# Further VERSF Deliberations on the Born Rule

A consolidated uniqueness theorem for the VERSF Born measure, with its physical premises made explicit

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## General reader summary

Quantum theory predicts how likely each outcome of an experiment is, and the recipe has a curious shape: a certain quantity is attached to each possible outcome, and the probability is proportional to the square of its size. The VERSF programme asks why squaring, rather than some other rule, is what nature uses.

This paper gives one complete answer within the programme's framework. Picture the possibilities open to a system as directions in a space, and the probability rule as a way of weighing each direction. We ask only three things of the weighing: it never goes negative, it ignores a feature of the description that has no physical meaning, and when a possibility is split into parts that cannot both happen, the weight of the whole equals the sum of the weights of the parts. Those three requirements leave exactly one rule, and it is the squaring rule. No additional smoothness assumption is needed. The argument works for systems of any size, and it says exactly how many splittings it needs: just enough to link every possibility to every other.

Two further arguments reach the same rule from the laboratory side. One compares every outcome against a single fixed reference outcome and asks only that filtering some outcomes out, and lumping others together, behave consistently; from those comparisons the weights, and with them the squaring rule, can be reconstructed rather than assumed. The other uses two distant experimenters: nothing one of them does locally may change what the other sees, and among rules of the kind considered only squaring respects that.

The paper also shows that the total weight stays fixed while a system evolves undisturbed, says how many repetitions are needed before observed frequencies reliably match the predicted probabilities, states which physical assumptions are being made and where each one enters, and repairs several steps in earlier VERSF papers that were either unnecessary or not fully justified.

## Abstract

We consolidate the VERSF Born-rule programme around a self-contained uniqueness theorem and an operational reconstruction. On a complex inner product space of dimension at least two, every finite-valued, nonnegative, nonzero, globally phase-invariant functional additive on complex-orthogonal vector pairs is a positive scalar multiple of the squared norm. No continuity,

measurability or boundedness hypothesis is required. In finite dimensions, additivity on one reference record basis and its balanced two-direction rotations suffices exactly when the associated graph is connected; d − 1 rotated bases is the minimum within that class. Restriction to finite-dimensional subspaces extends the measure theorem to arbitrary dimensions. Projected-branch readout gives the Born rule, and for three or more records its proportionality follows from universal weight dependence and grouping consistency. An alternative operational theorem reconstructs both the raw weight and proportional readout from a fixed reference record, consistency under ideal filtering, equivalence of coarse and fine records, and phase invariance of the calibration odds. The geometric theorem then fixes the form of the reconstructed weight; the calibration fixes its scale. These operational premises have explicit experimental content; their implementation in VERSF remains a named physical commitment. Independent repetitions yield a quantitative concentration bound for recorded frequencies. A complementary causal argument selects linear component weights without assuming regularity and identifies the geometric input in its reversible controls. We establish conservation of the canonical total under unitary transport, retain the denominator's endpoint convention explicitly, correct earlier multiplicativity, normalization and sector-weight arguments, and state the extra premises for mixtures and dilated measurements. A dependency ledger separates the geometric, operational, causal and frequency results.

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## 1 Purpose and main claim

The VERSF Born-rule papers approach probability through operational geometry, record consistency, path correlations, admissible transport and entropic unfolding [1-4, 7-13]. These connections motivate a common question: which probability measure can coexist with the proposed geometry and physical operations? A useful consolidation should answer that question with an explicit theorem, identify where its physical inputs enter, and say exactly how much of each input is used.

The main result concerns a measure on a specified operational Hilbert carrier. It does not require a separate derivation of that carrier to be repeated inside the probability proof. This is the usual structure of a conditional physical derivation: the inputs have physical content, and the theorem fixes what follows from them. The Hilbert construction, the operational interpretation of orthogonality and the record readout remain identifiable parts of the VERSF explanation, and each is named below as a premise.

We prove that the global orthogonal-additivity axiom written in Probability as Admissible Measure [1, Definition 3.1] is strong enough, with positivity and phase symmetry alone, to fix a common squared-norm measure. No regularity condition is needed. The proof removes a supposed residual freedom in sector coefficients, lists the exact orthogonal decompositions it uses, shows by a counterexample that a sector-restricted version of the axiom would leave that freedom in place, and characterizes exactly which families of balanced-rotation record bases restore it. The measure theorem is shown to hold in every dimension, and the readout premise that converts it into the Born rule is reduced to two weaker consistency statements when three or more records are available. It also supplies a route to the Born measure that does not depend on the problematic multiplicativity-to-rank-one inference used elsewhere in the programme.

Section 4.2 strengthens the physical interpretation by constructing a weight from calibration probabilities rather than assigning it an unexplained raw scale. Its theorem derives additivity and proportional readout together from operational consistency. Section 4.3 gives the separate statistical implication for recorded frequencies under repeated-preparation assumptions. Neither step silently treats an inherited Born instrument as an independent explanation of its own probabilities.

The result is an elementary measure reconstruction, presented here to make the VERSF argument complete and auditable. We make no priority claim for the general mathematical idea of characterizing quadratic weights through additivity on orthogonal pairs; that idea has a literature of its own [17, 18, 21]. Gleason's projector-measure theorem and Busch's effect-measure theorem are established reconstruction results with different domains and hypotheses [14, 15]. The present contribution is the precise theorem in the form VERSF needs, its regularity-free proof, its application to the stated VERSF axioms, the identification of an explicit sufficient family of physical commitments behind the additivity axiom, and the resulting correction of the programme's dependency structure.

Figure 1 maps these routes and shows what each result uses.

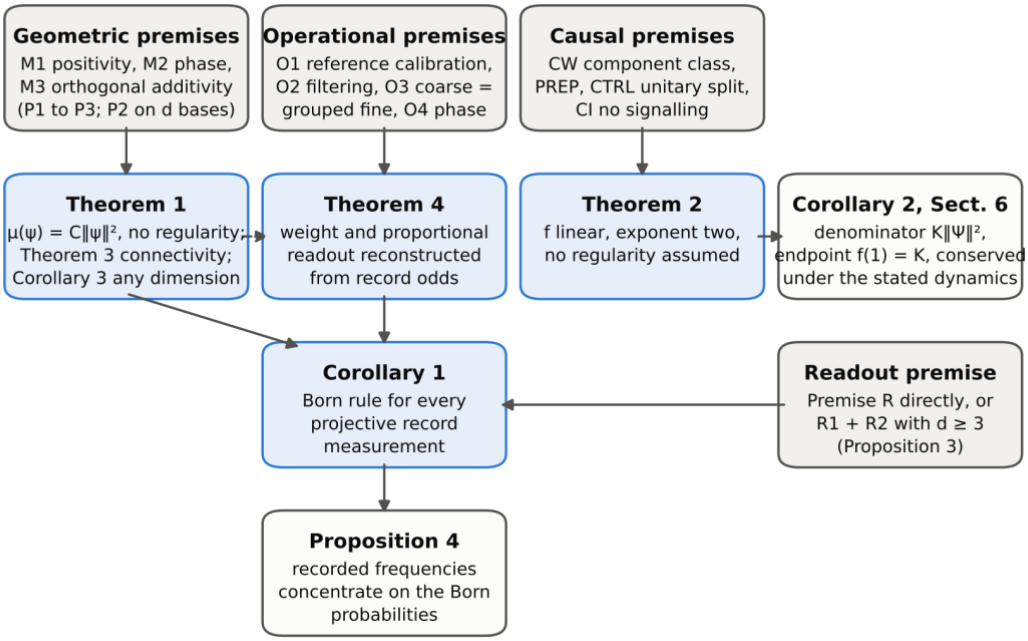



Figure 1. The routes of the paper and their dependencies. Three routes with explicitly stated premises lead to Theorem 1, Theorem 4 and Theorem 2; the operational route also invokes the geometric theorem. The Born rule (Corollary 1) is reached from Theorem 1 together with a readout premise, or from Theorem 4 directly; Theorem 1 determines the quadratic form of the weight that Theorem 4 defines operationally, while the reference calibration fixes its scale. Theorem 2 identifies the denominator, with Corollary 2's endpoint condition $f(1) = K$ and the stated dynamics needed for its conservation, and Proposition 4 connects the probabilities to recorded frequencies under independent repetition.

## 2 Geometric setting and measure premises

### 2.1 The carrier

Let $H$ be a complex Hilbert space of finite dimension $d \geq 2$, with inner product $\langle \cdot, \cdot \rangle$ linear in its second argument and conjugate-linear in its first. Its vectors describe pre-commitment configurations. The inner product and orthogonality are supplied geometrically, before an outcome probability is assigned. No identification of squared norm with physical packing volume is required anywhere in this paper.

### 2.2 Measure conditions

Let $\mu: H \to [0, \infty)$ be finite-valued and not identically zero. We impose three conditions.

**M1 Positivity and nontriviality.** $\mu(\psi) \geq 0$ for every $\psi$, and some vector has positive measure.

**M2 Global phase invariance.** For every real $\theta$ and every vector $\psi$,

$$\mu(e^{i\theta}\psi) = \mu(\psi). \tag{1}$$

This is a subgroup of the admissible-unitary invariance assumed in [1]. Only this subgroup is used in the uniqueness proof; invariance under every unitary is derived.

