closes SM-2 at leading order — and it does so by replacing two informal appeals with two named, falsifiable symmetry premises whose consequences are then theorems.

13.0 Closure standard and the two new premises

The closure standard for SM-2 is:

SM-2 is closed when the full $\mathfrak{su}(8)$ weak-doublet embedding is shown not to introduce any admissible leading-order deformation of the minimal C_3 CKM curvature.

This is narrower than deriving the entire microscopic substrate Hessian, which remains part of the deeper substrate programme. The question here can be closed by an exhaustive mode inventory plus two symmetry arguments — provided the symmetries are stated as premises, not smuggled in as intuitions. They are:

P-phase — Root-phase covariance. Before the phase-lift and mass/readout projection, the closure functional and the admissible background are invariant under conjugation by the generation root-phase torus

$T^2 = \{ \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \otimes I_{role} \oplus I_{anchor} : \theta_1 + \theta_2 + \theta_3 = 0 \} \subset SU(8)$.

Substrate justification: pre-readout, the relative phases of uncommitted generation root directions are unresolved gauge data of the branch structure; a functional that distinguished them would constitute a phase readout upstream of P_Y, contradicting the readout layering. This is the precise content of the informal statement "no admissible structure distinguishes the two quadratures before the phase-lift" — made exact, and made falsifiable.

Falsifier: any pre-readout substrate structure carrying nonzero root-phase charge (equivalently, any admissible term in the closure functional not invariant under T^2). See F13.

P-role — Committed role-swap parity. Before mass/readout, the closure functional on the committed quark doublet is invariant under the $\mathbb{Z}_2$ role swap u ↔ d (conjugation by $I_{gen} \otimes \tau_1$ on the committed block).

Substrate justification: this is the $\mathbb{Z}_2$ form of the committed role-isotropy already used in Theorem 3 — both weak-role partners are closure-committed with isotropic role metric, $Q_{role} = \kappa_{role}(|X_u|^2_K + |X_d|^2_K)$, which is manifestly swap-invariant. P-role does not assert that u and d are physically identical; it asserts that nothing distinguishes them *at this layer*, before the completion interface and readout act. Everything that does distinguish them lives downstream.

Falsifier: pre-readout role-metric anisotropy on the committed doublet, $\kappa_u \neq \kappa_d$. See F14.

P-bg — Background matching. The full admissible background, restricted to the committed branch support, coincides with the minimal-cell background, so the branch-diagonal second variations of the full closure functional reproduce the minimal-cell entries: $\kappa = \kappa_{C_3}$.

Substrate justification: the minimal cell was constructed as the restriction of the full admissible configuration, not as an independent model, so its background is the branch restriction of the full one by construction — provided no full-embedding structure feeds additional curvature into the branch diagonal. P-bg is needed because P-phase and P-role cannot supply it: they are invariance statements and fix the *form* of the branch block, never its *value*. A T^2-invariant, swap-even substrate term — a neutral quartic feeding anchor-background expectation values into the branch-diagonal second variation, for instance — shifts κ while satisfying both symmetries. P-bg is the premise that any such contribution is already included in the minimal-cell computation because the backgrounds agree.

Falsifier: $\kappa_Z^{dir} \neq \kappa_{C_3}$ in the Debt-1 block table — the direct-reproduction row, which is precisely the test of this premise and of nothing else. See F16.

All three premises are used only at leading pre-readout order. None touches the readout operator P_Y, the phase-lift, or the completion interface, all of which break the symmetries by design — downstream.

13.1 The Full Mode Inventory

The audit must begin from an exhaustive enumeration, not from a sector list constructed to exclude the danger. With the inherited weak-doublet support

$8 = (2 \text{ role}) \otimes (3 \text{ generation}) \oplus (2 \text{ anchor})$,

the Hermitian mode space u(8) has 64 real directions ($\mathfrak{su}(8)$ removes the global trace), decomposing exactly as

9 (role-even ⊗ gen) + 9 (role-odd τ_3 ⊗ gen) + 18 (role-changing τ_1, τ_2 ⊗ gen) + 4 (anchor ⊗ anchor) + 24 (committed ↔ anchor) = 64,

and the C_3 isotype count over the full space is 24 trivial ⊕ 20 ω ⊕ 20 $\bar{\omega}$ (verified by character computation; counts over u(8) — the $\mathfrak{su}(8)$ trace removal deletes one trivial mode, leaving 23 ⊕ 20 ⊕ 20). The branch orbit $\boldsymbol{\mathcal{B}}$ accounts for only 2 of the 20 ω modes. The audit table must therefore account for the other 18 — including two sectors absent from any $\boldsymbol{\mathcal{B}} \oplus \boldsymbol{\mathcal{S}} \oplus \boldsymbol{\mathcal{M}} \oplus \boldsymbol{\mathcal{R}}$ partition: the role-odd generation-off-diagonal sector and the generation-Cartan sector. Each mode class is assigned the quadruple

(C_3 character, weak-role grade, root-phase charge, gap status),

where the root-phase charge is the T^2 character under conjugation: a branch plane B_ij carries charge θ_i − θ_j; a committed↔anchor mode with generation index i carries charge θ_i; generation-diagonal, anchor, and trace modes are neutral.

Mode class	dim	C_3 character	role grade	root charge	Gap	Coupling to $\boldsymbol{\mathcal{B}}$	Mechanism
$\boldsymbol{\mathcal{B}}$ (role-even ⊗ off-diag)	6	triv ⊕ ω ⊕ ω̄	committed, even	θ_i − θ_j	—	is $\boldsymbol{\mathcal{B}}$	—
Gen-Cartan (role-even ⊗ diag)	3	triv ⊕ ω ⊕ ω̄	committed, even	**0**	ungapped allowed	**0**	P-phase: charge mismatch
Role-odd off-diag (τ_3 ⊗ off-diag)	6	triv ⊕ ω ⊕ ω̄	committed, odd	**θ_i − θ_j**	ungapped allowed	**0**	P-role: swap-odd cross term
Role-odd Cartan (τ_3 ⊗ diag)	3	triv ⊕ ω ⊕ ω̄	committed, odd	0	ungapped allowed	0	P-phase and P-role, independently
$\boldsymbol{\mathcal{M}}$ (τ_1, τ_2 ⊗ gen)	18	mixed	role-changing	mixed	Axiom-B gapped	O(Ω_mix)	inherited role gap
Committed ↔ anchor	24	3 ⊕ 3̄ type	interface	θ_i	model-dependent	**0**	P-phase: θ_i ≠ θ_j − θ_k on T^2, all cases (verified)
Anchor ⊗ anchor ($\boldsymbol{\mathcal{S}}$ proper)	4	trivial	anchor	0	ungapped allowed	0	P-phase: charge mismatch
Trace (removed)	1 of the above	trivial	—	0	—	0	𝔰𝔲(8) restriction

Three rows deserve emphasis, because they are the rows a definition-first audit misses.

Row 2 (Gen-Cartan). These modes carry the *same* C_3 character content and the *same* role grade as $\boldsymbol{\mathcal{B}}$. Schur's lemma permits their coupling to $\boldsymbol{\mathcal{B}}$_ω. They are excluded not by representation theory but by charge: they are root-phase neutral while every branch plane is charged, and a T^2-invariant bilinear cannot couple sectors of unequal charge.

Row 3 (Role-odd off-diagonal). This is the most dangerous sector in the inventory, and the one the flavour stack itself inhabits: the inherited generator Ω_q lives here. Its modes carry rotating C_3 character, committed (role-diagonal, not role-changing) grade, and — critically — *exactly the branch-plane root charges* θ_i − θ_j. Neither Schur nor P-phase forbids its coupling to $\boldsymbol{\mathcal{B}}$. It is excluded by P-role alone: the cross-Hessian between role-even and role-odd committed directions is odd under u ↔ d ($\tau_1 \tau_3 \tau_1 = -\tau_3$ while $\tau_1 I \tau_1 = I$, verified), so swap invariance forces it to vanish identically at pre-readout order. If P-role fails, this sector couples, and SM-2 does not close symbolically.

Row 6 (Committed↔anchor). These carry fundamental-type charges θ_i. On the torus $\{\theta_1 + \theta_2 + \theta_3 = 0\}$, no functional θ_i coincides with any plane charge $\theta_j - \theta_k$, in any sign combination (verified case-by-case). P-phase therefore decouples the entire interface block from $\mathcal{B}$ without any gap assumption.

Inventory conclusion. After P-phase and P-role, the only complement sector with a surviving coupling to $\mathcal{B}$ is $\mathcal{M}$, and that coupling is $O(\Omega_{mix})$ by the inherited Axiom-B gap. There is no complement mode carrying the dangerous quadruple (ω character, committed grade, matching root charge, ungapped, swap-even). The same-representation spectator loophole (F4, Debt 2) is closed — by enumeration, with each exclusion assigned a mechanism and a premise.

13.2 Lemma 7 — Root-Phase Rigidity

Under P-phase and pre-readout C_3 equivariance, the leading branch block is exactly scalar:

$G_{\mathcal{BB}} = \kappa \cdot I_6$.

In particular, quadrature isotropy (premise Q of Theorem 2) and isotype matching ($\kappa_0 = \kappa_\omega$) are both *derived*, not assumed, and

$\eta_{quad} = 0$, $\eta_{SA} = 0$, $\eta_{iso} = 0$

at leading order.

Proof

Conjugation by $\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ acts on the branch plane B_{ij} as the rotation of the (S_{ij}, A_{ij}) quadrature pair by angle $\theta_i - \theta_j$ (verified by explicit computation of the induced 6×6 orthogonal action). P-phase makes the branch Hessian invariant under this two-parameter family.

Within a plane: a real symmetric 2×2 form invariant under rotation by a generic angle is scalar (the commutant of a rotation is span{I, J} with J the quarter-turn; J is antisymmetric, so the symmetric part is $a \cdot I$). This delivers quadrature isotropy plane-by-plane — closing the off-diagonal (mixing) and diagonal (S/A anisotropy) failure channels in one stroke, and with no appeal to a simultaneous global quarter-turn, which is not realisable by any inner transformation, since the three plane angles satisfy $(\theta_1 - \theta_2) + (\theta_2 - \theta_3) + (\theta_3 - \theta_1) = 0$ and cannot all equal $\pi/2$.

Across planes: distinct planes carry distinct charges for generic $(\theta_1, \theta_2, \theta_3)$, so any cross-plane block is an intertwiner between rotations of independent angles and vanishes.

Hence the T^2-invariant symmetric commutant on $\mathcal{B}$ is exactly the three per-plane scalars (verified: computed commutant dimension is 3). Imposing C_3 equivariance equalises the three scalars (verified: joint commutant dimension is 1). Therefore $G_{\mathcal{B}\mathcal{B}} = \kappa \cdot I_6$. ■

Reading

P-phase is strictly stronger than premise Q: it delivers Q, and it also closes the isotype dial that Theorem 2 left open. Under P-phase the strong pass of §12.1 is not merely available — it is forced. Conversely, this sharpens the meaning of a failed diagnostic: $\eta_{quad} \neq 0$, $\eta_{SA} \neq 0$, or $\eta_{iso} \neq 0$ in the computed block table would falsify P-phase itself (F13), i.e. would demonstrate a root-phase-charged structure upstream of readout.

13.3 Lemma 8 — Charge-Conservation Decoupling

Under P-phase, the branch–complement coupling to every root-phase-neutral sector vanishes identically:

$G_{\mathcal{B}\mathcal{S}} = 0$, $G_{\{\mathcal{B},\mathrm{Cartan}\}} = 0$, $G_{\{\mathcal{B},\mathrm{anchor}\}} = 0$,

and under P-phase alone the committed↔anchor block also decouples. Consequently the Schur correction to the branch return from all non-$\mathcal{M}$ sectors is zero — not zero on the democratic channel only, but zero:

$G_{\mathcal{B}\perp}\, G_{\perp\perp}^{-1}\, G_{\perp\mathcal{B}} = 0 + O(\Omega_{mix}^2)$,

hence the branch return is identical across readings,

$G_{\mathcal{B}}^{eff} = G_{\mathcal{B}}^{dir} = G_{\mathcal{B}\mathcal{B}} + O(\Omega_{mix}^2)$, $\kappa_Z^{eff} = \kappa_Z^{dir}$, $\rho_8^{eff} = \rho_8^{dir}$,

and the full embedding cannot renormalise the branch curvature through any complement channel. The *value* of the shared scalar is a separate matter: under P-bg it equals κ_{C_3}, giving $\rho_8 = 1$. Without P-bg, invariance alone leaves κ free, and the Debt-1 direct-reproduction row $\kappa_Z^{dir} = \kappa_{C_3}$ is exactly its test (F16).

Proof

A T^2-equivariant linear coupling from a charge-neutral sector into $\mathcal{B}$ must satisfy $R(\theta)\cdot C = C$ for every torus element; for generic θ the induced rotation $R(\theta)$ on $\mathcal{B}$ has no unit eigenvalue (every plane charge is nonzero; verified), so $C = 0$. The committed↔anchor sector is not neutral, but its charges θ_i never match a plane charge on the torus (§13.1, row 6), and equivariance between mismatched characters again forces the coupling to zero. The only charged, charge-matched complement sector is role-odd off-diagonal, which vanishes by P-role (Lemma 8′ below), and $\mathcal{M}$, which is gapped. The Schur correction is a sum over complement channels of terms quadratic in these couplings; all vanish except the $\mathcal{M}$ channel at $O(\Omega_{mix}^2)$, so the direct and effective returns coincide and $\kappa_Z^{eff} = \kappa_Z^{dir}$. Invariance fixes no value: a T^2-invariant, swap-even substrate contribution shifts the shared scalar while satisfying every symmetry premise of this section. The identification $\kappa\_Z^{dir} = \kappa\_C_3$ therefore requires P-bg, and only P-bg. ■

Corrective note (replacing the support-orthogonality argument). An earlier draft of this audit derived $\Pi_0 G\_\mathcal{BS} = 0$ from metric orthogonality of the sectors under the Killing/closure inner product. That argument is invalid and is withdrawn: metric orthogonality of two subspaces never constrains the Hessian cross-block between them — the Hessian is an independent bilinear form, and every off-diagonal Hessian block everywhere couples metrically orthogonal directions. The correct mechanism is charge conservation under P-phase, which is both stronger (it kills the full coupling, not its democratic projection) and honestly priced (it fails exactly when F13 triggers).

13.3′ Lemma 8′ — Role-Parity Decoupling of the Same-Charge Sector

Under P-role, the cross-Hessian between the branch orbit (role-even, off-diagonal) and the role-odd off-diagonal sector vanishes identically at pre-readout order.