**M3 Global orthogonal additivity.** For every pair $u, v$ in $H$,

$$\langle u, v \rangle = 0 \implies \mu(u + v) = \mu(u) + \mu(v). \tag{2}$$

The word global fixes the quantifier: Equation (2) applies to all complex-orthogonal pairs, including vectors whose components occupy several substrate sectors. It is not restricted to the components of one preferred decomposition. Taking $u = v = 0$ gives $\mu(0) = 0$.

**No regularity condition.** We impose no continuity, measurability, monotonicity or boundedness condition on μ. Theorem 1 shows that continuity in the norm topology, and invariance under every unitary operator, follow from M1 to M3. This matters for the physical reading: a regularity axiom would be a statement about how the measure responds to small changes of a configuration, and no such statement has to be defended.

An overall calibration may subsequently set $\mu(\psi_{\mathrm{ref}}) = 1$ for one chosen unit vector. Theorem 1 shows that this fixes the entire measure scale. Thus the single-reference normalization in [1, A7] supplies both nontriviality and a choice of units; normalization on every unit vector is a consequence.

### 2.3 Operational premises behind M3

M3 is the principal physical commitment of the route, and its content should be stated in operational terms rather than left as a formal axiom. Three premises suffice.

**P1 Vector-intrinsic weight.** The raw weight of a pre-commitment configuration is a function μ of its vector alone. It does not depend on which record basis is used to describe the configuration, nor on which other alternatives appear in the description.

**P2 Fine-to-coarse consistency.** In any admissible fine record basis $e_1, \ldots, e_d$, the raw weight of a coarse alternative $E \subseteq \{1, \ldots, d\}$, whose branch is $P_E\psi = \Sigma_{i\in E}\, \langle e_i, \psi\rangle e_i$, equals the sum of the raw weights of its fine branches:

$$\mu(P_E\psi) = \Sigma_{i\in E}\, \mu(\langle e_i, \psi\rangle e_i). \tag{3}$$

For $E$ empty this reads $\mu(0) = 0$.

**P3 Record-basis admissibility.** Every orthonormal basis of $H$ is an admissible fine record basis.

**Lemma 1.** P1 to P3 imply M3.

**Proof.** Let $\langle u, v\rangle = 0$. If $u = 0$ or $v = 0$, Equation (2) reduces to $\mu(0) = 0$, which is Equation (3) with $E$ empty. Otherwise $u/\|u\|$ and $v/\|v\|$ are orthonormal; extend them to an orthonormal basis $e_1$, $e_2$, …, $e_d$, admissible by P3. For $\psi = u + v$ the fine branches are $\langle e_1, \psi\rangle e_1 = u$ and $\langle e_2, \psi\rangle e_2 = v$, and the coarse branch for $E = \{1, 2\}$ is $u + v$. Equation (3) gives $\mu(u + v) = \mu(u) + \mu(v)$. P1 guarantees that the function μ appearing in Equation (3) is the same for every basis, so the conclusion is a statement about the vectors alone. ■

P1 and P2 express what it means for μ to be the raw weight of an exclusive alternative. P3 is the premise with physical reach: it asserts that superposed alternatives are legitimate record alternatives in some context. Proposition 1 in Section 3 shows that P3 can be weakened to a short, explicit list of bases without loss.

## 3 Complete proof of the common quadratic measure

**Theorem 1.** Under M1 to M3 there is a unique constant $C > 0$ such that

$$\mu(\psi) = C\|\psi\|^2 \quad \text{for every } \psi \text{ in } H. \tag{4}$$

Conversely, every functional of this form satisfies M1 to M3; it is continuous and invariant under every unitary operator. A unit reference of measure one fixes $C = 1$.

**Lemma 2 Finite additivity.** If $w_1, \ldots, w_n$ are mutually orthogonal then $\mu(w_1 + \ldots + w_n) = \mu(w_1) + \ldots + \mu(w_n)$.

**Proof.** Induction on $n$: the partial sum $w_1 + \ldots + w_{n-1}$ is orthogonal to $w_n$, so Equation (2) applies. ■

### 3.1 Decomposition into coordinate weights

Choose any orthonormal basis $e_1, \ldots, e_d$ and define $f_j(r) = \mu(r\, e_j)$ for $r \geq 0$. Lemma 2 applied to the coordinate vectors, followed by M2 applied to each single vector $z_j e_j = |z_j| e^{i\theta} e_j$ (with $\theta$ the phase of $z_j$), gives

$$\mu(\Sigma_j\, z_j e_j) = \Sigma_j\, \mu(z_j e_j) = \Sigma_j\, f_j(|z_j|). \tag{5}$$

Each $f_j$ is nonnegative and satisfies $f_j(0) = 0$. Applying phase invariance separately when evaluating each summand does not assume that independent coordinate-phase controls are physically available: M2 is applied to one vector at a time.

### 3.2 Equality of all directional functions

Fix distinct indices $j$, $k$ and $r \geq 0$. The vectors $u = r(e_j + e_k)$ and $v = r(e_j - e_k)$ are orthogonal, and their sum is $2r\, e_j$. Equations (2) and (5) give

$$f_j(2r) = 2f_j(r) + 2f_k(r). \tag{6}$$

The pair $u, -v$ is also orthogonal and has sum $2r\, e_k$. Since Equation (5) evaluates $\mu(-v)$ and $\mu(v)$ through the same moduli,

$$f_k(2r) = 2f_j(r) + 2f_k(r) = f_j(2r). \tag{7}$$

Since $2r$ ranges over all nonnegative numbers, $f_j = f_k$. Every coordinate has the same function $f$. The argument uses any pair of orthogonal basis directions and therefore works in dimension two. It applies across different sectors as well as inside a sector because M3 is global. No regularity is used.

### 3.3 The common function is quadratic

For arbitrary $a, b \geq 0$ set

$$x = a\, e_1 + b\, e_2,\ y = ib\, e_1 - ia\, e_2,\ \langle x, y\rangle = a(ib) + b(-ia) = 0. \tag{8}$$

The sum is $x + y = (a + ib)e_1 + (b - ia)e_2$, and both coordinates have modulus $\sqrt{(a^2 + b^2)}$. Equation (5) gives $\mu(x + y) = 2f(\sqrt{(a^2 + b^2)})$ and $\mu(x) = \mu(y) = f(a) + f(b)$. Equation (2) therefore yields

$$f(\sqrt{(a^2 + b^2)}) = f(a) + f(b) \ \text{ for all } a, b \geq 0. \tag{9}$$

Define $h(s) = f(\sqrt{s})$ for $s \geq 0$. Equation (9) states that

$$h(s + t) = h(s) + h(t) \ \text{ for all } s, t \geq 0, \tag{10}$$

and $h \geq 0$ by M1.

**Lemma 3.** A nonnegative function $h$ on $[0, \infty)$ satisfying Equation (10) is of the form $h(s) = Cs$ with $C = h(1) \geq 0$.

**Proof.** For $0 \le s < t$, $h(t) = h(s) + h(t - s) \ge h(s)$, so $h$ is nondecreasing. Repeated use of Equation (10) gives $h(m/n) = (m/n)h(1)$ for all nonnegative integers $m$ and positive integers $n$. For irrational $s$ choose rationals $q_1 < s < q_2$; monotonicity gives $q_1 h(1) \le h(s) \le q_2 h(1)$, and letting $q_1$, $q_2 \to s$ gives $h(s) = s\,h(1)$. ■

Hence $f(r) = h(r^2) = Cr^2$ with $C = f(1) \ge 0$, and Equation (5) becomes Equation (4). If $C = 0$ the measure is identically zero, contrary to M1; hence $C > 0$. The constant is unique because $C = \mu(e)$ for any unit vector $e$. A unit reference of measure one gives $C = 1$.

For the converse, phase invariance of $C\|\psi\|^2$ is immediate, the Pythagorean identity proves orthogonal additivity, positivity and nontriviality follow from $C > 0$, and continuity and invariance under every unitary are properties of the squared norm. This completes the proof of Theorem 1. ■

### 3.4 The real-rotation route and the role of the complex structure

With the real pair $x = a\,e_1 + b\,e_2$, $y = b\,e_1 - a\,e_2$ in place of Equation (8), the same computation gives the Jordan–von Neumann quadratic equation

$$f(a + b) + f(|a - b|) = 2f(a) + 2f(b), \tag{11}$$

whose solutions are $f(r) = Cr^2$ under a continuity hypothesis [17, 20]. Without that hypothesis Equation (11) also admits nonnegative discontinuous solutions, for example $f = g^2$ with $g$ a discontinuous additive function on the reals. The complex pair of Equation (8) excludes them with no regularity assumption: at $a = b = 1$ it requires $f(\sqrt{2}) = 2f(1)$, and a discontinuous additive $g$ can be chosen with $g(\sqrt{2}) = g(1)$. In this proof the complex structure of the carrier does the regularity work: the quarter-phase pair turns the quadratic functional equation into the Cauchy equation (10), for which positivity alone is enough. A real carrier with additivity on all real-orthogonal pairs would also be regularity-free, by [18] together with evenness and Lemma 3, and directly by [21]; the complex pair is the instance used here, and it needs only one plane. For comparison, the general theory of orthogonally additive maps on real inner product spaces [18] yields, in general, an additional additive term that vanishes only for even maps; here M2, which includes $\mu(-\psi) = \mu(\psi)$, and the complex pairs leave the quadratic form as the only possibility.