Proof

Under conjugation by $I\_gen \otimes \tau_1$, role-even committed directions are invariant ($\tau_1 I \tau_1 = I$) and role-odd committed directions flip sign ($\tau_1 \tau_3 \tau_1 = -\tau_3$), while the generation structure is untouched. A Hessian cross term between the two sectors is therefore odd under the swap, and P-role invariance forces it to vanish. ■

Reading

This lemma is where the honest inventory earns its keep. The role-odd off-diagonal sector shares the branch orbit's C_3 character *and* its root charge; representation theory and phase covariance are both silent about it. Without P-role, the inherited quark-generator sector itself would be an admissible source of rotating-channel deformation, and the symbolic closure would fail at exactly the point a definition-first audit declares safe. The exclusion costs one $\mathbb{Z}_2$ premise — the same committed role isotropy Theorem 3 already consumes, now stated as a symmetry with its own falsifier.

13.4 Theorem 9 — Symbolic Full-Block Return

Under P-phase, P-role, P-bg, the inherited Axiom-B role gap, pre-readout C_3 equivariance, and the inherited support 8 = (2 role) ⊗ (3 gen) ⊕ (2 anchor), the symbolic full-block audit returns:

$G\_\mathcal{BB} = \kappa \cdot I_6$ with $\kappa = \kappa\_C_3$ (Lemma 7 for the scalar form; P-bg for the value), $G\_\mathcal{BS} = G\_\{\mathcal{B},\text{Cartan}\} = G\_\{\mathcal{B},\text{anchor}\} = G\_\{\mathcal{B},\text{interface}\} = 0$ (Lemma 8), $G\_\{\mathcal{B},\text{role-odd}\} = 0$ (Lemma 8′), $G\_\mathcal{BM} = O(\Omega\_mix)$ (inherited gap),

and hence, under both the direct-compression and the Schur-complement readings,

$$G_{\mathcal{B}}^{(8)} = \kappa_{C_3}\cdot I_6 + O(\Omega_{mix}^2),$$

with every §12 diagnostic taking its pass value symbolically:

$\eta_{iso} = 0$, $\eta_{quad} = 0$, $\eta_{SA} = 0$, $\eta_{anis} = 0$, $\ell_{spec} = O(\Omega_{mix})$, $\rho_8^{eff} = \rho_8^{dir}$ (Lemma 8), $=$ 1 under P-bg.

Combining with Theorems 4 and 5 (readout survivor and phase branch, unaffected by this section):

$$\zeta_8 = \zeta_{C_3} = (b/\sqrt{3})\cdot e^{5\pi i/6},\ \Delta_{13}^{(8)} = \tfrac{1}{2}\cdot a\cdot \zeta_{C_3}.$$

The full weak-doublet $\mathfrak{su}(8)$ embedding returns the same branch Hessian as the minimal committed C_3 cell at SM-2 order. ■

13.5 What this section pays, and what it newly owes

Paid.

1. Premise Q of Theorem 2 is discharged: quadrature isotropy is now a consequence of P-phase (Lemma 7), and F12 is re-pointed at P-phase.
2. The isotype hypothesis $\kappa_0 = \kappa_\omega$ of the strong pass is discharged by the same lemma; F11's conditional route survives only as the record of what a P-phase violation would look like.
3. Complement non-renormalisation is discharged as a theorem: $\rho_8^{eff} = \rho_8^{dir}$ (Lemma 8). The value statement $\rho_8 = 1$ is *not* derivable from invariance premises; it holds under P-bg, whose direct test is the $\kappa_Z^{dir} = \kappa_{C_3}$ row of the Debt-1 table. That row therefore remains owed — not as a gap in the audit but as the P-bg falsifier — and Theorem 6's premise 6 is discharged conditionally on P-bg.
4. Debt 2 (same-representation spectator audit) is discharged by the full inventory of §13.1: every one of the 20 complement-side ω modes is enumerated and assigned an exclusion mechanism.
5. Debt 1 is demoted from load-bearing to confirmatory: the numerical block table is no longer required to *establish* the pass values, only to confirm the symmetry premises' leading-order validity and to bound the $O(\Omega_{mix})$ residuals (Debt 3 unchanged).

Newly owed.

1. **P-phase substrate derivation.** The root-phase torus invariance is justified above by readout layering, but a substrate-native proof — that no admissible pre-readout closure term carries root-phase charge — is owed. Until then, the closure is conditional on P-phase.
2. **P-role substrate derivation.** Likewise for the committed role-swap parity; its current justification is the role-isotropy of Theorem 3 promoted to a symmetry.

3. **Support audit.** The inventory of §13.1 assumes the inherited support 8 = (2 role) ⊗ (3 gen) ⊕ (2 anchor). If the weak-doublet embedding's support differs, the inventory recomputes; the lemmas' mechanisms (charge, parity, gap) are support-independent, but the table is not.
4. **P-bg substrate derivation.** The identification of the full admissible background's branch restriction with the minimal-cell background is owed as a substrate result. Until it is supplied, the *value* $\kappa = \kappa\_C_3$ — as opposed to the *form* $\kappa \cdot I_6$ — is a premise, tested by the direct-reproduction row.

New falsifiers.

F13 — Root-phase charge upstream of readout. If any admissible pre-readout structure carries nonzero root-phase charge, P-phase fails; Lemmas 7 and 8 lose their base, premise Q and $\kappa_0 = \kappa\_\omega$ revert to hypotheses, and the diagnostics η_quad, η_SA, and η_iso become live again. Conversely, a computed η_quad ≠ 0, η_SA ≠ 0, or η_iso ≠ 0 falsifies P-phase directly.

F14 — Committed role-metric anisotropy. If the pre-readout role metric distinguishes u from d (κ_u ≠ κ_d), P-role fails, Lemma 8′ loses its base, and the role-odd off-diagonal sector — same character, same charge as $\boldsymbol{\mathcal{B}}$ — becomes an admissible deformation source. This is the single most consequential failure mode of the symbolic closure.

F15 — Support mismatch. If the weak-doublet 𝔰𝔲(8) support is not (2 ⊗ 3) ⊕ 2, the inventory table of §13.1 is void as printed and must be recomputed before the closure certificate is read.

F16 — Background mismatch. If the full admissible background does not restrict to the minimal-cell background on the branch support, P-bg fails and the shared scalar departs from κ_C_3: $\rho_8 \neq 1$ with every symmetry diagnostic still passing. The Debt-1 direct-reproduction row κ_Z^dir $= \kappa\_C_3$ is the direct test.

Closure grade. SM-2 closes at grade **Closed — Conditional/symbolic (leading order)**: exact given P-phase, P-role, P-bg, the Axiom-B gap, and the inherited support; conditional on those five named premises, with the numerical block table retained as confirmation — the direct-reproduction row being precisely the P-bg test — and as the bound on O(Ω_mix) residuals.

13.6 Referee note on method

Two arguments from the working draft of this audit were rejected during adversarial review and are recorded here so they are not resurrected.

First, the mode-audit table originally partitioned the complement as $\boldsymbol{\mathcal{S}} \oplus \boldsymbol{\mathcal{M}} \oplus \boldsymbol{\mathcal{R}}$ and concluded that no mode carried the dangerous triple — but the partition was not exhaustive: it omitted the role-odd off-diagonal sector (six modes, rotating character, committed grade, branch-matching root charge — the sector the inherited Ω_q inhabits) and the generation-Cartan sector (three modes, branch-matching character and grade). The count is checkable: the complement carries eighteen ω-isotype modes beyond $\boldsymbol{\mathcal{B}}$_ω, and a closure argument must name the mechanism excluding each. The corrected inventory does.

Second, the democratic-channel non-renormalisation was originally derived from "support orthogonality" — metric orthogonality of the spectator and branch sectors. That inference is invalid in general: Hessian cross-blocks are not constrained by metric orthogonality of the sectors they connect. The correct and stronger mechanism is root-phase charge conservation under P-phase.

The moral is the audit's own: symbolic closure is available, but only at the price of naming the symmetries that do the work, deriving their consequences honestly, and attaching a falsifier to each.

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14. What Has Actually Been Derived

Conditionally on the full-embedding premises, this paper derives:

1. the full $\mathfrak{su}(8)$ branch/complement block decomposition, with the branch orbit resolved into C_3 isotypes $\mathcal{B}_0 \oplus \mathcal{B}_\omega \oplus \mathcal{B}_{\bar{\omega}}$;
2. the exact distinction between direct compression and effective Schur-complement return, with well-posedness, Loewner monotonicity ($G_{\mathcal{B}}^{eff} \preccurlyeq G_{\mathcal{B}\mathcal{B}}$), and the quantitative leakage bound $\|G_{\mathcal{B}\mathcal{B}} - G_{\mathcal{B}}^{eff}\| \leq \|G_{\mathcal{B}\perp}\|^2/\lambda_{min}(G_{\perp\perp})$ proven (Lemma 0);
3. the isotype-resolved spectator orthogonality theorem: rotating channels are Schur-protected against singlet, trace/radial, and gapped role-changing leakage; the democratic channel is exposed by exactly one non-negative scalar;
4. the full-embedding two-scalar branch form $\kappa_0 \cdot \Pi_0 + \kappa_\omega \cdot (\Pi_\omega + \Pi_{\bar{\omega}})$ under the explicit quadrature-isotropy premise Q, with strong pass $\kappa_0 = \kappa_\omega$, and the honest statement of the C_3 commutant as twelve-dimensional (seven-dimensional on the symmetric family a Hessian inhabits), with within-isotype quadrature mixing and diagonal S/A anisotropy identified as the residual equivariant freedom when Q fails;
5. the two-tier norm diagnostics ρ_8^{dir}, ρ_8^{eff} built on the Rayleigh amplitude κ_Z along the returned curvature vector, with the theorem-backed ordering $\rho_8^{eff} \leq \rho_8^{dir}$, the residue-facing role of the democratic channel stated as conditional on $Z \in \mathcal{B}_0$, and the outright bound $\rho_8 \leq 1$ demoted to a checkable hypothesis ($\kappa_Z^{dir} = \kappa_{C_3}$) rather than claimed;
6. the condition under which $\rho_8 = 1$ (committed role-isotype matching);
7. the demonstration that equal per-branch weights do not certify the minimal return — rotating-channel deformations and within-isotype quadrature mixing can both hide behind exact weight democracy, the latter behind the trace-averaged isotype norms as well — so pass conditions must be isotype- and quadrature-resolved;
8. the full readout survivor condition returning 2↔3;
9. the phase-branch survival condition $\sigma_\varphi^q = +1$, with the antipodal branch shown to be the exact $\mathbb{Z}_2$ flip $\zeta_8 \to -\zeta_8$, $\Delta_{13} \to -\Delta_{13}$, admitting no intermediate deformation;
10. the full $\mathfrak{su}(8)$ survival theorem $\zeta_8 = \zeta_{C_3}$;
11. the deformation ledger $\zeta_8 = \rho_8 \cdot (b/\sqrt{3}) \cdot e^{i\theta_8} + \delta_{leak}$ with all six deformation channels priced (§§10.1–10.6);
12. the propagated CKM residue $\Delta_{13}^{(8)} = ½ \cdot a \cdot \zeta_8$.

This paper does **not** derive:

1. the substrate free-energy functional F;
2. the full numerical 𝔰𝔲(8) Hessian from microscopic substrate dynamics;
3. the values a, b, c, φ;
4. the PMNS leakage trace;
5. the octant sign;
6. the quark masses or Yukawa eigenvalues;
7. the full CKM matrix;
8. quantum renormalisation of the flavour sector.

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15. Open Debts

Debt 1 — Numerical full-block computation

The actual 𝔰𝔲(8) Hessian block table must be computed and printed, including the isotype norms κ_0, κ_ω, the full 2×2 quadrature blocks of each isotype (not only their traces), the quadrature diagnostic η_{quad}, the Rayleigh amplitude κ_Z, the direct-reproduction row $\kappa_Z^{dir} = \kappa_{C_3}$ (the direct test of P-bg — the one pass value the symbolic audit cannot derive, F16), and the complement gaps $\lambda_{min}(G_{\perp\perp})$. Following §13, this computation is confirmatory: the pass values are derived symbolically, and the table's role is to test P-phase and P-role — any η_{quad}, η_{SA}, η_{iso}, or rotating-leak deviation falsifies P-phase or P-role directly, and $\kappa_Z^{dir} \neq \kappa_{C_3}$ falsifies P-bg — and to bound the $O(\Omega_{mix})$ residuals.

Debt 2 — Same-representation spectator audit

The key danger is an ungapped complement mode carrying both the ω isotype of C_3 and the committed weak-role grade. Such a mode evades both Schur orthogonality and the role gap, and could deform the rotating channels. The full embedding basis must be enumerated and each complement mode's (C_3 character, role grade, gap) triple listed.

Status: discharged by §13.1. The full 64-mode inventory enumerates every complement direction with its (character, role grade, root charge, gap) quadruple and an exclusion mechanism, conditionally on the inherited support (F15). What remains owed is not the audit but the substrate derivations of the premises it consumes (Debts 7 and 8).

Debt 3 — Role-gap numerical bound

The weak-doublet projection theorem assumes Ω_{mix} is gapped or subleading. The full 𝔰𝔲(8) calculation must bound

$$\|\Omega_{mix}\| / (\|\Omega_{even}\| + \|\Omega_{odd}\|)$$

and fix the ε_role budget referenced in §12.3.

Debt 4 — Full readout operator

The branch-complement form of P_Y must be constructed in the full embedding, not inherited from the minimal cell, and the three survivor projections of §12.4 verified from that construction.

Debt 5 — Phase-branch invariant

The sign σ_φ^q must be computed as a full $\mathfrak{su}(8)$ holonomy/phase-return invariant, not selected by convention. Theorem 5 reduces the computation to a $\mathbb{Z}_2$ decision.

Debt 6 — CKM ledger propagation

If $\zeta_8 \neq \zeta_{C_3}$, the CKM ledger must be recomputed with the returned ζ_8, using the deformation decomposition of §10 to localise the change.

Debt 7 — P-phase substrate derivation

Root-phase covariance is justified in §13.0 by readout layering: a pre-readout structure distinguishing unresolved root phases would constitute a phase readout upstream of P_Y. A substrate-native proof — that no admissible pre-readout closure term carries root-phase charge — is owed. Until it is supplied, Lemmas 7 and 8, and with them the symbolic closure, are conditional on P-phase.

Debt 8 — P-role substrate derivation

Committed role-swap parity is the $\mathbb{Z}_2$ form of the role isotropy already consumed by Theorem 3. A substrate-native proof that nothing at pre-readout order distinguishes the two committed weak-role partners is owed. This is the single most consequential exposure of the symbolic closure (F14): if it fails, the role-odd off-diagonal sector — same C_3 character and same root charge as the branch orbit — becomes an admissible deformation source.

Debt 9 — P-bg substrate derivation

Background matching is justified in §13.0 by construction — the minimal cell is the branch restriction of the full admissible configuration — but a substrate proof that no full-embedding structure feeds additional T^2-invariant, swap-even curvature into the branch diagonal is owed. Until then, the value $\kappa = \kappa_{C_3}$ is a premise, and the Debt-1 direct-reproduction row is its standing test (F16).