### 3.5 An explicit sufficient family of decompositions

**Proposition 1 Instance form.** Fix one orthonormal basis $B = (e_1, \ldots, e_d)$. The conclusion of Theorem 1 holds if M1 and M2 hold and Equation (2) holds for the following pairs only:

- **I1 Coordinate additivity.** $\mu(\Sigma_j z_j e_j) = \Sigma_j \mu(z_j e_j)$ for all coefficients $z_j$.
- **I2 Balanced real rotations against the reference direction.** For each $k = 2, \ldots, d$ and every $r \ge 0$, the pairs $r(e_1 + e_k) \perp r(e_1 - e_k)$ and $r(e_1 + e_k) \perp -r(e_1 - e_k)$.
- **I3 Quarter-phase pairs in one plane.** For all $a, b \ge 0$, the pair $(a\,e_1 + b\,e_2) \perp (\mathrm{i}b\,e_1 - \mathrm{i}a\,e_2)$. This is a one-parameter family of decompositions of the plane spanned by $e_1$ and $e_2$, indexed by the ratio $a : b$, not a single pair.

**Proof.** Section 3.1 uses only I1 and M2. Section 3.2 with $j = 1$ uses only I2 and gives $f_k = f_1$ for every $k$. Section 3.3 uses only I3 in the plane spanned by $e_1$ and $e_2$. Nothing else is used. ■

The instance form can be restated in terms of admissible record bases, which is the form in which VERSF can most directly commit to it.

**Proposition 1B Basis form.** For $k = 2, \ldots, d$ let $B_k$ be the orthonormal basis obtained from $B$ by replacing $e_1$ and $e_k$ with $(e_1 + e_k)/\sqrt{2}$ and $(e_1 - e_k)/\sqrt{2}$. Assume M1 and M2. If P1 holds and P2 holds for the $d$ bases $B, B_2, \ldots, B_d$, then the conclusion of Theorem 1 holds. No separate quarter-phase assumption is needed.

**Proof.** P2 on $B$ gives I1. P2 on $B_k$ gives Equation (2) for every pair $u = \alpha(e_1 + e_k)$, $v = \beta(e_1 - e_k)$ with complex $\alpha, \beta$, because $u$ and $v$ are the two fine branches of $\psi = u + v$ in $B_k$; taking $\alpha = r$ and $\beta = \pm r$ gives I2, so by Section 3.2 all coordinate functions equal one function $f$. Now take $k = 2$, $\alpha = a$ and $\beta = ib$ with $a, b \geq 0$. Then $u + v = (a + ib)e_1 + (a - ib)e_2$, both coordinates of modulus $\sqrt{(a^2 + b^2)}$, and by I1 and M2, $\mu(u) = 2f(a)$, $\mu(v) = 2f(b)$ and $\mu(u + v) = 2f(\sqrt{(a^2 + b^2)})$. Equation (2) for this pair is Equation (9), and the proof of Theorem 1 proceeds unchanged from there. ■

The physical reading is direct. VERSF must recognize $d$ record bases: the reference basis and its $d - 1$ balanced rotations. Branch-local additive weights in each of them (P2), together with P1, is the entire physical input of Theorem 1 beyond M1 and M2. Every other basis, and every unitary, is covered by the conclusion without being assumed. Propositions 1 and 1B assert sufficiency: they list an explicit family that works. They do not assert that this family is the weakest possible.

### 3.6 Sharpness of the hypotheses

**Proposition 2.**

- (a) In complex dimension one, $\mu(z) = |z|^4$ satisfies M1 to M3, because the only orthogonal pairs involve the zero vector. Dimension at least two is essential.
- (b) If M3 is removed, $\mu(\psi) = \|\psi\|^4$ satisfies M1, M2 and continuity in every dimension. Orthogonal additivity is essential.
- (c) Let $H = \bigoplus_s H_s$ be a sector decomposition with projectors $P_s$, and suppose Equation (2) is required only for sectorwise-orthogonal pairs, those with $\langle P_s u, P_s v\rangle = 0$ for every $s$. Then for arbitrary positive constants $C_s$ the sector-weighted measure $\mu(\psi) = \Sigma_s C_s \|P_s\psi\|^2$ satisfies M1, M2 and the restricted condition. Sector-restricted additivity does not fix the sector coefficients.

**Proof of (c).** For a sectorwise-orthogonal pair,

$$\mu(u + v) = \Sigma_s C_s(\|P_s u\|^2 + \|P_s v\|^2 + 2\,\mathrm{Re}\langle P_s u, P_s v\rangle) = \mu(u) + \mu(v), \tag{12}$$

since every cross term vanishes. Positivity and phase invariance are evident. ■

The same measure violates the cross-sector instance I2. With two one-dimensional sectors, $C_1 = 1$, $C_2 = 2$, $u = (1, 1)$ and $v = (1, -1)$,

$$\mu(u + v) = \mu(2, 0) = 4, \;\; \mu(u) + \mu(v) = 3 + 3 = 6. \tag{13}$$

So some cross-sector input is necessary for the uniform conclusion; sectorwise additivity alone does not supply it. Proposition 2(c) does not show that interference pairs are the only possible cross-sector input (a postulated symmetry relating sectors would be another); it shows that the input cannot be omitted, and I2 is the form in which this paper supplies it. Theorem 3 makes the quantity of input exact within the class of balanced rotations.

### 3.7 How much cross-sector input is needed

Proposition 2(c) shows that some cross-sector input is required, and Proposition 1B shows that $d - 1$ balanced rotations supply it. Within the class of balanced two-direction rotations the gap between these statements can be closed exactly.

Let $B = (e_1, \ldots, e_d)$ be the reference basis. For distinct $j, k$ let $B_{jk}$ be the orthonormal basis obtained from $B$ by replacing $e_j$ and $e_k$ with $(e_j + e_k)/\sqrt{2}$ and $(e_j - e_k)/\sqrt{2}$. Let $A$ be a set of such bases and let $G_A$ be the graph on the index set $\{1, \ldots, d\}$ with an edge $\{j, k\}$ for each $B_{jk}$ in $A$.

**Theorem 3 Connectivity criterion.** Assume M1, M2 and P1, and assume P2 for $B$ and for every basis in $A$. If $G_A$ is connected, then $\mu(\psi) = C\|\psi\|^2$ for a unique $C > 0$. If $G_A$ is disconnected, there are measures satisfying all of these assumptions that are not multiples of the squared norm. In particular at least $d - 1$ rotated bases are required, and any $d - 1$ of them forming a spanning tree suffice.

**Proof.** Connected case. P2 on $B$ gives I1, so Equation (5) holds with coordinate functions $f_j$. For each edge $\{j, k\}$, P2 on $B_{jk}$ gives Equation (2) for the pairs $r(e_j + e_k) \perp \pm r(e_j - e_k)$, and the argument of Section 3.2 gives $f_j = f_k$. Connectivity propagates this equality along paths, so every coordinate function equals one function $f$. Since $d \geq 2$ and $G_A$ is connected, $A$ contains at least one basis $B_{jk}$; P2 on it, applied to the pair $a(e_j + e_k) \perp ib(e_j - e_k)$, gives Equation (9) exactly as in the proof of Proposition 1B, and Lemma 3 completes the argument. Disconnected case. Let $S_1, \ldots, S_m$ with $m \geq 2$ be the vertex sets of the components, let $Q_c$ be the projector onto the span of $\{e_j : j \in S_c\}$, and choose distinct constants $C_c > 0$. Set $\mu(\psi) = \Sigma_c C_c\|Q_c\psi\|^2$. M1, M2 and P1 hold. Every fine branch of any $\psi$ in $B$, and every fine branch in any $B_{jk}$ with $\{j, k\}$ an edge (so $j$ and $k$ in one component), lies in a single component subspace, on which $\mu$ is $C_c$ times the squared norm. A coarse branch is an orthogonal sum of fine branches, so its weight is the sum of theirs by the Pythagorean identity within each component. Hence P2 holds on $B$ and on every basis in $A$, while $\mu$ is not a multiple of the squared norm. A connected graph on $d$ vertices has at least $d - 1$ edges, and a spanning tree has exactly $d - 1$. ■

Theorem 3 answers the minimality question within its class: among balanced-rotation record bases, the admitted edges must connect every coordinate direction to every other. It does not exclude other kinds of cross-sector input, such as a postulated symmetry between sectors or rotated bases at other angles, so the general minimality question remains open. Disconnection permits unequal quadratic coefficients on the connected components and can leave additional freedom on isolated coordinates. For example, with only $B$ and $B_{12}$ admitted in three dimensions, $\mu(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 + |z_3|^4$ satisfies the restricted premises. The theorem establishes the connectivity criterion; it does not classify every disconnected case as quadratic. Figure 2 shows the three cases.