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16. Falsification Conditions

F1 — Branch block not equivariant

If $G\_\mathcal{B}^{(8)}$ is not of the form $\kappa_0 \cdot \Pi_0 + \kappa\_\omega \cdot (\Pi\_\omega + \Pi\_\bar{\omega})$ within the O(Ω_mix) budget, C_3 equivariance fails and the entire branch mechanism is void, not merely deformed.

F2 — Branch weights split

If w_{12}, w_{23}, w_{31} are unequal before readout, equivariance has already failed (F1) and the $b/\sqrt{3}$ amplitude fails with it. Note the asymmetry: unequal weights falsify, but equal weights do not confirm (§2.2).

F3 — Rotating-channel leakage

If the Schur correction $G\_\mathcal{B}\perp\ G\_\perp\perp^{-1}\ G\_\perp\mathcal{B}$ contains Π_ω or $\Pi\_\bar{\omega}$ components above the O(Ω_mix) budget, the isotype-orthogonality theorem is evaded and the minimal C_3 result is deformed beyond ρ_8 accounting.

F4 — Same-representation spectator found

If $\mathcal{B}^\perp$ contains an ungapped mode with ω-isotype C_3 character at the committed role grade, Theorem 1(a) does not apply to it and Debt 2's audit table must drive a recomputation.

F5 — Role-mixing not gapped

If ‖Ω_mix‖ is comparable to ‖Ω_even‖ + ‖Ω_odd‖, the clean role-even CKM curvature regime fails.

F6 — Norm factor not unity

If $\rho_8 \neq 1$, the minimal-cell amplitude is renormalised and CKM predictions must be updated by the multiplicative factor ρ_8. The theorem-backed sanity check is ρ_8^eff ≤ ρ_8^dir: a Reading-B result violating that ordering contradicts Lemma 0(iii) and indicates computational error. By contrast, ρ_8^dir ≠ 1 is not an error condition; it is the discovery that direct compression does not reproduce the minimal cell — the failure of the named κ_Z^dir = κ_C_3 row of the Debt-1 table.

F7 — Wrong readout survivor

If P_Y returns 12 or 31 instead of 23, Cabibbo protection or indirect triangle generation fails.

F8 — Phase branch flips

If $\sigma_\varphi^q = -1$, the CKM CP residue flips exactly: $\Delta_{13} \to -\Delta_{13}$. No partial flip is possible.

F9 — Direct target insertion

If the full embedding returns a direct 31 common curvature, the CKM triangle correction is inserted rather than derived.

F10 — CKM ledger mismatch after propagation

If the returned full ζ_8 produces CKM observables outside the accepted ledger tolerances, the minimal C_3 mechanism is not enough.

F11 — Isotype split without leakage

If $\kappa_\omega \neq \kappa_0$ with all leakage diagnostics passing, the strong pass fails: the full embedding structurally distinguishes the rotating channels. The quark residue survives (only the democratic channel reaches it, given $Z \in \mathcal{B}_0$), but the deviation must be carried into the PMNS-side gates, which read the rotating channels.

F12 — Quadrature deformation

If $\eta_{quad} \neq 0$ or $\eta_{SA} \neq 0$ — the full branch block mixes the S and A quadratures within an isotype, or weights them anisotropically ($\kappa_S \neq \kappa_A$) — premise Q fails. The two channels are distinct blind spots: mixing is C_3-equivariant, weight-democratic, invisible to every trace-based diagnostic, and at first order invisible even to κ_Z, registering only on η_{quad}; diagonal anisotropy is invisible to η_{quad}, η_{iso}, and η_{anis} alike, and registers on η_{SA} and as $\kappa_Z \neq \kappa_0$. The residue must then be rebuilt from κ_Z along the actually returned curvature vector, and the minimal cell's quadrature symmetry is shown to have been local rather than representative of the full embedding. Under the §13 audit, $\eta_{quad} \neq 0$ or $\eta_{SA} \neq 0$ additionally falsifies P-phase directly (F13): either quadrature deformation is a root-phase-charged structure upstream of readout.

F13 — Root-phase charge upstream of readout

If any admissible pre-readout structure carries nonzero root-phase charge, P-phase fails; Lemmas 7 and 8 lose their base, premise Q and $\kappa_0 = \kappa_\omega$ revert to hypotheses of Theorem 2, $\rho_8 = 1$ reverts to the role-isotype-matching hypothesis of Theorem 3, and the diagnostics η_{quad}, η_{SA}, and η_{iso} become live again. Conversely, a computed $\eta_{quad} \neq 0$, $\eta_{SA} \neq 0$, or $\eta_{iso} \neq 0$ in the Debt-1 table falsifies P-phase directly.

F14 — Committed role-metric anisotropy

If the pre-readout role metric distinguishes u from d ($\kappa_u \neq \kappa_d$), P-role fails, Lemma 8′ loses its base, and the role-odd off-diagonal sector — same character and same root charge as $\mathcal{B}$ — becomes an admissible deformation source. This is the most consequential failure mode of the symbolic closure: it opens the one channel that neither Schur orthogonality nor root-phase charge conservation can close.

F15 — Support mismatch

If the weak-doublet $\mathfrak{su}(8)$ support is not (2 role) ⊗ (3 generation) ⊕ (2 anchor), the inventory table of §13.1 is void as printed and must be recomputed before the closure certificate is read. The mechanisms of Lemmas 7, 8, and 8′ — charge, parity, gap — are support-independent; the table is not.

F16 — Background mismatch

If the full admissible background does not restrict to the minimal-cell background on the branch support, P-bg fails and the shared scalar departs from κ_{C_3}: $\rho_8 \neq 1$ with every symmetry diagnostic still at its pass value ($\eta_{quad} = \eta_{SA} = \eta_{iso} = 0$, no leakage, no rotating content). That signature is unique among the paper's failure modes — no other falsifier moves the residue while leaving all symmetry diagnostics passing — which is what makes the Debt-1 direct-reproduction row $\kappa_Z^{dir} = \kappa_{C_3}$ a clean test of P-bg and of nothing else: the falsifier set is diagnostically orthogonal, with $\eta_{quad}/\eta_{SA}/\eta_{iso}$ answering to P-phase, rot_leak and the role-odd cross-block to P-role, and κ_Z^{dir} to P-bg, no diagnostic serving two premises. The failure itself is purely multiplicative on the CKM residue (§10.1).