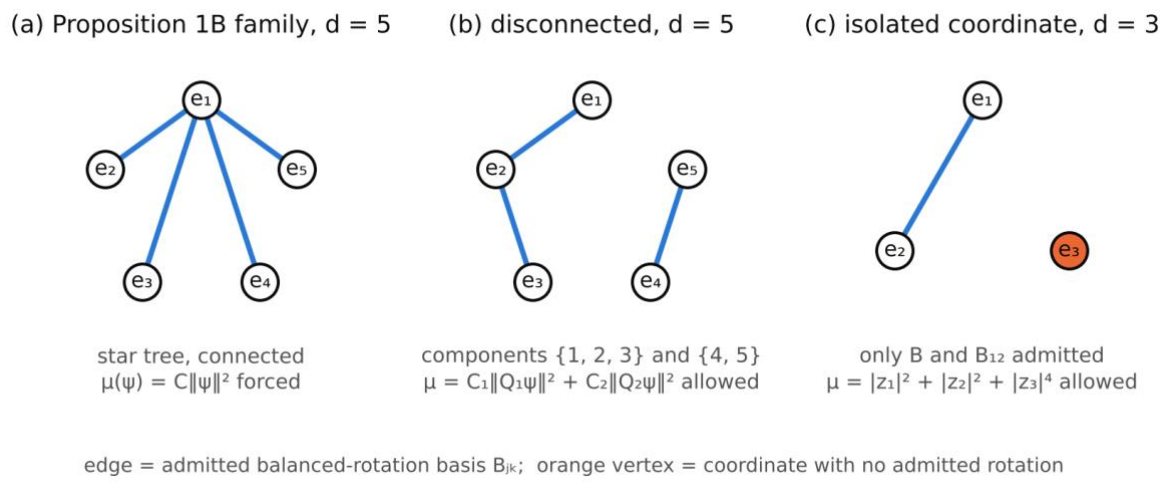



Figure 2. Rotation graphs for Theorem 3. Vertices are the coordinate directions of the reference basis, and an edge joins $e_j$ and $e_k$ when the balanced-rotation basis $B_{jk}$ is an admitted record basis. (a) The star tree of Proposition 1B is connected, so the measure is forced to be $C\|\psi\|^2$. (b) Two components allow two quadratic coefficients. (c) An isolated coordinate is constrained only by coordinate additivity and phase invariance, so a quartic weight survives there.

### 3.8 Carriers of arbitrary dimension

**Corollary 3.** Theorem 1 holds, with the same proof, on every complex inner product space of dimension at least two, finite or infinite, complete or not.

**Proof.** Fix a unit vector $e_0$ and set $C = \mu(e_0)$. For any $\psi$ choose a finite-dimensional subspace $V$ of dimension at least two containing $e_0$ and $\psi$. Every step of Sections 3.1 to 3.3 concerns finitely many vectors of $V$, and M2 and M3 hold on $V$ because they hold on the whole space. Those steps give $\mu(\varphi) = C_V\|\varphi\|^2$ for all $\varphi$ in $V$, with $C_V = \mu(e_0) = C \geq 0$. Hence $\mu(\psi) = C\|\psi\|^2$ for every $\psi$. If $C = 0$ then $\mu$ vanishes identically, contrary to M1, so $C > 0$. ■

The finite-dimensionality assumed in Section 2 is therefore a convenience, not a hypothesis. On a Hilbert space the Born rule for a countable complete family of orthogonal projectors follows from Corollary 1 below together with the Pythagorean identity $\Sigma_i \|P_i\psi\|^2 = \|\psi\|^2$, and for a projection-valued measure $E$ the probability $p(\Delta \mid \psi) = \|E(\Delta)\psi\|^2/\|\psi\|^2$ is countably additive because $E$ is. Continuous spectra require no new measure axiom. What remains model-specific is the operational reading of $E(\Delta)$ as a record, which is Premise R applied to the projectors $E(\Delta)$.

### 3.9 Consequences for the sector argument

Theorem 1 applies to the channel-adapted carrier of [1] whenever its A2 has the stated global meaning. A3 includes M2 and A7 fixes the nonzero scale; the regularity in A5 is not needed. All quadratic sector coefficients must then agree, including when every channel is one-dimensional. A separate channel-bridging axiom is not required, but bridging is not free either: Proposition 1 shows that the bridging content used by the proof is exactly the instances I2, Proposition 2(c) shows that without some cross-sector input the coefficients remain free, and Theorem 3 shows that the admitted rotations must connect the sectors, no more and no less.

If only a restricted collection of decompositions is physically admissible, the mathematical hypothesis must be restricted accordingly, and Proposition 2(c) then stands. The consolidated route adopts the global reading, through P1 to P3 or through the reduced basis form of Proposition 1B, as a stated physical commitment. It does not convert a restricted-sector axiom

into the global one silently; it names the additional instances and their operational meaning, and Section 8 discusses what VERSF must supply to justify them.

## 4 From the measure to the Born rule

The measure theorem and its measurement interpretation are separate, explicit steps. Section 4.1 states projected-branch readout and reduces it when three or more records are available. Section 4.2 supplies an alternative operational route to both the common weight and its probability interpretation. Section 4.3 addresses recorded frequencies.

### 4.1 Projected branch readout

**Premise R Projected-branch readout.** For a complete projective record measurement, the mutually exclusive record alternatives correspond to orthogonal projectors $P_i$ with $\Sigma_i P_i = I$. The branch associated with record $i$ is $P_i\psi$, and its probability is its share of the common measure. For $\psi \neq 0$,

$$p(i \mid \psi) = \mu(P_i\psi) \,/\, \mu(\psi). \tag{14}$$

By Lemma 2 the denominator equals $\Sigma_i\, \mu(P_i\psi)$, so Premise R is equivalent to the statement that probabilities are proportional to branch weights and sum to one over any complete set of records.

**Corollary 1.** Theorem 1 and Premise R imply the Born rule for every complete projective measurement:

$$p(i \mid \psi) = \|P_i\psi\|^2 \,/\, \|\psi\|^2. \tag{15}$$

For a normalized input and a rank-one outcome projector $P_i = |e_i\rangle\langle e_i|$,

$$p(i \mid \psi) = |\langle e_i, \psi\rangle|^2. \tag{16}$$

The constant $C$ cancels. Orthogonal additivity proves completeness and consistency under grouping of outcomes:

$$\Sigma_i\, p(i \mid \psi) = 1,\ \ p(E \mid \psi) = \Sigma_{i\in E}\, p(i \mid \psi),\ \ P_E = \Sigma_{i\in E}\, P_i. \tag{17}$$

The theorem also handles projectors of higher rank. Rank one is the geometry of a fine outcome, not a universal restriction imposed on every positive measurement operator. A record's probability depends on its projector and the preparation; changing the complementary projectors leaves it unchanged. This is the resulting probability noncontextuality for projective contexts.

Premise R can itself be reduced. Two weaker statements suffice when at least three records are available.

**R1 Weight dependence.** There is a function $g$: $[0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(x) > 0$ for $x > 0$ such that, for every complete projective record measurement,

$$p(i \mid \psi) = g(\mu(P_i\psi)) \,/\, \Sigma_j\, g(\mu(P_j\psi)). \tag{18}$$

**R2 Grouping consistency.** For every set $E$ of records, the probability of the coarse record $E$ in the measurement consisting of $P_E$ together with the $P_i$ for $i \notin E$ equals $\Sigma_{i\in E}\, p(i \mid \psi)$ in the fine measurement.

**Proposition 3 Readout reduction.** If $d \geq 3$, then Theorem 1, R1 and R2 imply Premise R.

**Proof.** Write $\mu_i = \mu(P_i\psi)$ and $\mu_E = \Sigma_{i\in E}\,\mu_i = \mu(P_E\psi)$, the last equality by Lemma 2. Let $S = \Sigma_{i\in E}\,g(\mu_i)$ and $T = \Sigma_{i\notin E}\,g(\mu_i)$. R1 for the fine and the coarse measurement, together with R2, gives $S/(S + T) = g(\mu_E)/(g(\mu_E) + T)$, hence

$$g(\mu_E) = \Sigma_{i\in E}\,g(\mu_i) \quad \text{whenever } T > 0. \tag{19}$$

Given $x, y \geq 0$, Theorem 1 allows the preparation $\psi = \sqrt{(x/C)}\,e_1 + \sqrt{(y/C)}\,e_2 + \sqrt{(1/C)}\,e_3$, with $\mu_1 = x$, $\mu_2 = y$ and $\mu_3 = 1$, so $T > 0$. Taking $E = \{1, 2\}$ in Equation (19) gives $g(x + y) = g(x) + g(y)$ for all $x$, $y \geq 0$. By Lemma 3, $g(x) = Kx$ with $K > 0$, and Equation (18) becomes Equation (14). ■

R1 says that a record's propensity depends on its branch only through the branch's weight, with one universal function; R2 says that declining to resolve a group of records preserves its probability. Both are noncontextuality statements of the kind used in [5], and neither fixes the form of Equation (14) in advance. In dimension two R2 is vacuous, so this route takes Premise R directly or embeds the qubit in a larger carrier. The reference-assisted route of Section 4.2 also covers a two-dimensional system, with its additional reference branch made explicit. The three-record restriction of Proposition 3 parallels Gleason's dimension restriction, although the hypotheses differ.

The dimension-two result uses vector-level additivity on arbitrary orthogonal vectors. Gleason's theorem concerns measures on projectors in dimension at least three [14], and Busch uses additivity on the larger set of effects [15]. These are distinct hypotheses. A frame function in dimension two assigns weights to projectors with no constraint linking different bases, which is why Gleason's theorem fails there; M3 links the weights of different vectors across all bases through the single function μ, which is why the qubit case is covered here. The qubit result does not contradict the two-dimensional limitation of the projector-only theorem.

### 4.2 Weights from calibrated record probabilities

The following construction gives the common weight a direct operational meaning and derives proportional readout at the same time. It uses probabilities of recorded alternatives, but assumes no formula relating those probabilities to squared amplitudes. The representation by stable probability ratios has a classical antecedent in Luce's choice axiom [22]; the additional content here is the orthogonal branch geometry and its coarse-record compatibility. No priority claim for the general ratio representation is made.

**O1 Record representation and reference calibration.** Let $H$ be the complex inner product space of the measure theorem, with dimension at least two. A finite record experiment is represented by mutually orthogonal nonzero branch vectors $S = (v_1, \ldots, v_n)$, prepared as the ray of $\Sigma_i\, v_i$, together with a specified ideal readout. The probabilities $p_S(v_i)$ are positive and sum to one; zero branches have probability zero and may be omitted. Physically equivalent realizations of the same branch list and readout have the same probabilities. This is an operational equivalence premise, including the preparation and apparatus conditions, not a claim about arbitrary detector implementations. A common nonzero reference branch $r$, orthogonal to $H$ in an enlarged carrier $H \oplus \mathbb{C}r$, is available. The calibration experiments $(v, r)$, the augmented experiments $S^+ = (v_1, \ldots, v_n, r)$, and the restrictions and coarse experiments used below are admissible. The reference realization is fixed across comparisons. Branch lists are unnormalized representatives; a common nonzero scalar changes only the representative of their prepared ray.

**O2 Consistency under ideal filtering.** For any nonempty retained sublist $A \subseteq S$, including the augmented lists in O1, an ideal successful filter prepares the retained configuration with the same subsequent readout. Its outcome law agrees with conditioning the original experiment:

$$p_A(v_i) = p_S(v_i) \,/\, \Sigma_{j\in A}\, p_S(v_j),\ \ v_i \in A. \tag{20}$$

The right-hand side is ordinary conditioning; its equality to the separately prepared or filtered experiment on the left is the physical premise. It requires a compatible filter, preservation of the retained relative amplitudes, and unchanged readout response. It does not follow merely from discarding entries in an already completed data set.

**O3 Equivalence of coarse and fine records.** For a nonempty set $E$ of orthogonal branches, put $w_E = \Sigma_{i\in E}\, v_i$ and let $S/E$ replace those branches by the single coarse branch $w_E$. The directly recorded coarse experiment is operationally equivalent, for these event probabilities, to the fine experiment followed by grouping its labels:

$$p_{S/E}(w_E) = \Sigma_{i\in E}\, p_S(v_i). \tag{21}$$

Every ungrouped branch, including the reference when present, retains its probability. The identification of the coarse branch with $w_E$ is part of the geometric record correspondence. For a fixed completed record list, grouped counts add identically. O3 additionally equates that grouped event to a directly implemented coarse record. It applies to exclusive records, not to paths whose recombination changes interference at an outcome.

**O4 Phase invariance of calibration odds.** Replacing $v$ by $e^{i\theta}v$ in the calibration experiment ($v$, $r$) leaves its odds against the reference unchanged. This is the operational condition that supplies M2 for the reconstructed weight. The calibration readout distinguishes the two records without recombining them; the condition does not deny that their relative phase can be observed in a different, interference-sensitive experiment.

**Theorem 4 Operational reconstruction.** Under O1 to O4 there is a finite-valued positive weight $w$ on nonzero vectors of $H$, with $w(0) = 0$, that satisfies M1 to M3. For every admitted finite record experiment in $H$, its outcome probabilities are the normalized weights. Consequently $w(v) = C\|v\|^2$ for one $C > 0$, and the outcome law is the Born rule. The calibration fixes the scale of $w$; the scale cancels from the probabilities.

**Proof.** For $v \neq 0$, define its weight through the binary calibration:

$$w(v) = p_{(v,r)}(v) \,/\, p_{(v,r)}(r),\ \ w(0) = 0. \tag{22}$$

O1 makes this a single function of the calibrated branch configuration, finite and positive for $v \neq 0$. O4 gives phase invariance. For $S = (v_1, \ldots, v_n)$ write $q_i = p_{S+}(v_i)$ and $q_r = p_{S+}(r)$. Apply O2 to the sublist $(v_i, r)$. Taking the ratio of its two conditional probabilities gives

$$q_i \,/\, q_r = w(v_i),\ \ q_r = 1 \,/\, (1 + \Sigma_j\, w(v_j)). \tag{23}$$

Apply O2 again, now retaining $S$ and excluding the reference. It follows that

$$p_S(v_i) = q_i \,/\, \Sigma_j\, q_j = w(v_i) \,/\, \Sigma_j\, w(v_j). \tag{24}$$

This derives proportional readout before specifying the amplitude dependence of $w$. For nonzero orthogonal $u$, $v$ in $H$, compare the fine experiment ($u$, $v$, $r$) with the coarse experiment ($u + v$, $r$). By O3 the coarse numerator is $q_u + q_v$ and its reference denominator is $q_r$. Hence

$$w(u + v) = (q_u + q_v) \,/\, q_r = w(u) + w(v). \tag{25}$$

Pairs containing zero satisfy the same identity by definition. Thus M1, M2 and M3 hold for $w$. Theorem 1, or Corollary 3 in arbitrary dimension, supplies the unique $C > 0$. Substitution into the derived readout gives

$$w(v) = C\|v\|^2, \; p_S(v_i) = \|v_i\|^2 \,/\, \|\Sigma_j \, v_j\|^2. \tag{26}$$

For a complete projective record measurement take $v_i = P_i\psi$ and omit zero branches. This is Equation (15). No preassigned raw measure, universal function $g(\mu_i)$, continuity hypothesis or Born trace rule was used. The probabilities and the operational consistency conditions O1 to O4 were used explicitly. ■

**Scope of the reference and the bases.** A two-dimensional system is covered through its reference-assisted experiments in an enlarged carrier; the argument does not claim to extract this result from an isolated binary probability table. In finite dimensions, the measure reconstruction can use P2 only on the reference basis and a connected collection of the balanced rotations in Theorem 3, with the corresponding augmented calibration and coarse experiments. O3 and the ratio formula then give P2 for $w$ on those bases, and Theorem 3 applies. Applying the derived readout to any additional measurement still requires O1 and O2 for that measurement. The finite-basis reduction of the measure proof does not certify every apparatus or operation as physically available.

**Relation to the other readout premises.** Under O1's positivity and common-reference assumptions, and given the availability of the comparison experiments it names, O2 yields a context-independent score whose normalized values give the outcome probabilities; this is the content of Luce's choice axiom and its representation theorem [22]. O2 therefore plays the role that R1 plays in Section 4.1, with the score defined operationally through calibration odds rather than postulated. The content that neither R1 nor O2 contains is additivity of the score, which O3 supplies here and R2 supplies there. Theorem 4 does not reduce the number of premises; it replaces a postulated raw measure with an operationally defined one.

**A non-Born rule that tests the distinction.** For a family of branch scores $\|v_i\|^{2q}$ with $q > 0$, conditional restriction and phase invariance hold for every $q$. Thus O2 and O4 do not insert the Born exponent. For $q = 2$, take three mutually orthogonal unit branches $u, v, r$. The fine experiment gives each probability $1/3$, whereas direct coarse recording of $u + v$ gives

$$p_{(u,v,r)}(u) + p_{(u,v,r)}(v) = 2/3, \; p_{(u+v,r)}(u + v) = 4/5. \tag{27}$$

The disagreement violates O3. It is an observable failure of coarse-record equivalence, not a failure to normalize either experiment. More generally, O3 forces addition of the calibrated weights, and the geometric theorem fixes their squared-amplitude form.

**Operational tests and their boundary.** The conditions can be tested by comparing a branch's odds against the same reference before and after ideal filtering, and by comparing directly recorded coarse probabilities with grouped fine probabilities while holding the preparation and response calibration fixed. The connected basis family specifies which cross-direction comparisons suffice in finite dimensions. Finite data test these conditions within uncertainty; they do not establish their exact universal form. Deriving their implementation from VERSF's microscopic record dynamics is an additional physical justification, not a step supplied by the algebra above.

### 4.3 From probabilities to recorded frequencies

An outcome law and a statement about repeated counts are distinct. The following standard statistical result makes their connection explicit without adding another quantum weighting rule.

**Proposition 4 Frequency concentration.** Suppose the same preparation and measurement are repeated independently $N$ times, with the same outcome law $p_i$ on every trial. Let $N_i$ count the occurrences of outcome $i$. For every $\varepsilon > 0$,

$$\Pr(|N_i/N - p_i| \geq \varepsilon) \leq 2 \exp(-2N\varepsilon^2). \tag{28}$$

**Proof.** The indicators $X_k = 1$ when trial $k$ records outcome $i$ and $X_k = 0$ otherwise are independent, lie in [0, 1], and have mean $p_i$. Since $N_i = \Sigma_k X_k$, the two-sided Hoeffding inequality [23] gives the result. It also implies convergence of $N_i/N$ to $p_i$ almost surely: for each fixed positive rational $\varepsilon$ the tail bounds are summable, and the first Borel–Cantelli lemma applies. ■

For a single outcome and target failure probability $\delta$ in (0, 1), a sufficient number of repetitions is

$$N \geq \ln(2/\delta) \,/\, (2\varepsilon^2). \tag{29}$$

For example, $N$ = 18 445 is sufficient for a deviation below 0.01 with probability at least 0.95, under the proposition's assumptions. This is a conservative theoretical bound, not a report of experimental data. Simultaneous bounds for several outcomes require a multiple-outcome allowance. Correlated trials, drifting preparations and changing detector responses are outside this statement. A VERSF repeated-record model can establish the required stability and independence, but neither follows merely from the persistence of a committed fact.