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17. SM-2 Closure Certificate

SM-2 pass condition	Status in this paper
Full $\mathfrak{su}(8)$ embedding basis stated	closed structurally
Branch/complement split defined	closed
Branch orbit isotype-resolved ($\mathcal{B}_0 \oplus \mathcal{B}_\omega \oplus \mathcal{B}_{\bar{\omega}}$)	closed; projectors verified
Direct and Schur-complement readings separated	closed
Well-posedness, Loewner order, leakage bound	closed (Lemma 0; bound verified incl. saturation)
Spectator leakage condition stated	closed, isotype-resolved, with quantitative bound
Full branch equivariant-return theorem proven	conditional

SM-2 pass condition	Status in this paper
Quadrature isotropy promoted to explicit premise Q with diagnostics η_quad and η_SA	closed; commutant dimensions verified (12 full / 7 symmetric)
Two-tier norm diagnostics ρ_8^dir/ρ_8^eff defined on the Rayleigh amplitude κ_Z	closed
Direct-reproduction row $\kappa\_Z\text{^}dir = \kappa\_C_3$	owed (Debt 1); now identified as the P-bg test (F16)
Weight-democracy and quadrature blind spots identified	closed (equal weights and trace-averaged norms necessary, not sufficient)
Readout survivor condition stated	closed
Phase branch condition stated	closed; deformation shown to be exact $\mathbb{Z}_2$ flip
CKM propagation formula given	closed
Numerical block table printed	still owed (Debt 1)
Same-representation audit table printed	still owed (Debt 2)
Full 64-mode inventory with per-mode exclusion mechanism	closed (§13.1)
Premise Q and $\kappa_0 = \kappa\_\omega$ derived from P-phase	closed (Lemma 7; commutant computation verified)
Complement non-renormalisation ρ_8^eff = ρ_8^dir derived	closed (Lemma 8)
$\rho_8 = 1$ under background matching	conditional (P-bg; tested by the direct-reproduction row, F16)
Role-odd same-charge sector decoupled	closed (Lemma 8′, under P-role)
P-phase / P-role / P-bg substrate derivations	owed (Debts 7–9)
SM-2 closure grade	Closed — Conditional/symbolic (leading order), on five named premises; numerics confirmatory

Closure status: SM-2 is closed by the symbolic audit of §13 at leading order, conditionally on the five named premises: P-phase, P-role, P-bg, the Axiom-B gap, and the inherited support. The numerical block tables (Debt 1) are retained as confirmation of those premises and as the bound on the O(Ω_mix) residuals; they are no longer what the closure waits on. This manuscript supplies the full survival theorem, the isotype-resolved pass conditions with quantitative bounds, the exact failure map — including the three failure modes the minimal language could not express: isotype splitting, rotating-channel deformation hidden behind democratic weights, and within-isotype quadrature mixing invisible to every trace-based diagnostic — and the symbolic discharge of the pass conditions themselves.

---

18. Conclusion

The minimal Projected C_3 Hessian Return was a major advance. It showed that, in the committed-quark branch cell, the closure, transport, and record-composition second variations return a scalar C_3 branch Hessian. From that scalar return came the democratic branch split, $|\zeta| = b/\sqrt{3}$, the 2↔3 survivor, the phase $5\pi/6$, and the CKM triangle residue

$$\Delta_{13} = \tfrac{1}{2}\cdot a\cdot\zeta\_C_3.$$

But the programme could not stop there. The Master Ledger correctly identifies the full $\mathfrak{su}(8)$ weak-doublet Hessian return as a Tier A risk. If the full embedding contradicts the minimal cell, the CKM branch may need revision.

This paper extends the calculation to the full embedding, and it does so with the resolution the question actually requires. The full weak-doublet $\mathfrak{su}(8)$ block decomposes into the C_3 branch orbit and its complement, and the branch orbit itself decomposes into a democratic channel and a conjugate pair of rotating channels. Schur orthogonality of C_3 characters, together with the inherited role gap, disconnects every singlet, trace/radial, and gapped role-changing complement mode from the rotating channels: the full embedding can renormalise the democratic quadrature block — one scalar under quadrature isotropy — and Loewner monotonicity fixes the direction of the leakage step relative to the direct block. Under committed role-isotype matching, Cabibbo-protected readout, and Hermitian minimal phase lift, the full embedding returns

$$\zeta_8 = \zeta\_C_3 = (b/\sqrt{3})\cdot e^{5\pi i/6}.$$

The result is not a new fit. It is a full-embedding survival theorem with its failure map priced in advance: a norm shift is multiplicative, with its leakage step directionally constrained and its direct-block component a named checkable row; a phase failure is an exact sign flip with no intermediate; an isotype split leaves the quark residue intact but must be carried to the PMNS gates; rotating-channel leakage is bounded arithmetically by the measured coupling norm and complement gap.

The closing audit of §13 then performs the decisive step symbolically. Under the named premises — root-phase covariance and committed role-swap parity forcing the form, background matching pinning the value, each carrying its own falsifier — the full 64-mode inventory excludes every complement coupling, the branch block is forced to $\kappa\cdot I_6$ with $\kappa = \kappa\_C_3$, and complement non-renormalisation ($\rho_8\text{\^{}eff} = \rho_8\text{\^{}dir}$) is a theorem: SM-2 closes at Conditional/symbolic grade on five named premises. The remaining work is sharply defined and confirmatory: compute the full $\mathfrak{su}(8)$ block tables to test P-phase, P-role, and P-bg directly — any η_quad, η_SA, η_iso, or rotating-leak deviation falsifies the symmetries, and $\kappa\_Z\text{\^{}dir} \neq \kappa\_C_3$ falsifies P-bg — and to bound the $O(\Omega\_mix)$ residuals, and supply the substrate derivations of the premises (Debts 7–9). If a deviation appears, the failure will be local, explicit, isotype-tagged, and immediately propagated through the CKM ledger.

In one sentence:

The minimal C_3 CKM curvature survives the full $\mathfrak{su}(8)$ weak-doublet embedding exactly when the full Hessian leaves the branch orbit C_3-equivariant with matched isotype norms and unmixed quadratures, rotating-channel-orthogonal to the complement, Cabibbo-protected in readout, and phase-positive — conditions this paper first reduces to finite computable tests with quantitative bounds, and then discharges symbolically under root-phase covariance, role-swap parity, and background matching.

That is the intended milestone.

Appendix A — Minimal Returned Values

$a = 9/40 = 0.225$, $b = 81/2000 = 0.0405$.

$|\zeta_{C_3}| = b/\sqrt{3} = 0.0233827$.

$\zeta_{C_3} = -81/4000 + i\cdot(27\sqrt{3}/4000) = -0.0202500 + 0.0116913\cdot i$.

$\Delta_{13} = \frac{1}{2}\cdot a\cdot\zeta_{C_3} = -729/320000 + i\cdot(243\sqrt{3}/320000) = -0.00227813 + 0.00131528\cdot i$.

$|\Delta_{13}| = 0.00263055$.

Antipodal phase branch (for the ledger): $\zeta_8 = +0.0202500 - 0.0116913\cdot i$, $\Delta_{13} = +0.00227813 - 0.00131528\cdot i$.

Appendix B — Full $\mathfrak{su}(8)$ Block Checklist

The final computation must print:

$G_{\mathcal{BB}}$, $G_{\mathcal{B}\perp}$, $G_{\perp\perp}$, $G_{\mathcal{B}}^{eff}$,

and the diagnostics:

$\ell_{spec} = \|G_{\mathcal{B}\perp}\| / \|G_{\mathcal{BB}}\|$ (leakage ratio),

$\beta_{leak} = \|G_{\mathcal{B}\perp}\|^2 / \lambda_{min}(G_{\perp\perp})$ (Lemma 0(iv) deformation bound),

$\kappa_0 = \mathrm{Re}\,\mathrm{tr}(\Pi_0 G \Pi_0)/\mathrm{tr}(\Pi_0)$, $\kappa_\omega = \mathrm{Re}\,\mathrm{tr}(\Pi_\omega G \Pi_\omega)/\mathrm{tr}(\Pi_\omega)$ (isotype norms; real parts taken — the traces are guaranteed real only for equivariant G),

$\eta_{quad} = \max_k \|\mathrm{offdiag}(B_k)\| / \bar{\kappa}$, B_k the 2×2 quadrature block of isotype k (within-isotype quadrature mixing),

$\eta\_SA = \max_k |B_k[S,S] − B_k[A,A]| / \bar{\kappa}$ (diagonal S/A anisotropy — the second premise-Q blind spot, invisible to η_quad),

$\kappa\_Z = \langle Z, G Z\rangle / |Z|^2$ (Rayleigh amplitude along the returned curvature vector; = κ_0 under premise Q),

$\eta\_iso = (\kappa_0 − \kappa\_\omega) / ((\kappa_0 + 2\kappa\_\omega)/3)$ (isotype splitting),

$\eta\_anis = (\max(w_{12}, w_{23}, w_{31}) − \min(w_{12}, w_{23}, w_{31})) / ((w_{12} + w_{23} + w_{31})/3)$ (weight anisotropy; necessary-only),

$\rho_8\hat{}dir = \sqrt{(\kappa\_Z\hat{}dir/\kappa\_C_3)}$, $\rho_8\hat{}eff = \sqrt{(\kappa\_Z\hat{}eff/\kappa\_C_3)}$,

$\sigma\_\varphi\hat{}q = \mathrm{sgn}\ \mathrm{Im}\ \zeta_8$.

Pass targets:

$(\Pi\_\omega + \Pi\_\bar{\omega})\cdot G\_\boldsymbol{\mathcal{B}}\perp = 0$ (or within the O(Ω_mix) budget), with β_leak reported; η_iso = 0 (strong pass) or η_iso ≠ 0 recorded and carried to PMNS gates (conditional pass A, F11); η_quad = 0 and η_SA = 0 (premise Q; either nonzero falsifies P-phase, F13) or the residue rebuilt from κ_Z (conditional pass B, F12); η_anis = 0 (necessary; not sufficient — see §2.2 and §2.5); ρ_8^eff = 1, with ρ_8^eff ≤ ρ_8^dir enforced as the Reading-B sanity check and κ_Z^dir = κ_C_3 reported as the direct-reproduction row (Debt 1); σ_φ^q = +1.

Appendix C — Reproducibility Skeleton (validated)

The following code records the required numerical structure once explicit $\mathfrak{su}(8)$ Hessian matrices are available. It has been validated for this manuscript on synthetic block families: (i) the pure-scalar case returns κ_0 = κ_ω and all residuals zero; (ii) a singlet-spectator coupling shifts κ_0 only, leaves κ_ω intact to machine precision, respects G_eff ≼ G_BB, and saturates the Lemma 0(iv) bound exactly in the rank-one case; (iii) a C_3-equivariant ω-isotype correction deforms κ_ω while leaving the three per-branch weights exactly equal — the weight-democracy blind spot of §2.2 (conditional pass A, not a fail); and (iv) a uniform within-plane quadrature splitting κ·I + ε·σ_x per plane is C_3-equivariant, weight-democratic, invisible to η_iso, η_anis, and κ_Z exactly, and registers on η_quad as $\varepsilon/\bar{\kappa}$, with the soft returned direction snapping to the $(S − A)/\sqrt{2}$ quadrature diagonal for any ε ≠ 0 — the §2.5 blind spot (conditional pass B); and (v) a per-plane diagonal anisotropy diag(κ_S, κ_A) is C_3-equivariant with η_iso = η_quad = η_anis = 0 exactly, detected only by η_SA (returning $|\kappa\_S − \kappa\_A|/\bar{\kappa}$) and by κ_Z ≠ κ_0 — the second Q blind spot.

```
import numpy as np

# --- C3 structure on the 6-dim branch space; plane order (12),(23),(31),
```

```
# --- coordinates (s12,a12,s23,a23,s31,a31)
R3 = np.zeros((6, 6))
for i in range(3):
    j = (i + 1) % 3
    R3[2*j, 2*i] = 1.0
    R3[2*j + 1, 2*i + 1] = 1.0

w = np.exp(2j * np.pi / 3)
Pi0  = sum(np.linalg.matrix_power(R3, k) for k in range(3)) / 3
Piw  = sum(w**(-k) * np.linalg.matrix_power(R3.astype(complex), k) for k in
range(3)) / 3
Piwb = Piw.conj()

# democratic symmetric-quadrature unit vector (the minimal cell's Z
direction)
Z_dem = np.zeros(6); Z_dem[[0, 2, 4]] = 1 / np.sqrt(3)

def schur_effective(G_BB, G_BX, G_XX):
    """Effective branch return; uses solve, not inv, and presumes
    lambda_min(G_XX) > 0 (Axiom-B gap) — check gap() first."""
    return G_BB - G_BX @ np.linalg.solve(G_XX, G_BX.conj().T)

def gap(G_XX):
    return float(np.min(np.linalg.eigvalsh((G_XX + G_XX.conj().T) / 2)))

def isotype_norms(G):
    # real parts taken: the traces are guaranteed real only for equivariant G
    k0 = np.real(np.trace(Pi0 @ G @ Pi0)) / np.real(np.trace(Pi0))
    kw = np.real(np.trace(Piw @ G @ Piw)) / np.real(np.trace(Piw))
    return k0, kw

def isotype_blocks(G):
    """2x2 quadrature blocks of G per isotype k = 0 (democratic), 1 (omega),
2 (omega-bar).
    Basis: u_k^S, u_k^A with (u_k^X)_j = w^(k*j)/sqrt(3) on quadrature X of
plane j."""
    blocks = []
    for k in range(3):
        uS = np.zeros(6, complex); uA = np.zeros(6, complex)
        for j in range(3):
            uS[2*j]     = w**(k*j) / np.sqrt(3)
            uA[2*j + 1] = w**(k*j) / np.sqrt(3)
        U = np.stack([uS, uA], axis=1)
        blocks.append(U.conj().T @ G @ U)
    return blocks

def eta_quad(G, kbar):
    return max(abs(B[0, 1]) for B in isotype_blocks(G)) / kbar

def eta_SA(G, kbar):
    # diagonal S/A anisotropy: the second premise-Q blind spot — invisible to
    # eta_quad, eta_iso, and eta_anis; registers here and as kappa_Z !=
kappa0
    return max(abs(np.real(B[0, 0] - B[1, 1])) for B in isotype_blocks(G)) /
kbar

def kappa_Z(G, Z):
```

```
    return float(np.real(Z.conj() @ (G @ Z)) / np.real(Z.conj() @ Z))

def branch_weights(G):
    return np.array([np.real(np.trace(G[2*i:2*i+2, 2*i:2*i+2])) / 2 for i in
range(3)])

def diagnostics(G_BB, G_BX, G_XX, kappa_C3, Z=Z_dem):
    assert np.allclose(G_BB, G_BB.conj().T), "G_BB not Hermitian"
    g = gap(G_XX)
    assert g > 0, "complement not gapped: Reading B inadmissible for this
sector"
    G_eff = schur_effective(G_BB, G_BX, G_XX)
    # Lemma 0(iii): Loewner sanity check (leakage step only)
    assert np.min(np.linalg.eigvalsh(G_BB - G_eff)) > -1e-10, "G_eff > G_BB:
computation error"
    k0, kw = isotype_norms(G_eff)
    kbar = (k0 + 2 * kw) / 3
    wts = branch_weights(G_eff)
    rho_dir = np.sqrt(kappa_Z(G_BB, Z) / kappa_C3)
    rho_eff = np.sqrt(kappa_Z(G_eff, Z) / kappa_C3)
    # theorem-backed ordering (Lemma 0(iii)); rho_dir != 1 is physics, not
error
    assert rho_eff <= rho_dir + 1e-10, "rho8_eff > rho8_dir: computation
error"
    D = G_BB - G_eff
    return {
        "kappa0": k0, "kappa_omega": kw,
        "eta_iso": (k0 - kw) / kbar,
        "eta_quad": eta_quad(G_eff, kbar),          # F12 detector; trace-
invisible channel
        "eta_SA": eta_SA(G_eff, kbar),              # F12 detector, diagonal
channel; Q pass needs both zero
        "weights": wts,
        "eta_anis": (wts.max() - wts.min()) / wts.mean(),
        "ell_spec": np.linalg.norm(G_BX, 2) / np.linalg.norm(G_BB, 2),
        "beta_leak": np.linalg.norm(G_BX, 2)**2 / g,   # Lemma 0(iv) bound
        "leak_actual": np.linalg.norm(D, 2),           # must be <= beta_leak
        # one-sided rotating-channel check suffices: D >= 0 (PSD), so
        # (Piw+Piwb) D = 0  <=>  (Piw+Piwb) D (Piw+Piwb) = 0  <=>
D^(1/2)(Piw+Piwb) = 0
        "rot_leak": np.linalg.norm((Piw + Piwb) @ D, 2),
        "kappa_Z_dir": kappa_Z(G_BB, Z),               # direct-reproduction
row: compare to kappa_C3
        "kappa_Z_eff": kappa_Z(G_eff, Z),
        "rho8_dir": rho_dir,
        "rho8_eff": rho_eff,
    }
```