## 5 Causal consistency and the exponent

Theorem 1 states a geometric measure condition. The supplements [5, 6] explain how probability consistency can constrain the numerical weights operationally. We restate the causal argument here, with its premises named and with the regularity assumption removed, so that the consolidated paper contains a complete alternative test of the exponent and the denominator.

**Premise CW Universal component-weight class.** For a complete joint record basis with amplitudes $c_i$, the outcome probabilities have the form

$$a_i = |c_i|^2, \;\; p_i = f(a_i) \,/\, \Sigma_j f(a_j), \;\; f(0) = 0, \;\; f(a) > 0 \text{ for } a > 0, \tag{30}$$

with one function $f$: [0, 1] → [0, ∞) used for every preparation and control considered. No continuity of $f$ is assumed. The inputs are geometrically normalized amplitudes, so $a_i \in [0, 1]$ and $\Sigma_i a_i = 1$.

**Premise PREP.** Alice holds two orthogonal states $|0\rangle$, $|1\rangle$ and Bob holds three orthogonal states $r$, $b_1$, $b_2$. For every $b$ in (0, 1) the preparation

$$|\Psi_b\rangle = \sqrt{(1 - b)}\; |0\rangle_A |r\rangle_B + \sqrt{b}\; |1\rangle_A |b_1\rangle_B \tag{31}$$

is available.

**Premise CTRL.** For every $t$ in [0, 1] Bob can apply the local reversible rotation, unitary in the Hilbert inner product,

$$U_t r = r, \;\; U_t b_1 = \sqrt{t}\, b_1 + \sqrt{(1 - t)}\, b_2, \;\; U_t b_2 = -\sqrt{(1 - t)}\, b_1 + \sqrt{t}\, b_2. \tag{32}$$

**Premise CI Causal independence.** Bob's freely chosen local control cannot change Alice's unconditioned frequencies outside their causal connection. Local record probabilities are sums of the corresponding joint probabilities. This is an independent physical premise, not a consequence obtained by assuming a Born trace rule.

The rotation (32) leaves Alice's reference branch unchanged while dividing the other branch coherently. No outcome is discarded; the complete product basis is the same before and after, with zero-weight components retained. Alice's outcome 0 has probabilities

$$p_0^{\text{before}} = f(1-b) \,/\, (f(1-b) + f(b)),\ \ p_0^{\text{after}} = f(1-b) \,/\, (f(1-b) + f(tb) + f((1-t)b)). \quad (33)$$

**Theorem 2.** Under CW, PREP, CTRL and CI there is a constant $K > 0$ with $f(a) = Ka$ for every $a$ in $[0, 1)$. Consequently, for every normalized complete preparation,

$$p_i = a_i \,/\, \Sigma_j\, a_j = a_i. \quad (34)$$

The value $f(1)$ never enters a probability: if some $a_i = 1$ then every other component vanishes and $p_i = 1$ whatever $f(1)$ is.

**Proof.** CI equates the two expressions in Equation (33). Their common numerator is positive, so their denominators agree:

$$f(b) = f(tb) + f((1-t)b) \ \text{ for all } b \text{ in } (0, 1),\ t \text{ in } [0, 1]. \quad (35)$$

For $x, y \geq 0$ with $0 < x + y < 1$ choose $b = x + y$ and $t = x/(x + y)$; Equation (35) gives $f(x + y) = f(x) + f(y)$ on that domain, and the case $x + y = 0$ holds because $f(0) = 0$.

**Lemma 4.** A nonnegative function $f$ on $[0, 1)$ with $f(x + y) = f(x) + f(y)$ whenever $x, y \geq 0$ and $x + y < 1$ is of the form $f(a) = Ka$ with $K = 2f(1/2)$.

**Proof.** For $0 \leq x < y < 1$, $f(y) = f(x) + f(y - x) \geq f(x)$, so $f$ is nondecreasing. For $n \geq 2$, the points $1/(2n)$, $2/(2n)$, …, $n/(2n)$ all lie below 1, so $f(1/2) = n\, f(1/(2n))$ and $f(1/n) = 2f(1/(2n))$; hence $f(1/n) = 2f(1/2)/n = K/n$. For $0 \leq m < n$, $f(m/n) = m\, f(1/n) = Km/n$. Monotonicity then gives $f(a) = Ka$ for every $a$ in $[0, 1)$ by squeezing between rationals. ■

Since $f(1/2) > 0$, $K > 0$. Equation (34) follows from Equation (30). ■

For the power family $f(a) = a^q$, $q > 0$, a balanced split in Equation (35) gives $2^{1-q} = 1$, so $q = 1$ and the amplitude exponent is two. For $q = 2$ and $b = t = 1/2$, Alice's probability changes from $1/2$ to $2/3$, a direct violation of CI. Figure 3 shows the test at these values.

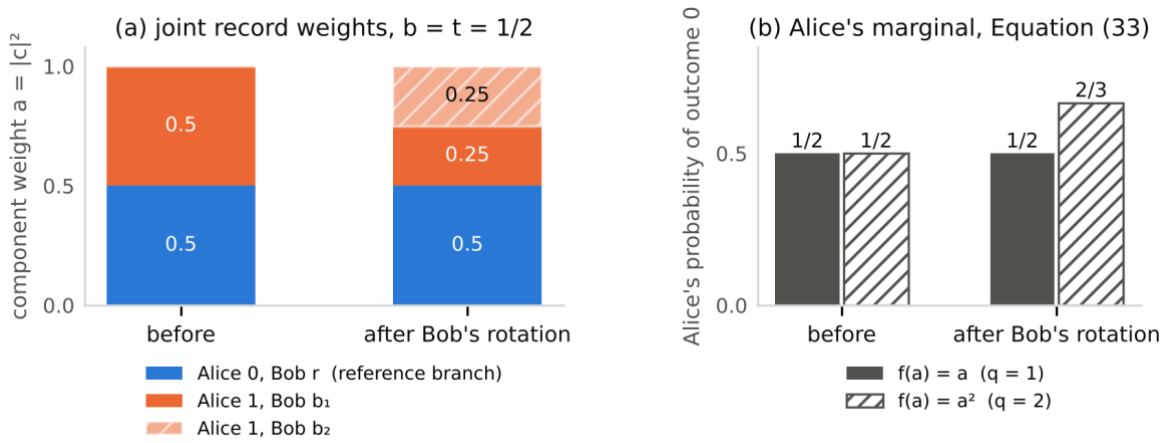



Figure 3. The remote splitting test of Theorem 2 at $b = t = 1/2$. (a) Bob's rotation divides the Alice-1 branch into two equal components while the reference branch is untouched, and the total weight is unchanged. (b) Alice's marginal probability of outcome 0 is 1/2 before and after under the linear rule; under the quadratic rule $f(a) = a^2$ it moves from 1/2 to 2/3, which would let Bob signal to Alice.

**Where the squared amplitude enters.** Theorem 2 does not assume the Born exponent, but it does inherit it from CTRL. The rotation (32) is unitary in the Hilbert inner product, so it preserves $\Sigma_i |c_i|^2$, and the theorem shows that probability must be linear in exactly that conserved quantity. Had the admissible reversible controls preserved some other $p$-norm, the argument could not run: for $p \neq 2$ the only norm-preserving linear maps of a finite-dimensional space are generalized permutations, so no continuous coherent splitting of the kind in Equation (32) exists [16, 19]. The squared-amplitude input of Theorem 2 is therefore the Hilbert-space geometry of reversible operations, exactly as the input of Theorem 1 is the Hilbert inner product. Neither route assumes the probability rule; both assume the carrier.

Theorem 2 addresses the universal component-weight class and the specified accessible controls. It is not an exclusion theorem for every imaginable modification of quantum dynamics. It complements Theorem 1 without reusing M3 as a premise. The general connection between altered weighting, normalization and signalling has an established literature [16]. Here it supplies a concrete physical reason for consistency of the VERSF denominator.

**Corollary 2.** Under the premises of Theorem 2, for every preparation with all $a_j < 1$ the denominator of Equation (30) is $\Sigma_j f(a_j) = K\|\Psi\|^2$: the normalizing total of the component rule is the quadratic total whose conservation is established in Section 6. Setting $f(1) = K$, a convention that changes no probability, extends the identity to every preparation. The convention is needed for conservation, not for probabilities: with $f(a) = a$ on $[0, 1)$ and $f(1) = 2$, every probability is still Born, but the denominator is 2 for weights (1, 0) and 1 for weights (1/2, 1/2), and a unitary connects those two preparations.