A genuine SM-2 close requires the explicit $\mathfrak{su}(8)$ block matrices to be inserted into this calculation and the pass/fail diagnostics reported — including η_iso, η_quad, κ_Z, the direct-reproduction row κ_Z^dir vs κ_C_3, and rot_leak, not only η_anis — before any CKM interpretation is made. Following §13, the calculation confirms rather than decides: η_quad, η_SA, and η_iso are direct falsifiers of P-phase, rot_leak together with the role-odd cross-block is the falsifier of P-role, the direct-reproduction row κ_Z^dir vs κ_C_3 is the falsifier of P-bg, and the remaining diagnostics bound the O(Ω_mix) residuals.

Appendix D — Referee Objections and Responses

Objection 1 — "This still does not compute the full microscopic Hessian."

Response: Correct, and the claim-level firewall (§0.8) says so. This paper supplies the full-embedding survival theorem, the isotype-resolved pass conditions, the quantitative leakage bound, and the block-return checklist. It does not claim the final numerical block table has been produced; Debts 1 and 2 own that computation. Its value is to reduce SM-2 to a finite computation with arithmetic pass/fail outputs and pre-priced failure modes.

Objection 2 — "You assumed the minimal C_3 result survives."

Response: No. The paper parameterises the full return as $\zeta_8 = \rho_8 \cdot (b/\sqrt{3}) \cdot e^{i\theta_8} + \delta_{leak}$, then states the exact conditions under which $\rho_8 = 1$, $\theta_8 = 5\pi/6$, and $\delta_{leak} = 0$ — and, where the structure permits, constrains the failure directions in advance ($\rho_8^{eff} \leq \rho_8^{dir}$ across readings; θ_8 binary). If the full block table violates those conditions, the minimal result is revised along the priced deformation channels of §10.

Objection 3 — "Spectator modes could deform the branch orbit."

Response: They can — through exactly one door. Theorem 1 resolves the branch orbit into C_3 isotypes and shows that singlet, trace/radial, and gapped role-changing spectators reach only the democratic channel, where their entire effect is a PSD block on the democratic quadrature plane — a single non-negative scalar under premise Q — bounded by Lemma 0(iv) and recorded by ρ_8; under P-phase (§13) the coupling vanishes identically. The rotating channels are Schur-protected. The residual danger — an ungapped ω-isotype mode at the committed role grade — is stated as F4 and assigned an explicit audit table (Debt 2), not waved away.

Objection 4 — "The readout survivor 2↔3 is still chosen by hand."

Response: The readout rule is inherited from the minimal C_3 logic: remove the protected Cabibbo doorway and the direct target plane. This paper requires the full P_Y block to be constructed in the full embedding (Debt 4) and the three projections $\Pi_Y^B Z_{12} = 0$, $\Pi_Y^B Z_{31}$

$= 0$, $\Pi\_Y^B Z_{23} = Z_{23}$ verified from that construction (§12.4). If P_Y does not return 23, F7 triggers and the CKM mechanism is revised, not patched.

Objection 5 — "The phase 5π/6 is still a convention."

Response: The paper treats the phase as a return condition, not a convention, and Theorem 5 sharpens the stakes: the phase branch is a $\mathbb{Z}_2$ holonomy datum — the choice of square root of the C_3 eigenvalue ω — so the full embedding must return σ_φ^q as a computed invariant (Debt 5), and the only possible deformation is the exact flip $\Delta_{13} \to -\Delta_{13}$. A convention could be redefined away; a sign that reverses a CP residue cannot.

Objection 6 — "Your Schur argument is broken: the branch orbit contains a trivial C_3 component, so singlet spectators are *not* orthogonal to it."

Response: Exactly right, and this paper is built around that observation rather than against it. The regular action of C_3 on the three branch planes contains the trivial isotype — the democratic channel $\mathcal{B}_0$ — and singlet spectators can and generically do couple to it. Theorem 1 therefore does not claim blanket orthogonality; it claims isotype-resolved orthogonality: the rotating channels $\mathcal{B}\_\omega \oplus \mathcal{B}\_\bar{\omega}$ are protected by Schur, while the democratic channel's exposure is a PSD quadrature block — one non-negative scalar under premise Q — absorbed into ρ_8 with the leakage direction fixed by Loewner monotonicity, and vanishing identically under P-phase (§13). The unresolved form of the claim would indeed have been false; the resolved form is a theorem.

Objection 7 — "The Schur complement may not exist: G_⊥⊥ could be singular."

Response: Lemma 0(ii) addresses this directly. The Schur complement is used only on complement sectors satisfying the Axiom-B gap, $\lambda\_min(G\_{\perp\perp}) > 0$, which is the same condition that licenses integrating those modes out at all. A gapless complement mode may not be integrated out; for such a mode only the direct-compression reading is admissible, and the mode must instead appear in the Debt-2 audit table, where its C_3 character and role grade determine whether it threatens the rotating channels (F4). The code of Appendix C enforces the gap check before any Schur inversion.

Objection 8 — "Quadrature isotropy is inherited from the minimal cell — but the minimal cell is exactly what this paper is testing. Your commutant is not two scalars."

Response: Correct on both counts, and the present version is built to accept the criticism in full rather than deflect it. Each C_3 isotype carries quadrature multiplicity two, so the commutant of the generation cycle on the branch space is twelve-dimensional, its symmetric part seven-dimensional — verified by explicit nullspace computation — not the two-parameter family $\kappa_0\Pi_0 + \kappa\_\omega(\Pi\_\omega + \Pi\_\bar{\omega})$. Nothing in this paper shrinks that family by inheritance. The collapse to two scalars is performed by premise Q (quadrature isotropy), stated as an explicit, testable condition of Theorems 2 and 6, with its own diagnostic (η_quad), its own falsifier (F12), and its own conditional-pass route (rebuild the residue from κ_Z along the actually returned curvature vector). The validation run makes the stakes concrete: a within-isotype quadrature splitting is C_3-equivariant, weight-democratic, and invisible to every trace-based diagnostic in the paper — including κ_Z itself — while the soft returned direction snaps onto the quadrature diagonal: the returned vector does not drift, it jumps. If the full block table returns η_quad ≠ 0, the minimal cell's quadrature symmetry is thereby shown to have been local, and the paper's own deformation ledger (§10.5), not its assumptions, carries the consequence.