## 6 The conserved denominator

Theorem 1 selects the canonical total measure $Q(\psi) = C\|\psi\|^2$. Any unitary transport $U$ preserves it:

$$Q(U\psi) = C\langle U\psi, U\psi\rangle = C\langle\psi, \psi\rangle = Q(\psi). \tag{36}$$

For a differentiable evolution with self-adjoint generator $H(\tau)$ and $i\hbar\, d\psi/d\tau = H\psi$, where $\tau$ denotes the chosen evolution parameter,

$$d\langle\psi, \psi\rangle/d\tau = (i/\hbar)\langle\psi, (H^\dagger - H)\psi\rangle = 0. \tag{37}$$

This is an invariant of the specified reversible dynamics, defined before calling its components probabilities. It is not obtained by renormalizing the state after each step. In an orthonormal basis the component transfer has the form

$$da_i/d\tau = \Sigma_j J_{ij},\ J_{ij} = (2/\hbar)\ \mathrm{Im}(c_i^* H_{ij} c_j),\ J_{ji} = -J_{ij},\ J_{ii} = 0. \tag{38}$$

Thus individual component weights can change while their total is conserved. This is the pooled-weight structure discussed in The Bath Criterion [12]. That paper's unitary premise is compatible with the present result; it is not needed to prove Theorem 1.

The numerical scale $C$ and the choice of a unit-norm representative are conventional. Multiplying every numerator and denominator by an arbitrary common positive function leaves normalized probabilities unchanged; an arbitrary expression written as a denominator is therefore not automatically a conserved physical quantity. Conservation here concerns the fixed, component-local quadratic representative selected by Theorem 1, identified by Corollary 2 on normalized preparations with its stated endpoint convention, and used consistently under transport [6].

Nor does Equation (37) by itself identify $Q$ with energy, a thermodynamic entropy, a literal count of substrate bits, or a particular packing volume. Such an identification requires a separate physical correspondence. The closed reversible interval is also distinct from postselection: conditioning on one record can change the retained branch norm while the complete outcome probabilities still sum to one.

## 7 Corrections to the earlier measure theoretic arguments

This section identifies the exact replacements supplied by the consolidated route. Corrections to an intermediate argument need not remove the physical motivation for the programme; their significance is determined by the conclusions that survive a valid proof.

**Sector coefficients.** The minimal-core insufficiency claim in [1, Lemma 6.2A] does not hold with the globally stated A2: Equation (13) shows that the proposed alternative measure already violates global additivity, and Sections 3.1 to 3.3 replace the insufficiency lemma and the bridging step with a direct proof of uniform quadraticity. Under a sector-restricted reading of A2 the insufficiency claim is correct, by Proposition 2(c). The two readings must not be mixed: the global reading is adopted here as a named commitment, and its bridging content is isolated in I2. Statements in [2] that inherit the insufficiency claim require the same change of scope.

**Multiplicativity and the exponent.** The multiplicative equation used in Admissibility Closure [4, Appendix A.1] allows power functions:

$$f(xy) = f(x)f(y),\ f(x) = x^q\ \ (q > 0). \tag{39}$$

It does not select $q = 1$ by itself. Theorem 1 supplies the missing geometric restriction through orthogonal additivity. Theorem 2 supplies an operational restriction through remote splitting and causal independence. Either proof states the extra content explicitly.

**Factorization and operator rank.** Product expectations factorize for arbitrary product operators. For example,

$$K_A = I_2/2,\ K_B = I_2/2,\ K_{AB} = K_A \otimes K_B = I_4/4. \tag{40}$$

This positive operator has rank four, and its expectation on every product vector equals the product of the two marginal expectations. Accordingly, the rank-one conclusion in [4, Appendix A.2] needs additional hypotheses. The consolidated Born-measure proof bypasses it. Rank-one projectors enter only when the measurement's fine outcomes have one-dimensional ranges, as in Equation (16).

**Raw weights and normalized probabilities.** Probability as Consistency [3, Theorem 1, Step 3] needs an additional condition when passing from invariant raw weights on untouched outcomes to an invariant total weight. Dividing arbitrary weights by their sum guarantees normalization in each context but does not guarantee that the sum itself is unchanged. With $q = 2$, the geometric weights (0.2, 0.3, 0.5) and (0.5, 0, 0.5) leave the last raw weight at 0.25 while changing its normalized probability from 25/38 to 1/2. The geometric supplement [5] uses probability noncontextuality to constrain this change; Equations (33) to (35) implement the corresponding physical test through causal independence, and Lemma 4 shows that no regularity assumption is needed for it.

**Entropic unfolding.** If maximum-caliber inference constrains the mean of $M_i = -\kappa \ln a_i + \lambda \Delta S_i$, its multiplier $\beta$ must be retained:

$$p_i \propto \exp(-\beta M_i) = a_i^{\beta\kappa} \exp(-\beta\lambda\Delta S_i). \tag{41}$$

The equal-cost exponent is $q = \beta\kappa$. Its selection requires a consistency argument such as [5] or Theorem 2. Alternatively, once Theorem 1 supplies the base measure $a_i$, minimizing the Kullback–Leibler divergence from that base measure at fixed expected cost gives

$$p_i = a_i \exp(-\gamma\Delta S_i) \,/\, \Sigma_j\, a_j \exp(-\gamma\Delta S_j). \tag{42}$$

This is a thermodynamic or inferential refinement of the selected base measure. Whether it describes a physical apparatus, a selected sample or a proposed fundamental deviation is an additional modelling question. The present paper establishes the ideal base measure and does not independently derive entropy-biased detector statistics [8].

## 8 Physical grounding and the wider programme

The principal physical commitment is M3, and Section 2.3 and Proposition 1 say precisely what it amounts to. In operational form it is P1 to P3: the weight of an alternative is a property of its vector; a coarse alternative weighs what its fine constituents weigh together; and superposed alternatives are legitimate record alternatives. In reduced form (Proposition 1B) it is admissibility, with branch-local additive weights, of $d$ record bases: the reference basis and its $d - 1$ balanced rotations. Theorem 3 fixes how many such bases are needed: enough to connect every coordinate direction to every other, and no fewer. Everything else in the measure layer is derived, including continuity of the measure, its invariance under every unitary, and its form in every dimension (Corollary 3). The geometric theorem alone establishes the form of the weight. The Born probabilities follow when it is combined with the readout conditions of Section 4.1 or with the operational reconstruction of Section 4.2; for three or more records the readout premise itself reduces to weight dependence and grouping consistency (Proposition 3).

The operational reconstruction gives these commitments a second formulation. Theorem 4 derives an intrinsic additive weight and proportional readout from O1 to O4, so its route postulates no raw measure and defines operationally the score that R1 postulates. Its physical content lies in the availability and equivalence of the calibration, filtering and grouping

experiments. The ideal filter must preserve the retained branch profile and the response of the readout. Direct coarse recording must agree with grouping the corresponding exclusive fine records. In [2, Remark 3.3.3], admissibility is explicitly inherited rather than generated by that paper; the required operations must therefore be supplied by the relevant VERSF construction. The theorem provides precise obligations and tests, without claiming that ordinary probability identities alone establish their physical realizability.

To apply the theorem, VERSF must therefore recognize the $d - 1$ balanced-rotation bases $B_k$ as admissible record bases alongside the reference basis. A decomposition may be a legitimate way of representing record alternatives without being a physically available transport between substrate sectors; those two notions must be distinguished. The theorem does not assume that every mathematical unitary is dynamically realizable, and Proposition 1B shows that it needs only $d$ of them. The causal route, in contrast, names the actual controls it needs in Equation (32), and Section 5 shows that those controls are where its squared-amplitude input resides.

Operational Distinguishability Geometry [2] makes the physical interpretation of labels and equivalence explicit. Its account is the natural place to argue that the alternatives $r(e_1 \pm e_k)$ are operationally distinguishable records in some context, which is what admissibility of $B_k$ asserts. A claimed equivalence must be supported by that content; simply defining two physically different arrangements to be equivalent would not establish a probability law. Likewise, finite resolution alone supplies neither all of M3 nor a quantitative identification of norm with packing volume. The latter is explicitly separated as an open identification in [1, Section 9 and Open Problem 10].

State sufficiency motivates the identification of equivalent record descriptions, but its application must include the relevant preparation, apparatus and environmental information. A relabelling of one experiment and a physical change of detector are different operations. An assertion that all observables depend on the committed record does not, by itself, show that changing an apparatus context leaves calibration odds unchanged: the apparatus can be part of that record. O1 to O3 identify the equivalences actually required. Likewise, a downstream instrument that already uses the Born kernel can illustrate these conditions but cannot independently derive that same kernel.

The phase papers [10, 11] address the upstream carrier. The continuous-phase paper distinguishes the completed phase group from the assignment of a phase to each individual path. Theorem 1 starts with the complex carrier and its phase equivalence; it makes no inference from the existence of U(1) to uniqueness of a path connection. Section 3.4 shows that a quarter-phase pair in the rotated basis $B_2$, available only because the carrier is complex, is the instance this proof uses to remove the need for a regularity axiom. Regularity-free uniqueness is also known on real inner product spaces under full orthogonal additivity [21], so the complex structure enables this short proof rather than being necessary for the result. This keeps the probability proof usable while those upstream constructions are developed.

The K7 closure-Born-TPB construction [13] addresses how allowed records, conditional outcome weights and event times can operate together. Its record-preserving renewal construction illustrates a dynamical use of the Born weights. It does not add an independent exponent theorem, since the conditional Born kernel is among its inputs. This separation allows the measure proof and the record dynamics to strengthen different parts of the programme.

The relationship among routes should therefore be stated by their actual dependencies. The common-measure account in [1, Section 8] is compatible with the present consolidation: geometric, entropic and dynamical descriptions can share one established measure. The stronger independence claims in [9] require their own demonstrations. Explanatory agreement remains useful without counting a reformulation or a downstream application as another independent proof.

## 9 Premise and dependency ledger

Table 1 records what each result uses and what it does not use. The second list is as important as the first: a reader auditing the route should be able to see that no hidden regularity, no sector-bridging postulate, no rank-one factorization and no Born trace rule enters any proof.

| Result | Uses | Does not use |
|---|---|---|
| Lemma 1 (M3 from operational premises) | P1, P2, P3 | Any regularity condition; any dynamics |
| Theorem 1 (common quadratic measure) | M1, M2, M3; of M3 only the instances I1, I2, I3 (Proposition 1), equivalently P1 and P2 on d record bases (Proposition 1B) | Continuity, measurability or boundedness; unitary invariance beyond global phase; a channel-bridging axiom; rank-one factorization; Gleason's or Busch's theorem; any probability premise |
| Proposition 2 (sharpness) | Explicit counterexamples, verified algebraically | Nothing beyond direct verification |
| Theorem 3 (connectivity criterion) | M1, M2, P1; P2 on the reference basis and the admitted balanced rotations | Continuity; any other basis; minimality outside the balanced-rotation class |
| Corollary 3 (arbitrary dimension) | Theorem 1 on finite-dimensional subspaces | Completeness of the carrier; continuity; countable additivity as a hypothesis |
| Corollary 1 (Born rule for projective records) | Theorem 1; Premise R (or R1 and R2 when d is at least three, by Proposition 3); alternatively the reconstructed weight and readout of Theorem 4 | Rank-one outcomes; dimension at least three; any regularity |
| Proposition 3 (readout reduction) | Theorem 1; R1; R2; at least three records | Premise R as a hypothesis; Gleason's theorem |
| Theorem 4 (operational reconstruction) | O1 to O4; Theorem 1 or Corollary 3; the reference-assisted experiments | A preassigned raw weight; Premise R or R1 as hypotheses; continuity; a Born trace rule |
| Proposition 4 (record frequencies) | A derived outcome law; independent trials with the same preparation and readout law; the Hoeffding inequality | An additional quantum weighting rule; independence inferred merely from record persistence |
| Theorem 2 (linear component weights) | CW, PREP, CTRL, CI | M3; continuity of f; the value f(1); a Born trace rule |
| Corollary 2 (denominator identification) | Theorem 2; additionally f(1) = K for the identity on every normalized preparation | Theorem 1 |
| Equations (36) to (38) (conserved total) | Theorem 1, to identify Q; unitary or self-adjoint dynamics | Probabilities; renormalization; identification of Q with energy, entropy or packing volume |

| Result | Uses | Does not use |
|---|---|---|
| Appendix A extensions | Corollary 1; classical mixing or isometric dilation premise | A unique instrument or its event timing |

*Table 1. Premise and dependency ledger.*

The route can be defeated only through its premises. Four defeaters are worth naming. First, withdrawing fine-to-coarse consistency from an admitted balanced-rotation basis removes its edge from the assumption graph of Theorem 3, and unequal component coefficients become possible exactly when that graph is disconnected. An observed violation on a basis whose edge is still needed falsifies at least one of the retained premises: it cannot coexist with assumptions that still form a connected graph, because those force the quadratic measure. Second, a physical reason to deny admissibility of the quarter-phase pair I3 would require, in its place, the full family of unbalanced real rotations $(a\ e_1 + b\ e_2) \perp (b\ e_1 - a\ e_2)$ of Section 3.4 together with a continuity hypothesis; under those two replacements the conclusion is unchanged, but regularity is again an assumption. The balanced pairs I2 alone would not suffice even with continuity, since $f(2r) = 4f(r)$ admits continuous non-quadratic solutions. Third, a failure of Premise R, that is, a record whose probability was not its branch's share of the common measure, would break Corollary 1 while leaving Theorem 1 intact; by Proposition 3 such a failure requires, for three or more records, either a propensity that depends on more than the branch weight or a violation of grouping consistency. For the operational route, a failure of proportional readout or of common additive weights would require a failure of O1 to O4 on the experiments used by Theorem 4. Fourth, an admissible reversible control that did not preserve the Hilbert inner product would detach Theorem 2 from Theorem 1; Section 5 notes that, for standard finite-dimensional $p$-norms with $p \neq 2$, norm-preserving linear transformations cannot implement the continuous coherent splitting considered here.

## 10 Conclusion

The VERSF probability-measure layer admits a complete conditional reconstruction. Positivity, global phase invariance and global orthogonal additivity force one common squared-norm measure on a finite complex Hilbert carrier of dimension at least two, with no regularity assumption; continuity and full unitary invariance are consequences. The proof uses additivity only on an explicitly listed sufficient family of decompositions, equivalent to admitting $d$ record bases; a sharpness proposition shows that some cross-sector input is necessary, and a connectivity theorem shows that within the class of balanced rotations the admitted bases must connect every direction to every other, which makes $d - 1$ the exact minimum. The theorem holds on carriers of every dimension. The projected-branch readout then gives the Born rule for complete projective measurements, including qubits and coarse outcomes, and for three or more records that readout is itself derived from weight dependence and grouping consistency. A complementary causal argument, also free of regularity assumptions, selects linear geometric component weights, locates its squared-amplitude input in the admissible-control premise, and makes denominator consistency operationally testable.

The consolidated proof closes the sector-coefficient, exponent and normalization gaps within this stated route and removes its dependence on an invalid rank-one inference. The physical grounding is concentrated in a short set of named commitments: the complex carrier, the admitted record bases of Proposition 1B, the correspondence between projected branches and

records, and, for the causal route, the Hilbert-space geometry of reversible controls. These commitments give the wider VERSF programme a precise theorem to support, apply and test, together with a ledger of exactly what would have to fail for it to fall.

The operational reconstruction reinforces the two links most directly exposed to physical scrutiny: a common additive raw weight across record bases, and its conversion into outcome probabilities. Under O1 to O4 the weight is fixed by reference odds, additivity follows from coarse-record equivalence, and proportional readout follows from filtering consistency. Theorem 1 then determines the exponent. These are proved consequences of stated operational conditions, whose realization is a task for the record dynamics. Under the separate repeated-preparation premise of Proposition 4, the resulting probabilities control the concentration of observed record frequencies.

## Appendix A Extensions and premise checks

**Classical mixtures.** Suppose a preparation is randomly selected from normalized states $\psi_k$ with ordinary classical probabilities $r_k$. With the law of total probability and $\rho = \Sigma_k r_k|\psi_k\rangle\langle\psi_k|$, Corollary 1 gives

$$p(i \mid \rho) = \Sigma_k r_k\langle\psi_k, P_i\psi_k\rangle = \mathrm{Tr}(\rho P_i). \tag{43}$$

The expression depends only on ρ, so equivalent mixture decompositions agree. This is an extension under classical mixing; it does not establish the microscopic dynamics that prepares a mixture.

**Finite effects through an isometric measurement model.** Let operators $M_i$ satisfy $\Sigma_i M_i^\dagger M_i = I$, and suppose the measurement admits the operational dilation

$$V\psi = \Sigma_i (M_i\psi) \otimes |i\rangle,\ V^\dagger V = I, \tag{44}$$

where the pointer states $|i\rangle$ are orthonormal records. Apply Corollary 1 to pointer projectors on the joint carrier. With $E_i = M_i^\dagger M_i$,

$$p_i = \|M_i\psi\|^2 / \|\psi\|^2 = \langle\psi, E_i\psi\rangle / \|\psi\|^2,\ p_i = \mathrm{Tr}(\rho E_i) \text{ for mixtures.} \tag{45}$$

Every finite positive effect family summing to the identity has a mathematical model of this form, for example $M_i = \sqrt{E_i}$. Identifying a particular physical apparatus with that model is an additional operational premise. The measure theorem does not select a unique instrument or its event timing.

**Checks on the hypotheses.** Proposition 2 records the three checks: dimension one admits $|z|^4$; removing M3 admits $\|\psi\|^4$ in every dimension; restricting M3 to sectorwise-orthogonal pairs admits unequal positive sector coefficients, which Equation (13) shows failing the cross-sector requirement. A fourth check concerns regularity: Section 3.4 shows that the nonnegative discontinuous solutions of the real-rotation equation (11) are excluded by the quarter-phase pair I3, so no regularity axiom is needed and none has been smuggled in.

**Exact algebraic checks.** The inner products in Equations (8), (12) and the real pair of Section 3.4 vanish identically. The quadratic function satisfies Equations (9) and (11). Replacing it with a fourth power produces the residuals

$$(a^2 + b^2)^2 - a^4 - b^4 = 2a^2b^2,\ (a + b)^4 + (a - b)^4 - 2a^4 - 2b^4 = 12a^2b^2, \tag{46}$$

both positive for nonzero $a$, $b$. These checks illustrate the exclusions; the universal statements follow from the proofs, not from numerical sampling. Infinite-dimensional carriers and continuous spectra are covered by Corollary 3; approximate consistency requires a separate extension.

**Operational checks.** The restriction identity (20) holds for every positive normalized component-score rule, so it does not select the exponent by itself. Equation (27) exhibits a fourth-power rule that passes restriction and phase checks but fails coarse-record equivalence. Theorem 4 uses both conditions, and its proof establishes the universal result without numerical fitting. Proposition 4 concerns repeated trials with the same probability law and independence; correlated or drifting preparations require their own frequency analysis.

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